



Duality as a guiding light in the genesis of Dedekind's Dualgruppen

Emmylou Haffner

► To cite this version:

Emmylou Haffner. Duality as a guiding light in the genesis of Dedekind's Dualgruppen. Ralf Krömer; Emmylou Haffner. Duality in 19th and 20th century mathematical thinking, Birkhäuser, inPress. hal-04352674

HAL Id: hal-04352674

<https://hal.science/hal-04352674>

Submitted on 19 Dec 2023

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

Chapter 4

Duality as a guiding light in the genesis of Dedekind's *Dualgruppen*

Emmylou Haffner

Postprint

Please do not cite this version

4.1 Introduction

4.1.1 The initial *impetus*

In (Dedekind, 1897b), Richard Dedekind recalled how he was led to the notion of *Dualgruppe*, formally equivalent to our lattices, which he introduced there for the first time:

For many years, I have been busy with these questions, but I was compelled to do so not by logic but by the theory of these systems of numbers which I call modules. Through the efforts to obtain this theory from the smallest number of fundamental laws, not without great difficulties, I recognized the [defining properties of *Dualgruppen*] (...) ¹ (Dedekind, 1897b, p. 113)

To understand what he meant by this, one has to go back to his famous 1871 *Supplement X* to Lejeune-Dirichlet's *Vorlesungen über Zahlentheorie*, ² in which

¹Unless stated otherwise, all translations are mine.

²(Dedekind, 1871), (Lejeune-Dirichlet, First edition 1863, re-ed. 1871, 1879, 1894)

he introduced the notion of module, a “system \mathfrak{a} of real or complex numbers α whose sums and differences themselves belong to \mathfrak{a} ” (Dedekind, 1871, p. 42).³ With this new notion, he also defined notions of divisibility (a module \mathfrak{a} is divisible by a module, \mathfrak{b} if $\mathfrak{a} \subset \mathfrak{b}$) and related arithmetical notions for modules, among which the least common multiple (LCM) of two modules \mathfrak{a} and \mathfrak{b} , which is $\mathfrak{a} \cap \mathfrak{b}$, and the greatest common divisor (GCD) of two modules \mathfrak{a} and \mathfrak{b} , which is the module composed by all the numbers $\alpha + \beta$ with α and β respectively running through all numbers of \mathfrak{a} and \mathfrak{b} . These notions can be applied to ideals as well, which is the basis of Dedekind's proof of the validity of the unique factorization theorem for algebraic number fields.

In (Dedekind, 1877), Dedekind introduced notations for these notions: \mathfrak{a} divides \mathfrak{b} is denoted by $\mathfrak{a} < \mathfrak{b}$ or $\mathfrak{b} > \mathfrak{a}$; the GCD of \mathfrak{a} and \mathfrak{b} by $\mathfrak{a} + \mathfrak{b}$; and their LCM by $\mathfrak{a} - \mathfrak{b}$. These notations facilitated the expression and proof of new theorems such as, for three modules $\mathfrak{a}, \mathfrak{b}, \mathfrak{c}$ with $\mathfrak{a} < \mathfrak{b}$ (Dedekind, 1877, p. 121):

$$\begin{aligned}(\mathfrak{a} + \mathfrak{b}) - (\mathfrak{a} + \mathfrak{c}) &= \mathfrak{a} + (\mathfrak{b} - (\mathfrak{a} + \mathfrak{c})) \\ (\mathfrak{a} - \mathfrak{b}) + (\mathfrak{a} - \mathfrak{c}) &= \mathfrak{a} - (\mathfrak{b} + (\mathfrak{a} - \mathfrak{c})).\end{aligned}$$

which correspond to what is now called the “modular law” in lattice theory.

Dedekind noted that these “characteristic theorems” for modules display a “dualism that holds throughout for the notions of GCD and LCM” (Dedekind, 1877, p. 121), that is, any true formula expressed in terms of $+$ and $-$ can be transformed into another true formula by switching these symbols.

Dedekind published a new version of his algebraic number theory in the fourth edition of Dirichlet's *Vorlesungen über Zahlentheorie* (Dedekind, 1894a), developing further the study of GCD and LCM of modules, and stating that

Between the concepts of greatest common divisor and least common multiple of arbitrary modules there exists a peculiar dualism, whose *ultimate* reason might be hard to recognize. (Dedekind, 1894a, p. 498)

Dedekind's interest in this “peculiar dualism” prompted him to investigate the laws obeyed by the operations defined between modules. This eventually led to the introduction of *Dualgruppen*.⁴

In footnotes, in (Dedekind, 1894a), Dedekind mentioned several times that he was working on a notion named *Modulgruppe*, which corresponds to a special case of *Dualgruppen*.⁵ Dedekind presented his *Dualgruppen* theory in (Dedekind,

³Dedekind's modules are \mathbb{Z} -modules. They were introduced as an extension of congruences.

⁴Although a *Dualgruppe* is formally equivalent to a lattice, the two concepts were developed independently, and are in fact very different in their conception and uses. Birkhoff's first works on lattices are independent of Dedekind's *Dualgruppen*. It was Øystein Ore, who at the time was editing Dedekind's *Gesammelte Werke*, who brought these papers to Birkhoff's attention (Birkhoff, 1934, p. 200). I will use Dedekind's terminology throughout this chapter. In general, I will keep Dedekind's vocabulary. In particular, I will follow his use of the word “group”, which he used with a relatively wide and fluid meaning.

⁵According to (Mehrtens, 1979, p. 87), this suggests that Dedekind had already thoroughly studied the “structure” in question. It is important to emphasize, however, that this does not mean that he had already formed the *general* definition of a *Dualgruppe*.

1897b) and (Dedekind, 1900). This was, in his words, obtained “not without great effort” (Dedekind, 1897b, p. 113) and indeed after two decades of work.

As emphasized by (Mehrtens, 1979, p. 83), Dedekind’s *Nachlass* reveals that the “dualism” between the operations was a strong “stimulus” for his research on modules and toward *Dualgruppen*, as he wrote several texts explicitly on the subject. Observing this, and taking seriously Dedekind’s remarks on the origins of *Dualgruppen* and the years of protracted labor preceding its conception, I will dive into what will reveal to be a long and complex story, so as to understand the role played by duality in the genesis of *Dualgruppen*.

4.1.2 Some possible external influences regarding duality

Since it is an aim of this book to clarify to what extent various conceptions of duality influenced each other, let me mention two notable such conceptions with which Dedekind came in contact.

The only two explicit references made by Dedekind in the drafts we will study are Ernst Schröder’s *Vorlesungen über Algebra der Logik* (vol. I, 1890) and Frobenius and Stickelberger’s 1879 paper on groups.⁶ The latter is not related to duality *per se*. The first certainly is, but Dedekind’s interest in the duality of the operations in module theory started well before he read it. Reading the *Algebra der Logik* was likely important for Dedekind’s research, but rather for its “logical” aspect than for duality.

We know that Dedekind also read (Schröder, 1873) before 1876⁷, in which duality is briefly addressed.⁸ While we have no textual evidence of him reading (bo:Schroeder1877a), in which Schröder started to study duality related properties, it is not unlikely that he could have also read this treatise as well. In particular, Dedekind numbers equations using (α) and (α') to denote the dual property, just as Schröder did.⁹

Dedekind also regularly uses the strategy to write dual results in two columns. This format, introduced by Gergonne in 1825, was widespread among geometers such as Steiner or Chasles to display the striking parallelism between dual theorems in projective geometry.¹⁰ And Dedekind, as we learn from a letter to Klein quoted in (Ferreirós, 2008), taught geometry when he was *Privatdozent* in Göttingen in 1854–55:

Dedekind had chosen geometry as the subject for his first lecture

⁶(Frobenius and Stickelberger, 1879)

⁷(Lipschitz, 1986, p. 74), (Scheel, 2014, p. 214).

⁸See Dirk Schlimm’s chapter in this volume.

⁹I will come back to Dedekind’s reading of Schröder in 4.4.3. For more on Schröder and Dedekind, see (Mehrtens, 1979, pp. 114–123). — As far as duality in 19th century logic is concerned, let me mention that Dedekind studied some of Boole’s work on probability theory (Boole, 1854b; Dedekind, 1855), and might have read (Boole, 1862) (as Boole, in a 1862 letter to Borchardt asks him to talk to Dedekind about it (Dugac, 1976, p. 222)). This does not however necessarily imply that he read either of Boole’s 1847 or 1854 treatises.

¹⁰Any theorem can be transformed into another one by exchanging the words “point” and “line” in plane projective geometry, or “point” and “plane” in spatial projective geometry. See also (EtwF2019).

course at Göttingen in 1854/55, when, as he said in a letter to Klein, he “made an effort to establish a parallelism between the modern analytic and synthetic methods” (Lorey, 1916, p. 82). To prepare that course, he borrowed from the Göttingen library Steiner’s book, together with works by Chasles, Plücker [1828 and 1835], and the *barycentrische Calcul* of Möbius. (Ferreirós, 2008, p. 24)

Dedekind’s *Nachlass* contains several sets of notes dated from the 1850s and 1860s on geometry, ideal geometry, *Geometrie der Lage*, suggesting additional reflections on the subject.

4.1.3 Dedekind’s drafts and the genesis of *Dualgruppen*

This chapter will concentrate on the place, role and status of duality in the invention of the concept of *Dualgruppe*. Its goal is to follow the slow and progressive march from the operations defined between modules in the mid-1870s to the introduction of a brand new and very general concept. It will thus be an exploration of the conceptual and textual genesis of the concept, which will spread over twenty years of drafts — and only drafts. These drafts also give us textual evidence to unfold the long, complex path followed by Dedekind, and show how duality played the role of a leading thread, of a guiding light, in the conception of *Dualgruppen*.

Description of the archive

We are lucky that in Richard Dedekind’s *Nachlass*, kept at the Niedersächsische Staats- und Universitätsbibliothek in Göttingen, nine files contain documents related to *Dualgruppen*: Cod. Ms. R. Dedekind III 14, X 9, X 10, X 11-1, X 11-2, XI 1, XI 2, XI 3-1 and XI 3-2. They contain a total of 568 pages. In addition, Cod. Ms. R. Dedekind III 30 contains 8 pages of reading notes on (Schröder, 1890).¹¹

The dating of these documents is difficult, as Dedekind himself rarely gives such indications on his drafts. Time intervals can be given using references to other works (in particular, Dirichlet’s *Vorlesungen* and Dedekind’s own works, occasionally Schröder’s or Frobenius’), or indications given by the sheets on which Dedekind was writing. Indeed, he often wrote on the back of advertisements, invoices, administrative papers, letters, or drafts of his own (published) papers. However, a discrepancy of up to ten years can be observed, for example on manuscripts containing both a reference to a published work and the date of an invoice, which prompts caution in dating. Other contentual aspects, such as results or notations, can help to determine an order for the manuscripts.

The establishment of Dedekind’s *Nachlass* adds to the difficulty: as explained at the beginning of the archives’ catalog¹², Dedekind’s *Nachlass* was

¹¹Some notes can also be found in Cod. Ms. R. Dedekind III 14 and XI 3-2.

¹²<http://hans.sub.uni-goettingen.de/nachlaesse/Dedekind.pdf>

donated to the Göttingen archive by Dedekind's heirs in 1931 (fifteen years after Dedekind's death) and there is no certainty that Dedekind himself put the folios together in each file. Despite the seemingly thematic organization of the *Nachlass*, some boxes seem to contain unrelated documents. In addition, many of the files appear to be very fragmentary. Hence, while it is evidence possible to date with a relative precision some of the folios, it cannot be systematically applied to those surrounding them.

Thus, the reference codes of the files do not tell us anything about the chronological order. My own reading of the archives suggests that the order is — broadly and, always, with some exceptions — the following:

- Cod. Ms. R. Dedekind X 11-1 (late 1870s to 1894);
- Cod. Ms. R. Dedekind III 14 (1890s up to 1897)¹³;
- Cod. Ms. R. Dedekind X 9 (after 1884 up to 1900);
- Cod. Ms. R. Dedekind X 10 (mid-1890s);¹⁴
- Cod. Ms. R. Dedekind XI 1 (1894 to 1897, and a couple of folios from the late 1870s);
- Cod. Ms. R. Dedekind XI 2 (late 1890s);
- Cod. Ms. Dedekind X 11-2 (late 1890s, with an early draft of (Dedekind, 1900));
- Cod. Ms. Dedekind XI 3-1 and XI 3-2 (mostly after 1900).

These last two will be of little use to us, since we are interested in the research Dedekind did before publishing his papers on *Dualgruppen*. As the tentative dating suggest, several of these files largely overlap, too.

These files mostly contain work documents, drafts. They consist of investigations and computations on modules, groups, the duality and various properties of the operations, of *Modulgruppen*, and of *Dualgruppen*, much of which can be found in both published papers.

Despite the difficulties described here, and the lack of context and commentary from Dedekind's hand, the archive is extremely rich and offers great insights on the research process through these material traces. Dedekind's drafts are also well preserved, very clean, few things are scratched out and his handwriting is very readable.

¹³This file is the only one that can be related to (Dedekind, 1897b) and exclusively to it.

¹⁴Cod. Ms. R. Dedekind X 9 and X 10 contain, in fact, documents from both before and after Cod. Ms. R. Dedekind XI 1.

Working with mathematical drafts

By studying mathematical drafts, we enter into the mathematician's workshop. They show us material traces of the scientific research, its trials, errors and detours. They allow us to understand the research process, the mathematics as it was being done, as it was being written. As such, they unveil entire parts of the mathematical practice that remain unknown if one only reads published papers: preliminary and intermediary steps and states of mathematical research; ways of writing specific to this private space, freed from publication constraints; artefacts, devices and elements of practice elaborated for the research and absent from publications. It is particularly interesting to see the extent to which his mathematical and writing practices *in drafts* can differ from his publications — and from his statements on how mathematics is best written —, in particular in his extensive practice of computations, his careful crafting of notations and even his use of visualisation devices. Indeed, Dedekind famously advocated for “conceptual” mathematics, freed from computations, not relying in any essential way on notations or “modes of representation”, and he openly spoke against using visual representations in mathematics, if it were not for purely pedagogical or heuristic reasons.¹⁵

Dedekind's drafts show us that he started by exploring the properties of modules and their operations, using computations as a way to experiment, to better understand the divisibility laws of modules by manipulating simple cases, testing on progressively more general cases. A slow process of generalisation arose through these repeated computations: numerical examples, resolution of problems, particular cases with arbitrary elements, more general particular cases, attempts at a completely general approach. . . This gave Dedekind key elements to observe the properties and laws verified by the operations between modules, and identify which properties are generally valid, and which laws are fundamental, i.e. cannot be deduced from other laws (e.g. associativity). In this process, the duality displayed by the operations, and put forward by the notations and layouts elaborated by Dedekind, is a tool for the conceptual clarification of (what will become) *Dualgruppen*. The reflections on duality and the computations enriched each other. Duality was a driving force for investigating the operations' properties, and at the same time, the computations allowed to experiment on the said operations and to observe their dualistic behavior. Duality is, thus, omnipresent in the computations themselves, and in the spatial organisation of writing.

Dedekind's research went through several phases of such explorations of the properties of modules and their operations through computations. Indeed, his research process here was not linear. It did not consist of a phase of research and a phase of writing. Several such phases succeeded and overlapped each other. The process does seem to follow a certain pattern: computations, documents such as tables summarizing the results, problems (*Aufgaben*), more complete

¹⁵Hence, drafts can also help understand how epistemological, disciplinary, sociological and rhetorical criteria shape the final published text. But this will fall outside of the scope of the present chapter.

documents stating — and sometimes proving — results, textualisation of the research, then back to computations.

Digital edition of the manuscripts

A digital edition of the manuscripts is available at

<http://eman-archives.org/Dedekind/>

I elaborated this edition with a triple aim in mind: to understand the long and winding road that led from GCD and LCM of modules to *Dualgruppen*; to unfold the writing and research processes in Dedekind’s mathematics, as a basis for a historical and critical characterisation of his mathematics *in the making*; and to lay the foundations of a reflection on a textual and conceptual genetic analysis inspired by similar works in literature and philosophy.¹⁶ The edition is organised into files corresponding to the catalog in Göttingen. For each material textual unit, I wrote a notice using Dublin Core and personalized metadata. Finally, I established a system of keywords and of relations between the texts.¹⁷ This allows the reader to have a global view of the corpus, in particular to grasp the internal organisation across the files, but also to be able to single out specific aspects (e.g., steps of the research, thematic units, research on a given result...). In particular, two keywords were associated to duality:

- <http://eman-archives.org/Dedekind/items/browse?tags=dualité>
- <http://eman-archives.org/Dedekind/items/browse?tags=dualisme>

The first one, “dualité”, corresponds to my assessment of a manuscript being related to duality. The second one, “dualisme”, corresponds to pages in which Dedekind explicitly refers to duality (as Dedekind favors the term “*Dualismus*” to “*Dualität*”). This distinction allows us to distinguish between my assessment of duality and Dedekind’s explicit mention of it, and at the same time to see the occurrence and circulation of duality in the corpus.

This edition complements the present chapter, but the chapter was written so that the reader doesn’t need to refer to the edition.

Structure of the chapter

This chapter consists of four main sections. First, I will consider two early texts on module theory and the dualism in module theory written in the mid-1870s, which set the machine in motion. Secondly, I will study several aspects of Dedekind’s practices in his drafts which show how duality is present throughout the mathematical research, in the way Dedekind elaborates new notations, specific layouts and devices. The *Modulgesetz*, identified early by Dedekind as a law of special interest, will also provide us with documents showing how his attention to duality might have, at some points, sent him in directions he later

¹⁶(Biasi, 2011; Barberousse and Pinon, 2003; Grésillon, 2016)

¹⁷<http://eman-archives.org/Dedekind/graphall/>

abandoned. Thirdly, I will consider the gradual generalisation of Dedekind's research, and the role it played in the clarification of *Dualgruppen* and a certain distance taken from Dedekind's initial interest in duality. Finally, I will present three manuscripts as the last steps towards *Dualgruppen*, showing how the different aspects of Dedekind's investigations are tied together into the general, unifying concept of *Dualgruppe*.

I do not have any claim of exhaustiveness in this reconstruction of the genesis of *Dualgruppen*. While the choice of the manuscripts to comment, here, remains a subjective one, I did my best to choose representative folios, taking into account that these drafts can be, at times, quite repetitive.

4.1.4 Dedekind's main results on *Dualgruppen*

Because the genesis of a concept can only be understood if one keeps the final product in mind, let me describe the main contents of Dedekind's two papers on *Dualgruppen*: "On the decomposition of numbers into their greatest common divisor" (1897), and "On the *Dualgruppe* generated by three modules" (1900).¹⁸

The 1897 paper

In (Dedekind, 1897b), Dedekind studies the possibility of generalising the decomposition of a finite system of numbers in their GCDs:

If we have a finite system of natural numbers, and if one forms all the greatest common divisors of two or more of these numbers, the latter are thereby divided into factors in a multitude of ways. Although these factors are known in general not to be prime numbers, they are of sufficient service for many investigations, and it is therefore well worth the effort to present the laws thus exhibited in this context. This is the first subject of the present paper, but at the same time the initial problem shall be generalised as much as possible and transferred to domains where there are no decomposition into prime factors at all. In doing so, the investigation admittedly loses its arithmetic character almost completely, so that it hardly requires mathematical knowledge, but at the same time the laws and their grounds become clearer, and I may hope, that in this respect my work may be welcome to some mathematicians. (Dedekind, 1897b, p. 103)

Dedekind starts by studying systems of 3 and 4 numbers, and observes that in order to extend these investigations to systems of n numbers, and to be able to get a more general approach, it becomes necessary to study the possible combinations (*Kombinationen*) of n elements.¹⁹ Dedekind studies two combinations, denoted by $+$ and $-$ equivalent to union and intersection (and called "*Summe*" and "*Durchschnitt*"), in a standard way. He shows that they verify

¹⁸For more on these papers, see (Corry2004; Mehrtens, 1979; Schlimm, 2011).

¹⁹Here, "combinations" designates subsets of a finite set of integers.

“six fundamental laws”: commutativity, associativity and absorption (for $+$ and $-$). Because of the absorption property, he writes, the operations “enter into a dualistic connexion”, from which follows the idempotence property (Dedekind, 1897b, p. 109). And there is also the “double law”:

$$\begin{aligned}(\alpha - \beta) + (\alpha - \gamma) &= \alpha - (\beta + \gamma) \\ (\alpha + \beta) - (\alpha + \gamma) &= \alpha + (\beta - \gamma)\end{aligned}\tag{4.1}$$

which is, in lattice theory, the distributive law (of which the modular law can be seen as a special case).

This leads to the notion of *Dualgruppe*, which Dedekind introduces by acknowledging that his previous remarks are not new, and that insofar as combinations are just systems of elements, they are parts of the theory of systems, which was treated in Schröder’s *Algebra der Logik* (Dedekind, 1897b, p. 112). Giving Schröder the priority for these investigations in (what Dedekind calls) logic (that is, a relatively naive set theory), Dedekind explains that he had himself been studying these questions for several years when working on module theory. It seems that reading Schröder’s book was a strong incentive for Dedekind to publish his own work.²⁰ We will see that, indeed, Schröder’s book might have had a significant impact on Dedekind’s reflections.

Recalling, as quoted above, that he was looking for the “smallest number of fundamental laws” on which to found his theory of modules, Dedekind eventually defined the concept of *Dualgruppe* :

A system \mathfrak{A} of things $\alpha, \beta, \gamma, \dots$ is called a *Dualgruppe*, if there are two operations \pm , such that they create from two things α, β , two things $\alpha \pm \beta$, that are also in \mathfrak{A} and that satisfy [commutativity for $+$ and $-$, associativity for $+$ and $-$, and $\alpha \pm (\alpha \mp \beta) = \alpha$ (absorption)]. (Dedekind, 1897b, p. 113), transl. in (Schlimm, 2011).

Because he wants to “show how multifarious the domains are to which this concept can be applied”, Dedekind gives a series of examples: Schröder’s logical calculus, the *Dualgruppen* formed by modules, ideals, the subgroups of an Abelian group, fields, and points of an n -dimensional space. He then studies more deeply some of these examples (ideals and modules).

Dedekind explains that some (but not all) *Dualgruppen* verify the previously stated equalities (4.1), and that this “double law” should be called *Idealgesetz* (ideal law) in reference to the fact that *Dualgruppen* generated by “ideals in a finite[ly generated] field” verify said law — such *Dualgruppen* are said to be “of the ideal type”.²¹ Similarly, a *Modulgesetz* (module law) can be stated

$$[\alpha + (\beta - \gamma)] - (\beta + \gamma) = [\alpha - (\beta + \gamma)] + (\beta - \gamma)\tag{4.2}$$

which is verified by *Dualgruppen* generated by modules — which are thus said to be “of the module type” (Dedekind, 1897b, p. 117) (and which are our modular

²⁰There were several other instances in which other people’s publications similarly prompted Dedekind to publish some of his own works, in particular (Dedekind, 1872).

²¹*Dualgruppen* of the ideal type are thus distributive lattices.

lattices). These laws cannot be derived from the “six fundamental laws”, a result which Dedekind proves. They should be considered each as an additional law complementing the fundamental laws to define a specific type of *Dualgruppe*. Dedekind proves that every *Dualgruppe* of the ideal type is also of the module type, that there exist *Dualgruppen* that are not of the module type, and that there exist *Dualgruppen* of the module type which are not of the ideal type. He also shows that to characterise a (finite) *Dualgruppe*, it is necessary and sufficient to give the laws defining the operations and the explicit list of its elements. These two laws, in particular the *Modulgesetz*, and the investigations surrounding them, play an important role in the genesis of *Dualgruppen*.

The next step is the definition of (what we would call) the order $<$ (or $>$) by setting that, for α, α_1 in a given *Dualgruppe*, $\alpha > \alpha_1$, $\alpha_1 < \alpha$ for $\alpha + \alpha_1 = \alpha = \alpha - \alpha_1$. Dedekind does not name this relation in (Dedekind, 1897b), but calls it “divisibility” in his number-theoretical works and in (Dedekind, 1900).

However, for Dedekind, his 1897 paper is not so much about *Dualgruppen*, which he calls a “digression”. *Dualgruppen* appear to have, here, the status once attributed to module theory²² of a useful auxiliary theory.²³ Coming back to the core subject of his paper, combinations of numbers, he explains that the sought after generalisation can be done in two ways: combinations of arbitrary many numbers, or combinations of elements of an Abelian (finite or infinite) group \mathfrak{G} . It is the latter that Dedekind wishes to explore. From there, Dedekind engages into a study of group theory, extending the definitions of product, unit, fraction, combinations, integral elements, divisibility and GCD (which entails the introduction of a second operation denoted by $+$ verifying the laws of $+$ in *Dualgruppen* and distributive over the multiplication) to elements of an Abelian group.

Finally, Dedekind comes back to *Dualgruppen* by showing, after defining the integral elements, that it is always possible to define another operation $-$ (which verifies the laws of $-$ in *Dualgruppen*) in \mathfrak{G} , which gives it the structure of a *Dualgruppe* with *Idealgesetz* (that is, a distributive lattice).

(Mehrtens, 1979) concludes his presentation of the paper with the following (very accurate) statement:

The work is remarkable in its abstractness. The title task is of little

²²At the beginning of the section entitled “Auxiliary theorems of module theory” of (Dedekind, 1876-1877), Dedekind states:

As I have emphasised in the Introduction, we shall often have to consider systems of numbers closed under addition and subtraction. The general properties of such systems form a theory quite extended, which can also be used in other research; but, for our purposes just the elements of this theory are sufficient. In order to avoid later interruption to the course of our exposition, and at the same time to make it easier to understand the scope of the concepts on which our theory of algebraic numbers is based, it seems appropriate to begin with a small number of very simple theorems, even though their interest lies mainly in their applications. (Dedekind, 1876-1877, 62, transl. modified)

²³In fact, the study of the drafts suggests that a large part of (Dedekind, 1897b) was developed without the use of *Dualgruppen*, which were rather added towards the end.

interest, and the solution is ultimately rather complicated. The generalisation of the problem to a group with an additional operation, the use of the *Dualgruppe* and arguably the solution by a combinatorial use of finite sets must have been strange to contemporaries. In addition, the essay appeared in a commemorative publication of the Technical University of Braunschweig, so that hardly much response was to be expected. (Mehrtens, 1979, p. 95)

The 1900 paper

In (Dedekind, 1900), Dedekind investigates the *Dualgruppe* generated by three modules (which, in modern terms, is the free modular lattice with three generators), and the *Dualgruppe* generated by three ideals (which is the free distributive lattice with three generators). Finally, he further investigates the *Dualgruppe* generated by modules, and proves some fundamental theorems about them.

The paper's first section gives the definition and general properties of *Dualgruppen* as given in (Dedekind, 1897b), including the definition and properties of the order. By analogy with number and module theories, Dedekind names the operations $+$ and $-$ respectively the “greatest common divisor” and “least common multiple”. It ends with the property:

If \mathfrak{d} is a divisor of \mathfrak{m} , that is $\mathfrak{d} < \mathfrak{m}$ and \mathfrak{p} is an arbitrary element, then we have

$$(\mathfrak{p} + \mathfrak{m}) - \mathfrak{d} < (\mathfrak{p} - \mathfrak{d}) + \mathfrak{m}$$

because each of the two elements $\mathfrak{p} + \mathfrak{m}$ and \mathfrak{d} is a divisor of each of the two elements $\mathfrak{p} - \mathfrak{d}$ and \mathfrak{m} . (Dedekind, 1900, p. 239)

Dedekind then studies \mathfrak{D} the “*Dualgruppe* generated by three modules” in the second section. He first states the *Modulgesetz*, already given in (Dedekind, 1897b), under the form:

If a module \mathfrak{d} is a divisor of a module \mathfrak{m} , that is $\mathfrak{d} < \mathfrak{m}$ and \mathfrak{p} an arbitrary module, then we have

$$(\mathfrak{p} + \mathfrak{m}) - \mathfrak{d} = (\mathfrak{p} - \mathfrak{d}) + \mathfrak{m}^{24}$$

(Dedekind, 1900, p. 239)

To study the said *Dualgruppe* \mathfrak{D} , Dedekind starts by listing the 28 elements of which it is composed, and then shows that it indeed verifies the laws defining a *Dualgruppe*. He emphasizes the duality of the operations, by pointing the possibility to transform (dual) equalities into each other by “permutation” (*Vertauschung*) of the operations. Dedekind then proves that \mathfrak{D} contains exactly 28 distinct elements. In §4, Dedekind studies the *Dualgruppe* generated by three

²⁴This corresponds to the law given in (Dedekind, 1897b) with $\mathfrak{p} = \mathfrak{a}$, $\mathfrak{d} = \mathfrak{b} + \mathfrak{c}$, $\mathfrak{m} = \mathfrak{b} - \mathfrak{c}$.

ideals. These paragraphs constitute a great part of Dedekind's research in his drafts.

Dedekind introduces three important notions, respectively in §§3, 5 and 7: $(\mathfrak{m}, \mathfrak{n})$ the number of congruence classes in \mathfrak{m} modulo \mathfrak{n} , chains and “levels” (*Stufen*). The “number of non-congruent numbers in \mathfrak{m} modulo \mathfrak{n} ” $(\mathfrak{m}, \mathfrak{n})$ was introduced in Dedekind's number-theoretical works. He himself cites (Dedekind, 1894a, §171), but the number of congruence classes already played an important role in (Dedekind, 1871), as it is used to define the norm of ideals — the notation used here was introduced in (Dedekind, 1876-1877). A number of properties of $(\mathfrak{m}, \mathfrak{n})$ were given in previous works,²⁵ of which Dedekind recalls the following:

$$\begin{aligned}(\mathfrak{m}, \mathfrak{n}) &= (\mathfrak{m} + \mathfrak{n}, \mathfrak{n}) \\(\mathfrak{m}, \mathfrak{n}) &= (\mathfrak{m}, \mathfrak{m} - \mathfrak{n})\end{aligned}$$

which he uses to study further the elements of \mathfrak{D} : he groups the $(\mathfrak{m}, \mathfrak{n})$ which are equal to each other, studies the values of $(\mathfrak{m}, \mathfrak{n})$ for the three generators of \mathfrak{D} in terms to the other members of the *Dualgruppe*, which eventually leads him to the equality (already “mentioned” in a footnote in (Dedekind, 1871)):

$$(\mathfrak{b}, \mathfrak{c})(\mathfrak{c}, \mathfrak{a})(\mathfrak{a}, \mathfrak{b}) = (\mathfrak{c}, \mathfrak{b})(\mathfrak{a}, \mathfrak{c})(\mathfrak{b}, \mathfrak{a}).$$

In order to “deepen the understanding of the *Modulgesetz*”, Dedekind introduces the notion of “chain” (*Kette*):²⁶

By a *chain* of the *Dualgruppe* \mathfrak{D} , we mean a finite sequence of at least two elements in \mathfrak{D} , which are each nearest divisor of the next one;²⁷ these elements are called the *links* of the chain, and the first and last links are to be called the beginning and the end of the chain; the number of links minus one is named the *length* of the chain. When two chains have the same beginning and the same end, they are said to be *equivalent*, and when all links of a chain \mathfrak{H} are also links of a chain \mathfrak{K} , we call \mathfrak{H} a *sub-chain* (*Theilkette*) of \mathfrak{K} .” (Dedekind, 1900, p. 253)

The *Kettengesetz* is then proved:

Any two equivalent chains have the same number of links, thus have the same length. (Dedekind, 1900, p. 254)

²⁵Dedekind cites (Dedekind, 1894a), the most recent version of his algebraic number theory, but these results are in (Dedekind, 1876-1877).

²⁶Note that in his number-theoretical works, Dedekind used the notion of “chain” of modules (or of ideals) to designate a finite ordered sequence of modules, for example p. 561 of the 1879 edition of Dirichlet's *Vorlesungen*, and p. 523sq. of the 1894 edition. It was likely the inspiration for the notion of chain in *Dualgruppen*. We will encounter several times such chains of modules in this chapter.

²⁷An element \mathfrak{d} in \mathfrak{D} is called a “nearest divisor” of \mathfrak{m} (resp. \mathfrak{m} is called a “nearest multiple” of \mathfrak{d}) when $\mathfrak{d} < \mathfrak{m}$, $\mathfrak{d} \neq \mathfrak{m}$, and there exists no element \mathfrak{n} of \mathfrak{D} such that $\mathfrak{d} < \mathfrak{n} < \mathfrak{m}$ (Dedekind, 1900, p. 252).

Following this, Dedekind studies “relationships between the *Modul-* and the *Kettengesetz*” (Dedekind, 1900, p. 255), so as to show that the chain condition is not as “self-evident” as it could seem (this being due, according to Dedekind, to the choice of studying this specific *Dualgruppe* generated by three modules). Here is a selection of the results he proves:

- If in a *Dualgruppe* \mathfrak{H} , the *Modulgesetz* is not valid for all the elements, then there exists in \mathfrak{H} a *Dualgruppe* \mathfrak{G} constituted of five distinct elements in which neither the *Modulgesetz* nor the *Kettengesetz* are valid (Dedekind, 1900, p. 255).²⁸
- If in a *Dualgruppe* \mathfrak{H} and in all its subgroups, the *Kettengesetz* is valid, then the *Modulgesetz* is also valid (Dedekind, 1900, p. 259).
- If \mathfrak{a} and \mathfrak{b} are two arbitrary elements of a *Modulgruppe* \mathfrak{M} , then there exists a one-to-one correspondence between the group of all the elements \mathfrak{b}' in \mathfrak{M} verifying the condition $\mathfrak{a} + \mathfrak{b} < \mathfrak{b}' < \mathfrak{b}$ and the group of all the elements \mathfrak{a}_1 in \mathfrak{M} such that $\mathfrak{a} < \mathfrak{a}_1 < \mathfrak{a} - \mathfrak{b}$, which can be expressed by the following relations (Dedekind, 1900, pp. 259–260):²⁹

$$\begin{aligned}\mathfrak{a}_1 &= \mathfrak{a} - \mathfrak{b}' \\ \mathfrak{b}' &= \mathfrak{b} + \mathfrak{a}_1\end{aligned}$$

- In any *Modulgruppe* \mathfrak{M} , any two equivalent chains have the same length, that is, the *Kettengesetz* is verified in any *Modulgruppe* (Dedekind, 1900, p. 264).

Motivated by the second and last of the theorems listed here, Dedekind turns to what he calls the “levels” in a finite *Modulgruppe* \mathfrak{M} (Dedekind, 1900, pp. 265–267).³⁰ In such a \mathfrak{M} , it is “obvious” that among its elements \mathfrak{m} , there exists one and only one element \mathfrak{p} which is a divisor of all \mathfrak{m} , and one and only one \mathfrak{q} which is a multiple of all \mathfrak{m} . If $\mathfrak{m} \neq \mathfrak{p}$ (resp. $\mathfrak{m} \neq \mathfrak{q}$), then there exists at least one nearest divisor (resp. multiple) of \mathfrak{m} . Dedekind proposes to group the elements into “levels”, with \mathfrak{p} being the lowest one and \mathfrak{q} the highest one. The level of \mathfrak{p} is denoted by S_p with p an arbitrary integer. For $\mathfrak{m} \neq \mathfrak{p}$ and a multiple of \mathfrak{p} , with h the length of all the chains beginning in \mathfrak{p} and ending in \mathfrak{m} , $m = p + h$ is called the “*Stufenzahl*” of \mathfrak{m} , and \mathfrak{m} is taken in the level S_m (so, p is the “*Stufenzahl*” of \mathfrak{p}). If $s(\mathfrak{m})$ designates the “*Stufenzahl*” of an element \mathfrak{m} , then for any two \mathfrak{a} , \mathfrak{b} :

$$s(\mathfrak{a}) + s(\mathfrak{b}) = s(\mathfrak{a} + \mathfrak{b}) + s(\mathfrak{a} - \mathfrak{b})$$

Finally, Dedekind turns to relations between $(\mathfrak{m}, \mathfrak{n})$ and the *Modulgesetz*. We know from module theory that $(\mathfrak{m}, \mathfrak{d}) = 1$ and $\mathfrak{m} > \mathfrak{d}$ are equivalent. Dedekind

²⁸These five distincts elements are \mathcal{N}_5 , the pentagon. Dedekind also showed that \mathcal{N}_5 is the smallest non-modular lattice.

²⁹This result was already in (Dedekind, 1894a, 499, footnote).

³⁰The notion of “level” cannot be formalized in general lattice theory.

proves that in a *Dualgruppe* \mathfrak{D} , if for any two $\mathfrak{m}, \mathfrak{n}$, $(\mathfrak{m}, \mathfrak{n})$ is non-zero and verifies

$$\begin{aligned} (\mathfrak{m}, \mathfrak{n}) &= (\mathfrak{m} + \mathfrak{n}, \mathfrak{n}) \\ (\mathfrak{m}, \mathfrak{n}) &= (\mathfrak{m}, \mathfrak{m} - \mathfrak{n}) \\ \text{If } \mathfrak{p} < \mathfrak{q} < \mathfrak{r}, \text{ then } (\mathfrak{p}, \mathfrak{r}) &= (\mathfrak{p}, \mathfrak{q})(\mathfrak{q}, \mathfrak{r}) \\ (\mathfrak{m}, \mathfrak{d}) = 1 &\Leftrightarrow \mathfrak{m} > \mathfrak{d} \end{aligned}$$

then this *Dualgruppe* also verifies the *Modulgesetz*.

A note on some of Dedekind's other works in the 1890s

Investigations on module theory constitute the ground for developments on *Modul-* and *Dualgruppen*. Some results of module theory, in particular on the arithmetic of modules and on their norms, were presented in (Dedekind, 1894a), in which a first reference to *Modulgruppen* was made. These results, as well as some notations used to study *Modulgruppen* are used in (Dedekind, 1895),³¹ whose goal is to generalise the notion of norm defined for modules and ideals. For both these works, drafts from early manuscript versions of the published text were re-used as draft paper.

In 1895, Dedekind also worked on group theory, which led to (Dedekind, 1897a),³² in which he studies Hamiltonian groups and proved that a finite Hamiltonian group can be decomposed into a direct product of an Abelian group and the group of quaternions. While there are no methods related to *Dualgruppen* in that paper, investigations around Abelian and non-Abelian groups appear often in Dedekind's drafts and played an important role in the genesis of (Dedekind, 1897b). This suggests that his reflections on group theory were intertwined with the development of *Modulgruppen* and later *Dualgruppen* — but rather in the sense that his investigations on groups influenced, or maybe helped, his research on *Dualgruppen*, than the opposite. Dedekind also had a regular correspondence with Frobenius, with whom he exchanged a lot on the topic of groups.³³ The correspondence with Heinrich Weber also includes exchanges on groups, in particular a letter from 30 October 1895 exposing his reflections on what would become (Dedekind, 1897a) and asking for Weber's advice.³⁴

These two papers also developed ideas presented in (Dedekind, 1894a), placing the latter at the root of many of the works Dedekind published after his retirement in 1894.

³¹In the footnote in (Dedekind, 1895, p. 77), Dedekind notes that he is using notations introduced to study *Modulgruppen*.

³²In the introduction of the paper, Dedekind writes that he worked on this research in the Fall of 1895 (Dedekind, 1897a, p. 88)

³³See (Hawkins, 2013).

³⁴(Scheel, 2014, pp. 285–287). It is the only letter available for that year — and in general, many letters between Dedekind and Weber were not retrieved for this period. We do know, from the rest of their correspondence, that Dedekind and Weber wrote to each other very regularly, sometimes weekly. It is thus reasonable to assume that they exchanged more on the subject. Let me mention that there are no traces of them discussing Dedekind's works on *Dualgruppen*, unfortunately, nor are there any discussions related to duality.

It is however relatively difficult, either from the drafts or from the papers, to state precisely how these works could have influenced the research on *Dualgruppen* and reciprocally — and clarifying that would be outside the scope of this chapter. So as to avoid any undue over-interpretation, let us restrict ourselves to the following statement: research on module theory and research on group theory, which coincided in time, intersected and both played significant roles in the genesis of *Dualgruppen*. This can be verified with the drafts studied in this chapter, which are, in the *Nachlass* catalog, explicitly related to “the theory of three modules”, *Modulgruppen* and *Dualgruppen*.

4.2 Earliest considerations on duality in module theory

4.2.1 “Theory of modules”

The roots of Dedekind’s *Dualgruppen* are to be found in his investigations on module theory, as he told us himself. One of the first texts dedicated to module theory, in Dedekind’s *Nachlass* — that is, on module theory independently of his research for (Dedekind, 1871) and later versions of it — is entitled “*Theorie der Moduln*” (Cod. Ms. Dedekind XI 1, p. 36-43). It can be dated from between 1871 and 1877.³⁵ It was never published as such, even if some of the results were integrated in later works using modules, such as (Dedekind, 1877) and the Supplements on algebraic number theory in Dirichlet’s *Vorlesungen*. This text is 14 pages long, written in a two columns layout. This layout is characteristic (albeit not systematically used) of Dedekind’s writing for next-to-last versions of a text. The left-side column contains the main text, and the right-side column contains additions, corrections and various notes (not always to be integrated to the text). The text is not completely written (for example, Dedekind only broadly refers to (Dedekind, 1871) for some of the definitions) and Dedekind indicated in the top right corner that another title could be “*General theorems on modules, orders and congruences*”.

“*Theorie der Moduln*” is a good starting point to understand the path Dedekind went down and the ground covered. Dedekind gives basic elements for the theory of modules of numbers, which continue the analogy with rational arithmetic that he introduced in his works on algebraic number theory. Here is broadly the content of the text:

- Definition of a module.

³⁵Dedekind makes references to the 1871 edition of Dirichlet’s *Vorlesungen*. As he systematically refers to the most recent version of that work, this suggests that it was written before (Dedekind, 1876-1877; Dedekind, 1879). The reference to “orders” (*Ordnungen*) in the title further suggests that it was written around the same time or just after as (Dedekind, 1876-1877). This hypothesis is supported by the content of the text, and in particular the fact that questions related to the dualism of the operations in module theory, apparently highlighted here for (one of the) first time(s), were first published in (Dedekind, 1877).

- Definition of divisibility for modules and what it means for a module \mathfrak{d} to be a divisor of a module \mathfrak{m} .
- Definition and properties of GCD of modules.
- Definition and properties of LCM of modules.
- Definition of congruence, classes, and number of classes for modules.
- Definition of the multiplication, quotient, inverse, and power for modules.

For the definition of modules and their divisibility, Dedekind only refers to the 1871 edition of Dirichlet's *Vorlesungen*. In the margin, he introduces, probably for the first time, the notation for \mathfrak{d} is a divisor of \mathfrak{a} :

The divisibility of \mathfrak{a} by \mathfrak{d} – i.e., \mathfrak{a} multiple of \mathfrak{d} , \mathfrak{d} divisor of \mathfrak{a} – may be indicated by

$$\mathfrak{a} \leftarrow \mathfrak{d}, \mathfrak{d} \rightarrow \mathfrak{a}$$

or maybe just the other way around, by

$$\mathfrak{a} > \mathfrak{d}, \mathfrak{d} < \mathfrak{a}$$

(Cod. Ms. Dedekind XI 1, p. 36r)

Let me address briefly the hesitation on the notation. Recall that a module \mathfrak{d} is said to divide (or be a divisor of) a module \mathfrak{a} if \mathfrak{a} is included in \mathfrak{d} . Dedekind's first attempt at a notation reflected the inclusion relation between \mathfrak{a} and \mathfrak{d} . However, as it has been noticed by Dedekind himself, the definition of divisibility as inclusion goes against the intuition we get from number theory, since the divisor contains more elements than the multiple. The notation eventually chosen by Dedekind seems, here, to try to restore the analogy with arithmetic.

The definitions of the GCD and the LCM of modules are the occasion to introduce notations for these notions — which were absent from (Dedekind, 1871; Dedekind, 1876-1877):

3. Definition of the greatest common divisor $\mathfrak{a} + \mathfrak{b}$ of two modules \mathfrak{a} and \mathfrak{b} ; $\mathfrak{a} + \mathfrak{b} = \mathfrak{b} + \mathfrak{a}$.

Theorem. Every common divisor of \mathfrak{a} and \mathfrak{b} is a divisor of $\mathfrak{a} + \mathfrak{b}$.

4. The divisibility between \mathfrak{a} and \mathfrak{d} is also expressed by $\mathfrak{a} + \mathfrak{d} = \mathfrak{d}$.

5. We have $\mathfrak{a} + \mathfrak{a} = \mathfrak{a}$.

6. We have $(\mathfrak{a} + \mathfrak{b}) + \mathfrak{c} = \mathfrak{a} + (\mathfrak{b} + \mathfrak{c})$. From this, (from Dirichlet's *Vorlesungen über Zahlentheorie*, §2³⁶), definition of the greatest common divisor of several modules $\mathfrak{a}, \mathfrak{b}, \mathfrak{c}, \mathfrak{d}, \dots \mathfrak{m}$ (in countable number?).

³⁶Interestingly, Dedekind refers here to the section generalising the properties of multiplication to an arbitrary number of *integers*.

7. Also for an uncountable number of modules, there exists a fully determined greatest common divisor: the system of all the numbers each of which is a sum of a finite numbers which are each contained in one of the given modules.

8. Definition of the least common multiple $\mathfrak{a} \vee \mathfrak{b} = \mathfrak{b} \vee \mathfrak{a}$ of two modules \mathfrak{a} and \mathfrak{b} .

Theorem. Every common multiple of \mathfrak{a} and \mathfrak{b} is a multiple of $\mathfrak{a} \vee \mathfrak{b}$.

9. We have also $(\mathfrak{a} \vee \mathfrak{b}) + \mathfrak{a} = \mathfrak{a}$ and from

$$\left\{ \begin{array}{l} \mathfrak{c} + \mathfrak{a} = \mathfrak{a} \\ \mathfrak{c} + \mathfrak{b} = \mathfrak{b} \end{array} \right\}$$

follows

$$\mathfrak{c} + (\mathfrak{a} \vee \mathfrak{b}) = \mathfrak{a} \vee \mathfrak{b}$$

and reciprocally.

10. We have $(\mathfrak{a} \vee \mathfrak{b}) \vee \mathfrak{c} = \mathfrak{a} \vee (\mathfrak{b} \vee \mathfrak{c})$ From this, (from Dirichlet's *Vorlesungen über Zahlentheorie*, §2), definition of the least common multiple of several modules $\mathfrak{a}, \mathfrak{b}, \mathfrak{c}, \mathfrak{d}, \dots \mathfrak{m}$ (in countable number).

11. Also for an uncountable number of modules, there exists a fully determined least common multiple: the system of all the numbers which are contained in each of the given modules.

(Cod. Ms. Dedekind XI 1, p. 36)

The notation for the LCM is different from the one eventually chosen by Dedekind. This text and the accompanying table seems to be the only manuscript left in which Dedekind uses this \vee notation. In fact, the bottom right corner of the page 36r, shows that Dedekind considered various notations. For the divisibility relation, he considered writing

$$\mathfrak{a} \succ \mathfrak{d}, \mathfrak{d} \prec \mathfrak{a}$$

For $\mathfrak{a} + \mathfrak{b}$ and $\mathfrak{a} \vee \mathfrak{b}$, he considered writing respectively

$$\mathfrak{a} \wedge \mathfrak{b} \text{ and } \mathfrak{a} \vee \mathfrak{b}.$$

Dedekind also made several attempts at drawing alternate notations: curlier versions of the above ones, or crossed over versions (which resemble a cursive letter A). The origins of Dedekind's use of the \vee and almost use of the \wedge notations are unclear.

I will consider the question of notations a little more in the next section. It should already be emphasized, though, that Dedekind did not use these notations in later works.

Properties such as

$$\begin{array}{ll} \mathfrak{a} + \mathfrak{a} = \mathfrak{a} & \mathfrak{a} \vee \mathfrak{a} = \mathfrak{a} \\ \mathfrak{a} + \mathfrak{b} = \mathfrak{b} + \mathfrak{a} & \mathfrak{a} \vee \mathfrak{b} = \mathfrak{b} \vee \mathfrak{a} \\ (\mathfrak{a} + \mathfrak{b}) + \mathfrak{c} = \mathfrak{a} + (\mathfrak{b} + \mathfrak{c}) & (\mathfrak{a} \vee \mathfrak{b}) \vee \mathfrak{c} = \mathfrak{a} \vee (\mathfrak{b} \vee \mathfrak{c}) \end{array}$$

are given as consequences of the interpretation of the operations as GCD and LCM of modules. For example, rather than stating that $\mathfrak{a} + \mathfrak{a} = \mathfrak{a}$ is a property of $+$, Dedekind presents it as coming from the fact the the GCD of \mathfrak{a} and itself is \mathfrak{a} . His focus is, thus, very much on the arithmetical interpretation of the operations.

The text consists mainly in elements of module theory which are of little relevance here. And while the text itself does not contain any mention of duality, it is accompanied by the following table, which I transcribe and translate in its entirety (Cod. Ms. Dedekind, XI 1 p. 35):³⁷

Duality in the theory of modules between	
Gr[eatest] com[mon] divisor $\mathfrak{a} + \mathfrak{b}$ of two modules	Le[ast] com[mon] multiple $\mathfrak{a} \vee \mathfrak{b}$ of two modules
1. We have $\mathfrak{a} + \mathfrak{b} < \mathfrak{a}$ and $\mathfrak{a} + \mathfrak{b} < \mathfrak{b}$ and from $\mathfrak{d} < \mathfrak{a}$ and $\mathfrak{d} < \mathfrak{b}$ follows $\mathfrak{d} < \mathfrak{a} + \mathfrak{b}$	1. We have $\mathfrak{a} \vee \mathfrak{b} > \mathfrak{a}$ and $\mathfrak{a} \vee \mathfrak{b} > \mathfrak{b}$ and from $\mathfrak{d} > \mathfrak{a}$ and $\mathfrak{d} > \mathfrak{b}$ follows $\mathfrak{d} > \mathfrak{a} \vee \mathfrak{b}$
2. We have $\mathfrak{a} + \mathfrak{a} = \mathfrak{a}$	2. We have $\mathfrak{a} \vee \mathfrak{a} = \mathfrak{a}$
3. We have $(\mathfrak{c} \vee \mathfrak{a}) + (\mathfrak{a} \vee \mathfrak{b}) = \mathfrak{a} \vee ((\mathfrak{c} \vee \mathfrak{a}) + \mathfrak{b})$	3. We have $(\mathfrak{c} + \mathfrak{a}) \vee (\mathfrak{a} + \mathfrak{b}) = \mathfrak{a} + ((\mathfrak{c} + \mathfrak{a}) \vee \mathfrak{b})$
4. We have $(\mathfrak{a} + \mathfrak{b}, \mathfrak{b}) = (\mathfrak{a}, \mathfrak{b}) = (\mathfrak{a}, \mathfrak{a} \vee \mathfrak{b})$	4. We have $(\mathfrak{b}, \mathfrak{a} \vee \mathfrak{b}) = (\mathfrak{b}, \mathfrak{a}) = (\mathfrak{a} + \mathfrak{b}, \mathfrak{a})$
5. We have $(\mathfrak{a}, \mathfrak{b} \vee \mathfrak{c}) = (\mathfrak{a}, \mathfrak{b})(\mathfrak{a} \vee \mathfrak{b}, \mathfrak{c})$	5. We have $(\mathfrak{b} + \mathfrak{c}, \mathfrak{a}) = (\mathfrak{b}, \mathfrak{a})(\mathfrak{c}, \mathfrak{a} + \mathfrak{b})$
6. We have $(\mathfrak{a}, \mathfrak{b}) = (\mathfrak{a}, (\mathfrak{c} \vee \mathfrak{a}) + \mathfrak{b})(\mathfrak{a} \vee \mathfrak{a}, \mathfrak{b})$	6. We have $(\mathfrak{b}, \mathfrak{a}) = ((\mathfrak{c} + \mathfrak{a}) \vee \mathfrak{b})(\mathfrak{b}, \mathfrak{c} + \mathfrak{a})$
7. We have (when \mathfrak{z} runs through all rational integers) $\mathfrak{a}\mathfrak{z} = \mathfrak{a}$	7. We have $\frac{\mathfrak{a}}{\mathfrak{z}} = \mathfrak{a}$
8. From $\mathfrak{a} > \mathfrak{b}$ follows $\mathfrak{a}\mathfrak{c} > \mathfrak{b}\mathfrak{c}$	8. From $\mathfrak{a} > \mathfrak{b}$ follows $\frac{\mathfrak{a}}{\mathfrak{c}} > \frac{\mathfrak{b}}{\mathfrak{c}}$ and $\frac{\mathfrak{c}}{\mathfrak{a}} < \frac{\mathfrak{c}}{\mathfrak{b}}$
9. We have $(\mathfrak{a} + \mathfrak{b})\mathfrak{c} = \mathfrak{a}\mathfrak{c} + \mathfrak{b}\mathfrak{c}$	9. We have $\frac{\mathfrak{a} \vee \mathfrak{b}}{\mathfrak{c}} = \frac{\mathfrak{a}}{\mathfrak{c}} \vee \frac{\mathfrak{b}}{\mathfrak{c}}$ and $\frac{\mathfrak{c}}{\mathfrak{a} + \mathfrak{b}} = \frac{\mathfrak{c}}{\mathfrak{a}} \vee \frac{\mathfrak{c}}{\mathfrak{b}}$

³⁷Let me emphasize that this is one of the rare texts in which Dedekind uses the word “*Dualität*” (duality) rather than “*Dualismus*” (dualism). It is worth noting that “*Dualismus*” is also the word used by Ernst Schröder (already in (**bo:Schroeder1877a**)). However, this does not allow us to draw any conclusion on his influence on Dedekind’s choice of words for two reasons. The first reason is that Dedekind uses “*Dualismus*” in (Dedekind, 1877) which was published the same year as Schröder’s book. The second reason is that while we have several sets of notes from Dedekind’s reading of Schröder’s *Vorlesungen über die Algebra der Logik*, we do not have any for *Der Operationskreis des Logikkalküls*, which although not excluding that Dedekind did read it, prompts caution.

10. We have $(\mathfrak{a} \vee \mathfrak{b})\mathfrak{c} > (\mathfrak{a}\mathfrak{c}) \vee (\mathfrak{b}\mathfrak{c})$	10. We have $\frac{\mathfrak{a}+\mathfrak{b}}{\mathfrak{c}} < \frac{\mathfrak{a}}{\mathfrak{c}} + \frac{\mathfrak{b}}{\mathfrak{c}}$ and $\frac{\mathfrak{c}}{\mathfrak{a}\vee\mathfrak{b}} < \frac{\mathfrak{c}}{\mathfrak{a}} + \frac{\mathfrak{c}}{\mathfrak{b}}$
11. We have $\frac{\mathfrak{c}}{\mathfrak{a}} + \frac{\mathfrak{c}}{\mathfrak{b}} > \frac{\mathfrak{c}}{\mathfrak{a}\vee\mathfrak{b}}$	
12. Definition of the quotient consists in that $\mathfrak{a}\mathfrak{c} > \mathfrak{b}$ and $\mathfrak{c} > \frac{\mathfrak{b}}{\mathfrak{a}}$ are identical conditions.	
13. From $\mathfrak{b} > \mathfrak{c}$ follows $\frac{\mathfrak{b}}{\mathfrak{a}} > \frac{\mathfrak{c}}{\mathfrak{a}}$ and $\frac{\mathfrak{a}}{\mathfrak{b}} < \frac{\mathfrak{a}}{\mathfrak{c}}$	

This table seems to be *taking note of* the duality displayed by the operations and the divisibility relation, rather than providing an analysis or use of these properties. Whether this table was written after or during “*Theorie der Moduln*” is difficult to assess. However, the act of establishing a table seems to be, for Dedekind, a way to put things into order and maybe, to a certain extent, a visualisation device.

4.2.2 “Dualism in the laws of modules of numbers”

In a second text, likely written before 1877,³⁸ we observe a distinct inflexion towards duality. It is entitled “Dualism in the laws of modules of numbers” and starts in the following way:

The following considerations presuppose no other mathematical knowledge than that of addition and subtraction, and refer to the concept of a number-module that I introduced,* whose properties, as far as they are of importance here, are to be developed first. (Cod. Ms. Dedekind, X 9, p. 3)

This introductory sentence, followed immediately by the definition of a module (of numbers), shows that Dedekind’s sole interest, in this short draft, is the theory of modules. The next properties concern more precisely the *operations* defined between modules. This is in line with Dedekind’s long lasting idea of developing an arithmetic of modules (and ideals), on which I have written elsewhere.³⁸ It is also clearly embedded in a very typically Dedekindian approach of wanting to lay clear, rigorous grounds for a given theory. Hence, Dedekind’s desire to develop the arithmetic of modules and clarify the foundation of module theory was at the root of his works, but the additional property of duality

³⁸Indeed, it does not contain the laws first published in (Dedekind, 1877) and considered, at that point, as a core result. It could be that the text is unfinished. In that case, let me also mention that it refers to the 1871 edition of Dirichlet’s *Vorlesungen*, which suggests that it was written at the latest before 1879.

*Dirichlet, *Lectures on number theory, second edition* [Footnotes with a symbol are Dedekind’s. Footnotes with a number are the author’s.]

³⁸(Haffner, 2017)

displayed by the operations took an increasing importance in these parts of his research.

It would be more accurate to describe the text as concerned with properties of operations which happen to be dual of each other, rather than as concerned with studying their duality properly speaking.³⁹ Dedekind starts by defining modules and finitely generated modules of numbers, as well as divisibility of modules (as he usually does, see above) and introduces the notation for \mathfrak{m} divisible by \mathfrak{d} :⁴⁰

$$\mathfrak{m} > \mathfrak{d}; \mathfrak{d} < \mathfrak{m}. \quad (1)$$

He then gives the definition of a sum of two modules $\mathfrak{a} + \mathfrak{b}$, as defined in his published number-theoretical works:

If α designates each number in the module \mathfrak{a} and β each number in the module \mathfrak{b} , then all the numbers of the form $\alpha + \beta$ evidently form again a module which we name the sum of \mathfrak{a} and \mathfrak{b} , and designate by $\mathfrak{a} + \mathfrak{b}$. (Cod. Ms. Dedekind X 9, p. 4r)

Dedekind completes the definition with the three following properties:

$$\mathfrak{a} + \mathfrak{a} = \mathfrak{a} \quad (2)$$

$$\mathfrak{a} + \mathfrak{b} = \mathfrak{b} + \mathfrak{a} \quad (3)$$

$$(\mathfrak{a} + \mathfrak{b}) + \mathfrak{c} = \mathfrak{a} + (\mathfrak{b} + \mathfrak{c}) \quad (4)$$

Following this, he justifies calling $\mathfrak{a} + \mathfrak{b}$ the *greatest common divisor* of \mathfrak{a} and \mathfrak{b} . Such a justification is also given in his number-theoretical works, in which the labeling and its justification are usually introduced *before* the operation (when the latter is indeed introduced, which is not always the case). Dedekind then underlines that

If \mathfrak{m} is divisible by \mathfrak{d} , then

$$\mathfrak{m} + \mathfrak{d} = \mathfrak{d} \quad (1')$$

(Cod. Ms. Dedekind X 9, p. 5r)

In later texts, Dedekind starts by introducing the two operations $+$ and $-$ and uses (1') to define the divisibility relationship, then denoted by the $<$ symbol.

Dedekind does the same for the difference of two modules $\mathfrak{a} - \mathfrak{b}$ (also already explicitly defined in some of his number-theoretical papers):

If \mathfrak{a} and \mathfrak{b} are any two modules, then the totality of all the numbers μ which belong to both modules \mathfrak{a} and \mathfrak{b} , such as the number 0 for example, always forms a module. [...] We want to designate this module by $\mathfrak{a} - \mathfrak{b}$ and name it the difference of \mathfrak{a} and \mathfrak{b} . (Cod. Ms. Dedekind X 9, p. 5r)

³⁹We will see that this is not the only occurrence of a text whose title gives a great importance to duality, while the text itself less so.

⁴⁰I kept Dedekind's numbering of the equations in the manuscripts from here on.

He completes it with the three properties:

$$\mathfrak{a} - \mathfrak{a} = \mathfrak{a} \quad (2')$$

$$\mathfrak{a} - \mathfrak{b} = \mathfrak{b} - \mathfrak{a} \quad (3')$$

$$(\mathfrak{a} - \mathfrak{b}) - \mathfrak{c} = \mathfrak{a} - (\mathfrak{b} - \mathfrak{c}) \quad (4')$$

Then, he remarks that divisibility between \mathfrak{m} and \mathfrak{d} is equivalent to:

$$\mathfrak{m} - \mathfrak{d} = \mathfrak{d} \quad (1'')$$

Note the way Dedekind numbers dual properties with (α) and (α') . This is an explicit way to put forward the dualism (or parallelism) between the operations.⁴¹

After this, Dedekind justifies again the arithmetical terminology used here, and introduces the product and quotient of two modules. These notions are, as Dedekind emphasizes, important for ideal theory. They do not, however, play a significant role in the development of *Dualgruppen*.

The text ends abruptly after the definition of the “number of classes in \mathfrak{a} according to \mathfrak{b} ” $(\mathfrak{a}, \mathfrak{b})$, and properties such as:

$$(\mathfrak{a}, \mathfrak{b}) = (\mathfrak{a} + \mathfrak{b}, \mathfrak{b}) \quad (5)$$

$$(\mathfrak{a}, \mathfrak{b}) = (\mathfrak{a}, \mathfrak{a} - \mathfrak{b}) \quad (5')$$

$$(\mathfrak{m}, \mathfrak{d}) = 1 \leftrightarrow \mathfrak{m} > \mathfrak{d} \quad (1''')$$

In this text, Dedekind’s considerations on “the dualism in module theory” seem to be, again, restricted to *observing* a certain property of symmetry or parallelism between the operations. The said dualism is not used at any point in the text. It is merely highlighted as an interesting property.

These texts are written fairly early in Dedekind’s investigations on these aspects of module theory. I will show, in the next section, that it is through the research process, the computations, devices and artefacts set up to study modules, that duality is exhibited, analysed and clarified.

4.3 How duality emerges in Dedekind’s mathematical research

Dedekind’s drafts suggest that the role played by the dualism is brought out by the research itself: through the writing and research practices, Dedekind followed the thread of duality. It is exhibited, put forward by the computations on modules and analyses of their properties. At the same time, it was a strong incentive for studying the operations more closely, and continues to be a guiding light for Dedekind’s investigations. This can be observed in particular in the choice of notations, of the organisation of writings on paper, and in the artefacts devised by Dedekind to support his research.

⁴¹A similar strategy was used by Ernst Schröder in (**bo:Schröder1877a**).

In a large portion of these drafts, Dedekind's research process consists in computations on modules. Such computations are explorations of the properties of modules and their operations, in an approach akin or likened to an experimentation with modules allowing to observe the laws governing the theory. Computations, as well as some of the layouts and devices used by Dedekind in drafts, are largely (if not entirely) absent from his publications. In that aspect, they appear to be ways of working designed for the research process, dedicated to work out the properties of modules (and later groups of modules, and *Dualgruppen*). I would like to consider, in the following sections, to which extent these devices and experimentations played a role in putting forward duality properties of the operations, and in clarifying their role in the definition-to-be of *Dualgruppen*. My goal will not be to study the details of Dedekind's computations — they are, most of the time, relatively simple and fairly repetitive⁴² — but rather to exhibit how Dedekind devised notations and layouts for his drafts that allowed to better bring to the fore the dualism in the properties of modules.

4.3.1 A short aside on groups

Before I expose some of the details of these research practices, it is important to mention that Dedekind's drafts show us that he was working on two aspects of module theory. On one hand, he studied the properties of the operations: finding the smallest number of fundamental laws, establishing and proving properties (e.g., the *Modulgesetz*). On the other hand, he was interested in what he called “groups of modules”, systems formed by modules and closed under the GCD and LCM operations.

Dedekind had been working on and with groups since the mid-1850s, when he gave the first class on Galois theory in Germany (Dedekind, 1856-1858). He adopted a very general conception of groups from the start, defining them in the following way:

The following investigations are based only on the two fundamental results proven above [associativity, right- and left-cancelability] and on the assumption that the number of substitutions is finite: The results are therefore valid for any finite domain of elements, things, concepts $\theta, \theta', \theta'', \dots$ admitting an arbitrarily defined composition $\theta\theta'$ for any two given elements θ, θ' , such that $\theta\theta'$ is itself a member of the domain, and such that this composition satisfies the laws expressed in the two main results. In many parts of mathematics, and especially in the theory of numbers and in algebra, one often finds examples of this theory; the same methods of proof are valid here as there. (Dedekind, 1856-1858, p. 63), transl. in (Corry2004)

In his 1894 Supplement X, in which he develops Galois theory further, the definition of a group is more concise:

⁴²See (Haffner, 2018).

A system Π of n different field permutations π is called a group if any one can be composed with any other, and the resultant is always contained in Π . (Dedekind, 1894a, p. 482)

The associativity and the existence of an inverse element are seen, here, as consequences of the definition of permutations themselves. Dedekind's uses of the word "group" suggest that the essential condition for a system (of numbers, permutations, or other mathematical objects, including arbitrary indefinite elements) to be a group is the closure by one or two binary operations. The properties of the operations would give defining properties of the "group" in question — properties such as associativity were not, for example, included in his general idea of a "group". It is the use made in his drafts on *Dualgruppen*. Besides, for Dedekind, there was no problem with forming sets of sets, as he thought of them as objects rather than collections. It is thus unsurprising that having defined two commutative operations for modules, he would look at "groups" formed by modules. I will use the term "group" as an actor's category and follow Dedekind's use of the word.⁴³

Throughout the drafts we are studying, from the theory of three modules to *Dualgruppen*, the methods used by Dedekind are strongly reminiscent of the methods used in group theory at the end of the 19th century: study of the internal organisation (or 'structure'), tables or list of elements... Such strategies were usual at the time in works on groups (they were used by Cayley, who Dedekind read, for example). Dedekind's drafts thus suggest that he was working in module theory and slowly taking the direction of setting up and studying a notion of "group", broadly conceived, of modules.

In Cod. Ms. Dedekind X 10, p. 8, written on an envelope dated from 01/02/1878 but followed by a page dated from 01/04/1889, for example, Dedekind considers two chains of modules \mathfrak{a} (of m links) and \mathfrak{b} (of n links), and builds all the $\mathfrak{c} = \mathfrak{a} - \mathfrak{b}$ (of mn links) and $\mathfrak{d} = \mathfrak{a} + \mathfrak{b}$ (of mn links). He then observes that

$$\left. \begin{array}{llll} \text{All} & \mathfrak{a} - \mathfrak{b} & \text{are themselves} & \mathfrak{c} \\ " & \mathfrak{b} - \mathfrak{c} & " & \mathfrak{c} \\ " & \mathfrak{a} + \mathfrak{d} & " & \mathfrak{d} \\ " & \mathfrak{b} + \mathfrak{d} & " & \mathfrak{d} \\ " & \mathfrak{a} \pm \mathfrak{a}' & " & \mathfrak{a} \\ " & \mathfrak{b} \pm \mathfrak{b}' & " & \mathfrak{b} \\ " & \mathfrak{c} - \mathfrak{c}' & " & \mathfrak{c} \\ " & \mathfrak{d} + \mathfrak{d}' & " & \mathfrak{d} \end{array} \right\}$$

Here, $\mathfrak{a}', \mathfrak{b}', \mathfrak{c}', \mathfrak{d}'$ seem to be respectively members of the chains of $\mathfrak{a}, \mathfrak{b}, \mathfrak{c}, \mathfrak{d}$.

⁴³Outside of Dedekind's works, at the end of the 19th century, the term "group" also covered several different meanings or understandings of the concept (Wussing, 1984; Ehrhardt, 2012). There was no "group theory" as we understand it today. Note that, in a draft of (Dedekind, 1888) reproduced in (Dugac, 1976, p. 296), we see that Dedekind's chains were first called "groups". It is one of many occurrences, in Dedekind's manuscripts, of changing terminology as part of the research process.

Dedekind then builds $\mathfrak{a} + \mathfrak{c}$, $\mathfrak{b} + \mathfrak{c}$, $\mathfrak{a} - \mathfrak{d}$, $\mathfrak{b} - \mathfrak{d}$, $\mathfrak{c} + \mathfrak{c}'$, $\mathfrak{d} - \mathfrak{d}'$, $\mathfrak{c} + \mathfrak{d}$, $\mathfrak{c} - \mathfrak{d}$ for cases when $\mathfrak{a} > \mathfrak{a}'$ and $\mathfrak{b} > \mathfrak{b}'$. He notes:

The gr[eatest] com[mon] divisors built by \mathfrak{a} , \mathfrak{b} , $\mathfrak{a} - \mathfrak{b}$ coincide with the sm[allest] com[mon] multiples built by \mathfrak{a} , \mathfrak{b} , $\mathfrak{a} + \mathfrak{b}$, and constitute consequently a group. (Cod. Ms. Dedekind X 10, p. 8r)

Following this, Dedekind does not investigate further the group aspect of this system, but considers the cases for $\mathfrak{a} < \mathfrak{a}'$ and $\mathfrak{b} < \mathfrak{b}'$, extends his computations to more factors and states a “general theorem” for the operations.⁴⁴ We observe, here, how the concept of group (of modules) enters into Dedekind's module-theoretic research: as an overarching concept of a system closed by a binary operation, but without Dedekind actually developing considerations pertaining to group theory standardly conceived (unlike, for example, in his (Dedekind, 1897a)).

We will see many examples of Dedekind's investigations on so-called groups of modules. Most of the time, Dedekind was interested in case studies, in which he gave himself three or four modules (arbitrary or finitely generated), sometimes with numerical cases or with initial conditions on the divisibility relations of those modules. In such case studies, of course, the examination of the properties of the operations holds a central place. The inverse is not true: a number of Dedekind's studies on the properties of the operations for modules did not consider questions related to groups. These two aspects are, thus, closely related, but were nevertheless first developed independently. Ultimately, they will be merged, in a crucial move for the emergence of the *Dualgruppe* concept.

4.3.2 Notations

As Dedekind himself emphasized, it was the introduction of a notation for the GCD and LCM of modules, that allowed him not only to go from mimicking arithmetic to actually *computing* with modules. By introducing these notations, the notions of GCD and LCM became *operations*. It also led to highlighting their duality. We saw that, in the 1870s, Dedekind tried a different symbol for the operation giving the LCM of modules, which he ultimately decided to write as $-$. In fact, Dedekind used several notations through the years, some of which did not appear in the published papers. Dedekind experimented several ways of writing his computations, which suggests a willingness to put forward — if not to rely on — the dualism.

⁴⁴“If

$$\begin{aligned}\mathfrak{a} &> \mathfrak{a}' > \mathfrak{a}'' > \dots > \mathfrak{a}^{(r)} \\ \mathfrak{b} &< \mathfrak{b}' < \mathfrak{b}'' < \dots < \mathfrak{b}^{(r)}\end{aligned}$$

then we have

$$\begin{aligned}(\mathfrak{a} - \mathfrak{b}) + (\mathfrak{a}' - \mathfrak{b}') + \dots + (\mathfrak{a}^{(r)} - \mathfrak{b}^{(r)}) &= (\mathfrak{a} + \mathfrak{b}') - (\mathfrak{a}' + \mathfrak{b}'') - \dots - (\mathfrak{a}^{(r-1)} - \mathfrak{b}^{(r)}) - \mathfrak{a}^{(r)} - \mathfrak{b} \\ (\mathfrak{a} + \mathfrak{b}) - (\mathfrak{a}' + \mathfrak{b}') - \dots - (\mathfrak{a}^{(r)} + \mathfrak{b}^{(r)}) &= (\mathfrak{a} - \mathfrak{b}') + (\mathfrak{a}' - \mathfrak{b}'') + \dots + (\mathfrak{a}^{(r)} - \mathfrak{b}^{(r-1)}) - \mathfrak{a} - \mathfrak{b}^{(r)}\end{aligned}$$

(Dedekind X 10, p. 8r)

This suggests that choosing a suitable notation can help visualize, and maybe even better understand, some properties and relations of the objects studied. As Dedekind repeatedly argued that notations should not hold a central place in mathematical inquiries, it is interesting that it was a decisive step in his study of module theory. In particular, in (Dedekind, 1894b), Dedekind quoted Gauss's famous remark on notations vs. notions:

Waring confessed that the demonstration [for the result: the product of all numbers less than a given prime number, when increased by unity is divisible by this prime number (EH)] seemed more difficult because no *notation* can be devised through which one could express a prime number. But in our opinion truths of this kind ought to be drawn out of notions [*notionibus*, translated in German by *Begriffen*] and not out of notations. (Gauss, 1801, §76, 50)

Dedekind considered this statement to be “a great scientific thought” (and an argument in favor of his ideal theory against Kronecker's divisor theory). For him, it was a question of giving the primacy to “internal characteristic qualities” rather than “external forms of representation” (Dedekind, 1894b, pp. 54–55). Of course, this statement should not be taken as Dedekind suggesting that mathematics should be rid of *any kind* of *Darstellungformen* — this would amount to mathematics without writing. He is rather insisting on what he considers as suited to be taken as a definition. More importantly, Dedekind's careful reflections on his own notations happen *after* the operations are defined, and they do not modify the definition in any way. In addition, looking for the best notation was likely related to questions of heuristics and to mathematics *in the making*. The mathematics developed in such contexts is not yet fixed, it is in constant evolution and it is freed of a number of constraints of writing for publication. It illustrates the research and writing processes, the genesis of the mathematical works, in ways difficult to find in publications.

First example of notation

The first notation introduced by Dedekind is the following. For three modules $\mathfrak{a}, \mathfrak{b}, \mathfrak{c}$:

$$\begin{array}{l} Gr[reatest] \\ c[ommon] \\ div[isor] \end{array} \left\{ \begin{array}{l} \mathfrak{a}' = \mathfrak{b} + \mathfrak{c} \\ \mathfrak{b}' = \mathfrak{c} + \mathfrak{a} \\ \mathfrak{c}' = \mathfrak{a} + \mathfrak{b} \end{array} \right. \quad \begin{array}{l} Le[ast] \\ c[ommon] \\ m[ultiple] \end{array} \left\{ \begin{array}{l} \mathfrak{a}_1 = \mathfrak{b} - \mathfrak{c} \\ \mathfrak{b}_1 = \mathfrak{c} - \mathfrak{a} \\ \mathfrak{c}_1 = \mathfrak{a} - \mathfrak{b} \end{array} \right.$$

(Cod. Ms. Dedekind X 11-1, p. 19v)

The operations used by Dedekind, here, correspond to the ones whose definition we gave on p. 2.⁴⁵ He works, as much as possible, with modules as objects, as indeterminate operands. This allows him to manipulate the operations only taking into account their fundamental laws (associativity, commutativity, idempotence) and properties such as the ones proved in (Dedekind, 1877) (see p. 2). Introducing designations such as $\mathfrak{a}' = \mathfrak{b} + \mathfrak{c}$ shortens the computations.

⁴⁵When working with numerical cases, they amount to addition and intersection of \mathbb{Z} -modules.

For some time, Dedekind computes with only these notations. Eventually, the need arises for more modes of representation of operations between modules, such as:

$$\begin{aligned} \mathfrak{a}_2 &= \mathfrak{a} + \mathfrak{a}_1 \\ \mathfrak{a}'' &= \mathfrak{a} - \mathfrak{a}' \\ \mathfrak{a}_3 &= \mathfrak{b}_1 + \mathfrak{c}_1 \\ \mathfrak{a}''' &= \mathfrak{b}' - \mathfrak{c}' \end{aligned}$$

And so on. Several manuscripts bear the traces of Dedekind's reflections on how to represent modules and their operations. A striking example of this is what seems to be the very first attempt at a systematic table of all the possible combinations of modules with $+$ and $-$ in Cod. Ms. Dedekind X 11-1, p. 18v. See p. 34.

Once these modes of representation are introduced, Dedekind works with them in a very systematic manner. This is particularly noticeable in the various lists and tables he draws, to which I will come back later. Not only do these ways of writing allow for a considerable gain of time and simplify computations, they are also useful to see results and patterns, such as⁴⁶

$$\begin{aligned} \text{I. } (\mathfrak{b} - \mathfrak{b}') + (\mathfrak{c} - \mathfrak{c}') &= (\mathfrak{c} - \mathfrak{c}') + (\mathfrak{a} - \mathfrak{a}') = (\mathfrak{a} - \mathfrak{a}') + (\mathfrak{b} - \mathfrak{b}') \\ &= (\mathfrak{a} - \mathfrak{a}') + (\mathfrak{b} - \mathfrak{b}') + (\mathfrak{c} - \mathfrak{c}') \\ &= \mathfrak{a}' - \mathfrak{b}' - \mathfrak{c}' \\ \text{II. } (\mathfrak{b} + \mathfrak{b}_1) - (\mathfrak{c} + \mathfrak{c}_1) &= (\mathfrak{c} + \mathfrak{c}_1) - (\mathfrak{a} + \mathfrak{a}_1) = (\mathfrak{a} + \mathfrak{a}_1) - (\mathfrak{b} + \mathfrak{b}_1) \\ &= (\mathfrak{a} + \mathfrak{a}_1) - (\mathfrak{b} + \mathfrak{b}_1) - (\mathfrak{c} + \mathfrak{c}_1) \\ &= \mathfrak{a}_1 + \mathfrak{b}_1 + \mathfrak{c}_1 \end{aligned}$$

which we will encounter several times in this chapter.

The symmetry between the primes and the indexes mirrors the duality of operations and clearly puts it forwards. This notation is used until the mid-1890s, when it is replaced by a slightly different notation, though still relying on the idea of a dualism between the primes and indexes (see below).

Second example of notation

Before this change, and in fact probably in parallel with the other ways of writing described here, Dedekind coined a notation with what we could call symbolic numbers in which he designates modules by numbers. For example, in (Cod. Ms. Dedekind X 10, p. 52), he takes four modules designated by 1, 2, 3 and 4, then:

$$\begin{array}{l|l} 12 = 1 + 2 & 34 = 3 + 4 \\ 13 = 1 + 3 & 24 = 2 + 4 \\ 14 = 1 + 4 & 23 = 2 + 3 \end{array}$$

⁴⁶In the equations below, recall that the primes designate GCDs of two modules, and the indexes designate LCMs of two modules.

12 is thus a notation for $1 + 2$, and so on. Then, he writes:

$$\begin{array}{c} \overline{123 = 12 + 3 = 13 + 2 = 23 + 1} \\ 124 = 12 + 4 = 14 + 2 = 24 + 1 \\ 134 = 13 + 4 = 14 + 3 = 34 + 1 \\ 234 = 23 + 4 = 24 + 3 = 34 + 2 \\ \\ \overline{123 = 123 + 4 = 124 + 3 = 134 + 2 = 234 + 1} \\ = 12 + 34 = 13 + 24 = 14 + 23 \end{array}$$

in which 123 is a notation for $12 + 3$ (i.e., for $(1 + 2) + 3$) and the subsequent equalities give the associativity of addition.

Of course, this kind of concatenation notation is difficult to use for two dual operations. Dedekind does not address this issue, in that manuscript. Rather, he introduces additional notations, namely:

$$\begin{array}{l} a_1 = 1 - 234, a_2 = 2 - 134, a_3 = 3 - 124, a_4 = 4 - 123 \\ 123 - 124 = 12 + a_3 = 12 + a_4 = a_{12} \\ 123 - 134 = 13 + a_2 = 13 + a_4 = a_{13} \\ \dots \\ b_1 = 123 - 124 - 134 \\ b_2 = 123 - 124 - 234 \\ \dots \end{array}$$

Similar notations were used by Cayley in a number of papers⁴⁷. We know that Dedekind read some of Cayley's works on group theory, and it is a fairly reasonable assumption to believe he may have read more.

We can find several such attempts at working with symbolic numbers in Dedekind's drafts. Many of them are abandoned relatively quickly. These seem to be related to reflections on combinatorics and in particular the decomposition of numbers into prime factors, and a version of the notation is used in (Dedekind, 1897b, p. 109), in which Dedekind explains that three "combinations" α, β, γ such that $\alpha = 2347, \beta = 1357, \gamma = 1267$,⁴⁸ then:

$$\begin{array}{lll} \beta + \gamma = 123567, & \gamma + \alpha = 123467, & \alpha + \beta = 123457 \\ \beta - \gamma = 17, & \gamma - \alpha = 27, & \alpha - \beta = 37 \end{array}$$

where $+$ is the union and $-$ the intersection. The combinations used in (Dedekind, 1897b) are said to be "just a system of elements" and thus to belong to "the theory of systems (*Systemlehre*)" (Dedekind, 1897b, p. 112). As such, they fit into the inquiries on *Dualgruppen* as particular cases. Drafts for (Dedekind, 1897b) suggest that, just like for module theory, research on combinations led Dedekind to identify them as part of a larger, more general theory. In his published paper (and in a number of drafts), Dedekind goes back to his usual $+, -$ notation when studying the general case.

⁴⁷(Ribrag, 2019). I would like to thank Hourya Benis-Sinaceur for pointing this out to me.

⁴⁸Here, what Dedekind calls "combinations" are subsets of the set of integers $(1, 2, 3, 4, 5, 6, 7)$.

This concatenation notation is difficult to handle and, as the above example illustrates, sometimes demands to be mixed with other notations. As a computing tool, it is not very efficient. And as a device to highlight duality, it is close to useless, since the symmetry between the operations is almost invisible. In several instances, Dedekind seems to try to work around this last difficulty with the introduction of additional ways of writing used elsewhere, such as the use of primes. There is considerably less consistency and systematicity with this notation than with the two others described above and below.

Third example of notation

Finally, and this is the notation for which we have the greatest amount of material, Dedekind uses a similar strategy as described on p. 25 but with a decreasing scale of primes and indexes. In Cod. Ms. Dedekind X 11-2, p. 15r (approximative date: mid-1890s), for example, Dedekind considers three modules $\mathfrak{a}, \mathfrak{b}, \mathfrak{c}$, and defines:

$$\begin{aligned} \mathfrak{d}'''' &= \mathfrak{a} + \mathfrak{b} + \mathfrak{c} ; \mathfrak{d}_4 = \mathfrak{a} - \mathfrak{b} - \mathfrak{c} \\ \mathfrak{a}''' &= \mathfrak{b} + \mathfrak{c}, \mathfrak{b}''' = \mathfrak{a} + \mathfrak{c}, \mathfrak{c}''' = \mathfrak{a} + \mathfrak{b} ; \mathfrak{a}_3 = \mathfrak{b} - \mathfrak{c}, \mathfrak{b}_3 = \mathfrak{a} - \mathfrak{c}, \mathfrak{c}_3 = \mathfrak{a} - \mathfrak{b} \\ \mathfrak{d}' &= \mathfrak{a}''' - \mathfrak{b}''' - \mathfrak{c}''' ; \mathfrak{d}_1 = \mathfrak{a}_3 + \mathfrak{b}_3 + \mathfrak{c}_3 \end{aligned}$$

$$\left\{ \begin{array}{l} \mathfrak{a}'' = \mathfrak{a} + \mathfrak{d}', \mathfrak{b}'' = \mathfrak{a} + \mathfrak{d}', \mathfrak{c}'' = \mathfrak{a} + \mathfrak{d}' ; \mathfrak{a}_2 = \mathfrak{b} - \mathfrak{d}_1, \mathfrak{b}_2 = \mathfrak{a} - \mathfrak{d}_1, \mathfrak{c}_2 = \mathfrak{a} - \mathfrak{d}_1 \\ \mathfrak{a}' = \mathfrak{a} + \mathfrak{d}_1, \mathfrak{b}' = \mathfrak{a} + \mathfrak{d}_1, \mathfrak{c}' = \mathfrak{a} + \mathfrak{d}_1 ; \mathfrak{a}_1 = \mathfrak{b} - \mathfrak{d}', \mathfrak{b}_1 = \mathfrak{a} - \mathfrak{d}', \mathfrak{c}_1 = \mathfrak{a} - \mathfrak{d}' \\ \mathfrak{a}_0 = \mathfrak{a}' - \mathfrak{d}' = \mathfrak{a}_1 + \mathfrak{d}_1 ; \mathfrak{b}_0 = \mathfrak{b}' - \mathfrak{d}' = \mathfrak{b}_1 + \mathfrak{d}_1 ; \mathfrak{c}_0 = \mathfrak{c}' - \mathfrak{d}' = \mathfrak{c}_1 + \mathfrak{d}_1 \end{array} \right.$$

Some basic computations allow to see that we have also:

$$\left\{ \begin{array}{l} \mathfrak{a}'' = \mathfrak{b}''' - \mathfrak{c}''', \mathfrak{b}'' = \mathfrak{c}''' - \mathfrak{a}''', \mathfrak{c}'' = \mathfrak{a}''' - \mathfrak{b}''' ; \mathfrak{a}_2 = \mathfrak{b}_3 + \mathfrak{c}_3, \mathfrak{b}_2 = \mathfrak{c}_3 + \mathfrak{a}_3, \mathfrak{c}_2 = \mathfrak{a}_3 + \mathfrak{b}_3, \\ \mathfrak{a}' = \mathfrak{a} + \mathfrak{a}_3, \mathfrak{b}' = \mathfrak{b} + \mathfrak{b}_3, \mathfrak{c}' = \mathfrak{c} + \mathfrak{c}_3 ; \mathfrak{a}_1 = \mathfrak{a} - \mathfrak{a}''', \mathfrak{b}_1 = \mathfrak{b} - \mathfrak{b}''', \mathfrak{c}_1 = \mathfrak{c} - \mathfrak{c}''' \\ \mathfrak{a}_0 = \mathfrak{a}_1 + \mathfrak{a}_3 = \mathfrak{a}' - \mathfrak{a}''', \mathfrak{b}_0 = \mathfrak{b}_1 + \mathfrak{b}_3 = \mathfrak{b}' - \mathfrak{b}''', \mathfrak{c}_0 = \mathfrak{c}_1 + \mathfrak{c}_3 = \mathfrak{c}' - \mathfrak{c}''' \end{array} \right.$$

This third notation, following a similar idea as the first presented above, is the one that is the most widely used in Dedekind's drafts, and the one used in the published papers. It is unclear why Dedekind preferred the decreasing scale of indexes and primes over his first notation. In any case, using this decreasing scale of indexes and primes implies that Dedekind knew the number of elements of the group of modules.

How the dualism is highlighted by the notation

Dedekind's choice of notations allows him to state results displaying the “dualism” in module theory, such as, in Cod. Ms. Dedekind X 11-2, p. 15v., following the above manuscript. This example allows us to see how Dedekind worked with duality, and how the notations he devised helped him in this enterprise. In the top part of the page, Dedekind states results for numbers of classes, such as:

$$(\mathfrak{d}''', \mathfrak{a}''') = (\mathfrak{b}''', \mathfrak{c}'') = (\mathfrak{a}'', \mathfrak{d}') = (\mathfrak{a}', \mathfrak{a}_0) = (\mathfrak{a}, \mathfrak{a}_1) = a$$

using the definition of the elements, and the divisibility relations between them. He does so for $(\mathfrak{b}, \mathfrak{b}_1) = b$, $(\mathfrak{a}', \mathfrak{a}) = a_1$, $(\mathfrak{b}', \mathfrak{b}) = b_1$, $(\mathfrak{c}', \mathfrak{c}) = c_1$. This leads him to:

$$\left. \begin{aligned} (\mathfrak{a}'', \mathfrak{a}') &= (\mathfrak{b}'', \mathfrak{b}') = (\mathfrak{c}'', \mathfrak{c}') = (\mathfrak{d}', \mathfrak{a}_0) = (\mathfrak{d}', \mathfrak{b}_0) = (\mathfrak{d}', \mathfrak{c}_0) = \\ (\mathfrak{a}_1, \mathfrak{a}_2) &= (\mathfrak{b}_1, \mathfrak{b}_2) = (\mathfrak{c}_1, \mathfrak{c}_2) = (\mathfrak{a}_0, \mathfrak{d}_1) = (\mathfrak{b}_0, \mathfrak{d}_1) = (\mathfrak{c}_0, \mathfrak{d}_1) = \end{aligned} \right\} = h \text{ or } d.^{49}$$

From which he gets:

$$\left. \begin{aligned} (\mathfrak{b}, \mathfrak{c}) &= hbc_1 \\ (\mathfrak{c}, \mathfrak{a}) &= hca_1 \\ (\mathfrak{a}, \mathfrak{b}) &= hab_1 \end{aligned} \right| \begin{aligned} (\mathfrak{c}, \mathfrak{b}) &= hcb_1 \\ (\mathfrak{a}, \mathfrak{c}) &= hac_1 \\ (\mathfrak{b}, \mathfrak{a}) &= hba_1 \end{aligned}$$

And finally:

$$(\mathfrak{b}, \mathfrak{c})(\mathfrak{c}, \mathfrak{a})(\mathfrak{a}, \mathfrak{b}) = (\mathfrak{c}, \mathfrak{b})(\mathfrak{a}, \mathfrak{c})(\mathfrak{b}, \mathfrak{a}) = h^3 abca_1 b_1 c_1.$$

This last theorem, he writes, “can also be proven without the complete theory of three modules (dualism) with only the three elementary theorems”:

$$\text{I. } (\mathfrak{a}, \mathfrak{b}) = (\mathfrak{a} + \mathfrak{b}, \mathfrak{b})$$

$$\text{II. } (\mathfrak{a}, \mathfrak{b}) = (\mathfrak{a}, \mathfrak{a} - \mathfrak{b})$$

$$\text{III. If } \mathfrak{p} \text{ divisor of } \mathfrak{n}, \text{ and } \mathfrak{n} \text{ divisor of } \mathfrak{m}, \text{ then } (\mathfrak{p}, \mathfrak{m}) = (\mathfrak{p}, \mathfrak{n})(\mathfrak{n}, \mathfrak{m})[. \dots]$$

In fact, with application of the above notations (not theorems), we have

$$(\mathfrak{b}, \mathfrak{c}) = (\mathfrak{b} + \mathfrak{c}, \mathfrak{c}) = (\mathfrak{a}''', \mathfrak{c}) \text{ according to I.}$$

The “complete theory of three modules (*dualism*)” seems to designate, here, the formation of all the elements generated by \mathfrak{a} , \mathfrak{b} , \mathfrak{c} . But Dedekind certainly does not move away from *duality* related observations in what follows:

Moreover, since $\mathfrak{c}'' = \mathfrak{a}''' - \mathfrak{b}'''$ is a multiple of \mathfrak{a}''' and a divisor of \mathfrak{c} , then we have

$$(\mathfrak{a}''', \mathfrak{c}) = (\mathfrak{a}''', \mathfrak{c}'')(\mathfrak{c}'', \mathfrak{c}) \text{ according to III;}$$

furthermore, we have

$$(\mathfrak{a}''', \mathfrak{c}'') = (\mathfrak{a}''', \mathfrak{a}''' - \mathfrak{b}''') = (\mathfrak{a}''', \mathfrak{b}''') \text{ according to II,}$$

but we have $\mathfrak{b} + \mathfrak{b}''' = \mathfrak{b} + (\mathfrak{c} + \mathfrak{a}) \mathfrak{a}''' + \mathfrak{b}''' = (\mathfrak{b} + \mathfrak{c}) + (\mathfrak{c} + \mathfrak{a}) = \mathfrak{b} + \mathfrak{c} + \mathfrak{a} = \mathfrak{b} + \mathfrak{b}'''$, consequently

$$(\mathfrak{a}''', \mathfrak{b}''') = (\mathfrak{a}''' + \mathfrak{b}''', \mathfrak{b}''') = (\mathfrak{b} + \mathfrak{b}''', \mathfrak{b}''') = (\mathfrak{b}, \mathfrak{b}''') \text{ according to I ;}$$

and it follows the first of the six equations

$$\text{IV. } \left\{ \begin{aligned} (\mathfrak{b}, \mathfrak{c}) &= (\mathfrak{b}, \mathfrak{b}''')(\mathfrak{c}'', \mathfrak{c}); (\mathfrak{c}, \mathfrak{a}) = (\mathfrak{c}, \mathfrak{c}''')(\mathfrak{a}'', \mathfrak{a}); (\mathfrak{a}, \mathfrak{b}) = (\mathfrak{a}, \mathfrak{a}''')(\mathfrak{b}'', \mathfrak{b}) \\ (\mathfrak{c}, \mathfrak{b}) &= (\mathfrak{c}, \mathfrak{c}''')(\mathfrak{b}'', \mathfrak{b}); (\mathfrak{a}, \mathfrak{c}) = (\mathfrak{a}, \mathfrak{a}''')(\mathfrak{c}'', \mathfrak{c}); (\mathfrak{b}, \mathfrak{a}) = (\mathfrak{b}, \mathfrak{b}''')(\mathfrak{a}'', \mathfrak{a}) \end{aligned} \right.$$

from which the remaining ones follow from permutations, and at the same time follows the above theorem

$$\text{V. } (\mathfrak{b}, \mathfrak{c})(\mathfrak{c}, \mathfrak{a})(\mathfrak{a}, \mathfrak{b}) = (\mathfrak{c}, \mathfrak{b})(\mathfrak{a}, \mathfrak{c})(\mathfrak{b}, \mathfrak{a}) = (\mathfrak{a}, \mathfrak{a}''')(\mathfrak{b}, \mathfrak{b}''')(\mathfrak{c}, \mathfrak{c}''')(\mathfrak{a}'', \mathfrak{a})(\mathfrak{b}'', \mathfrak{b})(\mathfrak{c}'', \mathfrak{c}).$$

⁴⁹The d is not used elsewhere in the manuscript. It is unclear what it refers to and, in fact, it could just be a change of heart regarding naming the result with the letter h .

Then, writes Dedekind, “dualistically, it follows also”:

$$(\mathfrak{b}, \mathfrak{c}) = (\mathfrak{b}, \mathfrak{b} - \mathfrak{c}) = (\mathfrak{b}, \mathfrak{a}_3) \text{ according to II.}$$

furthermore $\mathfrak{a}_2 = \mathfrak{c}_3 + \mathfrak{a}_3$ is a divisor of \mathfrak{a}_3 and a multiple of \mathfrak{b} , so we have

$$(\mathfrak{b}, \mathfrak{a}_2) = (\mathfrak{b}, \mathfrak{b}_2)(\mathfrak{b}_2, \mathfrak{a}_3) \text{ according to III.}$$

furthermore, we have

$$(\mathfrak{b}_2, \mathfrak{a}_3) = (\mathfrak{c}_3 + \mathfrak{a}_3, \mathfrak{a}_3) = (\mathfrak{c}_3, \mathfrak{a}_3) \text{ according to I.}$$

but we have $\mathfrak{c}_3 - \mathfrak{a}_3 = (\mathfrak{a} - \mathfrak{b}) - (\mathfrak{b} - \mathfrak{c}) = \mathfrak{a} - \mathfrak{b} - \mathfrak{c} = \mathfrak{c} - \mathfrak{c}_3$, consequently

$$(\mathfrak{c}_3, \mathfrak{a}_3) = (\mathfrak{c}_3, \mathfrak{c}_3 - \mathfrak{a}_3) = (\mathfrak{c}_3, \mathfrak{c} - \mathfrak{c}_3) = (\mathfrak{c}_3, \mathfrak{c}) \text{ according to II.,}$$

consequently

$$\text{VI. } \begin{cases} (\mathfrak{b}, \mathfrak{c}) = (\mathfrak{b}, \mathfrak{b}_2)(\mathfrak{c}_3, \mathfrak{c}); & (\mathfrak{c}, \mathfrak{a}) = (\mathfrak{c}, \mathfrak{c}_2)(\mathfrak{a}_3, \mathfrak{a}); & (\mathfrak{a}, \mathfrak{b}) = (\mathfrak{a}, \mathfrak{a}_2)(\mathfrak{b}_3, \mathfrak{b}) \\ (\mathfrak{c}, \mathfrak{b}) = (\mathfrak{c}, \mathfrak{c}_2)(\mathfrak{b}_3, \mathfrak{b}); & (\mathfrak{a}, \mathfrak{c}) = (\mathfrak{a}, \mathfrak{a}_2)(\mathfrak{c}_3, \mathfrak{c}); & (\mathfrak{b}, \mathfrak{a}) = (\mathfrak{b}, \mathfrak{b}_2)(\mathfrak{a}_3, \mathfrak{a}) \end{cases}$$

And finally:

$$\text{VII. } (\mathfrak{b}, \mathfrak{c})(\mathfrak{c}, \mathfrak{a})(\mathfrak{a}, \mathfrak{b}) = (\mathfrak{c}, \mathfrak{b})(\mathfrak{a}, \mathfrak{c})(\mathfrak{b}, \mathfrak{c}) = (\mathfrak{a}, \mathfrak{a}_2)(\mathfrak{b}, \mathfrak{b}_2)(\mathfrak{c}, \mathfrak{c}_2)(\mathfrak{a}_3, \mathfrak{a})(\mathfrak{b}_3, \mathfrak{b})(\mathfrak{c}_3, \mathfrak{c}).$$

(Cod. Ms. Dedekind X 11-2, p. 15v.)

For Dedekind, this equality is the dual of V. It but must be looked at diagonally to ‘see’ the duality. Indeed, with the equalities we gave on p. 28, $\mathfrak{a}'' = \mathfrak{a} + \mathfrak{d}' = \mathfrak{b}''' - \mathfrak{c}'''$ and $\mathfrak{a}_2 = \mathfrak{a} - \mathfrak{d}_1 = \mathfrak{a}_3 + \mathfrak{c}_3$ are dual of each other,⁵⁰ and so on for \mathfrak{b}'' , \mathfrak{c}'' and \mathfrak{b}_2 , \mathfrak{c}_2 . The equalities in V. and VII. are thus equivalent to:

$$(\mathfrak{b}, \mathfrak{c})(\mathfrak{c}, \mathfrak{a})(\mathfrak{a}, \mathfrak{b}) = \begin{cases} (\mathfrak{a}, \mathfrak{b} + \mathfrak{c})(\mathfrak{b}, \mathfrak{a} + \mathfrak{c})(\mathfrak{c}, \mathfrak{a} + \mathfrak{b})(\mathfrak{b}''' - \mathfrak{c}''', \mathfrak{a})(\mathfrak{a}''' - \mathfrak{c}''', \mathfrak{b})(\mathfrak{b}''' - \mathfrak{a}''', \mathfrak{c}) \\ (\mathfrak{a}, \mathfrak{b}_3 + \mathfrak{c}_3)(\mathfrak{b}, \mathfrak{c}_3 + \mathfrak{a}_3)(\mathfrak{c}, \mathfrak{b}_3 + \mathfrak{a}_3)(\mathfrak{b} - \mathfrak{c}, \mathfrak{a})(\mathfrak{a} - \mathfrak{c}, \mathfrak{b})(\mathfrak{a} - \mathfrak{b}, \mathfrak{c}) \end{cases}$$

where the colors (mine) highlight the dualism between each side⁵¹.

Despite the changes in the notations, the play of symmetry between the primes for the addition and the indexes for the subtraction remains a central feature. It gives Dedekind the possibility to systematically work with this symmetry, and to put forward what he sometimes calls “dual pairs”. For example,

⁵⁰ $\mathfrak{d}' = \mathfrak{a}''' - \mathfrak{b}''' - \mathfrak{c}'''$ and $\mathfrak{d}_1 = \mathfrak{a}_3 + \mathfrak{b}_3 + \mathfrak{c}_3$ are dual of each other.

⁵¹If we develop again the \mathfrak{a}''' , etc., and \mathfrak{a}_3 , etc., we get: $(\mathfrak{b}, \mathfrak{c})(\mathfrak{c}, \mathfrak{a})(\mathfrak{a}, \mathfrak{b}) =$

$$\begin{cases} (\mathfrak{a}, \mathfrak{b} + \mathfrak{c})(\mathfrak{b}, \mathfrak{a} + \mathfrak{c})(\mathfrak{c}, \mathfrak{a} + \mathfrak{b})((\mathfrak{a} + \mathfrak{c}) - (\mathfrak{a} + \mathfrak{b}), \mathfrak{a})((\mathfrak{b} + \mathfrak{c}) - (\mathfrak{a} + \mathfrak{b}), \mathfrak{b})((\mathfrak{a} + \mathfrak{c}) - (\mathfrak{b} + \mathfrak{c}), \mathfrak{c}) \\ (\mathfrak{a}, (\mathfrak{a} - \mathfrak{c}) + (\mathfrak{a} - \mathfrak{b}))(\mathfrak{b}, (\mathfrak{a} - \mathfrak{b}) + (\mathfrak{b} - \mathfrak{c}))(\mathfrak{c}, (\mathfrak{a} - \mathfrak{c}) + (\mathfrak{b} - \mathfrak{c}))(\mathfrak{b} - \mathfrak{c}, \mathfrak{a})(\mathfrak{a} - \mathfrak{c}, \mathfrak{b})(\mathfrak{a} - \mathfrak{b}, \mathfrak{c}) \end{cases}$$

in a later draft, in Cod. Ms. Dedekind XI 2, p. 54r, he wrote:

Primary divisibilities:		Dual pairs
$d'''' < b''', c'''$	$d_1 > c_0, a'$	d''''', d_1
$b''' < a'', c'$	$c_0 > a'', c'$	b''', c_0
$c''' < a''$	$a' > a''$	c''', a'
$a'' < a', c_0$	$a'' > c''', b'''$	a'', a''
$c' < c_0$	$c' > b'''$	c', c'
$a' < d_1$	$c''' > d''''$	
$c_0 < d_1$	$b''' > d''''$	

Changing notations?

Dedekind does not justify either the notations, nor their modifications. Nevertheless, we do find several drafts acting the change of notation as being a “better designation” (*bessere Bezeichnung*). Some of them consist in a complete rewriting of the results, especially the tables of all possible combinations for 3 modules, which I will describe in the next section. In other drafts, Dedekind changes notations while working, which confirms that the way of writing played an essential role in his research. For example, in Cod. Ms. Dedekind X 10, p. 1, Dedekind starts with a “simple *Modulgruppe* (or chain)”. “Chain” means simply an ordered sequence of modules, here denoted by $1 < 2 < 3 < \dots < n-1 < n$ (with always $r < s < t < u$).⁵² He then considers a module 0 such that

$$\begin{aligned}
 r' &= 0 + r \text{ of which there are } n \\
 r'' &= 0 + r \text{ of which there are } n \\
 rs &= r'' + s = s' - r, \text{ with } r < s \text{ of which there are } \frac{n(n-1)}{2}
 \end{aligned}$$

The multiplicative notation rs , albeit slightly unusual, is reminiscent of the one used in cases in which he denotes the modules with numbers. The lefthand side of the manuscript contains columns listing possible combinations of the group thus generated using letters $r, s, t, u, r', r'', \dots, s', s'', \dots$ (of which there are $(n+1) + n + n + \frac{n(n-1)}{2} = \frac{n^2+5n+2}{2}$). He then proposes another presentation:

$$\begin{aligned}
 \mathfrak{a}_r &\text{ instead of } r \\
 \mathfrak{b}_r &\text{ instead of } r' = n + a_r \\
 \mathfrak{c}_r &\text{ instead of } r'' = n - a_r \\
 \mathfrak{d}_{rs} &\text{ instead of } rs = c_r + a_s \\
 \mathfrak{m} &\text{ instead of } 0
 \end{aligned}$$

With this new notation — which seems more manageable —, he studies the order in this *Modulgruppe*. But by the bottom of the manuscript, after drawing

⁵²Dedekind used chains of modules in number theory. The fact that he considered *Modulgruppen* and chains together, here, gives us yet another indication that the consideration of “groups” of modules was a very progressive step taken by Dedekind.

a separation, he notes “Yet another presentation!!” and proposes that the n given modules be renamed:

$$\mathfrak{a}_{-n+1} < \mathfrak{a}_{-n+3} < \dots < \mathfrak{a}_{n-3} < \mathfrak{a}_{n-1}; \mathfrak{b}_{-n} < \mathfrak{b}_{-n+1} < \dots < \mathfrak{b}_n.$$

The module to be added is called \mathfrak{b} , and we have

$$\begin{aligned} \mathfrak{b} + \mathfrak{a}_r &= \mathfrak{b}_{\frac{r-n-1}{2}} \\ \mathfrak{b} + \mathfrak{a}_r &= \mathfrak{b}_{\frac{r+n+1}{2}} \end{aligned}$$

From there, Dedekind continues to investigate properties of the order for a couple of lines.⁵³ On a corner, he notes the following with the commentary “*Versuch!*”:

$$\left. \begin{aligned} (\mathfrak{b} - \mathfrak{a}_r) + \mathfrak{a}_{r+2} &= \\ (\mathfrak{b} + \mathfrak{a}_{r+2} - \mathfrak{a}_r &= \end{aligned} \right\} \mathfrak{a}'_{r+1}$$

The changes do not seem to be prompted by considerations on duality — at least, not on duality alone. Rather, they are motivated by how practical, manageable and readable the notation can be. In these considerations on usability and readability, duality comes into account as one of the questions to be investigated when looking for fundamental laws or general properties.⁵⁴

An additional change of notation happens later in the progress of the research, which is, this time, justified by Dedekind, namely replacing the signs $+$ and $-$ by the Greek letters φ and ψ when generalising the investigation to arbitrary binary operations verifying a set of properties such as associativity, absorption etc. This move, which I study in section 4.4.4, is a fairly usual one for Dedekind.

⁵³On the next page (written with a similar pencil, but on a different piece of paper), Dedekind studies a chain of modules $\mathfrak{b}_1 < \dots < \mathfrak{b}_n$ to which he adds a module \mathfrak{a} such that

$$\mathfrak{a} + \mathfrak{b}_1 < \dots < \mathfrak{a} + \mathfrak{b}_n < \mathfrak{a} < \mathfrak{a} - \mathfrak{b}_1 < \dots < \mathfrak{a} - \mathfrak{b}_n.$$

This additional hypothesis on \mathfrak{a} gives:

$$\begin{aligned} (\mathfrak{a} + \mathfrak{b}_r) - \mathfrak{b}_s &= (\mathfrak{a} - \mathfrak{b}_s) + \mathfrak{b}_r \text{ when } r \geq s \\ (\mathfrak{a} + \mathfrak{b}_r) - \mathfrak{b}_s &= \mathfrak{b}_s \text{ when } r \leq s \\ (\mathfrak{a} - \mathfrak{b}_s) + \mathfrak{b}_s &= \mathfrak{b}_r \text{ when } r \leq s \end{aligned}$$

thus simplifying the research. Dedekind then considers examples, in which the modules are denoted by the numbers 1, 2, 3 and 0.

⁵⁴Cod. Ms. Dedekind X 11-1, p. 42-43 is another good example of adapting notations to research (see <http://eman-archives.org/Dedekind/items/show/211>). The two pages Cod. Ms. Dedekind X 10, p. 20-21 show two similar studies on finitely generated modules with two different notations, one of them being labelled “old designation” (*alte Bezeichnung*) (see <http://eman-archives.org/Dedekind/items/show/242>). Cod. Ms. Dedekind X 10, p. 48, which is studied below p. 50 is also an example of changes in notations during the writing. See also the documents listed on the page <https://eman-archives.org/Dedekind/items/browse?tags=meilleure-presentation>, where are gathered the various documents in which Dedekind explicitly noted using a different (new, older or better) presentation.

4.3.3 Layouts and research devices

In Dedekind's drafts, we observe isolated computations, detailed case studies, tables, lists, partially written texts, even diagrams. These various manners of approaching the study of modules serve as an intensive, repeated exploration of the properties of modules and their operations. A striking feature of Dedekind's drafts is the way in which they are organised, how the writings are spatially arranged, which not only marks a significant departure from the linear, rectangular printed text, but also testifies of a reflection on how and where to position certain computations and/or results, how to organise them, and sometimes even how to provide a visual support for the research (showcasing duality, internal 'structure', order. . .). Depending on the step of the research presented in the draft, the spatial organisation of the writings can testify to successive operations or of a willingness to organise the results in a specific manner — both to support the very systematic way in which Dedekind works, and to showcase certain results, among which is the duality. It seems, as the following paragraphs will illustrate, that there is a reciprocal relation between the choices of notation and organisation of writings and the investigation of the dualistic properties of divisibility of modules: some of these choices played a role in the observation and clarification of properties of duality, and reciprocally the attention to duality certainly guided a number of such choices.

To give the reader a taste of the spatial organisation of Dedekind's work-in-progress manuscripts, I will focus on a series of pages in Cod. Ms. Dedekind X 11-1 (early to mid-1890s), which are particularly striking for two reasons. Firstly, the visual features: the organisation in columns, small frames, with and without diagrams, sometimes rewritten several times, the presence of very clear corrections show us the progression of the research. Secondly, they appear to constitute a delimited sequence towards an identifiable set of manuscripts, which contain cleaned up versions of the results (here, lists and tables for the theory of three modules, Cod. Ms. Dedekind X 11-1, pp. 40-41).⁵⁵ Thus, they show us different states of research and writing. It is important to keep in mind that the progression might not have been linear. It is however possible to distinguish some steps in the development of the investigation (by comparing to the final results, or again with erasures, displacements, etc.). The order in which the manuscripts are presented in the following section does not mean that a document was written before the next one (which can be difficult to assess precisely). I have rather chosen to present the manuscripts according to their form and contents.

The choice of these pages also comes from the fact that these documents' features are very representative of what can be found in Dedekind's drafts. The layouts are fairly consistent throughout Dedekind's years of research, up to the last stages, and even after the definition for the *Dualgruppe* was given.

⁵⁵See also

<http://eman-archives.org/Dedekind/exhibits/show/parcours-vers-document-de-trav/> which exposes (in French) the steps.

Tables

The first document I would like to present is the table in Cod. Ms. Dedekind X 11-1, p. 18v, which lists the modules generated by three modules a , b , and c (in other words, not used on this sheet by Dedekind, the members of the *Modulgruppe* generated by a , b , and c). This table, written on an old marking sheet for the winter semester of 1872-73, bears the traces of its own elaboration. Dedekind first wrote the elements themselves, then replaced them by the notation he designed. A transcription of the first nine columns and four lines gives us:

a	b	c	a'	b'	c'	a_1	b_1	c_1
a'	b'	c'	d	d	d	a'''	b'''	c'''
			$b' - e'$	$e' - a'$	$a' - b'$			
a_1	b_1	c_1	a_3	b_3	c_3	m	m	m
a''	b''	c''	$b_1 + c_1$	$c_1 + a_1$	$a_1 + b_1$			
$a - a_1$	$b - b_1$	$c - c_1$	m'	m'	m'	a_1	b_1	c_1

Crossed-out cells show the moment when Dedekind decided to introduce the supplementary designations. We can, here, observe the temporality of the writing of this table. The complete table suggests that Dedekind might have been looking for a pattern. Because an image is sometimes, indeed, worth a thousand words, a reproduction of this table can be seen as Figure 4.1, p. 35.⁵⁶

The many rewritings of similar tables for three modules also suggest that he was looking for a way to display the clean symmetry which can be seen in the finished table (as published in (Dedekind, 1900, pp. 246–247)). Indeed, Dedekind drew a dozen of such tables standing on their own, and many more (smaller ones) integrated into the research. We find tables of divisors, of GCD / LCM, of divisibility, several of which are 28 lines and 28 columns. Interestingly Dedekind did not, in fact, draw a lot of different tables, he drew several times the same table with different notations, different organisations. The prevalent kind of table gives the GCD of two modules on the upper right side and LCM of two modules on the lower left side. See Figure 4.2, p. 36. Dedekind draws several versions of this table, in particular as he changes the notation used.

Dedekind also drew a couple of tables dedicated to showcasing the divisibility relations between modules, in particular a 28×28 table in which the sign $+$ indicates that the module on the line is a divisor of the module in the column, and the sign $-$ indicates that the module on the line is a multiple of the module in the column. See Figure 4.3, p. 37.⁵⁷

In these two types of tables, the presentation and notation chosen emphasize the dualism, as they display a perfect symmetry between GCDs and LCMs (or between divisor and multiples).

⁵⁶Manuscripts are reproduced with the autorisation of the Niedersächsische Staats- und Universitätsbibliothek Göttingen.

⁵⁷Another attempt at such a table is done in Cod. Ms. Dedekind X 11-1, p. 45, in which the signs for being a divisor and being a multiple were respectively a O and an X .

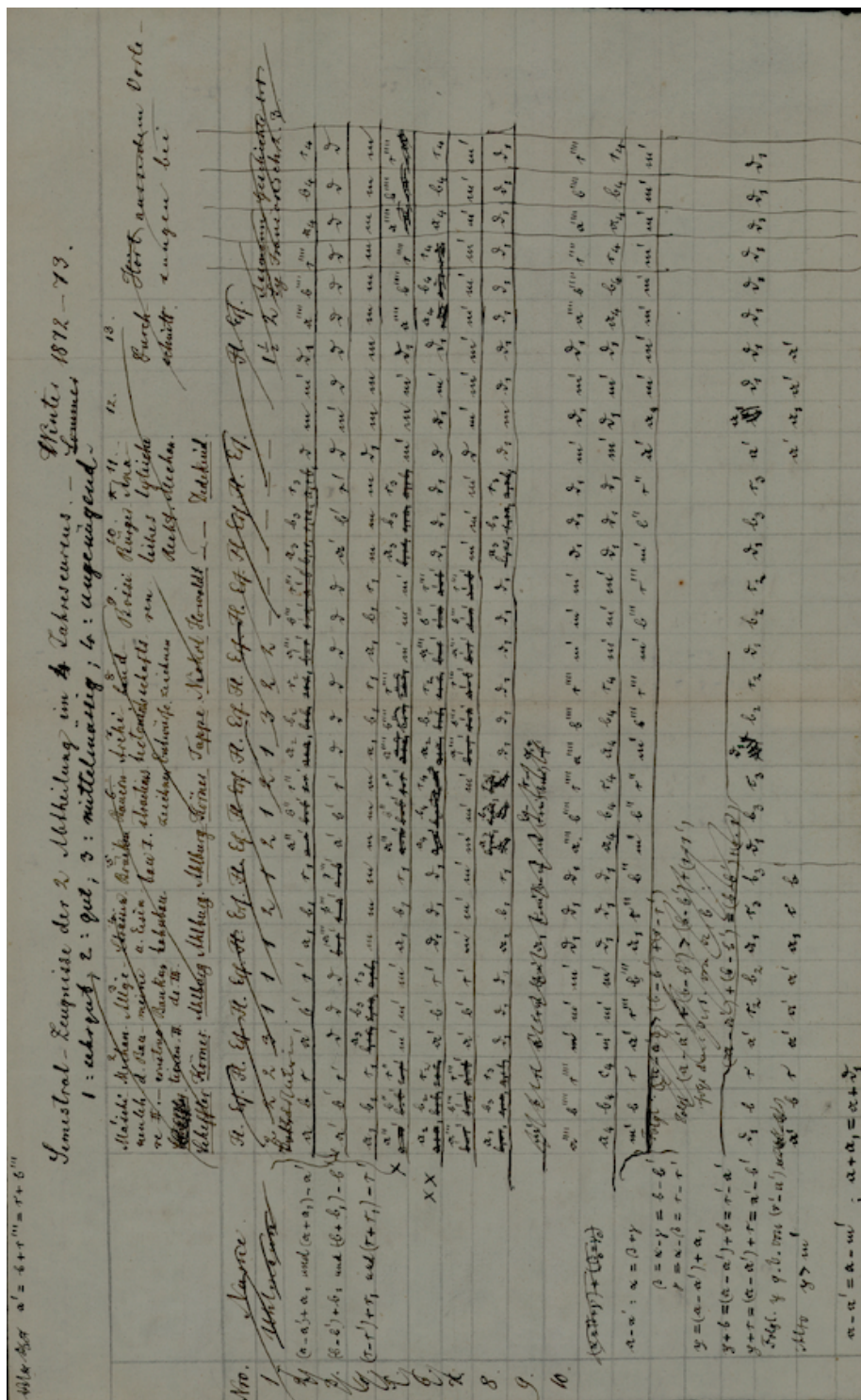


Figure 4.1: Cod. Ms. Dedekind X 11-1, p.18v.

Gründende gemeinschaftliche Theiler

Kleinste gemeinschaftliche Vielfache

Figure 4.2: Cod. Ms. Dedekind X 11-1, p. 41r.

[illegible]

Figure 4.3: Cod. Ms. Dedekind X 11-1, p. 46r.

Finally, Dedekind also drew several tables whose aim seems to have been to organize the results on chains and levels.

Spatial organisation(s) of writings and results

When computing all the possible combinations of three modules, Dedekind experiments with organizing them in different ways. We find, in his drafts, several variations of lists and tables for these modules. It is unclear whether these documents were developed simultaneously, and if not which one was written first. In any case, they testify to a moment of Dedekind's research process during which he was computing all possible combinations and looking for the best way to organize and/or present his results.

Cod. Ms. Dedekind X 11-1, p. 25 presents a relatively straightforward list of all possible combinations, written as the combination of two operands (the elements a' , a_1 , a'' , a_2 , etc. have already been defined, although their definition is not re-written on this page), and sorted according to the operation. Here is a transcription of the first five lines and columns:⁵⁸

$\underline{a + b = c'}$	$\underline{a - b = c_1}$	$\underline{a - b = c_1}$	$c' - m' = m'$	$a'' - m' = a''$
$\underline{a + c = b'}$	$\underline{a - c = b_1}$	$\underline{a - c = b_1}$	$b' - d = m'$	$a_1 - m = m$
$a + c' = c'$	$a - c' = a$	$a - c' = a$	$b' - c_1 = c_1$	$a_1 - b'' = a_1$
$a + b' = b'$	$a - b' = a$	$a - b' = a$	$b' - b_1 = b_1$	$a_1 - c'' = a_1$
$\underline{b + c = a'}$	$\underline{a - a' = a''}$	$\underline{a - a' = a''}$	$b' - a'' = a''$	$a_1 - a''' = a_1$
...

The crossings-out and a few arrows to move certain lines indicate that this list was written at a relatively early stage. Note also that some elements seem to be missing, as the *Modulgruppe* generated by three modules contains 28 elements, and Dedekind's list only gives 21 of them.

The first two columns suggest an attention to the dualism between the results, but the table appears to have been written too early in the research development for this to be fully accomplished. A complete and clean version (with the notation described p. 28) can be found in Cod. Ms. Dedekind X 11-1, p 10. In this clean version, the dualism is clearly highlighted by the columns carefully putting face to face the dual elements. For example, the first six lines:

$a + a = a$	$a - a = a$	$a + a'' = a''$	$a - a_2 = a_2$
$a + b = c'''$	$a - b = c_3$	$a + b'' = c'''$	$a - b_2 = c_3$
$a + c = b'''$	$a - c = c_3$	$a + c'' = b'''$	$a - c_2 = b_3$
$b + b = b$	$b - b = b$	$b + a'' = c'''$	$b - a_2 = c_3$
$b + c = a'''$	$b - c = a_3$	$b + b'' = b''$	$b - b_2 = b_2$
$c + c = c$	$c - c = c$	$b + c'' = a'''$	$b - a_2 = a_3$
...

The double line separating the first two and the last two columns is a delimiter signifying that each set of columns go together. The second set is only the con-

⁵⁸The underlined equalities correspond to the definition of the element.

tinuation of the list. The dualism here, is particularly striking and showcased beautifully by the symmetry in Dedekind's notation.

Cod. Ms. Dedekind X 11-1, p. 26 is a combination of a list and a table — or, to put it differently, a table of lists. Organised in 28 columns (and three unused) each corresponding to an element, the table gives the additions (and only the additions) for two operands between the elements naming each column and all the other elements. This organisation allows Dedekind to avoid repeating the same computations (thanks to the commutativity of the operation) and results in a triangular shaped table, of which I transcribe the first five lines and columns:

a	b	c	d	m	\dots
$a + b = c'$					
$a + c = b'$	$b + c = a'$				
$a + d = d$	$b + d = d$	$c + d = d$			
$a + m = a$	$b + m = b$	$c + m = c$	$d + m = d$		
$a + a' = d$	$b + a' = a'$	$c + a' = a'$	$d + a' = d$	$m + a' = a$	
\dots	\dots	\dots	\dots	\dots	\dots

At the bottom of each column are written the operations for the three additional elements a_4 , b_4 and c_4 (e.g., $a + a_4 =$). They were crossed out later. This suggests that Dedekind first wrote all the possible combinations and filled them afterwards, since the *Modulgruppe* generated by three modules only contains 28 elements. So, this document and the previous one show us that this fact was not clear at this point for Dedekind. After the table, Dedekind wrote some results on the order, on which I will come back later.

The third example in this series is Cod. Ms. Dedekind X 11-1, p. 27, the only one with a title: “*Modulgruppe* built by three arbitrary modules \mathfrak{a} , \mathfrak{b} , \mathfrak{c} ”. On the left side of the page, Dedekind wrote a numbered list of all the 28 modules in a column entitled “Definition”. On the right side, facing each line, under the title “Propositions” (*Sätze*), are listed all the combinations equal to the corresponding module in the “Definition” column. For example:

$$\begin{array}{l}
 \dots \\
 6. \quad \left\{ \begin{array}{l} a' = b + c \\ b' = c + a \\ c' = a + b \end{array} \right. \left| \begin{array}{l} a' = b''' + c''' = b + c''' = c + b''' = b + c_2 = c + b_2 = b_2 + c_2 \\ b' = c''' + a''' = c + a''' = a + c''' = c + a_2 = a + c_2 = c_2 + a_2 \\ c' = a''' + c''' = a + b''' = b + a''' = a + b_2 = b + a_2 = a_2 + b_2 \end{array} \right. \\
 7. \\
 8. \\
 \dots
 \end{array}$$

Again, the way the writings are arranged tells us that Dedekind wrote his “propositions” as his computations progressed, since he did not always plan enough room on some of the lines.

Such tables and lists are one of Dedekind's main tools in his investigations on the “theory of three modules”, and indeed part of a larger toolbox developed

to accompany his computations. They constitute a result in themselves insofar as they contain the results of Dedekind's many computations, which are subsequently ordered and put in the said table. Dedekind, when developing a more general theory, insists on the importance of having tables of all the elements of a finite *Modul-* or *Dualgruppe*. But the tables also appear to be a device for Dedekind, for conducting computations: they help shorten and verify the computations, and they also might help to better see the patterns. In particular, the duality which appears in the computations and can be difficult to fully visualize or understand, becomes striking with a graphical device such as this one.

Representing the order

At several points, after the lists and tables exposed above, Dedekind wrote some results on the order, with lists of “immediate neighbours” (*Unmittelbare Nachbarn*), which he will later call “nearest divisors” (or “nearest multiples”). To study this aspect, he also set up specific layouts. The small table showcasing “dual pairs” reproduced on p. 31 is an example. In the series of manuscripts we have been looking at in this section, considerations on the order are often written after the table. One page is dedicated to it, in a similar way as the above cited manuscripts were dedicated to the combinations: Cod. Ms. Dedekind X 11-1, p. 20. Dedekind separated the page in two sides: the left side for $<$, and the right side for $>$ (Figure 4.4, p. 40).

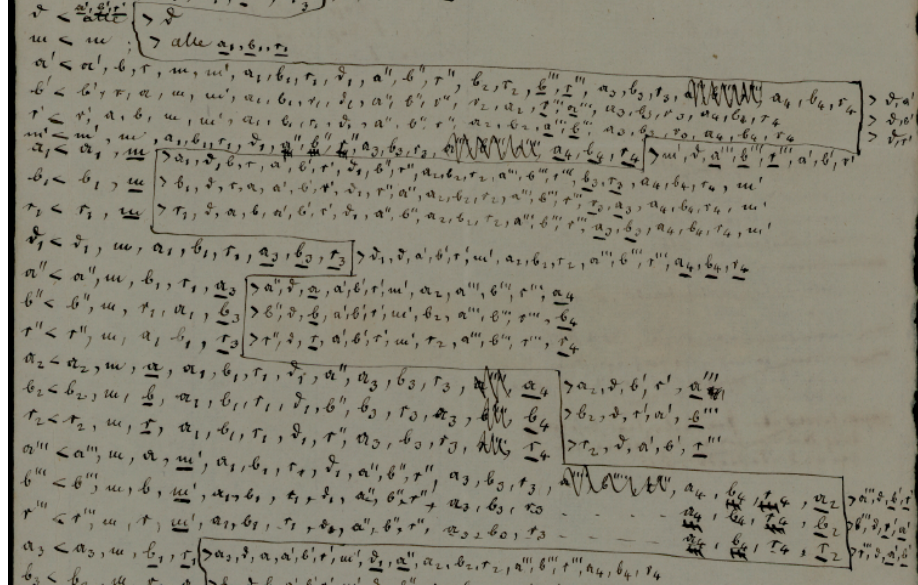


Figure 4.4: Cod. Ms. Dedekind X 11-1, p. 20v.

The disposition of the writings on this page suggests that Dedekind first wrote the left side, and completed the right side afterwards. Here, Dedekind

tried to state the order relation of each element in comparison to the others: all the modules $< \mathfrak{a}$ and all the modules $> \mathfrak{a}$, all the modules $< \mathfrak{b}$ and all the modules $> \mathfrak{b}$, etc. This exhaustiveness is not systematic in Dedekind's studies on the order relation. His interest rather lies in three key notions: nearest divisors/multiples, levels and chains.

For nearest divisors/multiples and chains, Dedekind usually adopted a simple presentation in columns, such as (Cod. Ms. Dedekind X 11-1, p. 27):⁵⁹

$$\begin{array}{c|c|c|c|c} d < a', b', c' ; & d < a' < b''', c''' & b', c' < a''' < m', a_2 & a''' < a_2 < a, a_4 & a_2 < a < a'' \\ & d < b' < c''', a''' & c', a' < b''' < m', b_2 & b''' < b_2 < b, b_4 & b_2 < b < b'' \\ & s < c' < a''', b''' & a', b' < c''' < m', c_2 & c''' < c_2 < c_1, c_4 & c_2 < c < c'' \\ \dots & & & & \end{array}$$

More interesting, as far as layouts are concerned, are the levels. Dedekind tries out several different dispositions to highlight their organisation and duality. The one used most often is this type of triangular list (in Cod. Ms. Dedekind X 11-1, p. 24):

- I. d
- II. a', b', c'
- III. a''', b''', c'''
- IV. a_2, b_2, c_2, m'
- V. a_4, b_4, c_4, a, b, c
- VI. a'', b'', c'', d_1
- VII. a_3, b_3, c_3
- VIII. a_1, b_1, c_1
- IX. m

Dedekind tries several horizontal versions (Figure 4.5, also in Cod. Ms. Dedekind X 11-1, p. 24):

d							
a'		b'		r'		a'	
b'''		r'''		a'''		b'''	
b ₂	m' ₂	r ₂	m' ₂	a ₂	m' ₂	b ₂	
b	b ₄	r	r ₄				

Figure 4.5: Cod. Ms. Dedekind X 11-1, p. 24r.

⁵⁹Note the presence of a_4, b_4 and c_4 . Dedekind noted above, probably later: $a'''' = a_4, b'''' = b_4, c'''' = c_4$. This shows that, here again, the clarification of the elements of the 'group' is not finished.

The diagrammatic aspect of both layouts is remarkable. The horizontal one doesn't seem to work as well as Dedekind wished, and appears only a few times. Diagrammatic representations are also used to represent chains, in diagrams strikingly resembling Hasse diagrams,⁶⁰ the modern representation of lattices (again, in Cod. Ms. Dedekind X 11-1, p. 24) :

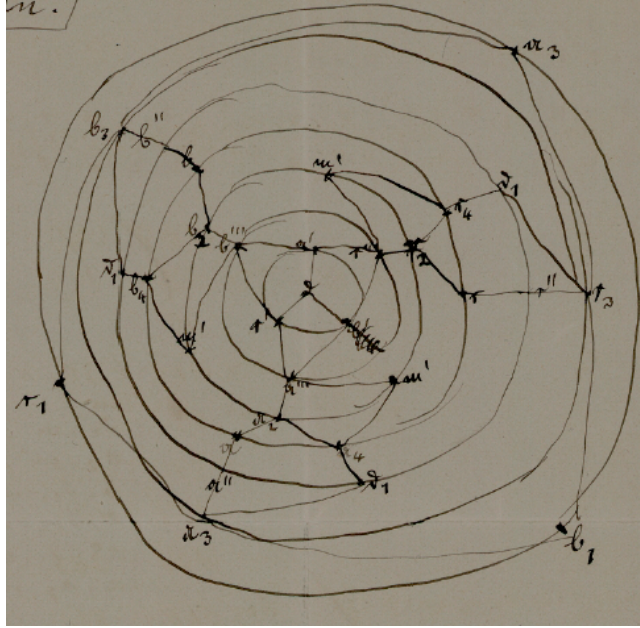


Figure 4.6: Cod. Ms. Dedekind X 11-1, p. 24r.

In this diagram, starting at the center, each link corresponds to the relation “being a divisor of”: $\mathfrak{d} = \mathfrak{a}' + \mathfrak{b}' + \mathfrak{c}'$, hence \mathfrak{d} is a divisor of \mathfrak{a}' , of \mathfrak{b}' and of \mathfrak{c}' , which Dedekind would write $\mathfrak{d} < \mathfrak{a}', \mathfrak{b}', \mathfrak{c}'$ as above. Following the successive links from the center to the upper left, we have a chain of modules:⁶¹

$$\mathfrak{d} < \mathfrak{a}' < \mathfrak{b}''' < \mathfrak{b}_2 < \mathfrak{b} < \mathfrak{b}'' < \mathfrak{b}_3 < \mathfrak{a}_1$$

⁶⁰According to (Birkhoff48), the name “Hasse diagram” does come from Helmut Hasse’s “effective use” of such diagrams, but “the scheme goes back at least to H. Vogt, *Résolution algébrique des équations*, Paris, 1895, p. 91, and has been used for many years in genealogy.” As a matter of fact.

⁶¹The \mathfrak{a}_3 indicated in the diagram is likely a mistake — as supported by the chain given a little further down in the manuscript.

with

$$\begin{aligned}
 \mathfrak{d} &= \mathfrak{a}' + \mathfrak{b}' + \mathfrak{c}' \\
 \mathfrak{a}' &= \mathfrak{b}''' + \mathfrak{c}''' \\
 \mathfrak{b}''' &= \mathfrak{b}_2 + \mathfrak{c}_2 \\
 \mathfrak{b}_2 &= \mathfrak{b} + \mathfrak{b}_1 \\
 &\quad \mathfrak{b} \\
 \mathfrak{b}'' &= \mathfrak{b} - \mathfrak{b}' \\
 \mathfrak{b}_3 &= \mathfrak{b}'' - \mathfrak{c}_2 = \mathfrak{b}'' - \mathfrak{a}_2 = \mathfrak{a}_1 + \mathfrak{c}_1 \\
 \mathfrak{a}_1 &= \mathfrak{b}_3 - \mathfrak{c}_3.
 \end{aligned}$$

Note that for a perfect symmetry, one would need to add $\mathfrak{m} = \mathfrak{a}_1 - \mathfrak{b}_1 - \mathfrak{c}_1$, but it does not appear in Dedekind's diagram. This may be for lack of space, or of a good way to draw all the links, since \mathfrak{m} does appear in the list of chains written on the same page. In any case, Dedekind's notation seems clearly more suited for computations than for diagrams. Many smaller instances of such diagrams appear throughout Dedekind's drafts.⁶²

Some drafts also contain diagrams which combine the diagrammatic representation of levels and chains (Figure 4.7):

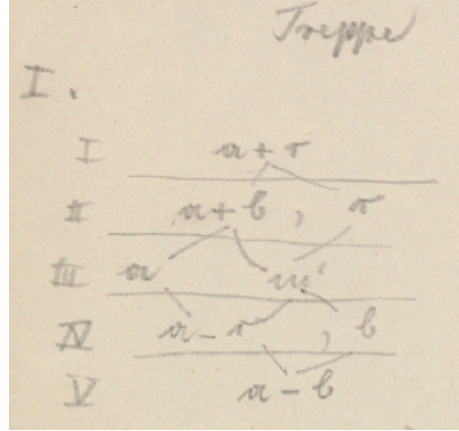


Figure 4.7: Cod. Ms. Dedekind X 10, p. 9r (early to mid-1890s).

Dedekind never reproduced the diagrams in his published papers, despite studying chains and levels.

Writing in small frames and how it helps display duality

The recto of the table Cod. Ms. Dedekind X 11-1, p. 18 (see p. 34) contains computations which are most probably *not* the ones that led to the table, as they study order (divisibility) properties of modules, but the two pages are likely complementary and part of the same research. The results laid on p. 18r of Cod. Ms. Dedekind X 11-1 do not use the notations introduced in the

⁶²See the “Diagrammes” tab on <https://eman-archives.org/Dedekind/emanindexpage?q=81&vide=0&type=Item>.

table. Dedekind finds some of the equalities (such as $(\mathbf{c}_1 + \mathbf{a}_1) = (\mathbf{b} - \mathbf{d}_1)$), which suggests that this might have been written before the table. This also suggests that the chronology of the use of paper was not necessarily linear — at least at that moment of research. Nevertheless, this page is interesting in itself, as it displays a recurring feature of Dedekind's drafts: the writing in series of small frames, with small lists facing each other and thus displaying the dualism between them. Here is a partial transcription of the top part of Cod. Ms. Dedekind X 11-1, p. 18r:

$\mathbf{b} > \mathbf{a}'$	$;$	$\mathbf{a}' < \mathbf{b}$	$\mathbf{b} \gg \mathbf{a}_1$	$ \mathbf{a}_1 > \mathbf{b} $	$\mathbf{m} > \mathbf{d}_1 > \mathbf{m}' > \mathbf{d}$
$\mathbf{c} > \mathbf{a}'$	$;$	$\mathbf{a}' < \mathbf{c}$	$\mathbf{c} \gg \mathbf{a}_1$	$ \mathbf{a}_1 > \mathbf{c} $	
$\mathbf{a} > \mathbf{b}'$	$;$	$\mathbf{b}' < \mathbf{a}$	$\mathbf{c} \gg \mathbf{b}_1$	$ \mathbf{b}_1 > \mathbf{c} $	
$\mathbf{c} > \mathbf{b}'$	$;$	$\mathbf{b}' < \mathbf{c}$	$\mathbf{a} \gg \mathbf{b}_1$	$ \mathbf{b}_1 > \mathbf{a} $	
$\mathbf{a} > \mathbf{c}'$	$;$	$\mathbf{c}' < \mathbf{a}$	$\mathbf{a} \gg \mathbf{c}_1$	$ \mathbf{c}_1 > \mathbf{a} $	
$\mathbf{b} > \mathbf{c}'$	$;$	$\mathbf{c}' < \mathbf{b}$	$\mathbf{b} \gg \mathbf{c}_1$	$ \mathbf{c}_1 > \mathbf{b} $	

$\mathbf{b} > \mathbf{b}' - \mathbf{c}'$	$ \mathbf{b}_1 + \mathbf{c}_1 > \mathbf{a} $	$ \mathbf{d}_1 < \mathbf{a}' $	$ \mathbf{d}_1 > \mathbf{m}' $	$ \mathbf{d}_1 > \mathbf{b}' - \mathbf{c}' $	$ \mathbf{a} + \mathbf{d}_1 > \mathbf{b}' - \mathbf{c}' $
$\mathbf{b} > \mathbf{c}' - \mathbf{a}'$	$ \mathbf{c}_1 + \mathbf{a}_1 > \mathbf{b} $	$ \mathbf{d}_1 < \mathbf{b}' $		$ \mathbf{d}_1 > \mathbf{c}' - \mathbf{a}' $	$ \mathbf{b} + \mathbf{d}_1 > \mathbf{c}' - \mathbf{a}' $
$\mathbf{c} > \mathbf{a}' - \mathbf{b}'$	$ \mathbf{a}_1 + \mathbf{b}_1 > \mathbf{c} $	$ \mathbf{d}_1 < \mathbf{c}' $		$ \mathbf{d}_1 > \mathbf{a}' - \mathbf{b}' $	$ \mathbf{c} + \mathbf{d}_1 > \mathbf{a}' - \mathbf{b}' $

This organisation of the computations and/or their results is very frequently used by Dedekind — in fact, we already encountered it in the drafts studied up to now in this chapter, and we will do so again. This seems to be, simply, a way to put them into order. See, for example, this small set of results on Cod. Ms. Dedekind X 11-1, p. 32. This is written on the back of an invoice dated from 1885, and seven other similar small folios have similar computations, the final version of which is likely Cod. Ms. Dedekind X 11-1, p. 31.⁶³ In this excerpt, Dedekind computes with finitely generated modules, with the basis elements written in square brackets, as he does usually in his number-theoretical works.

$$\begin{array}{l}
 \omega \text{ irrat[ional]} \\
 \mathbf{a} = [1] ; \mathbf{b} = [\omega] ; \mathbf{c} = [c, c_1 + c_2\omega] \quad c, c_1, c_2 \text{ integers} \\
 \quad \quad \quad c, c_2 \text{ pos[itive]} \\
 \mathbf{a}' = \mathbf{b} + \mathbf{c} = [\omega, c, c_1 + c_2\omega] = [c, c_1, \omega] = [a', \omega]; [a'] = [c, c_1] \\
 \mathbf{b}' = \mathbf{c} + \mathbf{a} = [1, c_2\omega] \\
 \mathbf{c}' = \mathbf{a} + \mathbf{b} = [1, \omega] \\
 \hline
 \mathbf{a}_1 = \mathbf{b} - \mathbf{c} = \left[\frac{cc_2}{a'}\omega\right] \\
 \mathbf{b}_1 = \mathbf{c} - \mathbf{a} = [c] \\
 \mathbf{c}_1 = \mathbf{a} - \mathbf{b} = 0 \\
 \hline
 \begin{array}{l|l}
 \mathbf{b}_1 + \mathbf{c}_1 = [c] & \mathbf{a} - \mathbf{a}' = [a'] \\
 \mathbf{c}_1 + \mathbf{a}_1 = \left[\frac{cc_2}{a'}\omega\right] & \mathbf{b} - \mathbf{b}' = [c_2\omega] \\
 \mathbf{a}_1 + \mathbf{b}_1 = \left[c, \frac{cc_2}{a'}\omega\right] & \mathbf{c} - \mathbf{c}' = [c, c_1 + c_2\omega] \\
 \hline
 \mathbf{b}' - \mathbf{c}' = [1, c_2\omega] & \mathbf{a} + \mathbf{a}_1 = \left[1, \frac{cc_2}{a'}\omega\right] \\
 \mathbf{c}' - \mathbf{a}' = [a', \omega] & \mathbf{b} + \mathbf{b}_1 = [c, \omega] \\
 \mathbf{a}' - \mathbf{b}' = [a', c_2\omega] & \mathbf{c} + \mathbf{c}_1 = [c, c_1 + c_2\omega]
 \end{array}
 \end{array}$$

⁶³See <http://eman-archives.org/Dedekind/graphitem/200>. Another one of these drafts is studied p. 48.

In general, we have:

$$\begin{array}{l}
 \mathfrak{b}' - \mathfrak{c}' = \mathfrak{a} + (\mathfrak{b} - \mathfrak{b}') = \mathfrak{a} + (\mathfrak{c} - \mathfrak{c}') \\
 \mathfrak{c}' - \mathfrak{a}' = \mathfrak{b} + (\mathfrak{c} - \mathfrak{c}') = \mathfrak{b} + (\mathfrak{a} - \mathfrak{a}') \\
 \mathfrak{a}' - \mathfrak{b}' = \mathfrak{c} + (\mathfrak{a} - \mathfrak{a}') = \mathfrak{c} + (\mathfrak{b} - \mathfrak{b}') \\
 \hline
 \mathfrak{b}_1 + \mathfrak{c}_1 = \mathfrak{a} - (\mathfrak{b} + \mathfrak{b}_1) = \mathfrak{a} - (\mathfrak{c} + \mathfrak{c}_1) \\
 \mathfrak{c}_1 + \mathfrak{a}_1 = \mathfrak{c} - (\mathfrak{c} + \mathfrak{c}_1) = \mathfrak{b} - (\mathfrak{a} + \mathfrak{a}_1) \\
 \mathfrak{a}_1 + \mathfrak{b}_1 = \mathfrak{c} - (\mathfrak{a} + \mathfrak{a}_1) = \mathfrak{c} - (\mathfrak{b} + \mathfrak{b}_1)
 \end{array}$$

(Cod. Ms. Dedekind X 11-1, p. 32)

The presentation of the computations or results in two columns is often used by Dedekind in his drafts, as we saw above. It is only one of the strategies he adopted to investigate and display the dualism. The choices of notation and spatial organisation, the specific layouts and devices developed during his research process suggest that the investigation of duality is indeed strongly related to such choices. With these examples, we also see clearly how the clarification of the dualism was done along and through the research process — computations, spatial organisation, rewriting and interpretation of results. . .

4.3.4 The *Modulgesetz*, the source of dualism?

Throughout Dedekind's investigations, one specific property retained his attention, which he would later call the *Modulgesetz*.⁶⁴ Recall that this law can be stated as follows: for three modules \mathfrak{p} , \mathfrak{m} , \mathfrak{d} , if $\mathfrak{d} < \mathfrak{m}$, then we have

$$(\mathfrak{p} + \mathfrak{m}) - \mathfrak{d} = (\mathfrak{p} - \mathfrak{d}) + \mathfrak{m}$$

or, with $\mathfrak{p} = \mathfrak{a}$, $\mathfrak{m} = \mathfrak{b} - \mathfrak{c}$ and $\mathfrak{d} = \mathfrak{b} + \mathfrak{c}$

$$(\mathfrak{a} + (\mathfrak{b} - \mathfrak{c})) - (\mathfrak{b} + \mathfrak{c}) = (\mathfrak{a} - (\mathfrak{b} + \mathfrak{c})) + (\mathfrak{b} - \mathfrak{c}).$$

It was first published in (Dedekind, 1894a, p. 499) as a property of modules. In the earliest drafts (in the 1870s and the 1880s) and in (Dedekind, 1877, p. 121), the following properties are given for three arbitrary modules \mathfrak{a} , \mathfrak{b} , \mathfrak{c} :

$$\begin{array}{l}
 (\mathfrak{a} + \mathfrak{b}) - (\mathfrak{a} + \mathfrak{c}) = \mathfrak{a} + (\mathfrak{b} - (\mathfrak{a} + \mathfrak{c})) \\
 (\mathfrak{a} - \mathfrak{b}) + (\mathfrak{a} - \mathfrak{c}) = \mathfrak{a} - (\mathfrak{b} + (\mathfrak{a} - \mathfrak{c})).
 \end{array}$$

But the *Modulgesetz* is not stated. It starts to appear in later drafts, in particular in computations. Such uses of the property seem to have prompted Dedekind to question its conditions of validity, from the late 1880s on. Still working in the module-theoretical framework, he tried to determine the conditions for the law to be generally valid. It is not the place, here, to look into his attempts to prove that $\mathfrak{d} < \mathfrak{m}$ is a sufficient and necessary condition for $(\mathfrak{p} + \mathfrak{m}) - \mathfrak{d} = (\mathfrak{p} - \mathfrak{d}) + \mathfrak{m}$, and whether this property can be proven from the “fundamental laws” (i.e., the

⁶⁴As far as conceptual development is concerned, it is very significant that the *Modulgesetz* is named so late. For readability purposes, I will use its name (and likewise for the *Idealgesetz*) even before Dedekind did.

defining laws) of the operations — which I have done elsewhere.⁶⁵ It suffices to say that Dedekind's approach, here again, was that of stepwise generalisation and proceeded largely by trials and errors.

Around 1890, his research focused on proving that for three arbitrary modules $\mathfrak{a}, \mathfrak{b}, \mathfrak{c}$ if $\mathfrak{b} > \mathfrak{c}$, then $(\mathfrak{a} + \mathfrak{b}) - \mathfrak{c} = (\mathfrak{a} - \mathfrak{c}) + \mathfrak{b}$, using only the axioms for $+$ and $-$ and the definition of divisibility — or, in Dedekind's own words, to prove it “without new principles”. This proof is, in fact, not possible,⁶⁶ a realisation to which Dedekind came slowly. The observation of the impossibility to prove this property from the fundamental laws alone was made well before introducing the concept of *Modulgruppe* — in fact, even before applying the operations to other operands. In this context, Dedekind's first conclusion about the *Modulgesetz* was not that it could be added to the fundamental laws of the operations as a distinctive feature of module theory. Rather, he attributed a particular status to it, calling it “indispensable”. In Cod. Ms. Dedekind X 11-1, p. 15, he took as initial condition

$$(\mathfrak{a} + \mathfrak{b}) - \mathfrak{c} < \mathfrak{a} + (\mathfrak{b} - (\mathfrak{c} + \mathfrak{a}))$$

which invalidates the equality. Note that it is always true, in *Dualgruppen*, that

$$(\mathfrak{a} + \mathfrak{b}) - \mathfrak{c} < \mathfrak{a} + (\mathfrak{b} - (\mathfrak{c} + \mathfrak{a}))$$

which can be deduced from the (also always valid) inequality (for $\mathfrak{m} < \mathfrak{d}$)

$$(\mathfrak{m} + \mathfrak{p}) - \mathfrak{d} < \mathfrak{m} + (\mathfrak{p} - \mathfrak{d}).$$

For *Modulgruppen* (i.e., not a modular lattice), one would need the equality. Dedekind gave an example which is not a *Modulgruppe*, and concluded:

17 November 1890. This example of an additive group shows the indispensability of the law (which does not hold here)

$$(\mathfrak{p} + \mathfrak{m}) - \mathfrak{d} = (\mathfrak{p} - \mathfrak{d}) + \mathfrak{m}$$

\mathfrak{m} divisible by \mathfrak{d} , that is, $\mathfrak{m} + \mathfrak{d} = \mathfrak{d}$. (Cod. Ms. Dedekind X 11-1, p. 15)

Labelling the law as “indispensable” suggests that Dedekind was still working solely with modules (especially since he did acknowledge later that the law is not necessary to *Dualgruppen*, but only a specificity of module-generated and module-like ones). Importantly enough, Dedekind seemed to conceive this “indispensability” in relation to forming “groups” of modules, which might justify

⁶⁵(Haffner, 2021)

⁶⁶In (Dedekind, 1900), he wrote:

But this *Modulgesetz* is, as I showed in §4 of my [(Dedekind, 1897b), i.e. the proof that there exist *Dualgruppen* which do not verify the modular law (EH)], unfortunately not derivable from the fundamental laws [commutativity, associativity and absorption for $+$ and $-$] and constitutes therefore an essential *extension* of them for module theory. (Dedekind, 1900, 239, Dedekind's emphasis)

that it was not added to the fundamental laws defining the operations. This led him to attribute a (temporary) specific status to it: the “source of dualism”.

In Cod. Ms. Dedekind X 11-1, p. 13-14, which can be dated from the early to mid-1890s,⁶⁷ Dedekind studied the divisibility laws for three modules \mathfrak{m} , \mathfrak{d} , \mathfrak{p} with \mathfrak{d} divisor of \mathfrak{m} . The first two pages are covered with lists of results and computations for the divisibility of the three modules (with all possible additions and some subtractions), tables of GCDs, LCMs, multiples, and the systematic verification of associativity. On the third (and last) page, in what seems to be the conclusion of the preceding investigations, Dedekind wrote a short paragraph stating:

The source of dualism in the theory of modules, that is, the following theorem:

I. If \mathfrak{m} is divisible by \mathfrak{d} , then $(\mathfrak{m} + \mathfrak{p}) - \mathfrak{d} = \mathfrak{m} + (\mathfrak{p} - \mathfrak{d})$ amounts to what follows (for the sign +):

II. If $\mathfrak{m} + \mathfrak{d} = \mathfrak{d}$, $\mathfrak{a} + \mathfrak{m} + \mathfrak{p} = \mathfrak{m} + \mathfrak{p}$, $\mathfrak{a} + \mathfrak{d} = \mathfrak{d}$ then there exists at least one element (module) \mathfrak{q} verifying the conditions

$$\mathfrak{q} + \mathfrak{p} = \mathfrak{p}, \mathfrak{q} + \mathfrak{d} = \mathfrak{d}, \mathfrak{a} + \mathfrak{m} + \mathfrak{q} = \mathfrak{m} + \mathfrak{q}.$$

Or again (if one sets $\mathfrak{d} = \mathfrak{m} + \mathfrak{p}'$).

III. If $\mathfrak{a} + \mathfrak{m} + \mathfrak{p} = \mathfrak{m} + \mathfrak{p}$, $\mathfrak{a} + \mathfrak{m} + \mathfrak{p}' = \mathfrak{m} + \mathfrak{p}'$, then we have at least one element \mathfrak{q} verifying the conditions:

$$\mathfrak{q} + \mathfrak{p} = \mathfrak{p}, \mathfrak{q} + \mathfrak{m} + \mathfrak{p}' = \mathfrak{m} + \mathfrak{p}', \mathfrak{a} + \mathfrak{m} + \mathfrak{q} = \mathfrak{m} + \mathfrak{q}.$$

But this theorem (I or II or III) is not in any way a necessary consequence of the three laws of pure addition:

IV. $\mathfrak{a} + \mathfrak{a} = \mathfrak{a}$, $\mathfrak{a} + \mathfrak{b} = \mathfrak{b} + \mathfrak{a}$, $(\mathfrak{a} + \mathfrak{b}) + \mathfrak{c} = \mathfrak{a} + (\mathfrak{b} + \mathfrak{c})$.

(Cod. Ms. Dedekind X 11-1, p. 14v)

Dedekind does not justify calling this law the “source of dualism”, but the likeliest hypothesis is related to the “indispensability” mentioned above, suggesting that the dualism would be, at that point, conceived as intimately related to forming “groups” of modules.⁶⁸ Note that, in some drafts, Dedekind used “*Dualismus*” as synonymous of “the theory of three modules” (see the example on p. 29), here. In later texts, Dedekind seems to use “*Dualismus*” to talk about duality, which suggests an ambiguity between *dualism* and *duality*.

While this is a significant moment in the investigation of the dualism in module theory — and, more broadly, of the genesis of *Dualgruppen* —, the *Modulgesetz*’s status as the “source of dualism” is short-lived. It is not mentioned again, even though the law still plays a central role in Dedekind’s research.

What Dedekind probably hadn’t seen when he attributed this status to the *Modulgesetz* is that he could extend these operations and the formation

⁶⁷It is written on the back of an advertisement that can be dated from after 1888.

⁶⁸But of course, the initial condition in Cod. Ms. Dedekind X 11-1, p. 15, is in fact stronger than just assuming that the *Modulgesetz* does not hold, although we cannot say when Dedekind actually saw that.

of “groups” to operands that did not verify the *Modulgesetz* but did verify the (strict) inequality

$$(\mathfrak{p} + \mathfrak{m}) - \mathfrak{d} < (\mathfrak{p} - \mathfrak{d}) + \mathfrak{m}.$$

This changed when Dedekind applied and defined the operations in a more general setting. Only then did it become clear that the law presented as the “source of dualism” is a property of modules, that is, related to the restricted context he was initially working in; that neither duality nor the possibility to form “groups” are dependent or related to it; and that non-modular *Dualgruppen* can indeed be defined. These three conclusions are entangled and certainly were not the result of a linear process.

4.4 Generalisation of the research

4.4.1 More generality in module theory

As we saw, in his mathematical experimentations and explorations, Dedekind often opted for a stepwise approach to the generalisation of his results, manipulating simple cases and testing properties on progressively more general cases. By doing so, Dedekind was able to observe the properties and laws verified by the operations between modules, and identify which properties are generally valid and which laws are fundamental, which was his initial purpose. Eventually, he also extended his investigations outside of module theory, reaching a higher level of generality that would be crucial for *Dualgruppen*.

From numerical cases to general results

Dedekind computed with finitely generated and arbitrary modules and generalising inductively. The series of drafts Cod. Ms. Dedekind X 11-1, p. 28-29 and p. 31-34 offers a clear example of this.⁶⁹ Dedekind uses an early version of his notation for sums and differences of three modules, and wrote on the back of several 1885 invoices, so these can be dated between the late 1880s and the early 1890s.

Dedekind's drafts follow mostly the same pattern. They begin with a numerical example for three finitely-generated modules $\mathfrak{a}, \mathfrak{b}, \mathfrak{c}$ (of the form $[x, x_1 + x_2\omega]$, with ω an irrational number, x, x_i integers) with the (more or less detailed) computation of (some or all of) $\mathfrak{a}', \mathfrak{b}', \mathfrak{c}', \mathfrak{b}' - \mathfrak{c}', \mathfrak{a}' - \mathfrak{b}', \mathfrak{c}' - \mathfrak{a}', \mathfrak{a} - \mathfrak{a}', \mathfrak{b} - \mathfrak{b}', \mathfrak{c} - \mathfrak{c}', \mathfrak{a}_1, \mathfrak{b}_1, \mathfrak{c}_1, \mathfrak{b}_1 + \mathfrak{c}_1, \mathfrak{c}_1 + \mathfrak{a}_1, \mathfrak{a}_1 + \mathfrak{b}_1, \mathfrak{a} + \mathfrak{a}_1, \mathfrak{b} + \mathfrak{b}_1, \mathfrak{c} + \mathfrak{c}_1$. Then, similar computations for three ideals (seen as a specific type of modules) are done. Dedekind is, here, testing the properties in a more restricted case of the theory. This suggests that he was experimenting in more or less restricted contexts. After that, he considers questions of divisibility between the computed modules, with varying degrees of generality. General results are stated for the first time in Cod. Ms. Dedekind X 11-1, p. 28. After the said computations, Dedekind works

⁶⁹One of these drafts was presented p. 44. See also <http://eman-archives.org/Dedekind/graphitem/200>.

with arbitrary modules, and starts with the following statements for numbers of classes:⁷⁰

$$\begin{aligned} (\mathfrak{a} - \mathfrak{a}', \mathfrak{b}_1 + \mathfrak{c}_1) &= (\mathfrak{a} - (\mathfrak{b} + \mathfrak{c}), (\mathfrak{c} - \mathfrak{a}) + (\mathfrak{a} - \mathfrak{b})) = (\mathfrak{a} - (\mathfrak{b} + \mathfrak{c}), \mathfrak{a} - (\mathfrak{b} + (\mathfrak{c} - \mathfrak{a}))) \\ (\mathfrak{b}' - \mathfrak{c}', \mathfrak{a} + \mathfrak{a}_1) &= ((\mathfrak{c} + \mathfrak{a}) - (\mathfrak{a} + \mathfrak{b}), \mathfrak{a} + (\mathfrak{b} - \mathfrak{c})) \end{aligned}$$

Since numbers of classes allow to deduce relations of divisibility between the modules considered, this gives:

Every number in $\mathfrak{a} - \mathfrak{a}' = \mathfrak{a} - (\mathfrak{b} + \mathfrak{c})$ is of the form:

$$\begin{aligned} \alpha &= \beta + \gamma; & \text{since } \beta &= \alpha - \gamma, \gamma = \alpha - \beta \text{ follows} \\ \text{so, we have } & \beta \text{ in } \mathfrak{b} - (\mathfrak{a} + \mathfrak{c}) = \mathfrak{b} - \mathfrak{b}' \\ & \gamma \text{ in } \mathfrak{c} - (\mathfrak{a} + \mathfrak{b}) = \mathfrak{c} - \mathfrak{c}'. \end{aligned}$$

Similar results for $\theta = \alpha_1 + \beta_1 + \gamma_1$ are written on the right side of the paper. The above means in terms of divisibility of modules, that:

$$\left. \begin{aligned} \mathfrak{a} - \mathfrak{a}' &> (\mathfrak{b} - \mathfrak{b}') + (\mathfrak{c} - \mathfrak{c}') \\ \mathfrak{b} - \mathfrak{b}' &> (\mathfrak{c} - \mathfrak{c}') + (\mathfrak{a} - \mathfrak{a}') \\ \mathfrak{c} - \mathfrak{c}' &> (\mathfrak{a} - \mathfrak{a}') + (\mathfrak{b} - \mathfrak{b}') \end{aligned} \right\} (\mathfrak{a} - \mathfrak{a}') + (\mathfrak{b} - \mathfrak{b}') > (\mathfrak{b} - \mathfrak{b}') + (\mathfrak{c} - \mathfrak{c}')$$

A circled “general” result, which was likely added towards the end (at least after the next set of computations for ideals), states:

$$\begin{aligned} \text{In general} & \qquad \qquad \qquad = \mathfrak{a}' - \mathfrak{b}' - \mathfrak{c}' \\ (\mathfrak{b} - \mathfrak{b}') + (\mathfrak{c} - \mathfrak{c}') &= (\mathfrak{c} - \mathfrak{c}') + (\mathfrak{a} - \mathfrak{a}') = (\mathfrak{a} - \mathfrak{a}') + (\mathfrak{b} - \mathfrak{b}') \\ (\mathfrak{b} + \mathfrak{b}_1) - (\mathfrak{c} + \mathfrak{c}_1) &= (\mathfrak{c} + \mathfrak{c}_1) - (\mathfrak{a} + \mathfrak{a}_1) = (\mathfrak{a} + \mathfrak{a}_1) - (\mathfrak{b} + \mathfrak{b}_1) \\ &= \mathfrak{a}_1 + \mathfrak{b}_1 + \mathfrak{c}_1 \end{aligned}$$

Both “ $= \mathfrak{a}' - \mathfrak{b}' - \mathfrak{c}'$ ” and “ $= \mathfrak{a}_1 + \mathfrak{b}_1 + \mathfrak{c}_1$ ” were added after the two main lines, suggesting that the “general” result was written in two times. This result is somewhat superposed with Dedekind’s computations for three ideals, for which he finds that⁷¹

$$\begin{array}{l|l} \mathfrak{b}' - \mathfrak{c}' = \mathfrak{a} + \mathfrak{a}_1 & \mathfrak{b}_1 + \mathfrak{c}_1 = \mathfrak{a} - \mathfrak{a}' \\ \mathfrak{c}' - \mathfrak{a}' = \mathfrak{b} + \mathfrak{b}_1 & \mathfrak{c}_1 + \mathfrak{a}_1 = \mathfrak{b} - \mathfrak{b}' \\ \mathfrak{a}' - \mathfrak{b}' = \mathfrak{c} + \mathfrak{c}_1 & \mathfrak{a}_1 + \mathfrak{b}_1 = \mathfrak{c} - \mathfrak{c}' \end{array}$$

He concludes by stating that “in general”, “only” the inequalities are valid:

$$\begin{array}{ll} \mathfrak{b}' - \mathfrak{c}' < \mathfrak{a} + \mathfrak{a}_1 & \text{and } \mathfrak{b}_1 + \mathfrak{c}_1 > \mathfrak{a} - \mathfrak{a}' \\ \mathfrak{c}' - \mathfrak{a}' < \mathfrak{b} + \mathfrak{b}_1 & \mathfrak{c}_1 + \mathfrak{a}_1 > \mathfrak{b} - \mathfrak{b}' \\ \mathfrak{a}' - \mathfrak{b}' < \mathfrak{c} + \mathfrak{c}_1 & \mathfrak{a}_1 + \mathfrak{b}_1 > \mathfrak{c} - \mathfrak{c}' \end{array}$$

The circled “general result” is proved in Cod. Ms. Dedekind X 11-1, p. 31. That folio is itself divided in two parts: on the top part, there are (similar) computations for three finitely generated modules; on the bottom part, Dedekind states the said result with a proof.

⁷⁰See p. 12.

⁷¹It is clear that Dedekind was investigating the same kind of relationships in the draft mentioned on p. 44.

From a finite number of modules to any number of modules

Another move towards a more general approach consisted in going from working with a small number of modules (three to five) and attempting to generalise to any number of modules. This is a recurring process in Dedekind's research, rather than a linear progression. He went back and forth between various degrees of generality in his computations.

An example that exhibits many of the traits I just described is Cod. Ms. Dedekind X 10, p. 48, which does not contain any indications for datation but in which the notations used suggest that it was written in the late 1880s to early 1890s. Dedekind starts by considering four modules, which he denotes by 1, 2, 3, 4. He does not compute all of the combinations but counts the number of modules “of the type $1 + (2 - 3 - 4)$ (...), $1 - (2 + 3 + 4)$ ”, and so on. He considers the following operation

$$\{(2 - 3 - 4) + (1 - 3 - 4) + (1 - 2)\} - \{3 + (1 - 2 - 4)\}$$

and immediately substitutes letters for some of the operands:

$$\left. \begin{array}{l} (2 - 3 - 4) + (1 - 3 - 4) = \mathfrak{p} \quad ; \quad (1 - 2) = \mathfrak{q}' \\ 3 = \mathfrak{p}' \quad ; \quad (1 - 2 - 4) = \mathfrak{q} \end{array} \right\} \begin{array}{l} \mathfrak{p} > \mathfrak{p}' \quad ; \quad \mathfrak{p} + \mathfrak{p}' = \mathfrak{p}' \\ \mathfrak{q} > \mathfrak{q}' \quad ; \quad \mathfrak{q} + \mathfrak{q}' = \mathfrak{q}' \end{array}$$

This leads him to reformulate the initial computation, and deduce the following equalities:⁷²

$$\begin{aligned} (\mathfrak{p} + \mathfrak{q}') - (\mathfrak{q} + \mathfrak{p}') &= (\mathfrak{p} + \mathfrak{q} + \mathfrak{q}') - (\mathfrak{p} + \mathfrak{q} + \mathfrak{q}') = (\mathfrak{p} + \mathfrak{q}) + (\mathfrak{q}' - (\mathfrak{p} + \mathfrak{q}')) \\ &= (\mathfrak{p} + \mathfrak{q}) + ((\mathfrak{p}' - \mathfrak{q}') + \mathfrak{q}) \\ &= \mathfrak{p} + \mathfrak{q} + (\mathfrak{p}' - \mathfrak{q}') \end{aligned}$$

In the right corner of the sheet, he notes the equalities he is going to use with one of his more usual notations:

$$\begin{aligned} &\{\mathfrak{a} + (\mathfrak{b} - \mathfrak{c})\} - \{\mathfrak{b} + (\mathfrak{a} - \mathfrak{c})\} \\ &= (\mathfrak{a} + \mathfrak{a}_1) - (\mathfrak{b} + \mathfrak{b}_1) \\ &= \mathfrak{a}''' - \mathfrak{b}''' = \mathfrak{d}_1 = \mathfrak{a}_1 + \mathfrak{b}_1 + \mathfrak{c}_1 \\ &= (\mathfrak{b} - \mathfrak{c}) + (\mathfrak{c} - \mathfrak{a}) + (\mathfrak{a} - \mathfrak{b}) \end{aligned}$$

He also draws what seems to be a first try at the above reformulation (which I do not reproduce, as the equalities are the same).

From there, he considers “the multiple for every choice of three modules”:

$$\left. \begin{array}{l} \{1 + (2 - 3 - 4)\} - \{2 + (1 - 3 - 4)\} - \{3 + (1 - 2 - 4)\} \\ (2 - 3 - 4) + (1 - 3 - 4) + (1 - 2 - 4) + (1 - 2 - 3) \end{array} \right\} \text{symmetrical}$$

This “symmetry” between the two lines noted by Dedekind is related to the already known property for three modules, namely:

$$(\mathfrak{a} + (\mathfrak{b} - \mathfrak{c})) - (\mathfrak{b} + (\mathfrak{a} - \mathfrak{c})) + (\mathfrak{c} + (\mathfrak{a} - \mathfrak{b})) = (\mathfrak{b} - \mathfrak{c}) + (\mathfrak{a} - \mathfrak{c}) + (\mathfrak{a} - \mathfrak{b}),$$

⁷²These computations follow easily from laws of module theory such as the ones stated in the introduction and the definitions of $\mathfrak{p}, \mathfrak{p}', \mathfrak{q}, \mathfrak{q}'$.

Or, with the notation used here:

$$(1 + (2 - 3)) - (2 + (1 - 3)) - (3 + (1 - 2)) = (2 - 3) + (1 - 3) + (1 - 2).$$

Dedekind follows with a “general theorem on n modules 1, 2, 3, \dots ($n - 1$), n ”:

$$(1 + \mathfrak{m}_1) - (2 + \mathfrak{m}_2) - \dots - ((n - 1) + \mathfrak{m}_{n-1}) = \mathfrak{m}_1 + \mathfrak{m}_2 + \mathfrak{m}_1 + \dots + \mathfrak{m}_{n-1} + \mathfrak{m}_n$$

where \mathfrak{m}_1 is the multiple of the $n - 1$ modules 2, 3, \dots , n all except 1

\mathfrak{m}_2 [is the multiple of the $n - 1$ modules] 1, 3, \dots , n all except 2

\mathfrak{m}_n [is the multiple of the $n - 1$ modules] 1, 2, \dots , $n - 1$ all except n
Other presentation :

$$n \text{ modules } \mathfrak{a}_1, \mathfrak{a}_2 \dots \mathfrak{a}_n$$

From there

$$\mathfrak{m}_s = \mathfrak{a}_1 - \mathfrak{a}_2 - \dots - \mathfrak{a}_{s-1} - \mathfrak{a}_{s+1} - \dots - \mathfrak{a}_{n-1} - \mathfrak{a}_n \text{ multiples of all the } \mathfrak{a} \text{ without } \mathfrak{a}_s$$

Then, theorem: multiples of only ($n - 1$) modules : already
symmetrical

$$(\mathfrak{a}_1 + \mathfrak{m}_1) - (\mathfrak{a}_2 + \mathfrak{m}_2) - \dots - (\mathfrak{a}_{n-1} + \mathfrak{m}_{n-1}) = \mathfrak{m}_1 + \mathfrak{m}_2 + \mathfrak{m}_1 \dots + \mathfrak{m}_{n-1} + \mathfrak{m}_n$$

(Cod. Ms. Dedekind X 10, p. 48r)

The proof is done by induction on n . Dedekind then proposes a “more general” result. With

\mathfrak{m}_1	divisible by	$\mathfrak{a}_1, \mathfrak{a}_2, \dots \mathfrak{a}_n$		and these divisible by	\mathfrak{d}_1
\mathfrak{m}_2	" "	$\mathfrak{a}_1, \mathfrak{a}_3, \dots \mathfrak{a}_n$		" " " "	\mathfrak{d}_2
\dots	\dots	\dots		\dots	\dots
\mathfrak{m}_n	" "	$\mathfrak{a}_1, \mathfrak{a}_2, \dots \mathfrak{a}_{n-1}$		" " " "	\mathfrak{d}_n

he concludes that

$$(\mathfrak{a}_1 - \mathfrak{d}_1) + (\mathfrak{a}_2 - \mathfrak{d}_2) + \dots + (\mathfrak{a}_n - \mathfrak{d}_n) = \mathfrak{c} - \mathfrak{d}_1 - \mathfrak{d}_2 - \dots - \mathfrak{d}_n$$

“where \mathfrak{c} is the gr[eatest] c[ommon] divisor or $\mathfrak{a}_1, \mathfrak{a}_2, \dots \mathfrak{a}_n$ ” (*ibid.*).

After the proof, which uses the previous result and, again, induction, Dedekind proposes a “simpler expression”: with \mathfrak{d}_r the divisors of \mathfrak{m}_s ($r \neq s$):

$$\left. \begin{array}{l} 1) (\mathfrak{d}_1 + \mathfrak{m}_1) - (\mathfrak{d}_2 + \mathfrak{m}_2) - \dots - (\mathfrak{d}_n + \mathfrak{m}_n) = \mathfrak{d} + \mathfrak{m} \\ 2) (\mathfrak{d}_1 - \mathfrak{m}_1) + (\mathfrak{d}_2 - \mathfrak{m}_2) + \dots + (\mathfrak{d}_n - \mathfrak{m}_n) = \mathfrak{d} - \mathfrak{m} \end{array} \right\} \text{ where } \begin{array}{l} \mathfrak{d} = \mathfrak{d}_1 - \mathfrak{d}_2 - \dots - \mathfrak{d}_n \\ \mathfrak{m} = \mathfrak{m}_1 + \mathfrak{m}_2 + \dots + \mathfrak{m}_n \end{array}$$

to which Dedekind adds that these results “in fact are dualistic of each other”. This confirms Dedekind’s interest in finding ways of expressing his results so as to showcase the duality between the operations.

The results stated by Dedekind, in this sheet, are a generalisation of a result he knew well for 3 modules, and which we already met several times. The step-wise aspect of the statement of general results is, here, clearly illustrated, especially if one considers these investigations as embedded in larger enquiries about the divisibility of modules, as illustrated in the previous section. This is not the only result which Dedekind proved for three modules and considered stating for a random number of modules. For example, other manuscripts show attempts at stating the *Modulgesetz* for a random number of modules.

4.4.2 Parallelism between modules and (Abelian) groups

As we saw on p. 49, Dedekind studied early on which kind of specific properties were exhibited by ideals, and possibly only by ideals. This comes from the fact that ideals were studied as a specific kind of modules. It led him to identify some properties that were specific to ideals (i.e., not generally true for modules), which in turn opened the possibility to study what he called “the theory of three modules (of ideal character)” (see for example Cod. Ms. Dedekind III 14, p. 12⁷³). In particular, he identified the so-called *Idealgesetz* (see p. 9).

Likewise, Dedekind presented modules as special cases of groups. As a conclusion of the section introducing modules (of numbers) in (Dedekind, 1876-1877), he indeed stated:

The research in this first section have been exposed in the special form suited to our goal, but it is clear that they do not cease to be true when the Greek letters do not denote *numbers* anymore, but arbitrary elements, subjects of our study, any two of which α, β produce a determinate third element $\gamma = \alpha + \beta$ of the same type, under a commutative and uniformly invertible operation (composition), taking the place of addition. The modules \mathfrak{a} become *groups* of elements, whose results (the *composites*) all belong to the same group. The rational integer coefficients indicate how many times an element contributes to the generation of another. (Dedekind, 1876-1877, 82, transl. modified, original emphasis)

Thus, just as he studied how applying the operations to ideals could affect their properties, Dedekind also considered to which extent his investigations on modules could transfer to the study of groups in general. This manifests itself in several ways.

Firstly, he draws explicit links between module theory and group theory. Here is the most explicit instance of it in (Cod. Ms. Dedekind X 9, p. 23):

⁷³<https://eman-archives.org/Dedekind/items/show/306>

Parallelism between

1) Modules $\mathfrak{a}, \mathfrak{b}$	Abelian groups \mathcal{A}, \mathcal{B}
2) $\mathfrak{a} + \mathfrak{b}$	$\mathcal{A} - \mathcal{B}$ [GCD] of \mathcal{A}, \mathcal{B}
3) $\mathfrak{a} - \mathfrak{b}$	$\mathcal{A}\mathcal{B}$ [LCM] of \mathcal{A}, \mathcal{B}
Divisibility, \mathfrak{d} divisor of \mathfrak{m}	
4) $\mathfrak{m} + \mathfrak{d} = \mathfrak{d}$; $\mathfrak{m} - \mathfrak{d} = \mathfrak{d}$	$\mathcal{M} - \mathcal{D} = \mathcal{D}$, $\mathcal{M}\mathcal{D} = \mathcal{M}$
Then theorem	
5) $\mathfrak{m} + (\mathfrak{a} - \mathfrak{d}) = (\mathfrak{m} + \mathfrak{a}) - \mathfrak{d}$	$\mathcal{M} - \mathcal{A}\mathcal{D} = (\mathcal{M} - \mathcal{A})\mathcal{D}$
If we set now	
6) $\mathfrak{m} = \mathfrak{b}$, $\mathfrak{d} = \mathfrak{b} + \mathfrak{c}$	$\mathcal{M} = \mathcal{B}$, $\mathcal{D} = \mathcal{B} - \mathcal{C}$
it follows	
7) $(\mathfrak{a} + \mathfrak{b}) - (\mathfrak{b} + \mathfrak{c}) = \mathfrak{b} + (\mathfrak{a} - (\mathfrak{b} + \mathfrak{c}))$	$(\mathcal{A} - \mathcal{B})(\mathcal{B} - \mathcal{C}) = \mathcal{B} - \mathcal{A}(\mathcal{B} - \mathcal{C})$
But if we set	
8) $\mathfrak{m} = \mathfrak{b} - \mathfrak{c}$, $\mathfrak{d} = \mathfrak{b}$	$\mathcal{M} = \mathcal{B}\mathcal{C}$, $\mathcal{D} = \mathcal{B}$
it follows	
9) $(\mathfrak{a} - \mathfrak{b}) + (\mathfrak{b} - \mathfrak{c}) = \mathfrak{b} - (\mathfrak{a} + (\mathfrak{b} - \mathfrak{c}))$	$\mathcal{A}\mathcal{B} - \mathcal{B}\mathcal{C} = \mathcal{B}(\mathcal{A} - \mathcal{B}\mathcal{C})$
Reciprocally, 5) follows from 7) when	
$\mathfrak{b} = \mathfrak{m}$, $\mathfrak{c} = \mathfrak{d}$, $\mathfrak{b} + \mathfrak{c} = \mathfrak{d}$	$\mathcal{B} = \mathcal{M}$, $\mathcal{C} = \mathcal{D}$, $\mathcal{B} - \mathcal{C} = \mathcal{D}$
and 5) follows from 9), when we set	
$\mathfrak{b} = \mathfrak{d}$, $\mathfrak{c} = \mathfrak{m}$, $\mathfrak{b} - \mathfrak{c} = \mathfrak{m}$	$\mathcal{B} = \mathcal{D}$, $\mathcal{C} = \mathcal{M}$, $\mathcal{B}\mathcal{C} = \mathcal{M}$
Symbol:	
10) $(\mathfrak{a}, \mathfrak{b}) = (\mathfrak{a} + \mathfrak{b}, \mathfrak{b}) = (\mathfrak{a}, \mathfrak{a} - \mathfrak{b})$	$(\mathcal{A}, \mathcal{B}) = (\mathcal{A} - \mathcal{B}, \mathcal{B}) = (\mathcal{A}, \mathcal{A}\mathcal{B})$

There are no sources cited for this table. It seems likely that Dedekind's comparison builds on both his own investigations on groups and his study of Frobenius' works. In particular, Dedekind also compares (without any additional commentary) some of his results to (Frobenius and Stickelberger, 1879), in Cod. Ms. Dedekind X 9, p. 9.⁷⁴

Dedekind's observation of the "parallelism" between groups and modules, here, is certainly nothing more than that at this point. The operations are not abstracted from their respective context of definition, and Dedekind is still far from seeing any sort of unifying concept of operation or structure.

Of course, Dedekind was well aware that a module is a specific type of group. By thinking of modules as arbitrary groups, Dedekind can abstract his research from modules of numbers, which in turn accentuates the fact that he works with modules as objects rather than collections of numbers. Working with modules as objects was usual for Dedekind — it was indeed a fundamental aspect of his definition of GCD and LCM of modules and, more broadly, of his development of an arithmetic of modules (and of ideals). This certainly played a significant role in the development of the conception of the operations as independent of the

⁷⁴See <http://eman-archives.org/Dedekind/items/show/144>.

operands, which seemed to be done, in parts, by the mediation of case studies on related objects, like ideals and groups.

Maybe prompted by the observations of the above table, Dedekind worked with groups inasmuch as they are a concept of which modules are an instance. Several *Aufgaben* are presented as working on “modules (or Abelian groups)”.⁷⁵ They highlight how properties of the operations do not depend on their initial definition as GCD and LCM of modules, but rather on the so-called fundamental laws (associativity, commutativity, absorption). In itself, this is far from an unusual approach to find in Dedekind's works. What is interesting, here, is the stepwise aspect revealed by his drafts.

This seems to be part of another aspect of the stepwise generalisation of his research: can the operations be applied to other objects, in particular objects more general than modules (e.g. arbitrary groups)? If so, what happens? This eventually gave him the possibility to consider the operations themselves as general, that is, independently of the operands. Slowly, Dedekind worked towards a conception of the operations in which they are detached (abstracted) from their interpretation in module theory. Although his position on this matter is not completely clear, as he often goes back to module theory,⁷⁶ there are distinct attempts in the direction of more general operations.

4.4.3 Reading Schröder and the “logical” theory

Dedekind's reading notes on the *Vorlesungen über die Algebra der Logik*

Dedekind left two sets of notes on and several references to the first volume of Ernst Schröder's *Vorlesungen über die Algebra der Logik* (Schröder, 1890) in his drafts.⁷⁷ Insofar as Schröder's book is quoted in “On the dualism in module theory” (1895-1897), which we will study below and which marks a next-to-final step in the genesis of the concept, we know that Dedekind read it before achieving the final definition of *Dualgruppen*. For this reason, it seems important to mention his reading of Schröder as part of the paths that led to *Dualgruppen* via the text “On the dualism in module theory”, and indeed to do so before studying that text, even though the second of these two sets of notes can be dated from after it.

The two sets of notes are broadly similar. It seems that Cod. Ms. Dedekind III 30 is a rewriting of the notes in Cod. Ms. Dedekind III 14, p. 1-2, in order to compare Schröder's and Dedekind's theories. Cod. Ms. Dedekind III 14, p. 1-2 also uses the symbols $+$ and $-$ rather than φ and ψ , suggesting that it is anterior

⁷⁵See for example Cod Ms Dedekind X 11-2, p. 14, in which he considers the lattice of normal subgroups. <https://eman-archives.org/Dedekind/items/show/301>

⁷⁶This is not surprising, since his (Dedekind, 1900) is indeed dedicated to *Dualgruppen* formed by modules. As I mentioned, Dedekind seems to show more interest in instances of *Dualgruppen* than in the abstract theory, which might be why he always goes back to modules. But it is also possible that, at that point of development of the work, in the absence of a solid foundation for the general theory, Dedekind prefers to go back to modules.

⁷⁷<http://eman-archives.org/Dedekind/items/browse?tags=Schröder>

to Dedekind's reflections on the matter. The latter symbols were introduced in the mid-1890s by Dedekind for the last step of generalisation of his work, and in particular for what he called a "logical theory" (for which Dedekind often refers to Schröder).⁷⁸ Let me concentrate on Cod. Ms. Dedekind III 14, p. 1-2. Dedekind's notes are concise, to say the least: only three pages on some points of interest in the first eight *Vorlesungen*, and in the *Anhang 6* on group theory. Dedekind's notes start with a couple of lines citing the *Anhang 6*:

P. 684 Enumeration for the possible types of groups in identical calculus.⁷⁹

P. 685-686 Proof of the theorem (statement in §12 p. [282]), that in the logical calculus with groups, there are cases in which the formula for the law of distributivity is only one-sidedly valid as a subordination (*Unterordnung*) (subsumption). (Cod. Ms. Dedekind III 14, 1)

Dedekind worked on distributivity related questions, as can be seen in the text presented in the next section, partly in relation with his investigations around the *Modulgesetz*, but also when investigating the properties of the operations as applied to Abelian groups.⁸⁰

From there, Dedekind's notes go back to the beginning of Schröder's book, presumably to better understand the previous notes. Most of them are not comments, but a brief statement of what he is reading: symbols used by Schröder (\neq corresponds to Dedekind's $<$; for the operations, Schröder uses $+$ and a multiplicative notation), reflexive property, transitive property, definition of equality, definition of 0 and 1... When noting the definition of Schröder's addition and multiplication, Dedekind notes the "similarity" with his own " $\mathfrak{G}(a, b)$ and $\mathfrak{M}(a, b)$ ", that is the union and intersection of sets as defined in (Dedekind, 1888). He continues with the commutativity and associativity of both operations, the "*Tautologiesetze*", i.e., the idempotence property, theorems on order, properties of 0 and 1 as respectively the unit and absorbing element of Schröder's addition and multiplication, results on the provability or unprovability of properties such as the distributivity law, proof "by intuition" of the distributivity law and the claim that it is impossible to prove it "on the basis of the existing" (which Dedekind punctuates with a "!"), and proof of said unprovability.

Dedekind's notes are slightly more developed when he arrives at Schröder's statement that there should be two calculi, that is, one in which both sides of the distributivity inequality are valid (identical calculus), and one in which only one side is valid (calculus with groups). Dedekind then states:

So, the logical calculus is the more general, weaker (*schwächere*)
the identical calculus is the more specialized, sharper (*schärfere*)

⁷⁸"Logical" is, for Dedekind, broadly synonymous with set-theoretical.

⁷⁹Schröder lists the "groups" that can be formed with the operations he defined for his identical calculus, e.g., the multiplicative group.

⁸⁰See below and (Dedekind, 1897b), §§5-6.

which is, he adds, his own expression of it. He continues:

P. 292 (§12) The two theorems 26_{\times} [$a(b+c) \notin ab+ac$] and 26_{+} [$(a+b)(a+c) \notin a+bc$]⁸¹ can be reduced to each other. [...]

P. 293 (§12) Principle III_{\times} : At least, when $bc=0$, it is certain that $a(b+c) < ab+ac$ (i.e. 26_{\times}).

P. 302 (§13) Definition of the negation.

P. 310 (§13) General proof of (26_{\times}) and of (27_{\times}) on the basis of Principle III and the negation.

P. 362 (§17) Theorem (40): From $\{ac \notin bc \text{ and } a+c \notin b+c\}$ follows $a \notin b$ ⁸²

Translation in my presentation:

From $\{(\alpha - \gamma) + (\beta - \gamma) = \beta - \gamma \text{ and } (\alpha + \beta) + (\beta + \gamma) = \beta + \gamma\}$ follows $\alpha + \beta = \beta$. [W]ith respect to the theorem $(\alpha - \gamma) + (\beta - \gamma) = (\alpha + \beta) - \gamma$, the hypotheses would be

$$\{(\alpha - \gamma) - \gamma = \beta - \gamma \text{ and } (\alpha + \beta) + \gamma = \beta + \gamma\}$$

or, when one sets $\alpha + \beta = \gamma$

$$\delta - \gamma = \beta - \gamma \text{ and } \delta + \gamma = \beta + \gamma,$$

the theorem (40) therefore amounts to the simple theorem:

From $\{\alpha\gamma = \beta\gamma \text{ and } \alpha + \gamma = \beta + \gamma\}$ follows $\alpha = \beta$. [In the margin: Schröder p. 362 mentions [it] in passing in Corollary 1.]

Or, in my presentation:

$\{\alpha - \gamma = \beta - \gamma \text{ and } \alpha + \gamma = \beta + \gamma\}$ follows $\alpha = \beta$. (Cod. Ms. Dedekind III 14, 3)

Without explicit comments from Dedekind, it is difficult to assess the role played by Schröder's book in his research on the *Modulgesetz*, its (un)provability and its status in his theory. Yet, it seems equally difficult to believe that the two are unrelated.

The first half of Dedekind's second (and later) set of notes is similar to this one, if slightly more developed. As it contains the word "*Dualgruppe*", it was likely written after "On the dualism in module theory". Dedekind also takes notes further in Schröder's book (up to Schröder's Theorem 50). He then develops considerations on the distributivity law, going further than Schröder — considerations which can be found in "On the dualism in module theory". Dedekind's second set of notes is written in columns designed to compare Schröder's theory and his own, but he ultimately barely does. The column dedicated to Schröder's work is itself written in columns highlighting the duality of the results.⁸³ Dedekind ends his notes with a table that summarises the

⁸¹In Dedekind's notation, these correspond to $a-(b+c) \notin (a-b)+(a-c)$ and $(a+b)-(a+c) < a+(b-c)$, i.e., one side of each of the dual equalities of the *Idealgesetz*. Schröder's 27_{\times} and 27_{+} are the *Idealgesetz* (Schröder, 1890, p. 292).

⁸²(Schröder, 1890, p. 362)

⁸³See Dirk Schlimm's chapter.

correspondences between Schröder's notations and definitions and Dedekind's, both in module theory and with the "new" notation φ, ψ .⁸⁴

The operations in set theory

Dedekind's laconic remarks do not allow to state with complete certainty what kind of impact the reading of Schröder had on his own works. Both in his published papers and in his drafts, Dedekind's (other) references to Schröder are placing his works as part of logic and as an instance of Dedekind's own theory (although, of course, Dedekind does not deny him priority). It seems likely that, at the very least, Schröder's book confirmed Dedekind's move towards building a theory that would be founded on the properties of the operations abstracted from their initial definition as GCD and LCM of modules, a theory that would encompass modules, ideals, groups and sets as special cases. Among the references to Schröder in Dedekind's drafts, several seem to point in that direction. Let me mention two of them.

In Cod. Ms. Dedekind XI 3-2 p. 38v, we find a reference to Schröder's *Algebra der Logik* at the end of two pages (pp. 37r and 38v⁸⁵) in which Dedekind studies the union and intersection of sets as operations corresponding to his $+$ and $-$ and Schröder's addition and multiplication. These sheets were written after 1894 (Dedekind refers to (Dedekind, 1894a)) but likely before "On the dualism in module theory". These investigations suggest that, even though he did know about the correspondence between union (resp. intersection) of sets and $+$ (resp. $-$) for modules,⁸⁶ Dedekind might not have considered (maybe before reading Schröder?) whether the validity of the laws he knew for modules also held for the intersection and union of systems.

Dedekind begins with the different manners of writing the intersection and union of two systems A and B :

$A + B$ the system compounded out of A, B [i.e., union (EH)],
 $\mathfrak{M}(A, B)$
 AB the intersection (*Gemeinheit*) of A, B , $\mathfrak{M}(A, B)$ or also $A - B$?
 (Cod. Ms. Dedekind XI 3-2 p. 37r)

Dedekind does not choose between Schröder's multiplicative notation and his own, and gives the two alternative notations throughout the draft. He defines "being a part of" (inclusion) in terms of union and intersection:

That M is a part of D will be expressed by $M + D = D$, similarly
 by $MD = M$ (that is, $M - D = M$). (*Ibid.*)

⁸⁴<https://eman-archives.org/Dedekind/items/show/308>

⁸⁵Dedekind writes on a large advertisement folded in two. On the two outer folios, he wrote on set theory. On the two inner folios, he wrote on modules. The two do not seem to be directly related.

⁸⁶(Dedekind, 1894a, 497–498, footnotes)

This leads him to state that “when M is a part of D , then we have”

$$M + (A - D) = (M + A) - D \text{ or } M + AD = (M + A)D$$

i.e., the modular law for sets, and for which he does give a proof. The absorption law is briefly noted in a corner. Dedekind is more interested in proving the generality of the distributive law (or *Idealgesetz*):

Is there generally $(A + B) - C = (A - C) + (B - C)$? otherwise $(A + B)C = AC + BC$? and $(A - B) + C = (A + C) - (B + C)$? otherwise $AB + C = (A + B)(B + C)$? (*Ibid.*)

For this, he starts by showing the by then well-known equalities

$$\begin{aligned} B + (A - (B + C)) &= (A + B) - (B + C) \\ (B - C) + (A - B) &= (A + (B - C)) - B. \end{aligned}$$

Dedekind's proof is set-theoretical.⁸⁷ He then uses these equalities to prove, in a similarly set-theoretical manner, that

$$(A + B) - C = (A - C) + (B - C) \text{ that is, } (A + B)C = AC + BC$$

next to which he refers to his Supplement XI in the 1894 edition of Dirichlet's *Vorlesungen über Zahlentheorie*, and more precisely to the sections on module theory and the laws of $+$ and $-$,⁸⁸ and to Schröder's *Algebra der Logik*, Lecture 6, that is, again, to the §12, mentioned in his notes.

Dedekind does not mention the link between the *Idealgesetz* and *Modulgesetz*, and the laws he proves here. While these investigations appear to be only marginally related to duality,⁸⁹ they are interesting as parts of the genesis of Dedekind's *Dualgruppen*, as part of the development of the idea that the operations could be abstracted from their context of definition and studied for themselves, as defined by the laws they verify rather than by the objects they act on.

Groups of arbitrary elements

A second notable reference to Schröder can be found in Cod. Ms. Dedekind XI 2, p. 10. There, Dedekind studies the “formation of a (φ, ψ) -group (logical calculus, Schröder S. 291) of three basis elements a, m, d under the hypothesis $m\varphi d = d$ and so also $m\psi d = \overline{m}$ ”.⁹⁰ This draft is not clearly datable, except for the reference to Schröder. It is written on the back of a dismissed text entitled “*Dualismus*” and numbered 17, which is likely not directly related to

⁸⁷The manuscript is available at <https://eman-archives.org/Dedekind/items/show/315>.

⁸⁸Among which, there is the *Modulgesetz* and the two dual laws $(a \pm b) \mp (b \pm c) = b \pm (a \mp (b \pm c))$.

⁸⁹Although it seems that the dualism in Schröder's operations was a key aspect of Dedekind's interest in them and, by extension, in investigations such as the one presented here.

⁹⁰Note that the hypothesis means that d is a divisor of m . See <https://eman-archives.org/Dedekind/items/show/264> for the full manuscript.

“On the dualism in module theory” (1896-1897), as the latter does not have the same presentation, is not written on the same paper, and does not miss a page 17. The present draft is likely to have been written before “On the dualism in module theory”, to which Dedekind eventually suggested a change of title.

Dedekind draws a table of the elements of this “group”. Below the tables, one finds the definition of the elements of the group:

$$\begin{aligned} d' &= m\varphi a, & m' &= a\psi d \\ m'' &= m\varphi m', & d'' &= d\psi d' \\ d''' &= a\varphi d, & m''' &= a\psi m \end{aligned}$$

Dedekind uses the same symmetrical display as he does for modules. On the right-hand side of the paper, he lists what he calls “*Folgerungen*”, which are all the possible combinations, thus giving him the contents of his table. On the left-hand side, he studies the order properties, briefly considering the ‘levels’ composed by the elements. The last lines state:

$$\begin{aligned} \text{From } a\varphi b = u \text{ follow } & \left\{ \begin{array}{ll} u\varphi a = u, & u\varphi b = u \\ u\psi a = a, & u\psi b = b \end{array} \right\} \\ \text{From } a\psi b = v \text{ follow } & \left\{ \begin{array}{ll} v\psi a = v, & v\psi b = v \\ u\varphi a = a, & u\varphi b = b \end{array} \right\} \end{aligned}$$

(Cod. Ms. Dedekind XI 2, p. 10r)

Dedekind thus seems to be stating general rules for the composition, but no proof is given.

The formation of this (φ, ψ) -group shows one of Dedekind’s attempts at applying his (initially) module-theoretical research to a different — here, logical — context. The choice of notation for the operations is not surprising in the context of a “logical” inquiry. Indeed, it is Dedekind’s preferred notation for mappings in a set-theoretical context (in particular since (Dedekind, 1888)). Dedekind adopts a similar approach in a number of other drafts, but most of them can be dated from after “On the dualism in module theory”.⁹¹ However, it is presented explicitly as a generalisation in an earlier draft, as we will see in the next section.

These investigations, together with that of set-theoretical properties and Schröder’s logical calculus, play a key role in the development of the idea that the operations initially defined for modules can apply to different operands — just as one can calculate with objects which are not numbers, which Dedekind used in many of his other works. We see, thus, a process of abstracting the operations from module theory, leading to Dedekind working with them more generally. We also see a clear inflection towards investigating not the laws of the operations, but the properties of the systems formed by closure under such operations, without taking into consideration the individual nature of their elements — another idea pregnant in many of his other works.

⁹¹See <http://eman-archives.org/Dedekind/items/browse?tags=notation-generale>

4.4.4 “Generalisation of a part of module theory”

In Cod. Ms. Dedekind X 11-2, p. 56-57, Dedekind proposes a “generalisation of a part of module theory” (the title of these two pages). These pages are difficult to date, but we can say that they were written before “On the dualism in module theory”⁹² and, considering the notations, around the same time as the texts presented above.

In this manuscript, Dedekind studies the conditions of possibility for the definition of a group defined by operations φ, ψ verifying the same laws as his module-theoretic $+$ and $-$, within a general framework. It is reminiscent of his notes on Schröder's *Algebra der Logik*. Dedekind pays particular attention to the (yet to be named) *Modulgesetz*, and shows that it is indeed possible to form a group that only verifies the inequality $m + (a - d) > (m + a) - d$ (with $m > d$).

Dedekind considers a system of elements a, b, \dots and defines two operations φ and ψ in the following way:

Operation φ with the laws (addition of modules; gr[eatest] com[mon] divisor)

$$(I) \quad a\varphi a = a ; a\varphi b = b\varphi a ; (a\varphi b)\varphi c = a\varphi(b\varphi c) = a\varphi b\varphi c = \varphi(a, b, c)$$

Operation ψ with the laws (sm[allest] com[mon] mult[iple])

$$(II) \quad a\psi a = a ; a\psi b = b\psi a ; (a\psi b)\psi c = a\psi(b\psi c) = \Psi(a, b, c) = a\psi b\psi c$$

Connection between φ and ψ

$$(III) \quad (a\varphi b)\psi a = a ; (a\psi b)\varphi a = a$$

We have, here, the operations defined by their laws solely, without taking into account the nature of the operands. Dedekind's first result is:

Theorem. Each of the two facts

$$(IV) \quad (a\varphi b) = a ; (a\psi b) = b$$

follows from each other. [...] We express the double relation contained in (IV) by

$$(V) \quad a < b \text{ or } b > a$$

from which he deduces the transitivity and antisymmetry of the order and that the operations preserve the order. More interesting is the consideration of (what he would later call) the *Modulgesetz*:

Theorem. If $m > d$, then we have

$$m\varphi(a\psi d) > (m\varphi a)\psi d$$

⁹²The clue for this datation is the use of the idempotence property in the so-called fundamental laws, as Dedekind realised, in the manuscript “On the dualism in module theory” itself, that it could be deduced from the absorption property, and was thus not “fundamental”.

Deletions show that Dedekind started writing the result using his usual $+$ and $-$ notations. The proof is short:

We have indeed always (that is, without hypothesis)

$$m > m\varphi a, \quad a\psi d > m\varphi a, \quad a\psi d > d$$

in addition the hypothesis $m > d$.

$$\left. \begin{array}{llll} \text{From} & m > m\varphi a & \text{and} & m > d \\ \text{"} & a\psi d > m\varphi a & \text{"} & a\psi d > d \end{array} \right\} \begin{array}{l} \text{follows} \\ \text{"} \end{array} \left. \begin{array}{l} m > (m\varphi a)\psi d \\ a\psi d > (m\varphi a)\psi d \end{array} \right\} \text{hence } m\varphi(a\psi d) > (m\varphi a)\psi d$$

But from the previous hypotheses, follows by no means the identity of these two ~~modules~~ elements!

On the back of the page, Dedekind lists inequalities valid without the hypothesis $m > d$.

On the second page, Dedekind attempts “to find the simplest example in which $m > d$ but $m\varphi(a\psi d)$ and $(m\varphi a)\psi d$ are different”. He makes the “claim” that

the five elements

$$a, m, d, m' = m\psi d, d' = m\varphi a$$

must all be different.⁹³

We see, here, that Dedekind initiates an explicit generalisation of his research, which in particular relies on the possibility to develop a *Modulgesetz*-less theory. Dedekind understood that the property is not related directly to the operations, but to the context in which they were initially defined, and it is thus not “indispensable” to study the operations (and the groups thus formed) if one wishes to generalise them. This is important as it shows how duality is eventually dissociated from modules for Dedekind, and thus how it leads him to understand that this dualism is not a property of *modules themselves*.

4.4.5 The case of the first draft of (Dedekind, 1897b)

The intricate links between Dedekind’s research on the operations in module theory and his research on groups are clearly visible in the drafts for (Dedekind, 1897b). These drafts can be tiresome to navigate and interpret, as they investigate combinatoric properties of divisors of numbers using Dedekind’s figurative numbers notation. But a series of notes in Cod. Ms. Dedekind III 14, p. 32-36⁹⁴ show early material for a paper whose contents correspond to parts of

⁹³See <https://eman-archives.org/Dedekind/items/show/304>. On page 58, Dedekind studies the group with seven elements $a, m, d, b = a\psi d, c = m\varphi a, h = m\varphi b, k = c\psi d$ which are “compatible” with the hypothesis $m > d$ but do not verify $m\varphi(a\psi d) = (m\varphi a)\psi d$. This page is not titled, unlike the two previous ones. Without further indication and since Dedekind used a different kind of paper, we can only guess that these three folios go together.

⁹⁴The following pages up to p. 52 certainly go together with these notes, but only contain results of computations.

(Dedekind, 1897b). These notes themselves do not contain many indications for their datation, but they were written on re-used sheets of paper bearing dates as late as 1896. The use of idempotence as a fundamental property suggests that it was written before “On the dualism in module theory”, and thus before Dedekind had the definition of a *Dualgruppe*.

Dedekind places these investigations in a group-theoretical context that seems removed from any considerations on *Dualgruppen*, which do not appear in these notes.⁹⁵ Dedekind is working through what will ultimately constitute the §§5-6 on Abelian groups and integral elements in (Dedekind, 1897b, pp. 120–135).

The notes, which start at §3 — presumably after two sections on case studies of the decomposition of numbers in their GCD, as in (Dedekind, 1897b) — first consider combinations. Dedekind gives the definition of a combination, its degree (number of distinct elements), the definition of the “combination composed of 2 combinations α and β denoted by $\alpha\beta = \beta\alpha$ ” which he later corrects with “Better: sum $\alpha + \beta$ ” with idempotence, commutativity, associativity, identity element, definition of the intersection (*Durchschnitt*) of two combinations denoted by $\alpha - \beta$, degree of the difference and sum, and as an addendum the absorption property, *Idealgesetz*, and idempotence, commutativity, associativity, identity element for $-$. This corresponds, broadly, to the contents of the §3 on combinations in (Dedekind, 1897b) — although the organisation is a little different.

Then, while in (Dedekind, 1897b) Dedekind defines *Dualgruppen* — to which he admittedly refers as a digression —, here he immediately goes to a multiplicative Abelian group, defined like in (Dedekind, 1897b). He introduces a second operation $+$ in the Abelian group, such that:

$$\mathfrak{a} + \mathfrak{a} = \mathfrak{a} \tag{1}$$

$$\mathfrak{a} + \mathfrak{b} = \mathfrak{b} + \mathfrak{a} \tag{2}$$

$$(\mathfrak{a} + \mathfrak{b}) + \mathfrak{c} = \mathfrak{a} + (\mathfrak{b} + \mathfrak{c}) \tag{3}$$

$$(\mathfrak{a} + \mathfrak{b})\mathfrak{c} = \mathfrak{a}\mathfrak{c} + \mathfrak{b}\mathfrak{c} \tag{4}$$

He adds:

By the hypothesis of the existence of the operation $+$ (addition), the generality of the Abelian group is restricted!

Question: is there inside an infinite Abelian group \mathfrak{A} , which consists of all the powers \mathfrak{a}^x (positive and negative exponents x and $\mathfrak{a}^0 = \mathfrak{a}$) of an element \mathfrak{a} , an addition in the above sense?

Answer: Yes, and indeed two (dual opposite [*dualistisch entgegengesetzte*]) additions.

(Cod. Ms. Dedekind III 14, p. 34v)

⁹⁵A discarded and very incomplete draft of (Dedekind, 1897b), albeit significantly different from the final version, does contain the notion of *Dualgruppe* either (Cod. Ms. Dedekind III 14, p. 14-29). See <https://eman-archives.org/Dedekind/items/show/311>

Further, he defines the second operation, again referred to as “dual opposite”, as

$$a - b = \frac{ab}{a + b}$$

He then proves the idempotence, commutative and associative properties, the distributivity of the multiplication over $-$, the absorption property and the *Idealgesetz*, and eventually defines the order (divisibility). This correspond to a condensed version to the §§5-6 of (Dedekind, 1897b) defining a structure of distributive lattice on the studied group — without, in these drafts, ever mentioning *Dualgruppen*.

These notes suggest that *Dualgruppen* were later superimposed on initially unrelated research — even more so in (Dedekind, 1897b) than in (Dedekind, 1900). Dedekind likely recognized afterwards that *Dualgruppen*, in their general definition, were a unifying concept for his investigations in group and module theory.

4.5 The last steps

The manuscript Cod. Ms. Dedekind XI 1, p. 1-27 was initially entitled “On the dualism in module theory”. Dedekind later corrected it to be “On *Dualgruppen*”. He also indicated that his text had to be merged together with a shorter manuscript entitled “Some propositions on *Modul-Gruppen*” (Cod. Ms. Dedekind XI 1, p. 29-32).⁹⁶ These two texts were visibly written independently and merged *a posteriori*. It is unclear whether they were written at the same time, but it they were likely both written in a time range of less than two years, as they can respectively be dated from between 1896 and 1897,⁹⁷ and between 1895 (reference to (Weber, 1895-1896)) and 1897. Together, these texts constitute an important step in the writing and clarification of Dedekind’s ideas on (would be) *Dualgruppen*. As a matter of fact, Dedekind suggested, for the new text, the title “On *Dualgruppen*”, adding that *Modulgruppen* (and *Idealgruppen*) were a “special case”. This, however, seems to have been added fairly late in the process, as none of these texts contain the definition of a *Dualgruppe* properly speaking.

This shows that Dedekind came to the concept of *Dualgruppen* in several steps, which were most likely related yet autonomous: on one hand, the generalisation of $+$ and $-$; on the other hand, a rigorous — systematic and general (i.e., not based on case studies) — presentation of his research on “groups of modules” as an integral part of module theory; and finally, the application of these two aspects to objects which are not modules and even to undetermined objects.

⁹⁶See respectively <http://eman-archives.org/Dedekind/items/show/171> and <http://eman-archives.org/Dedekind/items/show/172>.

⁹⁷Dedekind’s use of the idempotence as a fundamental law in one of the drafts of (Dedekind, 1897b) dated from around 1896 (see above), suggests that this was written afterwards.

4.5.1 “Some propositions on *Modul-Gruppen*”

After intensive investigations on the systems or “groups” generated by a finite number of modules, Dedekind eventually introduced the concept of “*Modul-Gruppe*” (most often written without the hyphen), which is defined formally in (Dedekind, 1894a). It was initially a part of module theory — as suggested by its first mention in (Dedekind, 1894a). “Some propositions on *Modul-Gruppen*” seems to be the outcome of his investigation of groups of modules, likely what he had in mind in (Dedekind, 1894a). The focus is on divisibility and order properties in *Modulgruppen*. The order and the properties of chains are considered with far more details than the *Modulgesetz* or the laws of the operations themselves.

This text is short (7 pages, written in two columns to allow addenda and modifications), and does not contain all the results published by Dedekind either on *Dualgruppen* of the module type, or even on *Modulgruppen*. In the margin, Dedekind indicated that it was to be “work[ed] into the paper ‘On the dualism in module theory’ (better: ‘On *Dualgruppen*’)” (which was followed by indications on the notation changes that had to be made). While the dualism is not addressed in the text, its presence is notable. Duality appears as one property of the operations among others. It is used, in particular, in the proofs.

In (Dedekind, 1894a), Dedekind gave the following indications on *Modulgruppen* in a footnote:

If one forms from three arbitrary modules new modules \mathfrak{ab} by always forming again the greatest common divisor and the smallest common multiple, then one obtains a finite *Modulgruppe*, which in general consists of 28 different modules. The particular properties of each group, which with two modules \mathfrak{a} , \mathfrak{b} also contains the modules $\mathfrak{a} \pm \mathfrak{b}$ is a topic for another place. Here, I would like to mention only the following theorem, often applied: if \mathfrak{a} , \mathfrak{b} are two arbitrary modules, then there exists a one-to-one correspondence between the group of all the modules \mathfrak{a}' which are divisors of \mathfrak{a} and likewise multiples of $\mathfrak{a} + \mathfrak{b}$, and the group of all the modules \mathfrak{b}_1 which are multiples of \mathfrak{b} and likewise divisors of $\mathfrak{a} - \mathfrak{b}$, which can be expressed by the two relations reciprocal of each other $\mathfrak{b}_1 = \mathfrak{b} - \mathfrak{a}'$, $\mathfrak{a}' = \mathfrak{a} + \mathfrak{b}_1$. (Dedekind, 1894a, 499, footnote)

The definition given at the beginning of the manuscript is more general and does not concern the special case of a *Modulgruppe* generated by three elements. It also includes both operations as operations of the group:

A system M of modules is called a group when the modules $\mathfrak{a} + \mathfrak{b}$, $\mathfrak{a} - \mathfrak{b}$ formed by any two modules \mathfrak{a} , \mathfrak{b} belong to the same system. (Cod. Ms. Dedekind XI 1, p. 29r.)

The name and the definition suggest that this concept results from the important conceptual shift from operations between modules to sets of modules closed by these operations.

In the text, Dedekind studies properties and divisibility laws of *Modulgruppen*, chains of modules and the related “nearest divisor”, “nearest multiple”, length of the group, levels (*Stufe*) and number of levels, which he had been studying in earlier drafts. Some of the results presented in this text are used for *Dualgruppen* in (Dedekind, 1900), in particular in the sections “§6 Relations between *Modulgesetz* and *Kettengesetz*” and “§7 Levels in finite *Modulgruppen*”.

After the definition of a *Modul-Gruppe*, Dedekind states that:

Every isolated module \mathfrak{a} forms a group by itself (\mathfrak{a}), because $\mathfrak{a} + \mathfrak{a} = \mathfrak{a} - \mathfrak{a} = \mathfrak{a}$.

The system \mathfrak{J} of all modules forms a group. (Cod. Ms. Dedekind XI 1, p. 29r)

He proposes a definition of the GCD of an arbitrary number of groups, also calling it the intersection of these groups (this intersection does not necessarily exist — however for a given system of modules, there exists a *Modulgruppe* containing them all); as well as the definition of LCM of an arbitrary number of groups corresponding to the union of these groups (but which is not named as such) denoted by $(\mathfrak{M}) = (\mathfrak{A}, \mathfrak{B}, \mathfrak{C})$. These notions are not in (Dedekind, 1900).

He then gives a more complete version along with a proof of the result stated in the (Dedekind, 1894a) footnote (to which he explicitly refers):

Let \mathfrak{a} , \mathfrak{b} be two arbitrary modules, then there exists between the group \mathfrak{A}' of all the modules \mathfrak{a}' , which verify the conditions

$$\mathfrak{a} + \mathfrak{b} < \mathfrak{a}' < \mathfrak{a} \quad (1)$$

and the group \mathfrak{B}_1 of all the modules \mathfrak{b}_1 , which verify the conditions

$$\mathfrak{a} - \mathfrak{b} > \mathfrak{b}_1 > \mathfrak{b} \quad (2)$$

a one-to-one correspondence, which can be expressed by the two reciprocal following relations

$$\mathfrak{b}_1 = \mathfrak{b} - \mathfrak{a}' \quad (3)$$

$$\mathfrak{a}' = \mathfrak{a} + \mathfrak{b}_1 \quad (4)$$

(Cod. Ms. Dedekind XI 1, p. 29v)⁹⁸

The proof relies on (what will later be called) the *Modulgesetz*, which was proved (for modules) in (Dedekind, 1894a, p. 498) (to which, again, Dedekind refers explicitly). In (Dedekind, 1900), it is given in the §6 on the relationships between *Modulgesetz* and *Kettengesetz*.

After this, Dedekind moves on to defining a “finite *Modulgruppe* \mathfrak{M} ” — which contains a finite number of modules — and its “degree” — the number of modules it contains. In \mathfrak{M} , he defines the notion of “nearest divisor” and “nearest multiple”:

⁹⁸In the margin, Dedekind gives the equations (3) and (4) in terms of numbers of congruence classes, which is the only occurrence of this notion in this text.

Furthermore, a module \mathfrak{m} is to be called a nearest divisor of \mathfrak{n} , and \mathfrak{n} a nearest multiple of \mathfrak{m} (in \mathfrak{M}), when

- 1) $\mathfrak{m} < \mathfrak{n}$,
- 2) $\mathfrak{m}, \mathfrak{n}$ are different,
- 3) \mathfrak{m} and \mathfrak{n} are the only two modules (in \mathfrak{M}), which are at the same time divisor of \mathfrak{n} and multiple of \mathfrak{m} . (Cod. Ms. Dedekind XI 1, p. 30r)

This leads Dedekind to two results:

If \mathfrak{m} is different from \mathfrak{g} [the LCM of all modules (contained in \mathfrak{M})], then there exists at least one nearest multiple of \mathfrak{m} ; if \mathfrak{m} is different from \mathfrak{o} [the GCD of all modules (contained in \mathfrak{M})]; then there exists at least a nearest divisor of \mathfrak{m} . [...]

If the module \mathfrak{m} possesses two distinct nearest multiples $\mathfrak{a}, \mathfrak{b}$, then we have

$$\mathfrak{a} + \mathfrak{b} = \mathfrak{m}$$

and $\mathfrak{a} - \mathfrak{b}$ is a nearest multiple of \mathfrak{a} as well as \mathfrak{b} . If the module \mathfrak{m} possesses two distinct nearest divisors $\mathfrak{a}, \mathfrak{b}$, then we have

$$\mathfrak{a} - \mathfrak{b} = \mathfrak{m}$$

and $\mathfrak{a} + \mathfrak{b}$ is the nearest divisor of \mathfrak{a} as well as \mathfrak{b} .

(Cod. Ms. Dedekind XI 1, p. 30)

For this last theorem, Dedekind explicitly states that the “second part” is dual to the first, but he nevertheless explicitly gives all the details. This result is also given, under a slightly different form, in §6 of (Dedekind, 1900).

The next set of notions concerns chains of modules (which itself is not defined, here — maybe because it was introduced in (Dedekind, 1894a, p. 523)). Dedekind gives:

- the definition of a complete chain (i.e., in a finite *Modulgruppe*), a chain that starts from \mathfrak{o} and finishes with \mathfrak{g} ;
- the definition of the length of the group, given as a theorem stating that in a finite *Modulgruppe* \mathfrak{M} , all complete chains are of the same length, which is named the length of the group \mathfrak{M} .

These two notions are not defined in (Dedekind, 1900). Finally, Dedekind defines “levels” and the “number of levels”, that is:

This results in the following arrangement of all modules \mathfrak{m} in a finite group \mathfrak{M} into levels, whose number is greater by 1 than the length of \mathfrak{M} . After one chooses an arbitrary rational integer o (e.g., zero), one must understand as number of levels of the module \mathfrak{m} , the number $s(\mathfrak{m}) = |o + l|$, where l is the length of the group $\mathfrak{M}_{\mathfrak{o}, \mathfrak{m}}$.⁹⁹ (Cod. Ms. Dedekind XI 1, p. 31v)

⁹⁹If $\mathfrak{a}, \mathfrak{b}, \mathfrak{c}, \dots$ are modules in the group \mathfrak{M} , then $\mathfrak{M}_{\mathfrak{a}, \mathfrak{b}, \mathfrak{c}, \dots}$ designates the group of all the modules \mathfrak{m} in \mathfrak{M} such that $\mathfrak{a} + \mathfrak{b} + \mathfrak{c} + \dots < \mathfrak{m} < \mathfrak{a} - \mathfrak{b} - \mathfrak{c} \dots$

He then gives and proves the following theorem:

If $\mathfrak{a}, \mathfrak{b}$ arbitrary modules in the group \mathfrak{M} , then we have

$$s(\mathfrak{a}) + s(\mathfrak{b}) = s(\mathfrak{a} + \mathfrak{b}) + s(\mathfrak{a} - \mathfrak{b})$$

(Cod. Ms. Dedekind XI 1, p. 31v)¹⁰⁰

In the rightside column, Dedekind gives several alternative notations for the result. The notions of levels, number of levels and the theorem constitute the content of the §7 “Levels in finite *Modulgruppen*” of (Dedekind, 1900) (in which he also gives an application of the result to the *Dualgruppe* generated by three modules).

4.5.2 “On the dualism in module theory”

Despite its initial title, “On the dualism in module theory” seems to try to dissociate the investigation from the specific nature of modules and adopt a more general approach.

I have often noticed a curious dualism, which appears in the theory of modules.* The same is repeated so frequently in other domains of research, that it seems beneficial to understand the general laws of combinations [*Verknüpfungs-Gesetze*] which prevail in this theory independently of the initial premises on which it is grounded. If such is the case, and that we replace the signs $+$ and $-$ used to designate the greatest common divisor and the least common multiple of two modules, by φ , ψ , then only the following hypotheses remain:

In a finite or infinite system S of elements a, b, c, \dots whose meaning is left completely indeterminate, there are two kinds of combinations φ and ψ , which from any two identical or distinct elements a, b always produce two completely determined elements of the same system S designated by $a\varphi b, a\psi b$. They obey the following six laws

$$\begin{array}{ll} \cancel{a\varphi a} = a & (1) \\ a\varphi b = b\varphi a & (2) (1) \\ (a\varphi b)\varphi c = a\varphi(b\varphi c) & (3) (2) \\ \cancel{a\psi a} = a & (1') \\ a\psi b = b\psi a & (2') (1') \\ (a\psi b)\psi c = a\psi(b\psi c) & (3') (2') \\ a\psi(a\varphi b) \cancel{(a\varphi b)\psi a} = a & (4) (3) \\ a\varphi(a\psi b) \cancel{(a\psi b)\varphi a} = a & (4') (3'). \end{array}$$

(Cod. Ms. Dedekind XI 1, p. 1r-2r)

¹⁰⁰An early version of these results can be found in Cod. Ms. Dedekind X 9, p. 21v (where the length is named “*Distanz*”).

*Dirichlet’s *Vorlesungen über Zahlentheorie*, fourth edition, §169.

For *Modulgruppen*, the closure was the defining property. Here, the six laws verified by the “combinations” are the defining property.

There are several things to point out, here. Firstly, the crossed out laws show us the exact moment when Dedekind realized that the idempotence could be deduced from the absorption law and was, thus, not a fundamental property, as he explains in a note on the back of the page:

The two last laws (3) and (3') contain [...] a connection between the operations φ, ψ and their combination leads — without relation to the earlier laws — to the two consequences

$$a\varphi a = a \tag{4}$$

$$a\psi a = a; \tag{4'}$$

when we replace the arbitrary element b in (3') by $a\varphi b$, in (3) by $a\psi b$. (Cod. Ms. Dedekind XI 1, p. 3v)

Secondly, this first quote illustrates clearly the move initiated towards a more general conception of operations. Dedekind, here, explicitly dissociates the operations from the module-theoretic interpretation. This move is indeed made explicit by changing the signs used, and abstracting the operations from their module-theoretic context.¹⁰³

Thirdly, the notation and layout for the laws put forward, as I mentioned previously, their dualism, and it is very explicitly stated by Dedekind:

Two of these laws, designated by (α) and (α') , correspond to each other dualistically in the sense that they can be changed one into the other by switching φ and ψ . Consequently, the dual transformation of the conditions of any proposition of our research always results in a second proposition, which is the dual equivalent of the first one, which itself, reciprocally, results from the second one in the same way, or can also very well be identical to it. (Cod. Ms. Dedekind XI 1, p. 2r)

Duality appears, in this text, as an essential property of the operations, and as such, it is considered at the center of the theory. Its status, however, stays relatively unclear. The initial title suggests that Dedekind set out to study the dualism itself, or at least that he would tell his reader how or maybe why it is significant. In fact, Dedekind is interested by the systems and the order. The dualism, while present and indeed important for the said systems, is barely addressed. Throughout the text, the duality of the definitions and theorems is underlined by stating systematically each law for both operations and numbering them (α) and (α') .

It is important for Dedekind to put forward how his general concept can be instantiated by many examples. These concrete cases, so to speak, are for him

¹⁰³It is interesting to observe, here, how Dedekind, who wrote against the use of so-called *Darstellungsformen*, put the emphasis on replacing the signs of the operations to carry out their generalisation.

a testimony of the usefulness of the concept, they show “how diverse are the domains to which our research can be applied” (Cod. Ms. Dedekind XI 1, p. 3, the same is written in (Dedekind, 1897b, p. 113)). Dedekind’s position is consistent with his usual statements: while it is best to give a definition as general as possible, the possible applications (e.g., in number theory) are what gives a concept its usefulness. In this draft, Dedekind lists the following examples: modules, groups (Abelian, Galois), sets, and points of a number-space, which all appear also in (Dedekind, 1897b).

As the order is defined by the operations, the dualism can be observed here again:

The two statements

$$m\varphi d = d \quad (5)$$

$$m\psi d = m \quad (5')$$

are equivalent, i.e., each is a consequence of the other.

If (5) is valid, then follows (5') from (3) when one sets $a = m$, $b = d$; and when (5') is valid, then (5) follows from (3') when one sets $a = d$, $b = m$.

These relations between two elements m, d designated by (5) or (5') appear so often that it is convenient to ~~name~~ express them in a simpler way; here I chose the two notations that I use in module theory[†]

$$m < d \quad (6)$$

$$m > d \quad (6')$$

which are completely equivalent to each other, as can be seen with (5) and (5'). Likewise, we will call m a multiple (*Vielfaches oder Multiplum*) of d , and reciprocally d a divisor (*Theiler oder Divisor*) of m . From these definitions follow the theorems whose dualistic character obviously comes from the fact that the inversion of φ , ψ always results in the same for the signs $>$, $<$. (Cod. Ms. Dedekind, XI 1, p. 5r-6r)

Note that Dedekind chooses to keep the terminology used in module theory. He also justifies introducing the designation of φ and ψ as GCD and LCM in a note added in the margin.¹⁰⁴

Dedekind proves a series of properties for $<$, starting with

$$a\varphi b < a \quad (7)$$

$$a\psi b > a \quad (7')$$

[†]In many other domains of application, the two signs $>$, $<$ would be reversed, which would correspond better to the meaning otherwise attached to them.

¹⁰⁴The justification is similar to the one given in his number-theoretical works.

and the reflexive, antisymmetric and transitive properties.

He then sets out to prove :

$$m\varphi(a\psi d) > (m\varphi a)\psi d \quad (16)$$

$$(a\varphi b)\psi(b\varphi c) > b\varphi(a\psi(b\varphi c)) \quad (17)$$

$$(a\psi b)\varphi(b\psi c) > b\psi(a\varphi(b\psi c)) \quad (17')$$

$$(a\varphi b)\psi(b\varphi c) > b\varphi(a\psi c) \quad (18)$$

$$(a\varphi b)\psi(b\varphi c) > b\psi(a\varphi c) \quad (18')$$

Dedekind notes that (17) and (17') can be deduced from (16) with the appropriate substitution, and that "reciprocally, the theorem (16) follows as well from (17) and also from (17')" with similar substitutions, and "the theorem (16) corresponds to itself dualistically" (Cod. Ms. Dedekind XI 1, p. 9). We recognize, here, the inequalities which correspond to the *Modulgesetz* (16) and *Idealgesetz* (18).

And finally, Dedekind proves these two inequalities:

$$b\varphi(a\psi(b\varphi c)) < b\varphi(a\psi c) \quad (19)$$

$$b\psi(a\varphi(b\psi c)) < b\psi(a\varphi c) \quad (19')$$

Dedekind endeavors to keep a general (not related to modules) approach throughout his manuscript. The properties of the operations and of the order relation are defined and proved mostly without references to the nature of the operations or operands. Module theory is often referred to as an example, but Dedekind's considerations on the so-called *Modulgesetz* suggest that it might be, here, more than a simple example:

We want now to consider again the theorem (16). In the module theory (and as well in the three domains of application mentioned in §1), it is possible to prove that from the premise $m > d$, this theorem also follows

$$(m\varphi a)\psi d > m\varphi(a\psi d) \quad (20)$$

from which, with respect to the theorem III, it follows that

$$m\varphi(a\psi d) = (m\varphi a)\psi d \quad (21)$$

For this proof, I have explicitly emphasized[‡] that the previous theorems, which are all based on the six fundamental laws stated in §1, do not suffice, and that it is rather required to go back, once again, to the concept of module. To justify this claim, we want now to set ourselves the task of finding the simplest example of a system S of elements in which the operations φ, ψ verify the fundamental laws of §1, but in which the premise $m > d$ does not by any means entail (20). (Cod. Ms. Dedekind XI 1, p. 10r)

[‡]Dirichlet's *Vorlesungen über Zahlentheorie*, fourth edition, §169, p. 499.

So as to find the “simplest” system not verifying (20), Dedekind adopts the same strategy as in the draft presented on p. 61: he considers three elements a , m , d *without* the hypothesis on the divisibility of m by d , and defines the two elements

$$m' = a\psi d \quad (22)$$

$$d' = m\varphi a \quad (23)$$

from which he deduces a series of order conditions for

$$d\psi d' > m\varphi m'^{105} \quad (20)$$

to be valid. Since he wants to build a system in which (20) is not valid, he looks for a system of elements for which the said conditions are *not* valid. He shows that such a system (and indeed the “simplest” one) can be the one constituted of a , m , d , a' and m' such that

	a	m	d	m'	d'
a			d'	d'	a
m	m'		d	m	d'
d	m'	m		d	d'
m'	m'	m'	m'		d'
d'	a	m	d	m'	

(in which the upper right part is constituted by the elements $x\varphi y$, the lower left part by the elements $x\psi y$, and the empty diagonal is for the idempotence property).

As it should be clear, since he worked in a more general setting, Dedekind did not see the *Modulgesetz* as either “indispensable” or “the source of dualism” anymore. Instead, he came to see it a property specific to modules — and later, the defining property for module-like *Dualgruppen*. While he hasn’t, yet, named the property the *Modulgesetz*, there are a few sheets following this text¹⁰⁶ and several more elsewhere¹⁰⁷ that investigate “non-modular” (*nicht modularartige*) and “modular” (*modularartige*) systems. Note, however, that there remains a certain imprecision in how this law is conceived, as it is not the same to simply state that this law is verified by modules, and to state that to prove this law one should “go back to” module theory. This imprecision is eventually lifted, in the additional research on the “general theory” which I will consider in the next section. In the published papers, Dedekind does not speak about *modular* systems, but of systems “of module type”.

The last paragraph of this text deals with the following:

¹⁰⁵This is indeed equivalent to the inequality (20) given the definition of d' and m' .

¹⁰⁶Cod. Ms. Dedekind XI 1, p. 25 (<http://eman-archives.org/Dedekind/items/show/176>).

¹⁰⁷In particular, Cod. Ms. Dedekind XI 2, p. 47-50, written on the back of a draft of (Dedekind, 1895).

The two operations φ and ψ in a system S are evidently completely defined if the combinations $a\varphi b$, $a\psi b$ are given for any two ~~distinct~~ elements a , b in S . Let us assume that they satisfy the laws of symmetry (1) and (1') and also the laws (4), (4'), then these definitions can be represented in a (finite or infinite) table of the same form as in the example at the end of §3. But if the cells of this table are arbitrarily occupied by elements of the system S , the verification of whether the associativity laws (2), (2') and the laws (3), (3') are really satisfied will require laborious work, even for a finite system S of a moderate number of elements. Or, if one insists on the correctness of these arbitrarily arranged tables, the demand that the last four laws should also apply will usually have the consequence that elements with different names should be identical with each other. (Cod. Ms. Dedekind XI 1, p. 15r)

Dedekind thus reformulates the laws of the order in set-theoretical terms and considers “for each element a of the system S the system $a\varphi'$ of all divisors of a ” (that is, all the elements d such that $d < a$), thus not involving the operations themselves:

- α . Each a of the system S corresponds to a fully determined system $a\varphi'$, which is a part of S .
- β . The element a is contained in $a\varphi'$, and so is an element of the system $a\varphi'$. [...]
- γ . From $a\varphi' = b\varphi'$ follows $a = b$. Then according to β ., a is in $a\varphi'$, and so ~~also~~ in $b\varphi'$, i.e., $a < b$, and since it ensues $b < a$ as well, it follows that $a = b$. [...]
- δ . If d is an element of $a\varphi'$, then $d\varphi'$ is a part of $a\varphi'$. Then, according to the hypothesis, we have $d < a$, and since each element in $d\varphi'$ is $e < d$, it follows that $e < a$. [...]
- ε . If a, b are arbitrary elements, there exists one (and according to γ only one) element d such that the system $d\varphi'$ is the intersection of $a\varphi'$ and $b\varphi'$, i.e., the collection of all the elements common to both systems. [...]
- ζ . If a, b are two arbitrary elements, then there always exist elements n such that $a\varphi'$ and $b\varphi'$ are parts of $n\varphi'$, and among these elements n , we find a single m such that ~~each such system $n\varphi'$ forms a part of $m\varphi'$~~ $m\varphi'$ forms a part of each such system $n\varphi'$.

(Cod. Ms. Dedekind XI 1, p. 16r-17r)

Dedekind deduces, from the fundamental laws of the operations φ and ψ given in §1, the existence of systems $a\varphi'$ verifying the laws $\alpha, \beta, \gamma, \delta, \varepsilon, \zeta$, and a reformulation of the laws of the order given in §2 “without any mention of the operations φ and ψ ”. His aim is, then, to “define these operations backwards”.

Note that only one side was studied, here, so the next step is “to show that on the basis of the six laws $\alpha, \beta, \gamma, \delta, \varepsilon, \zeta$, one can always oppose the systems $a\varphi'$ there occurring a dualistically corresponding second class of systems $a\psi'$, which obey exactly the same laws” (Cod. Ms. Dedekind XI 1, p. 17r).

If d is an arbitrary element, then according to β . there exists always at least one element m which has the property of being contained in the system $m\varphi'$, and we want to define the system $d\psi'$ as the collection of all these elements m . The two statements

$$\left. \begin{array}{l} d \text{ is an element of } m\varphi' \\ m \text{ is an element of } d\psi' \end{array} \right\} \quad (31)$$

are therefore perfectly equivalent. (Cod. Ms. Dedekind XI 1, p. 17r-18r)

Dedekind then defines six laws $\alpha', \beta', \gamma', \delta', \varepsilon', \zeta'$ for systems $m\psi'$ similarly to what he did previously, which he concludes by

Since these laws $\alpha', \beta', \gamma', \delta', \varepsilon', \zeta'$ derived from the laws $\alpha, \beta, \gamma, \delta, \varepsilon, \zeta$ differ from those essentially only in that the place of the signs φ' and ψ' are inverted, then it is obvious that reciprocally, the first can originate from the latter in the same way, when one considers that, following (31), the system $m\varphi'$ reciprocally is the collection of all these elements d which have the property that m is contained in $d\psi'$. (Cod. Ms. Dedekind XI 1, p. 20r)

On this ground, he (re)defines two operations φ and ψ , showing that if he took

a system S of elements a, b, \dots in which the laws $\alpha, \beta, \gamma, \delta, \varepsilon, \zeta$ (and consequently also the laws $\alpha', \beta', \gamma', \delta', \varepsilon', \zeta'$), then it is possible, as we wanted to prove conversely, to define one and only one pair of operations φ, ψ , which verify the six laws of §1 and from which originate, in the above manner, the systems $a\varphi'$ (und $a\psi'$), now considered as given. (Cod. Ms. Dedekind XI 1, p. 21r)

The manuscript ends with the proof of the unicity of the operations φ and ψ . The considerations of this last paragraph are not in (Dedekind, 1897b; Dedekind, 1900).

The definition given in this text is about a system in which are defined internal operations (or “combinations”). It is not equivalent to defining a system by the property that it is closed under internal operations, which is the case in the definition of a *Dualgruppe*. This is accomplished by joining together his investigations on the operations — as above — with those around “groups of modules”, which he eventually named *Modulgruppen*.

The work presented by Dedekind in this text is rooted in module theory. However, the text itself does not seem to pertain to module theory itself. It is

unclear whether the text is complete, so there is a possibility that the following parts would have been dedicated to module theory and *Modulgruppen*, but we do not have any textual proof of that. What we do know is that Dedekind eventually wanted to write a text composed of the two texts on *Modulgruppen* and on the dualism in module theory. This suggests that he wanted to merge the two following aspects: firstly, a very general theory based on the duality of operations defined between two sets (whose nature is indifferent, since, even though the links to module theory are still strong, the definitions and properties are general and Dedekind emphasizes the possibility of applications to other domains); secondly, an investigation into the theory of modules and groups formed by modules, which clearly builds on Dedekind's previous (not necessarily published) works on module theory and defines the concept of *Modulgruppe*. Dedekind's *Dualgruppen* are the result of this fusion: on one hand, the generalisation from modules to any elements satisfying the fundamental laws, and on the other hand, the passage to the level of sets. But before that, Dedekind goes through a last step of generalisation of his research, which allows him to effectively relate these with parts of his work.

4.5.3 “More general (logical) theory”

Dedekind's research accelerates after “On the dualism in module theory”. In the year or two following its writing, he investigated more closely a possible “generalisation” and eventually introduced the concept of *Dualgruppe*.

In Cod. Ms. Dedekind X-11-2, pp. 53, 54, 59 and (probably) 60,¹⁰⁸ which seem to have been written after “On the dualism in module theory”,¹⁰⁹ Dedekind very explicitly reveals his generalisation strategy. He starts with three modules $\mathfrak{a}, \mathfrak{b}, \mathfrak{c}$, on which he imposes conditions of divisibility, such that the generated group only has eight elements. He then states the following:

There are thus only eight modules left, that is, $\mathfrak{a}, \mathfrak{b}, \mathfrak{c}$ and

$$\left. \begin{array}{l} \mathfrak{b}''' = \mathfrak{a} + \mathfrak{c} \quad ; \quad \mathfrak{c}_3 = \mathfrak{a} - \mathfrak{b} \\ \mathfrak{c}''' = \mathfrak{a} + \mathfrak{b} \quad ; \quad \mathfrak{b}_3 = \mathfrak{a} - \mathfrak{c} \end{array} \right\} \quad \text{and } \mathfrak{d}' = \mathfrak{d}_1 \quad \left. \begin{array}{l} = (\mathfrak{a} + \mathfrak{b}) - \mathfrak{c} = (\mathfrak{a} - \mathfrak{c}) + \mathfrak{b} \\ = (\mathfrak{b} + \mathfrak{a}) - \mathfrak{c} = \mathfrak{b} + (\mathfrak{a} - \mathfrak{c}) \end{array} \right\} \text{ in the } \underline{\text{module}} \text{ theory.}$$

¹⁰⁸Dedekind wrote on the back of a large re-used university schedule, folded in two, with several folios inside it:

- p. 53r is one of the inner folios, but the third page of Dedekind's notes;
- p. 53v is the first part of the outer folio;
- p. 54r, on a separate sheet, is the first page of Dedekind's notes, it was written on what seems to be an old letter (p. 54v);
- pp. 55-58 are related studies, but most of them can be dated from before p. 54 (in particular, the pages studied in 4.4.4 and a case study);
- p. 59r is the second part of the outer folio;
- p. 59v, the second inner folio, is the second page of Dedekind's notes;
- p. 60 is also a separate sheet, and continues the studies from p. 53r.

The pages should be read in this order: p. 54r, p. 59v, p. 53r, p. 60.

¹⁰⁹Indeed, Dedekind uses the absorption property as a fundamental law and deduces the idempotence property from it.

Here, the *Modulgesetz* is clearly identified as a property of module theory. Dedekind goes on to the first step of a generalisation:

One replaces $\left. \begin{array}{l} +, -, a, b, c, b''', c''', b_3, c_3, d' = d_1 \\ \text{by } \varphi, \psi, a, m, d, d''', d', m', m'', \underline{d'' = m''} \end{array} \right\}$ NB! Consequence of the special module theory

The “Nota Bene” refers only to the underlined property which is a consequence of the *Modulgesetz*.

Changing the notation of the operations clearly sets this generalisation as an abstraction from module theory. Immediately below the passage just quoted, Dedekind starts what he calls a “more general (logical) theory”. He introduces the operations as indeterminate binary operations, a way to combine elements whose individual nature is also left indeterminate.

General (logical) theory.

Three elements a, b, c ; laws (general)

$$\left. \begin{array}{ll} (1) a\varphi b = b\varphi a ; & (1') a\psi b = b\psi a \\ (2) (a\varphi b)\varphi c = a\varphi(b\varphi c) ; & (2') (a\psi b)\psi c = a\psi(b\psi c) \\ (3) a\psi(a\varphi b) = a ; & (3') a\varphi(a\psi b) = a \end{array} \right\} \begin{array}{l} \text{from which (when } b \text{ in (3') is} \\ \text{replaced by } a\varphi b, \text{ and in (3)} \\ \text{by } a\psi b). \\ (4) a\varphi a = a ; (4') a\psi a = a. \end{array}$$

Dedekind defines the elements of the group, studies order properties, unfolds the laws for the group generated by the three elements a, b, c . As he needs to check which elements are equal to each other (e.g. a''' is defined as $b\varphi c$ but is also equal to a number of other combinations such as $b'\varphi c', b'\varphi c'' \dots$), he works in a very systematic, almost combinatorial way for three pages.¹¹⁰ I will not get into the details of his systematic checking but would like to mention that the layout of these pages follows Dedekind’s usual use of small frames organised in such a way that the dualism between the elements is clearly visible. In addition, he himself underlines the duality:

$b'\varphi c' = b\varphi b_3\varphi c\varphi a = (b\varphi c_3)\varphi(c\varphi b_3) = b\varphi c = a'''$ and since $a''' < b'' < b', a''' < c'' < c'$, it follows also $b''\varphi c'' = a'''$, $b''\varphi c' = a'''$, so together and dually

$b'\varphi c' = a'''$	$b'\varphi c'' = a'''$	$c'\varphi b'' = a'''$	$b''\varphi c'' = a'''$
$c'\varphi a' = b'''$	$c'\varphi a'' = b'''$	$a'\varphi c'' = b'''$	$c'\varphi a'' = b'''$
$a'\varphi b' = c'''$	$a'\varphi b'' = c'''$	$b'\varphi a'' = c'''$	$a''\varphi b'' = c'''$
$b_1\psi c_1 = a_3$	$b_1\psi c_2 = a_3$	$c_1\psi b_2 = a_3$	$b_2\psi c_2 = a_3$
$c_1\psi a_1 = b_3$	$c_1\psi a_2 = b_3$	$a_1\psi c_2 = b_3$	$c_2\psi a_2 = b_3$
$a_1\psi b_1 = c_3$	$a_1\psi b_2 = c_3$	$b_1\psi a_2 = c_3$	$a_2\psi b_2 = c_3$

These pages show clearly how Dedekind abstracted his general theory from module theory, as it was already taking shape in some of the earlier drafts we studied. We see, here, that Dedekind indeed completely merged the various aspects of his research: *Modulgruppen*, the general theory sketched in “On the dualism in module theory”, the “logical” theory which he studied apparently prompted by Schröder’s work. The dualism seems to have been, here, fully integrated as a property of the operations.

¹¹⁰<https://eman-archives.org/Dedekind/items/show/303>

4.5.4 *Dualgruppen*, finally

Unfortunately, there are no datable textual traces of what could with certainty be pointed as the first named definition for *Dualgruppen*. There are several drafts in which we find “*Dual-Gruppe*” and one in which Dedekind uses the word “*Dual-Modulgruppe*”, two words that display the process of concept formation. We see in such pages that Dedekind's investigation stabilized around the study of a system as defined in “On the dualism in module theory”. In the earliest drafts that use the words “*Dual-Gruppe*” or “*Dualgruppe*”, Dedekind starts by naming the operations (φ and ψ) and states the six dual laws defining these operations (with the usual dual presentation). He then proves the idempotence property, defines divisibility, studies laws as he did in “On the dualism in module theory”, considers the conditions of validity for the *Modulgesetz*, the *Idealgesetz*, etc.

Since we can't identify the very first written definition of a *Dualgruppe* — maybe because it was sufficient for Dedekind, *in his drafts*, to state the operations and their laws, rather than give a fully written definition — I propose a translation of the definition as given in the first draft of (Dedekind, 1900)¹¹¹ as one of the first definition of a *Dualgruppe*.¹¹² After referring to Dirichlet's *Vorlesungen über Zahlentheorie* in which he made several footnotes on *Modulgruppen*, Dedekind writes:

Let us designate (as in D[irichlet's *Vorlesungen über Zahlentheorie*] §169) with $\mathfrak{a} + \mathfrak{b}$ the greatest common divisor (or the sum), and with $\mathfrak{a} - \mathfrak{b}$ the least common multiple (or the intersection [*Durchschnitt*]) of two modules $\mathfrak{a}, \mathfrak{b}$, then for each of these two operations \pm the commutative and associative laws hold:

$$\mathfrak{a} + \mathfrak{b} = \mathfrak{b} + \mathfrak{a}, \quad \mathfrak{a} - \mathfrak{b} = \mathfrak{b} - \mathfrak{a} \quad (1)$$

$$(\mathfrak{a} + \mathfrak{b}) + \mathfrak{c} = \mathfrak{a} + (\mathfrak{b} + \mathfrak{c}), \quad (\mathfrak{a} - \mathfrak{b}) - \mathfrak{c} = \mathfrak{a} - (\mathfrak{b} - \mathfrak{c}) \quad (2)$$

with the known consequences that they hold for an arbitrary number of elements $\mathfrak{a}, \mathfrak{b}, \mathfrak{c}$ (D[irichlet's *Vorlesungen über Zahlentheorie*] §2).¹¹³ The two operations \pm are moreover linked to each other by the two laws

$$\mathfrak{a} + (\mathfrak{a} - \mathfrak{b}) = \mathfrak{a}, \quad \mathfrak{a} - (\mathfrak{a} + \mathfrak{b}) = \mathfrak{a} \quad (3)$$

from which it follows, without using (1) and (2)

$$\mathfrak{a} + \mathfrak{a} = \mathfrak{a}, \quad \mathfrak{a} - \mathfrak{a} = \mathfrak{a} \quad (4)$$

¹¹¹As mentioned above, the drafts of (Dedekind, 1897b) do not provide us with a definition of *Dualgruppen*.

¹¹²For a more detailed analysis of the first pages of that draft, see my paper upcoming in the 2024 issue of *Genesis* on scientific manuscripts. Below is a linearised translation, omitting some of the less significant corrections (e.g. \mathfrak{a} changed into \mathfrak{b}) for readability reasons. Passages between angle brackets were added in the margin.

¹¹³This is, again, a reference to the section generalising the properties of the multiplication to an arbitrary number of integers.

If we designate namely the first and second member of a double equation (n) with (n') and (n'') , then $(4')$ follows when we replace \mathfrak{b} in $(3')$ by $(\mathfrak{a} + \mathfrak{b})$ using $(3'')$, and as well, $(4'')$ follows when we replace \mathfrak{b} in $(3'')$ by $(\mathfrak{a} - \mathfrak{b})$ using $(3')$.

When two operations \pm of any two elements $\mathfrak{a}, \mathfrak{b}$ of a (finite or infinite) system $\mathfrak{M}\mathfrak{G}$ generate two elements $\mathfrak{a} \pm \mathfrak{b}$ of the same system $\mathfrak{M}\mathfrak{G}$ and at the same time satisfy the laws (1), (2), (3), then $\mathfrak{M}\mathfrak{G}$ is to be called a *Dualgruppe* with respect to these two operations \pm <whatever these elements may be>.

The totality of all modules is therefore a *Dualgruppe* with respect to the two operations which consist of the formation of greatest common divisors and least common multiples. <First, however, let us consider some properties which belong to each *Dualgruppe* \mathfrak{G} .>

Following (4), each ~~module~~ element¹¹⁴ \mathfrak{a} of a *Dualgruppe* \mathfrak{G} builds a *Dualgruppe* by itself.

For any two ~~modules~~ elements $\mathfrak{a}, \mathfrak{b}$ it follows also from (2) and (4) that

$$\mathfrak{a} + (\mathfrak{a} + \mathfrak{b}) = \mathfrak{a} + \mathfrak{b}, \quad \mathfrak{a} - (\mathfrak{a} - \mathfrak{b}) = \mathfrak{a} - \mathfrak{b} \quad (5)$$

Moreover, if we replace \mathfrak{c} by $(\mathfrak{a} - \mathfrak{b})$ in $(2')$ and by $(\mathfrak{a} + \mathfrak{b})$ in $(2'')$, then it follows also, using (3)

$$(\mathfrak{a} + \mathfrak{b}) + (\mathfrak{a} - \mathfrak{b}) = \mathfrak{a} + \mathfrak{b}, \quad (\mathfrak{a} - \mathfrak{b}) - (\mathfrak{a} + \mathfrak{b}) = \mathfrak{a} - \mathfrak{b} \quad (6)$$

Therefore, the four ~~modules~~ elements $\mathfrak{a}, \mathfrak{b}, \mathfrak{a} + \mathfrak{b}, \mathfrak{a} - \mathfrak{b}$ form a *Dualgruppe*, and the question is raised of how many are distinct from each other. Let us assume that $\mathfrak{a} + \mathfrak{b} = \mathfrak{a} - \mathfrak{b}$, then it follows from $(5')$ and $(3')$ that $\mathfrak{a} + \mathfrak{b} = \mathfrak{a}$, and since the hypotheses are symmetrical in \mathfrak{a} and \mathfrak{b} , it follows that $\mathfrak{a} + \mathfrak{b} = \mathfrak{b}$, hence $\mathfrak{a} = \mathfrak{b}$. And reciprocally, when $\mathfrak{a} = \mathfrak{b}$, then the <all> four ~~modules~~ elements are identical with each other.

If we make now the (more general) assumption that $\mathfrak{a} + \mathfrak{b}$ is identical with one of the two ~~modules~~ elements $\mathfrak{a}, \mathfrak{b}$, e.g. $\mathfrak{a} + \mathfrak{b} = \mathfrak{a}$, then it follows from $(3'')$, by exchanging \mathfrak{a} and \mathfrak{b} , also that $\mathfrak{a} - \mathfrak{b} = \mathfrak{b}$, and reciprocally, when the latter is the case, then from $(3')$ it follows also that $\mathfrak{a} + \mathfrak{b} = \mathfrak{b}$. <Since this case often occurs, we transfer the usual way of expressing and naming of module theory (D[irichlet's *Vorlesungen über Zahlentheorie*] §169) to all *Dualgruppen* $\mathfrak{M}\mathfrak{G}$ and we say that the element \mathfrak{b} > is divisible by the ~~module~~ element \mathfrak{a} , or \mathfrak{b} is called a multiple of \mathfrak{a} and \mathfrak{a} a divisor of \mathfrak{b} . This divisibility

¹¹⁴Dedekind's almost constant correction of the word "module" with "element" in these paragraphs is particularly interesting, as it is very telling about Dedekind's initial intent for his paper, and reflects his path to and conception of *Dualgruppen*.

(which does not exclude the identity of \mathfrak{a} and \mathfrak{b}) will be denoted by $\mathfrak{a} < \mathfrak{b}$ or $\mathfrak{b} > \mathfrak{a}$. And then, each of the four statements

$$\mathfrak{a} + \mathfrak{b} = \mathfrak{a}, \mathfrak{a} - \mathfrak{b} = \mathfrak{b}, \mathfrak{a} < \mathfrak{b}, \mathfrak{a} > \mathfrak{b} \quad (7)$$

is synonymous with the three others. Two such ~~modules~~ elements \mathfrak{a} , \mathfrak{b} build by themselves a *Dualgruppe*. (Cod. Ms. Dedekind X 11-2, p. 18-19)

4.6 Conclusion

The road from defining two operations for modules to defining *Dualgruppen* was long. I tried to show, in this chapter, how duality between the operations defined for modules, which prompted Dedekind's interest in the properties of these operations, was a leading thread in his research. We followed the slow emergence of Dedekind's *Dualgruppen* through four steps, which were broadly chronologically organized.

The first step studied Dedekind's earliest observations of duality in module theory, in the mid-1870s. Duality first appeared as an interesting property of GCDs and LCMs modules. Dedekind's drafts show that he took notice of it, in particular compiling his first observations in a table clearly displaying this duality. These early observations seem to have been compelling enough for Dedekind to pursue research in this direction.

The second step showed several aspects crucial to the genesis of *Dualgruppen*, in that it studied various aspects of Dedekind's research practices as displayed by his drafts. These investigations seem to have lasted well into the 1880s. I showed that duality played a prominent role in the research process, as it was a key component of Dedekind's choices in devising notations, tools and written artefacts, and in organising writing on the paper (in columns, little frames, etc., showcasing duality), all of which accompanied and even helped his investigation. It also explored a number of moments in the conceptual clarification of the theory of modules and groupes of modules, and I showed the intricate relations between the dualism and the (to-be-called) *Modulgesetz*, which was difficult for Dedekind to unknot. These are largely invisible in his publications, yet they are constitutive of the research process, crucial for the conceptual exploration and indeed show how important duality was for Dedekind's investigation.

The third step was that of the generalisation of Dedekind's research and mostly took place in the 1890s. It is a very important part of the genesis of the concept of *Dualgruppe*, a key aspect that only takes place very progressively. The major moments, which the drafts allowed me to identify, were a more general module theory, the study of the links with group theory, the study of (Schröder, 1890), and the generalisation of "a part of module theory" to a more abstract one working with indeterminate objects. It showed clearly what can be seen as a second step of conceptual development, the generalisation of the research from module theory to a more general, abstract and what Dedekind later called

“logical” theory. Although duality was not as present, here, as it was in the previous step, it hovers over the research and Dedekind refers to it regularly.

The fourth and last step led us to *Dualgruppen* properly speaking and happened between 1896 and 1897. The two next-to-last drafts of fully written texts showed that Dedekind arrived at the concept of *Dualgruppe* by merging to aspects of his investigation: the groups of modules and the generalisation of module theory, with a strong emphasis on the dualism in the theory. In doing so, Dedekind went from studying the duality of operations in module theory to defining a new concept of group presenting an internal duality – and which is thus named *Dualgruppe*. The resulting theory is not about a duality of operations anymore, but about a certain group whose operations present a property of dualism. This leads to Dedekind’s last step of research, in which he develops the general “logical” theory that effectively merges these two aspects.

Dedekind’s drafts are very rich, and offer us an outstanding view of his mathematics as he was doing them. Analyzing how Dedekind chose to write and organise his writing, the steps of the textual and conceptual development of his research, allows us to not only better understand the genesis of *Dualgruppen*, but also to put forward how decisive and essential duality was for it.

Acknowledgments

During the writing of this chapter, the following things happened: two people I loved dearly passed away; one of my fingers was partially amputated; I failed several job interviews; I started a new position; there was a variety of strikes in France; and there was the Covid-19 crisis and many weeks of slightly terrifying, completely isolated lockdown. What a ride. I would like to thank Ralf Krömer for his patience and help in many areas; David Waszek, Hourya Benis-Sinaceur and Dirk Schlimm for their feedbacks on early drafts. And I would also like to thank everybody who supported me while I complained, you know who you are.

This is dedicated to the memory of my grandmother (even though she really didn’t like mathematics).

Bibliography

- Barberousse, A. and L. Pinon, eds. (2003). *Genesis 20 « Écriture scientifique »*. Paris: Presses de l'Université Paris-Sorbonne.
- Biasi, P.-M. de (2011). *Génétique des textes*. Paris: CNRS éditions.
- Birkhoff, G. (1934). "Note on the paper "On the combination of subalgebras"". In: *Mathematical Proceedings of the Cambridge Philosophical Society* 30.2, p. 200.
- Boole, G. (1854a). *An Investigation of the Laws of Thought, on which are founded the Mathematical Theories of Logic and Probabilities*. London: Walton and Maberley.
- (1854b). "Solution of a Question in the Theory of Probabilities". In: *Philosophical Magazine* 7.4, pp. 29–32.
- (1862). "Some Observations by Professor Boole (contained in On a Question in the Theory of Probabilities, by A. Cayley)". In: *Philosophical Magazine* 23.4, pp. 361–365.
- Dedekind, R. (1855). "Bemerkungen zu einer Aufgabe der Wahrscheinlichkeitsrechnung". In: (**WerkeEn**). Vol. I, pp. 36–39.
- (1871). "Über die Composition der binären quadratischen Formen, Xth Supplement". In: Lejeune-Dirichlet's *Vorlesungen über Zahlentheorie*, 2nd edition, 380–497. Partially repr. in (**WerkeEn**) III, pages 223–262.
- (1872). "*Stetigkeit und irrationale Zahlen*". In: (**WerkeEn**) Vol. III, pp. 315–335.
- (1877). "Über die Anzahl der Ideal-Klassen in den verschiedenen Ordnungen eines endlichen Körpers". In: (**WerkeEn**). Vol. I, pp. 105–159.
- (1879). "Über die Theorie der algebraischen Zahlen, XIth Supplement". In: Lejeune-Dirichlet's *Vorlesungen über Zahlentheorie*, 3rd edition, 434–611. Partially repr. in (**WerkeEn**) III, pages 297–315.
- (1888). "*Was sind und was sollen die Zahlen?*" In: (**WerkeEn**) III, pp. 335–392.
- (1894a). "Über die Theorie der algebraischen Zahlen, XIth Supplement". In: Lejeune-Dirichlet's *Vorlesungen über Zahlentheorie*, 4th edition, 434–657. Repr. in (**WerkeEn**) III, pages 1–222.
- (1894b). "Zur Theorie der Ideale". In: (**WerkeEn**). Vol. II. 43–50.
- (1895). "Über eine Erweiterung des Symbols (**a**, **b**) in der Theorie der Moduln". In: *Nachrichten von der Königlischen Gesellschaft der Wissenschaften*

- zu Göttingen, *Mathem.-phys. Klasse*, pages 183–188. Repr. in (**WerkeEn**) II, pp. 59–86.
- (1897a). “Der Gruppen deren sämtliche Teiler Normalteiler sind”. In: *Mathematische Annalen* 48, 548–561. Repr. in (**WerkeEn**), pp. 87–102.
 - (1897b). “Über Zerlegungen von Zahlen durch ihre größten gemeinsamen Teiler”. In: (**WerkeEn**). Vol. II, pp. 103–147.
 - (1900). “Über die von drei Moduln erzeugte Dualgruppe”. In: (**WerkeEn**). Vol. II, pp. 236–271.
 - (1856–1858). “Eine Vorlesung über Algebra”. In: (**Scharlau**), pp. 59–108.
 - (1876–1877). “Théorie des nombres entiers algébriques”. In: Published in five parts in *Bulletin des Sciences Astronomiques et Mathématiques*: BSAM 11 (1876), pages 278–288; BSAM Sér. 2(1):1 (1877), pages 17–41, 69–92, 144–164, and 207–248. Partially repr. in (**WerkeEn**) III, 262–296. Engl. transl. in (**Stillwell1877**). All references to (**Stillwell1877**).
- Dugac, P. (1976). *Richard Dedekind et les fondements des mathématiques*. Paris: Vrin.
- Ehrhardt, C. (2012). *Itinéraire d'un texte mathématique. Réélaborsations d'un mémoire de Galois au XIXe siècle*. Paris: Hermann.
- Ferreirós, J. (2008). *Labyrinth of Thought: A History of Set Theory and Its Role in Modern Mathematics*. New York: 2d edition, Springer.
- Frobenius, F. G. and L. Stickelberger (1879). “Über Gruppen von vertauschbaren Elementen”. In: *Journal für die reine und angewandte Mathematik* 86, 217–262. Reprinted in G. Frobenius, *Gesammelte Abhandlungen*. J.-P. Serre (Ed). Springer-Verlag, Berlin, 1968, vol. 1, 545–590.
- Gauss, C. F. (1801). *Disquisitiones Arithmeticae*. En. transl. Arthur A. Clarke, Yale University Press, 1965.
- Grésillon, A. (2016). *Éléments de critique génétique - Lire les manuscrits modernes*. CNRS Collection : Textes & manuscrits.
- Haffner, E. (2017). “Strategical use(s) of arithmetic in Richard Dedekind and Heinrich Weber's *Theorie der algebraischen Funktionen einer Veränderlichen*”. In: *Historia Mathematica* 44.1, pp. 31–69.
- (2018). “From modules to lattices, insight into the genesis of Dedekind's *Dualgruppen*”. In: *British Journal for History of Mathematics* 34.1, pp. 23–42.
 - (2021). “The Shaping of Dedekind's Rigorous Mathematics: What Do Dedekind's Drafts Tell Us About His Ideal of Rigor?” In: *Notre Dame Journal of Formal Logic* 62.1, pp. 5–31.
- Hawkins, T. (2013). *The Mathematics of Frobenius in Context: A Journey Through 18th to 20th Century Mathematics*. Sources and Studies in the History of Mathematics and Physical Sciences. Springer London, Limited.
- Lejeune-Dirichlet, J. P. G. (First edition 1863, re-ed. 1871, 1879, 1894). *Vorlesungen über Zahlentheorie*. Braunschweig: Richard Dedekind, editor. Vieweg und Sohn.
- Lipschitz, R. (1986). *Rudolf Lipschitz: Briefwechsel mit Cantor, Dedekind, Helmholtz, Kronecker, Weierstrass und anderen*. Winfried Scharlau (ed.) Dokumente

- zur Geschichte der Mathematik. Braunschweig, Wiesbaden: Deutsche Mathematiker Vereinigung, Vieweg.
- Lorey, W. (1916). *Das Studium der Mathematik an den Deutschen Universitäten seit Anfang des 19. Jahrhunderts*. Leipzig/Berlin: Teubner.
- Mehrtens, H. (1979). *Die Entstehung der Verbandstheorie*. Gerstenberg: Hildesheim.
- Ribrag, P. (2019). “La pratique du nombre dans les mathématiques d’Arthur Cayley”. MA thesis. Université de Paris.
- Scheel, K. (2014). *Der Briefwechsel Richard Dedekind – Heinrich Weber*. Vol. 5. Hamburg: Abhandlungen der Akademie der Wissenschaften in Hamburg (5), De Gruyter Oldenbourg.
- Schlimm, D. (2011). “On the creative role of axiomatics. The discovery of lattices by Schröder, Dedekind, Birkhoff, and others”. In: *Synthese* 183.1, pp. 47–68.
- Schröder, E. (1873). *Lehrbuch der Arithmetik und Algebra*. Teubner.
- (1890). *Vorlesungen über die Algebra der Logik (Exakte Logik)* Vol. 1. Teubner.
- Weber, H. (1895-1896). *Lehrbuch der Algebra*. 2 vols. 2nd edition, vol. I (1898), vol. II (1899). vol. III (1908). Vieweg und Sohn, Braunschweig.
- Wussing, H. (1984). *The genesis of the abstract group concept. A contribution to the history of the origin of abstract group theory*. Cambridge, Mass., and London: MIT Press.