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# On the minimum number of arcs in 4-dicritical oriented graphs* 

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#### Abstract

The dichromatic number $\vec{\chi}(D)$ of a digraph $D$ is the minimum number of colours needed to colour the vertices of a digraph such that each colour class induces an acyclic subdigraph. A digraph $D$ is $k$-dicritical if $\vec{\chi}(D)=k$ and each proper subdigraph $H$ of $D$ satisfies $\vec{\chi}(H)<k$.

For integers $k$ and $n$, we define $d_{k}(n)$ (respectively $o_{k}(n)$ ) as the minimum number of arcs possible in a $k$-dicritical digraph (respectively oriented graph). Kostochka and Stiebitz have shown [10] that $d_{4}(n) \geqslant \frac{10}{3} n-\frac{4}{3}$. They also conjectured that there is a constant $c$ such that $o_{k}(n) \geqslant c d_{k}(n)$ for $k \geqslant 3$ and $n$ large enough. This conjecture is known to be true for $k=3$ (Aboulker et al. [2]).

In this work, we prove that every 4 -dicritical oriented graph on $n$ vertices has at least $\left(\frac{10}{3}+\frac{1}{51}\right) n-1$ arcs, showing the conjecture for $k=4$. We also characterise exactly the $k$-dicritical digraphs on $n$ vertices with exactly $\frac{10}{3} n-\frac{4}{3}$ arcs.


## 1 Introduction

Let $G$ be a graph. We denote by $V(G)$ its vertex set and by $E(G)$ its edge set; we set $n(G)=$ $|V(G)|$ and $m(G)=|E(G)|$. A $k$-colouring of $G$ is a function $\varphi: V(G) \rightarrow[k]$. It is proper if for every edge $u v \in E(G), \varphi(u) \neq \varphi(v)$. The smallest integer $k$ such that $G$ has a proper $k$-colouring is the chromatic number, and is denoted by $\chi(G)$. Since $\chi$ is non decreasing with respect to the subgraph relation, it is natural to consider the minimal graphs (for this relation) which are not $(k-1)$-colourable. Following this idea, Dirac defined $k$-critical graphs as the graphs $G$ with $\chi(G)=k$ and $\chi(H)<k$ for every proper subgraph $H$ of $G$. A first property of $k$-critical graph is

[^0]that their minimum degree is at least $k-1$. Indeed, if a vertex $v$ has degree at most $k-2$, then a $(k-1)$-colouring of $G-v$ can be easily extended to $G$, contradicting the fact that $\chi(G)=k$. As a consequence, the number of edges in a $k$-critical graph is at least $\frac{k-1}{2} n$. This bound is tight for complete graphs and odd cycles, but Dirac [4] proved an inequality of the form $m \geqslant \frac{k-1+\varepsilon_{k}}{2} n-c_{k}$ for every $n$-vertex $k$-critical graph with $m$ edges, for some $c_{k}$ and $\varepsilon_{k}>0$. This shows that, for $n$ sufficiently large, the average degree of a $k$-critical graph is at least $k-1+\varepsilon_{k}$. This initiated the quest after the best lower bound on the number of edges in $n$-vertex $k$-critical graphs. This problem was almost completely solved by Kostochka and Yancey in 2014 [11].

Theorem 1 (Kostochka and Yancey [11]).
Every $k$-critical graph on $n$ vertices has at least $\frac{1}{2}\left(k-\frac{2}{k-1}\right) n-\frac{k(k-3)}{2(k-1)}$ edges. For every $k$, this bound is tight for infinitely many values of $n$.

Kostochka and Yancey [12] also characterised $k$-critical graphs for which this inequality is an equality, and all of them contain a copy of $K_{k-2}$, the complete graph on $k-2$ vertices. This motivated the following conjecture of Postle [13].

Conjecture 2 (Postle [13]). For every integer $k \geqslant 4$, there exists $\varepsilon_{k}>0$ such that every $k$-critical $K_{k-2}$-free graph $G$ on $n$ vertices has at least $\frac{1}{2}\left(k-\frac{2}{k-1}+\varepsilon_{k}\right) n-\frac{k(k-3)}{2(k-1)}$ edges.

For $k=4$, the conjecture trivially holds as there is no $K_{2}$-free 4 -critical graph. Moreover, this conjecture has been confirmed for $k=5$ by Postle [13], for $k=6$ by Gao and Postle [6], and for $k \geqslant 33$ by Gould, Larsen, and Postle [7].

Let $D$ be a digraph. We denote by $V(D)$ its vertex set and by $A(D)$ its arc set; we set $n(D)=$ $|V(D)|$ and $m(D)=|E(D)|$. A $k$-colouring of $D$ is a function $\varphi: V(D) \rightarrow[k]$. It is a $k$ dicolouring if every directed cycle $C$ in $D$ is not monochromatic for $\varphi$ (that is $|\varphi(V(C))|>1$ ). Equivalently, it is a $k$-dicolouring if every colour class induces an acyclic subdigraph. The smallest integer $k$ such that $D$ has a $k$-dicolouring is the dichromatic number of $D$ and is denoted by $\vec{\chi}(D)$.

A digon in $D$ is a pair of opposite arcs between two vertices. Such a pair of arcs $\{u v, v u\}$ is denoted by $[u, v]$. We say that $D$ is a bidirected graph if every pair of adjacent vertices forms a digon. In this case, $D$ can be viewed as obtained from an undirected graph $G$ by replacing each edge $\{u, v\}$ of $G$ by the digon $[u, v]$. We say that $D$ is a bidirected $G$, and we denote it by $\overleftrightarrow{G}$. Observe that $\chi(G)=\vec{\chi}(\overleftrightarrow{G})$. Thus every statement on proper colouring of undirected graphs can be seen as a statement on dicolouring of bidirected graphs.

Exactly as in the undirected case, one can define $k$-dicritical digraphs to be digraphs $D$ with $\vec{\chi}(D)=k$ and $\vec{\chi}(H)<k$ for every proper subdigraph $H$ of $D$. It is easy to check that if $G$ is a $k$-critical graph, then $\overleftrightarrow{G}$ is $k$-dicritical. Kostochka and Stiebitz [10] conjectured that the $k$ dicritical digraphs with the minimum number of arcs are bidirected graphs. Thus they conjectured the following generalisation of Theorem 1 to digraphs.

Conjecture 3 (Kostochka and Stiebitz [10]). Let $k \geqslant 2$. Every $k$-dicritical digraph on $n$ vertices has at least $\left(k-\frac{2}{k-1}\right) n-\frac{k(k-3)}{k-1}$ arcs. Moreover, equality holds only if $D$ is bidirected.

In the case $k=2$, this conjecture is easy and weak as it states that a 2-dicritical digraph on $n$ vertices has at least two arcs, while, for all $n \geqslant 2$, the unique 2 -dicritical digraph of order $n$ is the directed $n$-cycle which has $n$ arcs. The case $k=3$ of the conjecture has been confirmed by Kostochka and Stiebitz [10]. Using a Brooks-type result for digraphs due to Harutyunyan and Mohar [ 8 ], they proved the following: if $D$ is a 3-dicritical digraph of order $n \geqslant 3$, then $m(D) \geqslant$ $2 n$ and equality holds if and only if $n$ is odd and $D$ is a bidirected odd cycle. The conjecture has also been proved for $k=4$ by Kostochka and Stiebitz [10]. However, the conjecture is open for every $k \geqslant 5$. Recently, this problem has been investigated by Aboulker and Vermande [3] who proved the weaker bound $\left(k-\frac{1}{2}-\frac{2}{k-1}\right) n-\frac{k(k-3)}{k-1}$ for the number of arcs in an $n$-vertex $k$-dicritical digraph.

For integers $k$ and $n$, let $d_{k}(n)$ denote the minimum number of arcs in a $k$-dicritical digraph of order $n$. By the above observations, $d_{2}(n)=n$ for all $n \geqslant 2$, and $d_{3}(n) \geqslant 2 n$ for all possible $n$, and equality holds if and only if $n$ is odd and $n \geqslant 3$. Moreover, if $n$ is even then $d_{3}(n)=2 n+1$ (see [2]).

Kostochka and Stiebitz [9] showed that if a $k$-critical graph $G$ is triangle-free (that is has no cycle of length 3 ), then $m(G) / n(G) \geqslant k-o(k)$ as $k \rightarrow+\infty$. Informally, this means that the minimum average degree of a $k$-critical triangle-free graph is (asymptotically) twice the minimum average degree of a $k$-critical graph. Similarly to this undirected case, it is expected that the minimum number of arcs in a $k$-dicritical digraph of order $n$ is larger than $d_{k}(n)$ if we impose this digraph to have no short directed cycles, and in particular if the digraph is an oriented graph, that is a digraph with no digon. Let $o_{k}(n)$ denote the minimum number of arcs in a $k$-dicritical oriented graph of order $n$ (with the convention $o_{k}(n)=+\infty$ if there is no $k$-dicritical oriented graph of order $n$ ). Clearly $o_{k}(n) \geqslant d_{k}(n)$.

Conjecture 4 (Kostochka and Stiebitz [10]). For any $k \geqslant 3$, there is a constant $\alpha_{k}>0$ such that $o_{k}(n)>\left(1+\alpha_{k}\right) d_{k}(n)$ for $n$ sufficiently large.

For $k=3$, this conjecture has been recently confirmed by Aboulker, Bellitto, Havet, and Rambaud [2] who proved that $o_{3}(n) \geqslant\left(2+\frac{1}{3}\right) n+\frac{2}{3}$.

In view of Conjecture 2, Conjecture 4 can be generalized to $\overleftrightarrow{K_{k-2}}$-free digraphs.
Conjecture 5. For any $k \geqslant 4$, there is a constant $\beta_{k}>0$ such that every $k$-dicritical $\overleftrightarrow{K_{k-2}}$-free digraph $D$ on $n$ vertices has at least $\left(1+\beta_{k}\right) d_{k}(n)$ arcs.

Together with Conjecture 3, this conjecture would imply the following generalisation of Conjecture 2 .

Conjecture 6. For every integer $k \geqslant 4$, there exists $\varepsilon_{k}>0$ such that every $k$-dicritical $\overleftrightarrow{K_{k-2}}$-free digraph $D$ on $n$ vertices has at least $\left(k-\frac{2}{k-1}+\varepsilon_{k}\right) n-\frac{k(k-3)}{k-1}$ arcs.

A $\overleftrightarrow{K_{2}}$-free digraph is an oriented graph, and there are infinitely many 4-dicritical oriented graphs. Thus, while Conjecture 2 holds vacuously for $k=4$, this is not the case for Conjecture 6 . In this paper, we prove that Conjectures 4, 5, and 6hold for $k=4$.

Theorem 7. If $\vec{G}$ is a 4-dicritical oriented graph, then

$$
m(\vec{G}) \geqslant\left(\frac{10}{3}+\frac{1}{51}\right) n(\vec{G})-1
$$

To prove Theorem 7, we use an approach similar to the proof of the case $k=5$ of Conjecture 2 by Postle [13]. This proof is based on the potential method, which was first popularised by Kostochka and Yancey [11] when they proved Theorem 1. The idea is to prove a more general result on every 4 -dicritical digraphs that takes into account the digons.

With a slight abuse, we call digon a subdigraph isomorphic to $\overleftrightarrow{K_{2}}$, the bidirected complete graph on two vertices. We also call bidirected triangle a subdigraph isomorphic to $\overleftrightarrow{K_{3}}$, the bidirected complete graph on three vertices. A packing of digons and bidirected triangles is a set of vertex-disjoint digons and bidirected triangles. To take into account the digons, we define a parameter $T(D)$ as follows.

$$
T(D)=\max \{d+2 t \mid \text { there exists a packing of } d \text { digons and } t \text { bidirected triangles }\}
$$

Clearly, $T(D)=0$ if and only if $D$ is an oriented graph.
Let $\varepsilon, \delta$ be fixed non-negative real numbers. We define the potential (with respect to $\varepsilon$ and $\delta$ ) of a digraph $D$ to be

$$
\rho(D)=\left(\frac{10}{3}+\varepsilon\right) n(D)-m(D)-\delta T(D) .
$$

Thus Theorem 7 can be rephrased as follows.
Theorem 7. Set $\varepsilon=\frac{1}{51}$ and $\delta=6 \varepsilon=\frac{2}{17}$. If $\vec{G}$ is a 4 -dicritical oriented graph, then $\rho(\vec{G}) \leqslant 1$.
In fact, we prove a more general statement which holds for every 4-dicritical digraph (with or without digons), except for some exceptions called the 4 -Ore digraphs. Those digraphs, which are formally defined in Section 2, are the bidirected graphs whose underlying graph is one of the 4 -critical graphs reaching equality in Theorem 1. In particular, every 4-Ore digraph $D$ has $\frac{10}{3} n(D)-\frac{4}{3}$ arcs. Moreover, the statement holds for all non-negative constants $\varepsilon$ and $\delta$ satisfying the following inequalities:

- $\delta \geqslant 6 \varepsilon$;
- $3 \delta-\varepsilon \leqslant \frac{1}{3}$;

Theorem 8. Let $\varepsilon, \delta \geqslant 0$ be constants satisfying the aforementioned inequalities. If $D$ is a 4dicritical digraph with $n$ vertices, then
(i) $\rho(D) \leqslant \frac{4}{3}+\varepsilon n-\delta \frac{2(n-1)}{3}$ if $D$ is 4-Ore, and
(ii) $\rho(D) \leqslant 1$ otherwise.

In order to provide some intuition to the reader, let us briefly describe the main ideas of our proof. We will consider a minimum counterexample $D$ to Theorem 8 , and show that every subdigraph of $D$ must have large potential. To do so, we need to construct some smaller 4-dicritical digraphs to leverage the minimality of $D$. These smaller 4-dicritical digraphs will be constructed by identifying some vertices of $D$. This is why, in the definition of the potential, we consider $T(D)$ instead of the number of digons: when identifying a set of vertices, the number of digons may be arbitrary larger in the resulting digraph, but $T(D)$ increases at most by 1 . Using the fact that every subdigraph of $D$ has large potential, we will prove that some subdigraphs are forbidden in $D$. Using this, we get the final contradiction by a discharging argument.

In addition to Theorem 7, Theorem 8 has also the following consequence when we take $\varepsilon=$ $\delta=0$.

Corollary 9. If $D$ is a 4-dicritical digraph, then $m(D) \geqslant \frac{10}{3} n(D)-\frac{4}{3}$. Moreover, equality holds if and only if $D$ is 4-Ore, otherwise $m(D) \geqslant \frac{10}{3} n(D)-1$.

This is a slight improvement on a result of Kostochka and Stiebitz [10] who proved the inequality $m(D) \geqslant \frac{10}{3} n(D)-\frac{4}{3}$ without characterising the equality case.

Another interesting consequence of our result is the following bound on the number of vertices in a 4 -dicritical oriented graph embedded on a fixed surface. Since a graph on $n$ vertices embedded on a surface of Euler characteristic $c$ has at most $3 n-3 c$ edges, we immediately deduce the following from Theorem 7 .
Corollary 10. If $\vec{G}$ is a 4-dicritical oriented graph embedded on a surface of Euler characteristic $c$, then $n(\vec{G}) \leqslant \frac{17}{6}(1-3 c)$.

The previous best upper bound was $n(\vec{G}) \leqslant 4-9 c$ [10].
In Section 2 we prove some first preliminary results on 4-Ore digraphs, before proving Theorem 8 in Section 3 . In Section 4 , we show that $o_{k}(n) \leqslant\left(2 k-\frac{7}{2}\right) n$ for every fixed $k$ and infinitely many values of $n$. The proof is strongly based on the proof of [2, Theorem 4.4], which shows $o_{k}(n) \leqslant(2 k-3) n$ for every fixed $k, n$ (with $n$ large enough). For $k=4$, the construction implies in particular that there is a 4-dicritical oriented graph with 76 vertices and 330 arcs, and there are infinitely many 4 -dicritical oriented graphs with $m / n \leqslant 9 / 2$.

## 2 The 4-Ore digraphs and their properties

We start with a few notations. We denote by $\llbracket x_{1}, \ldots, x_{n} \rrbracket$ the bidirected path with vertex set $\left\{x_{1}, \ldots, x_{n}\right\}$ in this order. If $x_{1}=x_{n}, \llbracket x_{1}, \ldots, x_{n} \rrbracket$ denotes the bidirected cycle of order $n$ with cyclic order $x_{1}, \ldots, x_{n}$. If $D$ is a digraph, for any $X \subseteq V(D), D-X$ is the subdigraph induced by $V(D) \backslash X$. We abbreviate $D-\{x\}$ into $D-x$. Moreover, for any $F \subseteq V(D) \times V(D), D \backslash F$ is the subdigraph $(V(D), A(D) \backslash F)$ and $D \cup F$ is the digraph $(V(D), A(D) \cup F)$

Let $D_{1}, D_{2}$ be two bidirected graphs, $[x, y] \subseteq A\left(D_{1}\right)$, and $z \in V\left(D_{2}\right)$. An Ore-composition $D$ of $D_{1}$ and $D_{2}$ with replaced digon $[x, y]$ and split vertex $z$ is a digraph obtained by removing $[x, y]$ of $D_{1}$ and $z$ of $D_{2}$, and adding the set of $\operatorname{arcs}\left\{x z_{1} \mid z z_{1} \in A\left(D_{2}\right)\right.$ and $\left.z_{1} \in Z_{1}\right\},\left\{z_{1} x \mid\right.$


Figure 1: An example of a 4-Ore digraph obtained by an Ore-composition of two smaller 4-Ore digraphs, with replaced digon $[x, y]$ and split vertex $z$.
$z_{1} z \in A\left(D_{2}\right)$ and $\left.z_{1} \in Z_{1}\right\},\left\{y z_{2} \mid z z_{2} \in A\left(D_{2}\right)\right.$ and $\left.z_{2} \in Z_{2}\right\},\left\{z_{2} y \mid z_{2} z \in A\left(D_{2}\right)\right.$ and $\left.z_{2} \in Z_{2}\right\}$, where $\left(Z_{1}, Z_{2}\right)$ is a partition of $N_{D_{2}}(z)$ into non-empty sets. We call $D_{1}$ the digon side and $D_{2}$ the split side of the Ore-composition. The class of the 4-Ore digraphs is the smallest class containing $\overleftrightarrow{K_{4}}$ which is stable under Ore-composition. See Figure 1 for an example of a 4-Ore digraph. Observe that all the 4-Ore-digraphs are bidirected.

Proposition 11 (Dirac [5], see also [12]). 4-Ore digraphs are 4-dicritical.
Proof. One can easily show that a bidirected digraph is 4 -dicritical if and only if its undirected underlying graph is 4 -critical. Then the result follows from the undirected analogous proved by [5].

Lemma 12. Let $D$ be a 4-dicritical bidirected digraph and $v \in V(D)$. Let $\left(N_{1}^{+}, N_{2}^{+}\right)$and $\left(N_{1}^{-}, N_{2}^{-}\right)$be two partitions of $N(v)$. Consider $D^{\prime}$ the digraph with vertex set $V(D) \backslash\{v\} \cup\left\{v_{1}, v_{2}\right\}$ with $N^{+}\left(v_{i}\right)=N_{i}^{+}, N^{-}\left(v_{i}\right)=N_{i}^{-}$for $i=1,2$ and $D^{\prime}\langle V(D) \backslash\{v\}\rangle=D-v$. Then $D^{\prime}$ has a 3 -dicolouring with $v_{1}$ and $v_{2}$ coloured the same except if $N_{1}^{+}=N_{1}^{-}$(that is $D^{\prime}$ is bidirected).

Proof. Suppose that $D^{\prime}$ is not bidirected. Consider a vertex $u \in N_{D}(v)$ such that $v_{1} u, u v_{2} \in A\left(D^{\prime}\right)$ or $v_{2} u$, $u v_{1} \in A\left(D^{\prime}\right)$. Without loss of generality, suppose $v_{1} u, u v_{2} \in A\left(D^{\prime}\right)$. As $D$ is 4-dicritical, $D \backslash[u, v]$ has a proper 3-dicolouring $\varphi$. We set $\varphi\left(v_{1}\right)=\varphi\left(v_{2}\right)=\varphi(v)$ and claim that $\varphi$ is a 3dicolouring of $D^{\prime}$. To show that, observe that $\varphi$ is a proper 3 -colouring of the underlying undirected graph of $D^{\prime} \backslash\left\{v_{1} u, u v_{2}\right\}$, and so $\varphi$ is a 3 -dicolouring of $D^{\prime}$ as wanted.

Lemma 13. Let $D$ be a digraph. If $v$ is a vertex of $D$, then $T(D-v) \geqslant T(D)-1$.
Proof. Let $M$ be a packing of $d$ digons and $t$ bidirected triangles in $H$ such that $d+2 t=T(D)$. If $v$ belongs to a digon $[u, v]$ in $M$, then $M \backslash\{[u, v]\}$ witnesses the fact that $T(D-v) \geqslant T(D)-1$. If $v$ belongs to a bidirected triangle $\llbracket u, v, w, u \rrbracket$, then $M \backslash\{\llbracket u, v, w, u \rrbracket\} \cup[u, v]$ witnesses the fact that $T(D-v) \geqslant T(D)-2+1$. Otherwise $T(D-v) \geqslant T(D)$.

Lemma 14. If $D_{1}, D_{2}$ are two digraphs, and $D$ is an Ore-composition of $D_{1}$ and $D_{2}$, then $T(D) \geqslant$ $T\left(D_{1}\right)+T\left(D_{2}\right)-2$. Moreover, if $D_{1}$ or $D_{2}$ is isomorphic to $\overleftrightarrow{K_{4}}$, then $T(D) \geqslant T\left(D_{1}\right)+T\left(D_{2}\right)-1$.

Proof. Let $D$ be the Ore-composition of $D_{1}$ (the digon side with replaced digon $[x, y]$ ) and $D_{2}$ (the split side with split vertex $z$ ). One can easily see that $T(D) \geqslant T\left(D_{1}-x\right)+T(D-z) \geqslant T\left(D_{1}\right)+$ $T\left(D_{2}\right)-2$ by Lemma 13. Moreover, if $D_{1}$ (resp. $\left.D_{2}\right)$ is a copy of $\overleftrightarrow{K_{4}}$, then $T\left(D_{1}-x\right)=2=T\left(D_{1}\right)$ (resp. $T\left(D_{2}-z\right)=2=T\left(D_{2}\right)$ ) and therefore $T(D) \geqslant T\left(D_{1}\right)+T\left(D_{2}\right)-1$.

Lemma 15. If $D$ is 4 -Ore, then $T(D) \geqslant \frac{2}{3}(n(D)-1)$.
Proof. If $D$ is $\overleftrightarrow{K_{4}}$, then the result is clear. Suppose now that $D$ is an Ore-composition of $D_{1}$ and $D_{2}$. Then $n(D)=n\left(D_{1}\right)+n\left(D_{2}\right)-1$ and, by Lemma 14, $T(D) \geqslant T\left(D_{1}\right)+T\left(D_{2}\right)-2$. By induction, $T\left(D_{1}\right) \geqslant \frac{2}{3}\left(n\left(D_{1}\right)-1\right)$ and $T\left(D_{2}\right) \geqslant \frac{2}{3}\left(n\left(D_{2}\right)-1\right)$, and so $T(D) \geqslant \frac{2}{3}\left(n\left(D_{1}\right)+n\left(D_{2}\right)-1-1\right)=$ $\frac{2}{3}(n(D)-1)$.

Let $D$ be a digraph. A diamond in $D$ is a subdigraph isomorphic to $\overleftrightarrow{K_{4}}$ minus a digon $[u, v]$, with vertices different from $u$ and $v$ having degree 6 in $D$. An emerald in $D$ is a subdigraph isomorphic to $\overleftrightarrow{K_{3}}$ whose vertices have degree 6 in $D$.

Let $R$ be an induced subdigraph of $D$ with $n(R)<n(D)$. The boundary of $R$ in $D$, denoted by $\partial_{D}(R)$, or simply $\partial(R)$ when $D$ is clear from the context, is the set of vertices of $R$ having a neighbour in $V(D) \backslash R$. We say that $R$ is Ore-collapsible if the boundary of $R$ contains exactly two vertices $u$ and $v$ and $R \cup[u, v]$ is 4-Ore.

Lemma 16. If $D$ is 4 -Ore and $v \in V(D)$, then there exists either an Ore-collapsible subdigraph of $D$ disjoint from $v$ or an emerald of $D$ disjoint from $v$.

Proof. If $D$ is a copy of $\overleftrightarrow{K_{4}}$, then $D-v$ is an emerald. Otherwise, $D$ is the Ore-composition of two 4-Ore digraphs: $D_{1}$ the digon side with replaced digon $[x, y]$, and $D_{2}$ the split side with split vertex $z$. If $v \in V\left(D_{2}-z\right)$, then $D_{1} \backslash\{x, y\}$ is an Ore-collapsible subdigraph with boundary $\{y, z\}$. Otherwise $v \in V\left(D_{1}\right)$ and we apply induction on $D_{2}$ to find an emerald or an Ore-collapsible subdigraph in $D_{2}$ disjoint from $z$.

Lemma 17. If $D \neq \overleftrightarrow{K_{4}}$ is 4-Ore and $T$ is a copy of $\overleftrightarrow{K_{3}}$ in $D$, then there exists either an Orecollapsible subdigraph of $D$ disjoint from $T$ or an emerald of $D$ disjoint from $T$.

Proof. As $D$ is not $\overleftrightarrow{K_{4}}$, it is an Ore-composition of two 4-Ore digraphs: $D_{1}$ the digon side with replaced digon $[x, y]$, and $D_{2}$ the split side with split vertex $z$. As $x$ and $y$ are non adjacent, we have either $T \subseteq D_{1}, T \subseteq D_{2}-z$, or $T$ contains a vertex $w \in\{x, y\}$ and two vertices in $V\left(D_{2}-z\right)$.

If $T \subseteq D_{1}$, then by Lemma 16, in $D_{2}$ there exists either an Ore-collapsible subdigraph $O$ or an emerald $E$ disjoint from $z$. In the former case $O$ is an Ore-collapsible subdigraph of $D$ disjoint from $T$, and in the later one $E$ is an emerald in $D$ disjoint from $T$.

If $T \subseteq D_{2}-z$, then $D_{1} \backslash\{x, y\}$ is an Ore-collapsible subdigraph disjoint from $T$.
Assume now that $T$ contains a vertex $w \in\{x, y\}$ and two vertices in $V\left(D_{2}-z\right)$. Without loss of generality, we may assume that $y \notin T$. Let $z_{1}$ and $z_{2}$ be the two vertices of $T$ disjoint from $w$. Then $\left\{z, z_{1}, z_{2}\right\}$ induces a bidirected triangle $T^{\prime}$ in $D_{2}$. If $D_{2} \neq \vec{K}_{4}$, then by induction in $D_{2}$, there exists either an Ore-collapsible subdigraph $O$ or an emerald $E$ disjoint from $T^{\prime}$. In the former case
$O$ is an Ore-collapsible subdigraph of $D$ disjoint from $T$, and in the later one $E$ is an emerald in $D$ disjoint from $T$.

Henceforth we may assume that $D_{2}=\overleftrightarrow{K_{4}}$. This implies that $y$ has exactly one neighbour in $D_{2}-z$ and so its degree is the same in $D_{1}$ and $D$. By Lemma 16, in $D_{1}$ there exists either an Ore-collapsible subdigraph $O$ or an emerald $E$ disjoint from $x$. In the former case $O$ is an Orecollapsible subdigraph of $D$ disjoint from $T$, and in the later one $E$ is an emerald in $D$ disjoint from $T$ even if $y \in V(E)$ because $y$ has the same degree in $D_{1}$ and $D$.

Lemma 18. If $R$ is an Ore-collapsible induced subdigraph of a 4-Ore digraph $D$, then there exists a diamond or an emerald of $D$ whose vertices lie in $V(R)$.

Proof. Let $D$ be a digraph. Let $R$ be a minimal counterexample to this lemma, and let $\partial(R)=$ $\{u, v\}$ and $H=D\langle R\rangle \cup[u, v]$. If $H=\overleftrightarrow{K_{4}}$, then $R$ is a diamond in $D$. Suppose now that $H$ is the Ore-composition of two 4-Ore digraphs $H_{1}$ (the digon side with replaced digon $[x, y]$ ) and $H_{2}$ (the split side with split vertex $z$ ). If $\{u, v\} \not \subset V\left(H_{2}\right)$, then by Lemma 16 there exists an Orecollapsible subdigraph in $H_{2}$ disjoint from $z$. As it is smaller than $H$, it contains an emerald or a diamond as desired, a contradiction.

Now assume that $\{u, v\} \subset V\left(H_{2}\right)$, then $H_{1}$ is an Ore-collapsible subdigraph of $D$ smaller than $H$, and by induction, $H_{1}$ contains a diamond or an emerald in $D$.

Lemma 19. If $D$ is a 4-Ore digraph and $v$ is a vertex in $D$, then $D$ contains a diamond or an emerald disjoint from $v$.

Proof. Follows from Lemmas 16 and 18 .
Lemma 20. If $D$ is a 4-Ore digraph and $T$ is a bidirected triangle in $D$, then either $D=\overleftrightarrow{K_{4}}$ or $D$ contains a diamond or an emerald disjoint from $T$.

Proof. Follows from Lemmas 17 and 18 .
The following theorem was formulated for undirected graphs, but by replacing every edge by a digon, it can be restated as follows:

Theorem 21 (Kostochka and Yancey [12], Theorem 6). Let D be a 4-dicritical bidirected digraph. If $\frac{10}{3} n(D)-m(D)>1$, then $D$ is 4 -Ore and $\frac{10}{3} n(D)-m(D)=\frac{4}{3}$.
Lemma 22. If $D$ is a 4-Ore digraph with $n$ vertices, then $\rho(D) \leqslant \frac{4}{3}+\varepsilon n-\delta \frac{2(n-1)}{3}$.
Proof. Follows from Theorem 21 and Lemma 15 .
Lemma 23 (Kostochka and Yancey [12], Claim 16). Let $D$ be a 4-Ore digraph. If $R \subseteq D$ and $0<n(R)<n(D)$, then $\frac{10}{3} n(R)-m(R) \geqslant \frac{10}{3}$.

Lemma 24. Let $D$ be a 4-Ore digraph obtained from a copy $J$ of $\overleftrightarrow{K}_{4}$ by successive Orecompositions with 4-Ore digraphs, vertices and digons in J being always on the digon side. Let $[u, v]$ be a digon in $D\langle V(J)\rangle$. For every 3-dicolouring $\varphi$ of $D \backslash[u, v]$, vertices in $V(J)$ receive distinct colours except $u$ and $v$.

Proof. We proceed by induction on $n(D)$, the result holding trivially when $D$ is $\overleftrightarrow{K_{4}}$. Now assume that $D$ is the Ore-composition of $D_{1}$, the digon side containing $J$, and $D_{2}$, with $D_{1}$ and $D_{2}$ being 4-Ore digraphs. Let $[x, y] \subseteq A\left(D_{1}\right)$ be the replaced digon in this Ore-composition, and let $z \in$ $V\left(D_{2}\right)$ be the split vertex. Let $\varphi$ be a 3-dicolouring of $D \backslash[u, v]$. Then $\varphi$ induces a 3-dicolouring of $D\left\langle V\left(D_{2}-z\right) \cup\{x, y\}\right\rangle$. Necessarily $\varphi(x) \neq \varphi(y)$, for otherwise $\varphi_{2}$ defined by $\varphi_{2}(v)=\varphi(v)$ if $v \in V\left(D_{2}-z\right)$ and $\varphi_{2}(z)=\varphi(x)$ is a 3-dicolouring of $D_{2}$, contradicting the fact that 4-Ore digraphs have dichromatic number 4 by Lemma 11 . Hence $\varphi$ induces a 3 -dicolouring of $D_{1} \backslash[u, v]$. So, by the induction hypothesis, vertices in $V(J)$ have distinct colours in $\varphi$, except $u$ and $v$.

Lemma 25. Let $D$ be a 4-Ore digraph obtained from a copy $J$ of $\overleftrightarrow{K_{4}}$ by successive Orecompositions with 4-Ore digraphs, vertices and digons in $J$ being always on the digon side. Let $v$ be a vertex in $V(J)$. For every 3-dicolouring $\varphi$ of $D-v$, vertices in $J$ receive distinct colours.
Proof. We proceed by induction on $n(D)$, the result holding trivially when $D$ is $\overleftrightarrow{K_{4}}$. Now assume that $D$ is the Ore-composition of $D_{1}$, the digon side containing $J$, and $D_{2}$, with $D_{1}$ and $D_{2}$ being 4-Ore digraphs. Let $[x, y] \subseteq A\left(D_{1}\right)$ be the replaced digon in this Ore-composition, and let $z \in$ $V\left(D_{2}\right)$ be the split vertex. Let $\varphi$ be a 3 -dicolouring of $D-v$. If $v \in\{x, y\}$, then $\varphi$ is a 3dicolouring of $D_{1}-v$ and the result follows by induction. Now assume $v \notin\{x, y\}$. Then $\varphi$ induces a 3-dicolouring of $D\left\langle V\left(D_{2}-z\right) \cup\{x, y\}\right\rangle$. Necessarily $\varphi(x) \neq \varphi(y)$, for otherwise $\varphi_{2}$ defined by $\varphi_{2}(v)=\varphi(v)$ if $v \in V\left(D_{2}-z\right)$ and $\varphi_{2}(z)=\varphi(x)$ is a 3-dicolouring of $D_{2}$, contradicting the fact that 4-Ore digraphs have dichromatic number 4 by Lemma 11 . Hence $\varphi$ induces a 3-dicolouring of $D_{1}-v$. So, by the induction hypothesis, vertices in $V(J)$ have distinct colours in $\varphi$.

## 3 Proof of Theorem 8

Let $D$ be a 4-dicritical digraph, $R$ be an induced subdigraph of $D$ with $4 \leqslant n(R)<n(D)$ and $\varphi$ a 3-dicolouring of $R$. The $\varphi$-identification of $R$ in $D$, denoted by $D_{\varphi}(R)$ is the digraph obtained from $D$ by identifying for each $i \in[3]$ the vertices coloured $i$ in $V(R)$ to a vertex $x_{i}$, adding the digons $\left[x_{i}, x_{j}\right]$ for all $1 \leqslant i<j \leqslant 3$. Observe that $D_{\varphi}(R)$ is not 3 -dicolourable. Indeed, assume for a contradiction that $D_{\varphi}(R)$ has a 3 -dicolouring $\varphi^{\prime}$. Since $\left\{x_{1}, x_{2}, x_{3}\right\}$ induces a $\overleftrightarrow{K_{3}}$, we may assume without loss of generality that $\varphi^{\prime}\left(x_{i}\right)=i$ for $i \in[3]$. Consider the 3 -colouring $\varphi^{\prime \prime}$ of $D$ defined by $\varphi^{\prime \prime}(v)=\varphi^{\prime}(v)$ if $v \notin R$ and $\varphi^{\prime \prime}(v)=\varphi(v)$ if $v \in R$. One easily checks that $\varphi^{\prime \prime}$ is a 3-dicolouring of $D$, a contradiction to the fact that $\vec{\chi}(D)=4$.

Now let $W$ be a 4-dicritical subdigraph of $D_{\varphi}(R)$ and $X=\left\{x_{1}, x_{2}, x_{3}\right\}$. Then we say that $R^{\prime}=D\langle(V(W) \backslash X) \cup R\rangle$ is the dicritical extension of $R$ with extender $W$. We call $X_{W}=$ $X \cap V(W)$ the core of the extension. Note that $X_{W}$ is not empty, because $W$ is not a subdigraph of $D$. Thus $1 \leqslant\left|X_{W}\right| \leqslant 3$. See Figure 2 for an example of a $\varphi$-identification and a dicritical extension.

Let $D$ be a counterexample to Theorem 8 with minimum number of vertices. By Lemma 22 , $D$ is not 4-Ore. Thus $\rho(D)>1$.

Claim 1. If $\tilde{D}$ is a 4-dicritical digraph with $n(\tilde{D})<n(D)$, then $\rho(\tilde{D}) \leqslant \frac{4}{3}+4 \varepsilon-2 \delta$.


Figure 2: A 4-dicritical digraph $D$ together with an induced subdigraph $R$ of $D$ and $\varphi$ a 3dicolouring of $R$, the $\varphi$-identification $D_{\varphi}(R)$ of $R$ in $D$ and the dicritical extension $R^{\prime}$ of $R$ with extender $W$ and core $X_{W}$. For clarity, the digons are represented by undirected edges.

Proof of claim. If $\tilde{D}$ is not 4-Ore, then $\rho(\tilde{D}) \leqslant 1$ by minimality of $D$. Thus $\rho(\tilde{D}) \leqslant \frac{4}{3}+4 \varepsilon-2 \delta$ because $4 \varepsilon-2 \delta \geqslant \frac{-1}{3}$. Otherwise, by Lemma 22, $\rho(\tilde{D}) \leqslant \frac{4}{3}+\varepsilon n(\tilde{D})-\delta \frac{2(n(\tilde{D})-1)}{3} \leqslant \frac{4}{3}+4 \varepsilon-2 \delta$ because $\delta \geqslant \frac{3}{2} \varepsilon$ and $n(\tilde{D}) \geqslant 4$.

Claim 2. Let $R$ be a subdigraph of $D$ with $4 \leqslant n(R)<n(D)$. If $R^{\prime}$ is a dicritical extension of $R$ with extender $W$ and core $X_{W}$, then

$$
\rho\left(R^{\prime}\right) \leqslant \rho(W)+\rho(R)-\left(\rho\left(\overleftrightarrow{K_{\left|X_{W}\right|}}\right)+\delta \cdot T\left(\overleftrightarrow{K_{\left|X_{W}\right|}}\right)\right)+\delta \cdot\left(T(W)-T\left(W-X_{W}\right)\right)
$$

and in particular

$$
\rho\left(R^{\prime}\right) \leqslant \rho(W)+\rho(R)-\frac{10}{3}-\varepsilon+\delta
$$

Proof of claim. We have

- $n\left(R^{\prime}\right)=n(W)-\left|X_{W}\right|+n(R)$,
- $m\left(R^{\prime}\right) \geqslant m(W)+m(R)-m\left(\overleftrightarrow{K_{\left|X_{W}\right|}}\right)$,
- $T\left(R^{\prime}\right) \geqslant T\left(W-X_{W}\right)+T(R)$
and by summing these inequalities, we get the first result.
Now observe that $T(W)-T\left(W-X_{W}\right) \leqslant\left|X_{W}\right|$ by Lemma 13, and that the maximum of $-\left(\rho\left(\overleftrightarrow{K_{\left|X_{W}\right|}}\right)+\delta T\left(\overleftrightarrow{K_{\left|X_{W}\right|}}\right)\right)+\delta\left|X_{W}\right|$ is reached when $\left|X_{W}\right|=1$, in which case it is equal to $-\frac{10}{3}-\varepsilon+\delta$. The second inequality follows.

Claim 3. If $R$ is a subdigraph of $D$ with $4 \leqslant n(R)<n(D)$, then $\rho(R) \geqslant \rho(D)+2-3 \varepsilon+\delta>$ $3-3 \varepsilon+\delta$.

Proof of claim. We proceed by induction on $n-n(R)$. Let $R^{\prime}$ be a dicritical extension of $R$ with extender $W$ and core $X_{W}$. By Claim 2, we have

$$
\rho\left(R^{\prime}\right) \leqslant \rho(W)+\rho(R)-\frac{10}{3}-\varepsilon+\delta
$$

Either $V\left(R^{\prime}\right)=V(D)$ and so $\rho\left(R^{\prime}\right) \geqslant \rho(D)$ or $V\left(R^{\prime}\right)$ is a proper subset of $V(D)$ and, since $R$ is a proper subdigraph of $R^{\prime}$, by induction $\rho\left(R^{\prime}\right) \geqslant \rho(D)+2-3 \varepsilon+\delta \geqslant \rho(D)$. In both cases, $\rho\left(R^{\prime}\right) \geqslant \rho(D)$. Now $W$ is smaller than $D$ so $\rho(W) \leqslant \frac{4}{3}+4 \varepsilon-2 \delta$ by Claim 1 . Thus

$$
\rho(D) \leqslant \rho\left(R^{\prime}\right) \leqslant \frac{4}{3}+4 \varepsilon-2 \delta+\rho(R)-\frac{10}{3}-\varepsilon+\delta .
$$

This gives $\rho(R) \geqslant \rho(D)+2-3 \varepsilon+\delta>3-3 \varepsilon+\delta$, because $\rho(D)>1$.
As a consequence of Claim 3, any subdigraph (proper or not) of size at least 4 has potential at least $\rho(D)$.

We say that an induced subdigraph $R$ of $D$ is collapsible if, for every 3-dicolouring $\varphi$ of $R$, its dicritical extension $R^{\prime}$ (with extender $W$ and core $X_{W}$ ) is $D$, has core of size 1 (i.e. $\left|X_{W}\right|=1$ ), and the border $\partial_{D}(R)$ of $R$ is monochromatic in $\varphi$.

Claim 4. Let $R$ be an induced subdigraph of $D$ and $\varphi$ a 3-dicolouring of $R$ such that $\partial(R)$ is not monochromatic in $\varphi$. If $D$ is a dicritical extension of $R$ dicoloured by $\varphi$ with extender $W$ and core $X_{W}$ with $\left|X_{W}\right|=1$, then

$$
\rho(R) \geqslant \rho(D)+3-3 \varepsilon+\delta .
$$

Proof of claim. Assume $D$ is a dicritical extension of $R$ dicoloured by $\varphi$ with extender $W$ and core $X_{W}$ with $\left|X_{W}\right|=1$. Observe that each of the following inequalities holds:

- $n(D)=n(W)-\left|X_{W}\right|+n(R)=n(X)+n(R)-1$,
- $m(D) \geqslant m(W)+m(R)-m\left(\overleftrightarrow{K_{\left|X_{W}\right|}}\right)+1=m(W)+m(R)+1$ because $\partial_{D}(R)$ is not monochromatic in $\varphi$, and
- $T(D) \geqslant T\left(W-X_{W}\right)+T(R) \geqslant T(W)+T(R)-1$ by Lemma 13 .

By Claim 1, we have

$$
\rho(D) \leqslant \rho(W)+\rho(R)-\left(\frac{10}{3}+\varepsilon\right)-1+\delta \leqslant\left(\frac{4}{3}+4 \varepsilon-2 \delta\right)+\rho(R)-\frac{13}{3}-\varepsilon+\delta
$$

and so $\rho(R) \geqslant \rho(D)+3-3 \varepsilon+\delta$.

Claim 5. If $R$ is a subdigraph of $D$ with $4 \leqslant n(R)<n(D)$ and $R$ is not collapsible, then $\rho(R) \geqslant \rho(D)+\frac{8}{3}-\varepsilon-\delta>\frac{11}{3}-\varepsilon-\delta$.

Proof of claim. Let $R^{\prime}$ be a dicritical extension of $R$ dicoloured by $\varphi$ with extender $W$ and core $X_{W}$.
(i) If $R^{\prime}$ is not $D$, then it has a dicritical extension $R^{\prime \prime}$ with extender $W^{\prime}$. By (the consequence of) Claim 3, we have $\rho(D) \leqslant \rho\left(R^{\prime \prime}\right)$; by Claim 2 (applied twice), $\rho\left(R^{\prime \prime}\right) \leqslant \rho(R)+\rho\left(W^{\prime}\right)+\rho(W)+$ $2\left(-\frac{10}{3}-\varepsilon+\delta\right)$; both $W$ and $W^{\prime}$ are smaller than $D$, so, by Claim $1, \rho(W), \rho\left(W^{\prime}\right) \leqslant \frac{4}{3}+4 \varepsilon-2 \delta$. Those three inequalities imply

$$
\rho(D) \leqslant \rho\left(R^{\prime \prime}\right) \leqslant \rho(R)+2\left(\frac{4}{3}+4 \varepsilon-2 \delta\right)+2\left(-\frac{10}{3}-\varepsilon+\delta\right)=\rho(R)-4+6 \varepsilon-2 \delta
$$

and so $\rho(R) \geqslant \rho(D)+4-6 \varepsilon+2 \delta \geqslant \rho(D)+\frac{8}{3}-\varepsilon-\delta$.
(ii) If $R^{\prime}=D$ and $\left|X_{W}\right|=2$, then $\rho\left(\overleftrightarrow{K_{\left|X_{W}\right|}}\right)+\delta T\left(\overleftrightarrow{K_{\left|X_{W}\right|}}\right)=\frac{14}{3}+2 \varepsilon$, and, by Lemma 13 , $T(W)-T\left(W-X_{W}\right) \leqslant\left|X_{w}\right|=2$. Thus, by Claim 2 ,

$$
\rho(D) \leqslant \rho(W)+\rho(R)-\frac{14}{3}-2 \varepsilon+2 \delta
$$

Now, since $W$ is smaller than $D, \rho(W) \leqslant \frac{4}{3}+4 \varepsilon-2 \delta$ by Claim 1 . Thus

$$
\rho(D) \leqslant \rho(R)+\frac{4}{3}+4 \varepsilon-2 \delta-\frac{14}{3}-2 \varepsilon+2 \delta=\rho(R)-\frac{10}{3}+2 \varepsilon
$$

and so $\rho(R) \geqslant \rho(D)+\frac{10}{3}-2 \varepsilon \geqslant \rho(D)+\frac{8}{3}-\varepsilon-\delta$.
(iii) If $R^{\prime}=D$ and $\left|X_{W}\right|=3$, then $\rho\left(\overleftrightarrow{K_{\left|X_{W}\right|}}\right)+\delta T\left(\overleftrightarrow{K_{\left|X_{W}\right|}}\right)=4+3 \varepsilon$, and, by Lemma 13 , $T(W)-T\left(W-X_{W}\right) \leqslant\left|X_{w}\right|=3$. Thus, by Claim2,

$$
\rho(D) \leqslant \rho(W)+\rho(R)-4-3 \varepsilon+3 \delta .
$$

Now, since $W$ is smaller than $D, \rho(W) \leqslant \frac{4}{3}+4 \varepsilon-2 \delta$ by Claim 1 . Thus

$$
\rho(D) \leqslant \rho(R)+\frac{4}{3}+4 \varepsilon-2 \delta-4-3 \varepsilon+3 \delta=\rho(R)-\frac{8}{3}+\varepsilon+\delta
$$

and so $\rho(R) \geqslant \rho(D)+\frac{8}{3}-\varepsilon-\delta$.
(iv) If $R^{\prime}=D,\left|X_{W}\right|=1$ and $\partial(R)$ is not monochromatic in $\varphi$, then, by Claim 4, we have $\rho(R) \geqslant \rho(D)+3-3 \varepsilon+\delta \geqslant \rho(D)+\frac{8}{3}-\varepsilon-\delta$.

If $R$ is not collapsible, then, by definition, it has a dicritical extension $R^{\prime}$ satisfying the hypothesis of one of the cases (i)-(iv). In any case, $\rho(R) \geqslant \rho(D)+\frac{8}{3}-\varepsilon-\delta$.

Recall that a $k$-cutset in a graph $G$ is a set $S$ of $k$ vertices such that $G-S$ is not connected. A graph is $k$-connected if it has more than $k$ vertices and has no $(k-1)$-cutset. A $k$-cutset in a digraph is a $k$-cutset in its underlying graph, and a digraph is $k$-connected if its underlying graph is $k$-connected.

Claim 6. $D$ is 2-connected.
Proof of claim. Suppose for contradiction that $\{x\}$ is a 1-cutset in $D$. Let $\left(A_{0}, B_{0}\right)$ be a partition of $V(D-x)$ into non-empty sets such that there is no edge between $A_{0}$ and $B_{0}$, and set $A=A_{0} \cup\{x\}$ and $B=B_{0} \cup\{x\}$.

Since $D$ is 4-dicritical, there exist a 3-dicolouring $\varphi_{A}$ of $D\langle A\rangle$ and a 3-dicolouring $\varphi_{B}$ of $D\langle B\rangle$. Free to swap the colours, we may assume $\varphi_{A}(x)=\varphi_{B}(x)$. Let $\varphi$ be defined by $\varphi(v)=\varphi_{A}(v)$ if $v \in A$ and $\varphi(v)=\varphi_{B}(v)$ if $v \in B$. Since $\vec{\chi}(D)=4, D$, coloured with $\varphi$, must contain a monochromatic directed cycle. Such a directed cycle must be contained in $D\langle A\rangle$ or $D\langle B\rangle$, a contradiction.

Claim 7. $D$ is 3 -connected. In particular, $D$ contains no diamond.
Proof of claim. Suppose for contradiction that $\{x, y\}$ is a 2-cutset of $D$. Let $\left(A_{0}, B_{0}\right)$ be a partition of $V(D) \backslash\{x, y\}$ into non-empty sets such that there is no edge between $A_{0}$ and $B_{0}$, and set $A=A_{0} \cup\{x, y\}$ and $B=B_{0} \cup\{x, y\}$.

Assume for a contradiction that there exists a 3-dicolouring $\varphi_{A}$ of $D\langle A\rangle$ and a 3-dicolouring $\varphi_{B}$ of $D\langle B\rangle$ such that $\varphi_{A}(x) \neq \varphi_{A}(y)$ and $\varphi_{B}(x) \neq \varphi_{B}(y)$. Free to swap the colours, we may assume $\varphi_{A}(x)=\varphi_{B}(x)$ and $\varphi_{A}(y)=\varphi_{B}(y)$. Let $\varphi$ be defined by $\varphi(v)=\varphi_{A}(v)$ if $v \in A$ and $\varphi(v)=\varphi_{B}(v)$ if $v \in B$. Every directed cycle either is in $D\langle A\rangle$, or is in $D\langle B\rangle$ or contains both $x$ and $y$. Therefore it cannot be monochromatic with $\varphi$ because $\varphi_{A}$ and $\varphi_{B}$ are 3-dicolourings of $D\langle A\rangle$ and $D\langle B\rangle$ respectively, and $\varphi(x) \neq \varphi(y)$. Thus $\varphi$ is a 3-dicolouring of $D$, a contradiction. Henceforth either $D\langle A\rangle$ or $D\langle B\rangle$ has no 3-dicolouring $\varphi$ such that $\varphi(x) \neq \varphi(y)$. Suppose without loss of generality that it is $D\langle A\rangle$.

Let $D_{A}=D\langle A\rangle \cup[x, y] . D_{A}$ is not 3-dicolourable because in every 3-dicolouring of $D\langle A\rangle$, $x$ and $y$ are coloured the same. Let $D_{B}$ be the digraph obtained from $D\langle B\rangle$ by identifying $x$ and $y$ into a vertex $z$. Assume for a contradiction that $D_{B}$ has a 3 -dicolouring $\psi_{B}$. Set $\psi(x)=$ $\psi(y)=\psi_{B}(z)$, and $\psi(u)=\psi_{B}(u)$ for every $u \in B \backslash\{x, y\}$. Then consider a 3-dicolouring $\psi_{A}$ of $D\langle A\rangle$ such that $\psi_{A}(x)=\psi(x)=\psi_{A}(y)=\psi(y)$ (such a colouring exists because $A$ is a proper subdigraph of $D$ ) and we set $\psi(u)=\psi_{A}(u)$ for ever $u \in V(A) \backslash\{x, y\}$. As $D$ is not 3-dicolourable, it contains a monochromatic directed cycle $C$ (with respect to $\psi$ ). The cycle $C$ is not included in $D\langle A\rangle$ nor in $D_{B}$. As a consequence, there is a monochromatic directed path from $\{x, y\}$ to $\{x, y\}$ in $B$, and so there is a monochromatic directed cycle in $D_{B}$ for $\psi_{B}$, a contradiction. Therefore $D_{B}$ is not 3 -dicolourable

Now $D_{A}$ has a 4-dicritical subdigraph $W_{A}$ which necessarily contains $\{x, y\}$, and $D_{B}$ has a 4-dicritical subdigraph $W_{B}$ which necessarily contains $z$. As $W_{A}$ and $W_{B}$ are 4-dicritical digraphs smaller than $D$, we have $\rho\left(W_{A}\right), \rho\left(W_{B}\right) \leqslant \frac{4}{3}+4 \varepsilon-2 \delta$ by Claim 1 . Let $H$ be the subdigraph of $D$ induced by $V\left(W_{A}\right) \cup V\left(W_{B}-z\right)$.

Note that $n(H)=n\left(W_{A}\right)+n\left(W_{B}\right)-1$ and $m(H) \geqslant m\left(W_{A}\right)+m\left(W_{B}\right)-2$. Moreover $T(H) \geqslant T\left(W_{A}-x\right)+T\left(W_{B}-z\right) \geqslant T\left(W_{A}\right)+T\left(W_{B}\right)-2$, by Lemma 13. Hence we have

$$
\begin{align*}
\rho(H) & \leqslant \rho\left(W_{A}\right)+\rho\left(W_{B}\right)-\left(\frac{10}{3}+\varepsilon\right)+\left(m\left(W_{A}\right)+m\left(W_{B}\right)-m(H)\right)+2 \delta \\
& \leqslant \rho\left(W_{A}\right)+\rho\left(W_{B}\right)-\frac{10}{3}-\varepsilon+2+2 \delta \\
& =\rho\left(W_{A}\right)+\rho\left(W_{B}\right)-\frac{4}{3}-\varepsilon+2 \delta  \tag{1}\\
& \leqslant 2\left(\frac{4}{3}+4 \varepsilon-2 \delta\right)-\frac{4}{3}-\varepsilon+2 \delta \\
& =\frac{4}{3}+7 \varepsilon-2 \delta
\end{align*}
$$

By Claim 3. if $n(H)<n(D)$ then $\rho(H)>3-3 \varepsilon+\delta$. As $10 \varepsilon-3 \delta \leqslant \frac{5}{3}$, we deduce that $H=D$. Hence $1<\rho(D)=\rho(H) \leqslant \frac{4}{3}+7 \varepsilon-2 \delta+\left(m\left(W_{A}\right)+m\left(W_{B}\right)-m(H)-2\right)$ and so $m(H)=m\left(W_{A}\right)+m\left(W_{B}\right)-2$ because $2 \delta-7 \varepsilon \leqslant \frac{2}{3}$. In particular, there is no arc between $x$ and $y$ in $D$. Moreover, no arc was suppressed when identifying $x$ and $y$ into $z$ to obtain $D_{B}$, so $x$ and $y$ have no common out-neighbour (resp. in-neighbour) in $B_{0}$.

We first show that either $W_{A}$ or $W_{B}$ is not 4-Ore. Assume for contradiction that both $W_{A}$ and $W_{B}$ are 4-Ore. If $H=D$ is not bidirected, then by Lemma 12, $D\langle B\rangle$ admits a 3-dicolouring $\varphi_{B}$ such that $\varphi_{B}(x)=\varphi_{B}(y)$. Now let $\varphi_{A}$ be a 3-dicolouring of $D\langle A\rangle$. We have $\varphi_{A}(x)=\varphi_{A}(y)$. Free to exchange colour we may assume, $\varphi_{A}(x)=\varphi_{A}(y)=\varphi_{B}(x)=\varphi_{B}(y)$. Hence we can define the 3 -colouring $\varphi$ of $D$ by $\varphi(v)=\varphi_{A}(v)$ if $v \in A$, and $\varphi(v)=\varphi_{B}(v)$ if $v \in B$. Observe that, since $A$ is bidirected, all neighbours of $x$ and $y$ in $D\langle A\rangle$ have a colour distinct from $\varphi(x)$. Therefore there is no monochromatic directed cycle in $D$ coloured by $\varphi$. Thus $\varphi$ is a 3-dicolouring of $D$, a contradiction. Therefore, $H=D$ is bidirected, and so $H$ is an Ore-composition of $W_{A}$ and $W_{B}$ (because $D$ is 2-connected by Claim 6), and so $D$ is 4-Ore, a contradiction. Henceforth, we may assume that either $W_{A}$ or $W_{B}$ is not 4-Ore.

If none of $W_{A}$ and $W_{B}$ are is a 4-Ore, then by minimality of $D, \rho\left(W_{A}\right) \leqslant 1$ and $\rho\left(W_{B}\right) \leqslant 1$. Together with Equation (1), this yields

$$
\rho(H) \leqslant \frac{2}{3}-\varepsilon+2 \delta \leqslant 1
$$

because $2 \delta-\varepsilon \leqslant \frac{1}{3}$, a contradiction.
If none of $W_{A}$ and $W_{B}$ is $\overleftrightarrow{K_{4}}$, then $\rho\left(W_{A}\right)+\rho\left(W_{B}\right) \leqslant 1+\left(\frac{4}{3}+7 \varepsilon-4 \delta\right)$ (recall that if a digraph is 4-Ore but not $\overleftrightarrow{K_{4}}$, then it has potential at most $\frac{4}{3}+7 \varepsilon-4 \delta$ by Lemma 22. Thus, with Equation (1), we get

$$
\rho(H) \leqslant 1+\left(\frac{4}{3}+7 \varepsilon-4 \delta\right)-\frac{4}{3}-\varepsilon+2 \delta=1+6 \varepsilon-2 \delta \leqslant 1
$$

because $\delta \geqslant 3 \varepsilon$.

Finally, if exactly one of $W_{A}$ or $W_{B}$ is isomorphic to $\overleftrightarrow{K_{4}}$, then either $T\left(W_{A}-x\right)=T\left(W_{A}\right)=2$ (if $W_{A}=\overleftrightarrow{K_{4}}$ ) or $T\left(W_{B}-z\right)=T\left(W_{B}\right)=2$ (if $W_{B}=\overleftrightarrow{K_{4}}$ ). Therefore $T(H) \geqslant T\left(W_{A}-x\right)+$ $T\left(W_{B}-z\right) \geqslant T\left(W_{A}\right)+T\left(W_{B}\right)-1$ by Lemma 13, and so

$$
\rho(H) \leqslant \rho\left(W_{A}\right)+\rho\left(W_{B}\right)-\left(\frac{10}{3}+\varepsilon\right)+2+\delta .
$$

Now the non 4-Ore digraph among $W_{A}, W_{B}$ has potential at most 1 and the other has potential $\rho\left(\overleftrightarrow{K_{4}}\right)=\frac{4}{3}+4 \varepsilon-2 \delta$. Thus

$$
\rho(H) \leqslant 1+\left(\frac{4}{3}+4 \varepsilon-2 \delta\right)-\left(\frac{10}{3}+\varepsilon\right)+2+\delta=1+3 \varepsilon-\delta \leqslant 1
$$

because $\delta \geqslant 3 \varepsilon$.
In all three cases, $\rho(D)=\rho(H) \leqslant 1$, which is a contradiction. Hence $D$ is 3-connected.

Claim 8. If $R$ is a collapsible subdigraph of $D, u, v$ are in the boundary of $R$ and $D\langle R\rangle \cup[u, v]$ is 4-Ore, then there exists $R^{\prime} \subseteq R$ such that
(i) either $R^{\prime}$ is an Ore-collapsible subdigraph of $D$, or
(ii) $R^{\prime}$ is an induced subdigraph of $R, n\left(R^{\prime}\right)<n(R)$, and there exist $u^{\prime}, v^{\prime}$ in $\partial_{D}\left(R^{\prime}\right)$ such that $R^{\prime} \cup\left[u^{\prime}, v^{\prime}\right]$ is 4-Ore.

Proof of claim. If $\partial(R)=\{u, v\}$, then $R$ is Ore-collapsible and we are done. Suppose now that there exists $w \in \partial(R)$ distinct from $u$ and $v$. Let $H=D\langle R\rangle \cup[u, v]$. Observe that $H \neq \overleftrightarrow{K_{4}}$ as $u, v$ and $w$ receive the same colour in any 3 -dicolouring of $D\langle R\rangle$ because $R$ is collapsible. Hence $H$ is the Ore-composition of two 4-Ore digraphs $H_{1}$ (the digon side with replaced digon [ $\left.x, y\right]$ ) and $H_{2}$ (the split side with split vertex $z$ ).

If $u$ or $v$ is in $V\left(H_{2}\right)$, then $R^{\prime}=D\left\langle V\left(H_{1}\right)\right\rangle$ with $u^{\prime}=x, v^{\prime}=y$ satisfies (ii). Now we assume that $u, v \in V\left(H_{1}\right) \backslash V\left(H_{2}\right)$. By repeating this argument successively on $H_{1}$, and then on the digonside of $H_{1}$, etc, either we find a subdigraph $R^{\prime}$ satisfying (ii) or $u$ and $v$ are in a copy $J$ of $\overleftrightarrow{K}_{4}$ such that $H$ is obtained by Ore-compositions between $J$ and some 4-Ore digraphs with $J$ being always in the digon side.

Observe that $w \notin V(J)$ because in any 3-dicolouring of $H \backslash[u, v]$, vertices in $J$ receive different colours by Lemma 24, except $u$ and $v$. Hence at one step in the succession of Ore-compositions, $w$ was in the split-side $S$ when a digon $e$ in $J$ has been replaced. However $e \neq[u, v]$, so either $u$ or $v$ is not in $e$. Suppose without loss of generality that $e$ is not incident to $v$.

We claim that $H^{\prime}=R-v \cup[u, w]$ is not 3-dicolourable. Otherwise, let $\varphi$ be a 3-dicolouring of $H^{\prime}$. Then $\varphi$ is a 3-dicolouring of $H-v$ with $H 4$-Ore, so vertices in $J-v$ must receive pairwise different colours by Lemma 25, Let $\varphi^{\prime}$ be a 3-dicolouring of $R$. Without loss of generality, we may assume that $\varphi(x)=\varphi^{\prime}(x)$ for every $x \in V(J-v)$. If $y \in S$, let $\varphi^{\prime \prime}(y)=\varphi(y)$, and let $\varphi^{\prime \prime}(y)=\varphi^{\prime}(y)$ if $y \notin S$. Then $\varphi^{\prime \prime}$ is a 3-dicolouring of $R$ but with $\varphi(u) \neq \varphi(w)$, contradicting the fact that $R$ is collapsible. This shows that $H^{\prime}=R-v \cup[u, w]$ is not 3-dicolourable.

Hence $R-v \cup[u, w]$ contains a 4-dicritical digraph $K$. By Lemma 23, $R^{\prime}=D\langle V(K)\rangle$, as a subdigraph of $H$ which is a 4-Ore, satisfies $\frac{10}{3} n\left(R^{\prime}\right)-m\left(R^{\prime}\right) \geqslant \frac{10}{3}$. This implies that $\frac{10}{3} n(K)-$ $m(K) \geqslant \frac{4}{3}$. Note also that $K$ is bidirected because $R-v$ is bidirected. Thus, by Theorem 21, $K$ is 4-Ore. Hence $R^{\prime}$ with $u$, $w$ satisfies (ii).

Claim 9. If $R$ is a subdigraph of $D$ with $n(R)<n(D)$ and $u, v \in V(R)$, then $R \cup[u, v]$ is 3-dicolourable. As a consequence, there is no collapsible subdigraph in $D$.

Proof of claim. Assume for a contradiction that the statement is false. Consider a smallest induced subdigraph $R$ for which the statement does not hold. Then $K=R \cup[u, v]$ is 4-vertex-dicritical, that is for every vertex $v \in V(K), \vec{\chi}(K-v)<4=\vec{\chi}(K)$. Note that 4-vertex-dicritical digraphs smaller than $D$ satisfy the outcome of Theorem 8 since adding arcs does not increase the potential. Note that $\rho(R) \leqslant \rho(K)+2+\delta$.

If $R$ is not collapsible, then, by Claim $5, \rho(R) \geqslant \rho(D)+\frac{8}{3}-\varepsilon-\delta>\frac{11}{3}-\varepsilon-\delta$. But we also have $\rho(R) \leqslant \rho(K)+2+\delta \leqslant \frac{10}{3}+4 \varepsilon-\delta$ by Claim 1 , which is a contradiction because $5 \varepsilon \leqslant \frac{1}{3}$. Hence $R$ is collapsible.

Let $\varphi$ be a 3-dicolouring of $R$. Observe that $\varphi(u)=\varphi(v)$ for otherwise $R \cup[u, v]$ would be 3 -dicolourable. Let $R^{\prime}$ be the dicritical extension of $R$ with extender $W$ and core $X_{W}$. We have $R^{\prime}=D$ and $\left|X_{W}\right|=1$. Since $R$ is collapsible, for every two vertices $u^{\prime}, v^{\prime}$ on the boundary of $R$, $R \cup\left[u^{\prime}, v^{\prime}\right]$ is not 3 -dicolourable. Hence, free to consider $u^{\prime}, v^{\prime}$ instead of $u, v$, we can suppose that $u$ and $v$ are on the boundary of $R$. If $K$ is 4 -Ore, then, by Claim 8 and by minimality of $R$, we have that $R$ is Ore-collapsible, and so has boundary of size 2 . This contradicts the fact that $D$ is 3 -connected. Hence $K$ is not 4-Ore.

By Claim 2, we have

$$
\begin{aligned}
1<\rho(D)=\rho\left(R^{\prime}\right) & \leqslant \rho(W)+\rho(R)-\frac{10}{3}-\varepsilon+\delta \\
& \leqslant \rho(W)+(\rho(K)+2+\delta)-\frac{10}{3}-\varepsilon+\delta
\end{aligned}
$$

and as $\rho(K) \leqslant 1$ (because it is not 4-Ore and by minimality of $D$ ) we get

$$
1<1+\rho(W)-\left(\frac{4}{3}+\varepsilon-2 \delta\right)
$$

that is $\rho(W)>\frac{4}{3}+\varepsilon-2 \delta$. But as $W$ is smaller than $D$, it satisfies Theorem 8 . Thus, since $\varepsilon-2 \delta \geqslant \frac{-1}{3}, W$ must be 4-Ore. Moreover, $W$ must be isomorphic to $\overleftrightarrow{K_{4}}$, for otherwise $\rho(W)$ would be at most $\frac{4}{3}+7 \varepsilon-4 \delta$, and $\frac{4}{3}+7 \varepsilon-4 \delta \geqslant \rho(W)>\frac{4}{3}+\varepsilon-2 \delta$ would contradict $\delta \geqslant 3 \varepsilon$. Hence $\rho(W)=\rho\left(\overleftrightarrow{K_{4}}\right)=\frac{4}{3}+4 \varepsilon-2 \delta$ and $T\left(W-X_{W}\right)=2=T(W)$. Thus, by Claim 2 and because $\delta \geqslant 3 \varepsilon$, we have

$$
1<\rho(D) \leqslant \rho(W)+\rho(K)+2+\delta-\frac{10}{3}-\varepsilon \leqslant \rho(K)+3 \varepsilon-\delta \leqslant \rho(K) \leqslant 1
$$

a contradiction.

This implies that $D$ does not contain any collapsible subdigraph. Indeed, assume for a contradiction that $D$ contains a collapsible subdigraph $R$, and let $u, v$ be two vertices in its boundary. Then there exists a 3-dicolouring $\varphi$ of $R \cup[u, v]$, for which $\partial(R)$ is not monochromatic, a contradiction.

Claim 10. If $R$ is a subdigraph of $D$ with $n(R)<n(D)$ and $u, v, u^{\prime}, v^{\prime} \in R$, then $R \cup\left\{u v, u^{\prime} v^{\prime}\right\}$ is 3-dicolourable. In particular, $D$ contains no copy of $\overleftrightarrow{K}_{4}$ minus two arcs.

Proof of claim. Assume for a contradiction that the statement is false. Consider a smallest subdigraph $R$ for which the statement does not hold. Then $K=R \cup\left\{u v, u^{\prime} v^{\prime}\right\}$ is 4-dicritical and smaller than $D$, so $\rho(K) \leqslant \frac{4}{3}+4 \varepsilon-2 \delta$ by Claim 1. By Claim $9, R$ is not collapsible, so, by Claim5. we have $\rho(R) \geqslant \rho(D)+\frac{8}{3}-\varepsilon-\delta>\frac{11}{3}-\varepsilon-\delta$. But $\rho(R) \leqslant \rho(K)+2+2 \delta \leqslant \frac{10}{3}+4 \varepsilon$, which is a contradiction as $5 \varepsilon+\delta \leqslant \frac{1}{3}$.

For any $v \in V(D)$, we denote by $n(v)$ its number of neighbours, that is $n(v)=\mid N^{+}(u) \cup$ $N^{-}(v) \mid$, and by $d(v)$ its number of incident arcs, that is $d(v)=d^{+}(v)+d^{-}(v)$.

Claim 11. Vertices of degree 6 in $D$ have either three or six neighbours.
Proof of claim. Let $x$ be a vertex of degree 6 .
If $n(x)=4$, then let $a, b, c, d$ be its neighbours such that $N^{+}(x)=\{a, b, c\}$ and $N^{-}(x)=$ $\{a, b, d\}$. Consider $D^{\prime}=D-x \cup d c$. By Claim 10, $D^{\prime}$ has a 3-dicolouring $\varphi$. If $\left|\varphi\left(N^{-}(x)\right)\right|<3$, then choosing $\varphi(x)$ in $\{1,2,3\} \backslash \varphi\left(N^{-}(x)\right)$, we obtain a 3 -dicolouring of $D$, a contradiction. Hence $\varphi\left(N^{-}(x)\right)=\{1,2,3\}$. We set $\varphi(x)=\varphi(d)$. As $D$ is not 3 -dicolourable, $D$ contains a monochromatic directed cycle $C$. This cycle $C$ must contain the arc $d x$, and an out-neighbour $z$ of $x$. Since $\varphi(a), \varphi(b)$ and $\varphi(d)$ are all distinct, necessarily $z=c$. But then $C-x \cup d c$ is a monochromatic directed cycle in $D^{\prime}$, a contradiction.

Similarly, if $n(v)=5$, let $N^{+}(x)=\{a, b, c\}$ and $N^{-}(x)=\{a, d, e\}$, and consider $D^{\prime}=$ $D-x \cup d b \cup d c$. By Claim 10, $D^{\prime}$ has a 3-dicolouring $\varphi$. If $\left|\varphi\left(N^{-}(x)\right)\right|<3$, then choosing $\varphi(x)$ in $\{1,2,3\} \backslash \varphi\left(N^{-}(x)\right)$, we obtain a 3 -dicolouring of $D$, a contradiction. Hence $\varphi\left(N^{-}(x)\right)=$ $\{1,2,3\}$. We set $\varphi(x)=\varphi(d)$. As $D$ is not 3 -dicolourable, there is a monochromatic directed cycle $C$, which must contain the arc $d x$ and an out-neighbour $z$ of $x$. Note that $z$ must be $b$ or $c$ because $\varphi(a) \neq \varphi(d)$. Then $C-x \cup d z$ is a monochromatic directed cycle in $D^{\prime}$, a contradiction.

Claim 12. There is no bidirected triangle containing two vertices of degree 6. In particular, $D$ contains no emerald.

Proof of claim. Suppose that $D\langle\{x, y, z\}\rangle=\overleftrightarrow{K_{3}}$ and $d(x)=d(y)=6$. By Claim 11 , $x$ and $y$ have exactly three neighbours, and $N[x] \neq N[y]$ because $D$ contains no copy of $\overleftarrow{K_{4}}$ minus two arcs by Claim 10. Let $u$ (resp. $v$ ) be the unique neighbour of $x$ distinct from $y$ and $z$ (resp. $x$ and $z$ ). Consider $D^{\prime}=D-\{x, y\} \cup[u, v]$. By Claim $9, D^{\prime}$ has a 3-dicolouring $\varphi$. Without loss of generality, suppose that $\varphi(u)=1$ and $\varphi(v)=2$. If $\varphi(z)=1$ (resp. $\varphi(z)=2, \varphi(z)=3$ ), we set
$\varphi(x)=2$ and $\varphi(y)=3($ resp. $\varphi(x)=3$ and $\varphi(y)=1, \varphi(x)=2$ and $\varphi(y)=1)$. In each case, this yields a 3 -dicolouring of $D$, a contradiction.

So now we know that $D$ contains no emerald, and no diamond by Claim 7 .
Claim 13. If $R$ is an induced subdigraph of $D$ with $4 \leqslant n(R)<n(D)$, then $\rho(R) \geqslant \rho(D)+3+$ $3 \varepsilon-3 \delta$, except if $D-R$ contains a single vertex which has degree 6 in $D$.

Proof of claim. Let $R$ be an induced subdigraph of $D$ with $4 \leqslant n(R)<n(D)$. By Claim $9, R$ is not collapsible. Let $\varphi$ be a 3-dicolouring of $R, R^{\prime}$ be a dicritical extension of $R$ with extender $W$ and core $X_{W}$ (with respect to $\varphi$ ). By (the consequence of) Claim 3 , we know that $\rho\left(R^{\prime}\right) \geqslant \rho(D)$.

Assume first that $R^{\prime} \neq D$. Then, by Claims 3 and 2 ,

$$
\rho(D)+2-3 \varepsilon+\delta \leqslant \rho\left(R^{\prime}\right) \leqslant \rho(W)+\rho(R)-\frac{10}{3}-\varepsilon+\delta .
$$

Since $\rho(W) \leqslant \frac{4}{3}+4 \varepsilon-2 \delta$ by Claim 1 , we have $\rho(R) \geqslant \rho(D)+4-6 \varepsilon+2 \delta \geqslant \rho(D)+3+3 \varepsilon-3 \delta$, because $1 \geqslant 9 \varepsilon-5 \delta$. In the following we suppose that $R^{\prime}=D$. We distinguish three cases depending on the cardinality of $\left|X_{W}\right|$.

- Assume first that $\left|X_{W}\right|=2$. Then, by Claim 2 and Lemma 13 ,

$$
\rho(D) \leqslant \rho\left(R^{\prime}\right) \leqslant \rho(W)+\rho(R)-\frac{20}{3}-2 \varepsilon+2+2 \delta
$$

and, as $\rho(W) \leqslant \frac{4}{3}+4 \varepsilon-2 \delta$ by Claim 1, we have $\rho(R) \geqslant \rho(D)+\frac{10}{3}-2 \varepsilon \geqslant \rho(D)+3+3 \varepsilon-3 \delta$ because $5 \varepsilon-3 \delta \leqslant \frac{1}{3}$.

- Assume now that $\left|X_{W}\right|=3$. If there is a vertex $v \in V(D-R)$ with two out-neighbours (resp. two in-neighbours) in $V(R)$ with the same colour for $\varphi$, then
- $n\left(R^{\prime}\right)=n(W)-\left|X_{W}\right|+n(R)$,
- $m\left(R^{\prime}\right) \geqslant m(W)+m(R)-m\left(\overleftrightarrow{K_{\left|X_{W}\right|}}\right)+1$ because $v$ has two in- or out-neighbour in $V(R)$ with the same colour for $\varphi$,
- $T\left(R^{\prime}\right) \geqslant T\left(W-X_{W}\right)+T(R)$.

It follows that

$$
\rho(D) \leqslant \rho\left(R^{\prime}\right) \leqslant \rho(W)+\rho(R)-(10+3 \varepsilon-6)+3 \delta-1
$$

and so $\rho(R) \geqslant \rho(D)-\frac{4}{3}-4 \varepsilon+2 \delta+5+3 \varepsilon-3 \delta \geqslant \rho(D)+\frac{11}{3}-\varepsilon-\delta \geqslant \rho(D)+3+3 \varepsilon-3 \delta$ because $4 \varepsilon-2 \delta \leqslant \frac{2}{3}$. Now we assume that there is no vertex with two out-neighbours (resp. two in-neighbours) in $R$ with the same colour for $\varphi$. In other words, the in-degrees and out-degrees of vertices in $D-R$ are the same in $D$ and in $W$.

If $W$ is not 4 -Ore, then by Claim 2

$$
\rho(D) \leqslant \rho\left(R^{\prime}\right) \leqslant \rho(W)+\rho(R)-(10+3 \varepsilon-6)+3 \delta
$$

and, as $\rho(W) \leqslant 1$, we have $\rho(R) \geqslant \rho(D)+3+3 \varepsilon-3 \delta$.
Now suppose $W$ is 4-Ore. If $W \neq \overleftrightarrow{K}_{4}$, then, by Lemma 20, $W$ contains a diamond or an emerald disjoint from $X$, and this gives a diamond or an emerald in $D$ because the degrees of vertices in $D-R$ are the same in $D$ and in $W$, which is a contradiction.
Now suppose that $W=\overleftrightarrow{K_{4}}$. Then $D-R$ has a single vertex of degree 6 in $D$.

- Assume finally that $\left|X_{W}\right|=1$. Since $R$ is not collapsible by Claim $9, \varphi$ may have been chosen so that $\partial(R)$ is not monochromatic in $\varphi$. Then, by Claim $4, \rho(R) \geqslant \rho(D)+3-3 \varepsilon+$ $\delta \geqslant \rho(D)+3+3 \varepsilon-3 \delta$, because $6 \varepsilon-4 \delta \leqslant 0$.

In $D$, we say that a vertex $v$ is a simple in-neighbour (resp. simple out-neighbour) if $v$ is a in-neighbour (resp. out-neighbour) of $u$ and $[u, v]$ is not a digon in $D$. If $v$ is a simple in-neighbour or simple out-neighbour of $u$, we simply say that $v$ is a simple neighbour of $u$.

Claim 14. Vertices of degree 7 have seven neighbours. In other words, every vertex of degree 7 has only simple neighbours.

Proof of claim. Let $x$ be a vertex of degree 7 . We suppose, without loss of generality, that $d^{-}(x)=$ 3 and $d^{+}(x)=4$.

If $n(x)=4$, then $x$ has a unique simple out-neighbour $a$. As $D$ is 4 -dicritical, $D \backslash x a$ has a 3 -dicolouring $\varphi$. But then every directed cycle is either in $D \backslash x a$ or it contains $x a$ and thus an inneighbour $t$ of $x$. In the first case, it is not monochromatic because $\varphi$ is a 3-dicolouring of $D \backslash x a$, and in the second case, it is not monochromatic because $[t, x]$ is a digon and so $\varphi(t) \neq \varphi(x)$. Hence $\varphi$ is a 3 -dicolouring of $D$, a contradiction.

If $n(x)=5$, let $N^{-}(x)=\{a, b, c\}$ and $N^{+}(x)=\{a, b, d, e\}$. By Claim 10, $D^{\prime}=D-x \cup$ $\{c d, c e\}$ has a 3 -dicolouring $\varphi$. If $\left|\varphi\left(N^{-}(x)\right)\right|<3$, then choosing $\varphi(x)$ in $\{1,2,3\} \backslash \varphi\left(N^{-}(x)\right)$ gives a 3 -dicolouring of $D$, a contradiction. If $\left|\varphi\left(N^{-}(x)\right)\right|=3$, then we set $\varphi(x)=\varphi(c)$. Suppose for a contradiction that there is a monochromatic directed cycle $C$ in $D$ (with $\varphi$ ). Necessarily $C$ contains $x$ (since $\varphi$ is a 3-dicolouring of $D-x$ ) and so it must contain $c$ and one vertex $y$ in $\{d, e\}$ because $\varphi(a), \varphi(b)$, and $\varphi(c)$ are all distinct. Then $C-x \cup c y$ is a monochromatic directed cycle in $D^{\prime}$, a contradiction. Therefore $\varphi$ is a 3 -dicolouring of $D$, a contradiction.

If $n(x)=6$, let $N^{-}(x)=\{a, b, c\}$ and $N^{+}(x)=\{a, d, e, f\}$. Consider $D^{\prime}=D-x \cup$ $\{b d, b e, b f\}$.

We first show that $D^{\prime}$ is not 3 -dicolourable. Assume for a contradiction that there is a 3dicolouring $\varphi$ of $D^{\prime}$. If $\left|\varphi\left(N^{-}(x)\right)\right|<3$, then choosing $\varphi(x)$ in $\{1,2,3\} \backslash \varphi\left(N^{-}(x)\right)$ gives a 3 -dicolouring of $D$, a contradiction. Hence $\left|\varphi\left(N^{-}(x)\right)\right|=3$. We set $\varphi(x)=\varphi(b)$. Since $D$ is not 3 -dicolourable, there exists a monochromatic directed cycle $C$ in $D$ (with $\varphi$ ). Necessarily
$C$ contains $x$ (since $\varphi$ is a 3-dicolouring of $D-x$ ) and so it must contain $b$ and one vertex $y$ in $\{d, e, f\}$ because $\varphi(a), \varphi(b)$, and $\varphi(c)$ are all distinct. Then $C-x \cup b y$ is a monochromatic directed cycle in $D^{\prime}$, a contradiction. This gives a 3 -dicolouring of $D$, a contradiction.

Henceforth $D^{\prime}$ is not 3 -dicolourable, and so it contains a 4 -dicritical digraph $\tilde{D}$, smaller than $D$. If $\tilde{D}$ does not contain the three arcs $b d, b e, b f$, then it can be obtained from a proper induced subdigraph of $D$ by adding at most two arcs, and so it is 3 -dicolourable by Claim 10, a contradiction.

Hence $\{b, d, e, f\} \subseteq V(\tilde{D})$. Now consider $U=D\langle V(\tilde{D}) \cup\{x\}\rangle$.

- Assume first that $a \notin V(U)$ or $c \notin V(U)$. Then we have
- $n(U)=n(\tilde{D})+1$,
- $m(U) \geqslant m(\tilde{D})+1$ and
- $T(U) \geqslant T(\tilde{D}-b) \geqslant T(\tilde{D})-1$ by Lemma 13 .

Hence

$$
\begin{array}{rlr}
\rho(U) & \leqslant \rho(\tilde{D})+\frac{10}{3}+\varepsilon-1+\delta & \\
& \leqslant \frac{4}{3}+4 \varepsilon-2 \delta+\frac{10}{3}+\varepsilon-1+\delta \quad \text { by Claim } 1 \\
& =1+\frac{8}{3}+5 \varepsilon-\delta & \\
& <\rho(D)+\frac{8}{3}+5 \varepsilon-\delta & \\
& \leqslant \rho(D)+3+3 \varepsilon-3 \delta \quad \text { because } \frac{1}{3} \geqslant 2 \delta+2 \varepsilon
\end{array}
$$

Hence by Claim 13, $D-U$ has a single vertex of degree 6 (in $D$ ), which must be either $a$ or $c$. Then we have

- $n(D)=n(\tilde{D})+2$,
- $m(D) \geqslant m(\tilde{D})-3+11$ and
- $T(D) \geqslant T(\tilde{D}-b) \geqslant T(\tilde{D})-1$.

Thus

$$
\begin{array}{rlrl}
\rho(D) & \leqslant \rho(\tilde{D})+2\left(\frac{10}{3}+\varepsilon\right)-8+\delta & \\
& \leqslant\left(\frac{4}{3}+4 \varepsilon-2 \delta\right)-\frac{4}{3}+2 \varepsilon+\delta & & \text { by Claim } 1 \\
& \leqslant 1 & & \text { because } 6 \varepsilon-\delta \leqslant 1 .
\end{array}
$$

This is a contradiction.

- Assume now that $a, c \in V(U)$, then we have
- $n(U)=n(\tilde{D})+1$,
- $m(U) \geqslant m(\tilde{D})+4$ and
- $T(U) \geqslant T(\tilde{D}-b) \geqslant T(\tilde{D})-1$ by Lemma 13 .

Thus

$$
\begin{array}{rlrl}
\rho(U) & \leqslant \rho(\tilde{D})+\frac{10}{3}+\varepsilon-4+\delta & \\
& \leqslant\left(\frac{4}{3}+4 \varepsilon-2 \delta\right)+\frac{10}{3}+\varepsilon-4+\delta & & \text { by Claim } 1, \\
& \leqslant 1 & & \text { because } 5 \varepsilon-\delta \leqslant \frac{1}{3} .
\end{array}
$$

Together with the consequence of Claim 3 , we get that $\rho(D) \leqslant \rho(U) \leqslant 1$, a contradiction.
The $8^{+}$-valency of a vertex $v$, denoted by $\nu(v)$, is the number of arcs incident to $v$ and a vertex of degree at least 8 .

Let $D_{6}$ be the subdigraph of $D$ induced by the vertices of degree 6 incident to digons. Let us describe the connected components of $D_{6}$ and their neighbourhoods. Remember that vertices of degree 7 are incident to no digon by Claim 14, and so they do not have neighbours in $V\left(D_{6}\right)$. If $v$ is a vertex in $D_{6}$, we define its neighbourhood valency to be the sum of the $8^{+}$-valency of its neighbours of degree at least 8 . We denote the neighbourhood valency of $v$ by $\nu_{N}(v)$.

Claim 15. If $[x, y]$ is a digon and both $x$ and $y$ have degree 6 , then either
(i) the two neighbours of $y$ distinct from $x$ have degree at least 8 , or
(ii) the two neighbours of $x$ distinct from $y$ have degree at least 8 and $\nu_{N}(x) \geqslant 4$.

Proof of claim. Let $[x, y]$ be a digon in $D$ with $d(x)=d(y)=6$. By Claim $11 n(x)=n(y)=3$. Let $u$ and $v$ be the two neighbours of $x$ different from $y$. By Claim 14, $u$ and $v$ have degree 6 or at least 8 .

If $u$ and $v$ are linked by a digon, then by Claim 12, $u$ and $v$ do not have degree 6 , so they have degree 8 . Moreover $\nu(u) \geqslant 2$ and $\nu(v) \geqslant 2$. Thus $\nu_{N}(x)=\nu(u)+\nu(v) \geqslant 4$ and (ii) holds. Henceforth, we may assume that $u$ and $v$ are not linked by a digon.

Let $D^{\prime}$ the digraph obtained by removing $x$ and $y$ and identifying $u$ and $v$ into a single vertex $u \star v$. We claim that $D^{\prime}$ is not 3-dicolourable. To see that, suppose for contradiction that there exists a 3-dicolouring $\varphi$ of $D^{\prime}$. Then set $\varphi(u)=\varphi(v)=\varphi(u \star v)$, choose $\varphi(y)$ in $\{1,2,3\} \backslash \varphi(N(y) \backslash\{x\})$, and finally choose $\varphi(x)$ in $\{1,2,3\} \backslash\{\varphi(u \star v), \varphi(y)\}$. One can easily see that $\varphi$ is now a 3dicolouring of $D$, a contradiction. This proves that $D^{\prime}$ is not 3 -dicolourable and so it contains a 4-dicritical digraph $\tilde{D}$, which must contain $u \star v$ because every subdigraph of $D$ is 3-dicolourable. Let $R$ be the subdigraph of $D$ induced by $(V(\tilde{D}) \backslash\{u \star v\}) \cup\{u, v, x\}$. We have

- $n(R)=n(\tilde{D})+2$,
- $m(R) \geqslant m(\tilde{D})+4$ and
- $T(R) \geqslant T(\tilde{D}-u \star v)+1 \geqslant T(\tilde{D})$ because $[x, u]$ is a digon, and by Lemma 13 .

If $\tilde{D}$ is not 4-Ore, then $\rho(\tilde{D}) \leqslant 1$ by minimality of $D$, and so

$$
\begin{aligned}
\rho(R) & \leqslant \rho(\tilde{D})+2\left(\frac{10}{3}+\varepsilon\right)-4 \\
& \leqslant 1+\frac{8}{3}+2 \varepsilon \\
& <\rho(D)+3+3 \varepsilon-3 \delta \quad \text { because } \varepsilon-3 \delta \geqslant-\frac{1}{3} .
\end{aligned}
$$

Similarly, if $\tilde{D}$ is 4-Ore but not $\overleftrightarrow{K_{4}}$, then

$$
\begin{aligned}
\rho(R) & \leqslant \rho(\tilde{D})+2\left(\frac{10}{3}+\varepsilon\right)-4 \\
& \leqslant\left(\frac{4}{3}+7 \varepsilon-4 \delta\right)+\frac{8}{3}+2 \varepsilon \quad \text { by Lemma } 22, \\
& =1+3+9 \varepsilon-4 \delta \\
& <\rho(D)+3+9 \varepsilon-4 \delta \\
& \leqslant \rho(D)+3+3 \varepsilon-3 \delta \quad \text { because } \delta \geqslant 6 \varepsilon .
\end{aligned}
$$

In both cases (that is when $\tilde{D}$ is not $\overleftrightarrow{K_{4}}$ ), by Claim $13, D-R$ is a single vertex of degree 6 , namely $y$. Then every neighbour $w$ of $y$ different from $x$ has degree at least 6 in $\tilde{D}$ (because $\tilde{D}$ is 3 -dicritical) and so has degree at least 8 in $D$ and (i) holds.

Assume now that $\tilde{D}$ is a copy of $\overleftrightarrow{K_{4}}$. Let us denote by $a, b, c$ the vertices of $\tilde{D}$ different from $u \star v$. Suppose for a contradiction that $u$ has degree 6 . Then $u$ has exactly three neighbours by Claim 11. If $|N(u) \cap\{a, b, c\}|=2$, then $D\langle\{u, a, b, c\}\rangle$ is a copy of $\vec{K}_{4}$ minus a digon, contradicting Claim 9 . If $|N(u) \cap\{a, b, c\}| \leqslant 1$, then $v$ must be adjacent to at least two vertices of $\{a, b, c\}$ with a digon, and so $D\langle\{v, a, b, c\}\rangle$ contains a copy of $\overleftrightarrow{K_{4}}$ minus a digon, contradicting Claim 9 . Hence $u$ has degree at least 8 , and by symmetry so does $v$. Moreover $D\langle\{a, b, c\}\rangle$ is a bidirected triangle, and so by Claim 12, at least two of these vertices have degree at least 8 (remember that vertices of degree 7 are in no digon by Claim 14). Hence at least four arcs between $\{u, v\}$ and $\{a, b, c\}$ are incident to two vertices of degree at least 8 . In other word, $\nu_{N}(x)=$ $\nu(u)+\nu(v) \geqslant 4$, so (ii) holds.

Claim 16. Let $C$ be a connected component of $D_{6}$. Then $C$ is one of the following (see Figure 3):
(i) a single vertex, or
(ii) a bidirected path on two vertices, or


Figure 3: The possible connected components of $D_{6}$.


Figure 4: An example of an out-chelou arc $x y$.
(iii) a bidirected path on three vertices, whose extremities have neighbourhood valency at least 4, or
(iv) a star on four vertices, whose non-central vertices have neighbourhood valency at least 4.

Proof of claim. First observe that $C$ does not contain a bidirected path $\llbracket x, y, z, w \rrbracket$ on four vertices, because otherwise, by Claim 15 applied on $[y, z]$, either $y$ or $z$ has two neighbours of degree at least 8, a contradiction. Observe also that $C$ contains no bidirected triangle by Claim 12 .

Moreover, if $\llbracket x, y, z \rrbracket$ is a bidirected path in $C$ on three vertices, then by Claim 15 applied both on $[y, z]$ and $[z, y], x$ and $z$ have both neighbourhood valency at least 4 . The statement of the claim follows.

An arc $x y$ is said to be out-chelou if
(i) $y x \notin A(D)$, and
(ii) $d^{+}(x)=3$, and
(iii) $d^{-}(y)=3$, and
(iv) there exists $z \in N^{-}(y) \backslash N^{+}(y)$ distinct from $x$.

Symmetrically, we say that an arc $x y$ is in-chelou if $y x$ is out-chelou in the digraph obtained from $D$ by reversing every arc. See Figure 4 for an example of an out-chelou arc.

Claim 17. There is no out-chelou arc and no in-chelou arc in $D$.
Proof of claim. By directional duality, it suffices to prove that $D$ has no out-chelou arcs.
Let $x y$ be an out-chelou arc with $z \in N^{-}(y) \backslash\left(N^{+}(y) \cup\{x\}\right)$. Consider $D^{\prime}=D-\{x, y\} \cup$ $\left\{z z^{\prime} \mid z^{\prime} \in N^{+}(y) \backslash N^{-}(y)\right\}$. We claim that $D^{\prime}$ is not 3-dicolourable. To see that, suppose
for contradiction that there is a 3 -dicolouring $\varphi$ of $D^{\prime}$. As $d^{+}(x)=3$, we can choose $\varphi(x)$ in $\{1,2,3\} \backslash \varphi\left(N^{+}(x) \backslash\{y\}\right)$ to obtain a 3-dicolouring of $D-y$. If $\left|\varphi\left(N^{-}(y)\right)\right|<3$, then choosing $\varphi(y)$ in $\{1,2,3\} \backslash \varphi\left(N^{-}(y)\right)$ gives a 3 -dicolouring of $D$, a contradiction. Hence $\mid \varphi\left(N^{-}(y) \mid=3\right.$. Set $\varphi(x)=\varphi(z)$. Suppose there is a monochromatic directed cycle $C$ in $D$. It must contain $y$ and thus $z$, its unique in-neighbour with its colour. Let $z^{\prime}$ be the out-neighbour of $y$ in $C$. It must be in $N^{+}(y) \backslash N^{-}(y)$, so $z z^{\prime}$ is an arc in $D^{\prime}$. Thus $C-y \cup z z^{\prime}$ is a monochromatic directed cycle in $D^{\prime}$, a contradiction. Therefore $\varphi$ is a 3 -dicolouring of $D$, a contradiction. Hence $D^{\prime}$ is not 3 -dicolourable.

Consequently, $D^{\prime}$ contains a 3 -dicritical digraph $\tilde{D}$, which is smaller than $D$ and contains $z$, for otherwise $\tilde{D}$ would be a subdigraph of $D$. Consider $U=D\langle V(\tilde{D}) \cup\{y\}\rangle$. We have

- $n(U)=n(\tilde{D})+1$,
- $m(U) \geqslant m(\tilde{D})+1$ and
- $T(U) \geqslant T(\tilde{D}-z) \geqslant T(\tilde{D})-1$ by Lemma 13

First if $\tilde{D}$ is not 4-Ore, then by minimality of $D$ we have $\rho(\tilde{D}) \leqslant 1$, so

$$
\rho(U) \leqslant \rho(\tilde{D})+\frac{10}{3}+\varepsilon-1+\delta \leqslant \frac{10}{3}+\varepsilon+\delta \leqslant \frac{11}{3}-\varepsilon-\delta
$$

$2 \varepsilon+2 \delta \leqslant \frac{1}{3}$.
Next if $\tilde{D}$ is 4-Ore, but not isomorphic to $\overleftrightarrow{K_{4}}$, then $\rho(\tilde{D}) \leqslant \frac{4}{3}+7 \varepsilon-4 \delta$ by Lemma 22, and

$$
\rho(U) \leqslant \rho(\tilde{D})+\frac{10}{3}+\varepsilon-1+\delta \leqslant \frac{11}{3}+8 \varepsilon-3 \delta \leqslant \frac{11}{3}-\varepsilon-\delta
$$

because $9 \varepsilon-2 \delta \leqslant 0$.
Finally if $\tilde{D}$ is isomorphic to $\overleftrightarrow{K_{4}}$, then we have $T(U) \geqslant T(\tilde{D}-z) \geqslant T(\tilde{D})$ and $\rho(\tilde{D})=$ $\frac{4}{3}+4 \varepsilon-2 \delta$. So the same computation yields

$$
\rho(U) \leqslant \rho(\tilde{D})+\frac{10}{3}+\varepsilon-1 \leqslant \frac{11}{3}+5 \varepsilon-2 \delta \leqslant \frac{11}{3}-\varepsilon-\delta
$$

because $6 \varepsilon-\delta \leqslant 0$.
In all cases, we have $\rho(U) \leqslant \frac{11}{3}-\varepsilon-\delta$. This contradicts Claim 5 because $U$ is not collapsible by Claim 9 .

We now use the discharging method. For every vertex $v$, let $\sigma(v)=\frac{\delta}{|C|}$ if $v$ has degree 6 and is in a component $C$ of $D_{6}$ of size at least 2 , and $\sigma(v)=0$ otherwise. Clearly $T(D)$ is at least the number of connected components of size at least 2 of $D_{6}$ so $\sum_{v \in V(D)} \sigma(v) \leqslant \delta T(D)$. We define the initial charge of $v$ to be $w(v)=\frac{10}{3}+\varepsilon-\frac{d(v)}{2}-\sigma(v)$. We have

$$
\rho(D) \leqslant \sum_{v \in V(D)} w(v) .
$$

We now redistribute this total charge according to the following rules:
(R1) A vertex of degree 6 incident to no digon sends $\frac{1}{12}-\frac{\varepsilon}{8}$ to each of its neighbours.
(R2) A vertex of degree 6 incident to digons sends $\frac{2}{d(v)-\nu(v)}\left(-\frac{10}{3}+\frac{d(v)}{2}-\varepsilon\right)$ to each neighbour $v$ of degree at least 8 (so $\frac{1}{d(v)-\nu(v)}\left(-\frac{10}{3}+\frac{d(v)}{2}-\varepsilon\right)$ via each arc of the digon).
(R3) A vertex of degree 7 with $d^{-}(v)=3$ (resp. $d^{+}(v)=3$ ) sends $\frac{1}{12}-\frac{\varepsilon}{8}$ to each of its inneighbours (resp. out-neighbours).
For every vertex $v$, let $w^{*}(v)$ be the final charge of $v$.
Claim 18. If $v$ has degree at least 8 , then $w^{*}(v) \leqslant 0$.
Proof of claim. Let $v$ be a vertex of degree at least 8 . If $v$ is not adjacent to a vertex of degree at most 7 , then $w^{*}(v)=w(v)=\frac{10}{3}+\varepsilon-\frac{d(v)}{2} \leqslant 0$ (because $\varepsilon \leqslant \frac{2}{3}$ ). Otherwise, $d(v)-\nu(v) \geqslant 1$ and

$$
\begin{aligned}
\frac{1}{d(v)-\nu(v)}\left(-\frac{10}{3}+\frac{d(v)}{2}-\varepsilon\right) & \geqslant \frac{1}{d(v)}\left(-\frac{10}{3}+\frac{d(v)}{2}-\varepsilon\right) \\
& \geqslant \frac{1}{12}-\frac{\varepsilon}{8}
\end{aligned}
$$

Thus $v$ receives at most $\frac{1}{d(v)-\nu(v)}\left(-\frac{10}{3}+\frac{d(v)}{2}-\varepsilon\right)$ per arc incident with a vertex of degree 6 or 7 . Since there are $d(v)-\nu(v)$ such arcs, $w^{*}(v) \leqslant w(v)-\frac{10}{3}-\varepsilon+\frac{d(v)}{2}=0$.

Claim 19. If $v$ has degree 7 , then $w^{*}(v) \leqslant 0$.
Proof of claim. By Claim 14, $v$ has seven neighbours. Without loss of generality, let us suppose that $d^{-}(v)=3$ and $d^{+}(v)=4$. By Claim 17, the in-neighbours of $v$ can not have out-degree 3 . In particular, they do not have degree 6 , and if they have degree 7 , they do not send anything to $v$ by Rule (R3). Hence $v$ receives at most four times the charge $\frac{1}{12}-\frac{\varepsilon}{8}$ by (R1) or (R3), and it sends three times this charge by (R3). Hence

$$
\begin{aligned}
w^{*}(v) & \leqslant w(v)+\frac{1}{12}-\frac{\varepsilon}{8} \\
& =-\frac{1}{12}+\frac{7}{8} \varepsilon
\end{aligned}
$$

and the result comes because $\varepsilon \leqslant \frac{2}{21}$.
Claim 20. If $v$ is a vertex of degree 6 incident to no digon, then $w^{*}(v) \leqslant 0$.
Proof of claim. The vertex $v$ sends $\frac{1}{12}-\frac{\varepsilon}{8}$ to each of its neighbours, and it receives no charge as all its in-neighbours (resp. out-neighbours) have out-degree (resp. in-degree) at least 4, by Claim 17 . As a consequence,

$$
w^{*}(v)=w(v)-6\left(\frac{1}{12}-\frac{\varepsilon}{8}\right)=-\frac{1}{6}+\frac{7 \varepsilon}{4}
$$

and the result comes because $\varepsilon \leqslant \frac{2}{21}$.

Claim 21. Let $v$ be a vertex in $D_{6}$ having at least two neighbours of degree at least 8. Then $w^{*}(v) \leqslant 0$. Moreover, if v is not an isolated vertex in $D_{6}$ and $\nu_{N}(v) \geqslant 4$, then $w^{*}(v) \leqslant-\frac{1}{9}+\frac{5}{3} \varepsilon-\frac{\delta}{4}$.

Proof of claim. Observe that $v$ receives no charge and sends the following charge to each of its neighbour $u$ with degree at least 8 :

$$
\begin{aligned}
\frac{2}{d(u)-\nu(u)}\left(-\frac{10}{3}-\varepsilon+\frac{d(u)}{2}\right) & \geqslant \frac{2}{d(u)}\left(-\frac{10}{3}-\varepsilon+\frac{d(u)}{2}\right) \\
& =1-\frac{2}{d(u)}\left(\frac{10}{3}+\varepsilon\right) \\
& \geqslant \frac{2}{8}\left(-\frac{10}{3}-\varepsilon+4\right) \\
& =\frac{1}{6}-\frac{\varepsilon}{4} .
\end{aligned}
$$

Assume first that $v$ is isolated in $D_{6}$. By Claim 14, its three neighbours do not have degree 7, and so have degree at least 8 . Thus $v$ sends three times at least $\frac{1}{6}-\frac{\varepsilon}{4}$, and so

$$
w^{*}(v) \leqslant w(v)-3\left(\frac{1}{6}-\frac{\varepsilon}{4}\right)=-\frac{1}{6}+\frac{7}{4} \varepsilon
$$

and the result comes because $\varepsilon \leqslant \frac{2}{21}$.
Assume now that $v$ is in a connected component $C$ of $D_{6}$ of size at least 2. By Claim 16, $\sigma(v) \geqslant \frac{\delta}{4}$, so $w(v) \leqslant \frac{1}{3}+\varepsilon-\frac{\delta}{4}$. Moreover it sends two times at least $\frac{1}{6}-\frac{\varepsilon}{4}$. Hence

$$
w^{*}(v) \leqslant\left(\frac{1}{3}+\varepsilon-\frac{\delta}{4}\right)-2\left(\frac{1}{6}-\frac{\varepsilon}{4}\right)=\frac{3}{2} \varepsilon-\frac{\delta}{4}
$$

and the result comes because $\delta \geqslant 6 \varepsilon$. This shows the first part of the statement.
We will now prove the second part of the statement. Assume that $v$ is not an isolated vertex in $D_{6}$ and $\nu_{N}(v) \geqslant 4$. Let $u_{1}$ and $u_{2}$ be the two neighbours of $v$ with degree at least 8 . For every $i \in\{1,2\}$ we have

$$
\frac{2}{d\left(u_{i}\right)-\nu\left(u_{i}\right)}\left(-\frac{10}{3}-\varepsilon+\frac{d\left(u_{i}\right)}{2}\right)=1-\frac{1}{d\left(u_{i}\right)-\nu\left(u_{i}\right)}\left(\frac{20}{3}+2 \varepsilon-\nu\left(u_{i}\right)\right)
$$

Case 1: $\nu\left(u_{i}\right) \geqslant 7$ for some $i \in\{1,2\}$. Without loss of generality suppose $i=1$. Then we have

$$
1-\frac{1}{d\left(u_{1}\right)-\nu\left(u_{1}\right)}\left(\frac{20}{3}+2 \varepsilon-\nu\left(u_{1}\right)\right) \geqslant 1
$$

because $\nu\left(u_{1}\right) \geqslant 7 \geqslant \frac{20}{3}+2 \varepsilon$ as $\varepsilon \leqslant \frac{1}{6}$. Then the total charge sent by $v$ is at least 1 , and thus

$$
w^{*}(v) \leqslant w(v)-1 \leqslant\left(\frac{1}{3}+\varepsilon-\frac{\delta}{4}\right)-1=-\frac{2}{3}+\varepsilon-\frac{\delta}{4}
$$

Thus, we have $w^{*}(v) \leqslant-\frac{1}{9}+\frac{5}{3} \varepsilon-\frac{\delta}{4}$ because $\varepsilon, \delta \geqslant 0$.
Case 2: $\nu\left(u_{1}\right), \nu\left(u_{2}\right) \leqslant 6$. Let $f:[0,6] \rightarrow \mathbb{R}$ be the function defined by

$$
f(x)=\frac{2}{8-x}\left(-\frac{10}{3}-\varepsilon+\frac{8}{2}\right)=1-\frac{1}{8-x}\left(\frac{20}{3}-2 \varepsilon-x\right)
$$

for every $x \in[0,6]$. Observe that $f$ is non decreasing and convex on $[0,6]$ because $-\frac{10}{3}-\varepsilon+\frac{8}{2} \geqslant 0$. For $i=1,2$, we have

$$
\frac{2}{d\left(u_{i}\right)-\nu\left(u_{i}\right)}\left(-\frac{10}{3}-\varepsilon+\frac{d\left(u_{i}\right)}{2}\right) \geqslant f\left(\nu\left(u_{i}\right)\right)
$$

because the function $d \mapsto 1-\frac{1}{d-\nu\left(u_{i}\right)}\left(\frac{20}{3}+2 \varepsilon-\nu\left(u_{i}\right)\right)$ is non decreasing on $\left[8,+\infty\left[\right.\right.$ as $\nu\left(u_{i}\right) \leqslant$ $6 \leqslant \frac{20}{3}+2 \varepsilon$. Hence the charge sent by $v$ to $u_{i}$ is at least $f\left(\nu\left(u_{i}\right)\right)$. By hypothesis we have $\nu_{N}(v)=\nu\left(u_{1}\right)+\nu\left(u_{2}\right) \geqslant 4$. It follows that the total charge sent by $v$ is at least

$$
\begin{aligned}
f\left(\nu\left(u_{1}\right)\right)+f\left(\nu\left(u_{2}\right)\right) & \geqslant 2 f\left(\frac{\nu\left(u_{1}\right)+\nu\left(u_{2}\right)}{2}\right) & & \text { by convexity of } f \\
& \geqslant 2 f(2) & & \text { because } f \text { is non decreasing } \\
& =\frac{4}{9}-\frac{2}{3} \varepsilon . & &
\end{aligned}
$$

Hence

$$
w^{*}(v) \leqslant w(v)-\left(\frac{4}{9}-\frac{2}{3} \varepsilon\right) \leqslant\left(\frac{1}{3}+\varepsilon-\frac{\delta}{4}\right)-\frac{4}{9}+\frac{2}{3} \varepsilon=-\frac{1}{9}+\frac{5}{3} \varepsilon-\frac{\delta}{4}
$$

showing the second part of the statement.
Claim 22. If $C$ is a connected component of $D_{6}$, then $\sum_{v \in V(C)} w^{*}(v) \leqslant 0$.
Proof of claim. If $C$ has a unique vertex $v$, then, by Claim 21, we have $w^{*}(v) \leqslant 0$ as wanted.
If $C$ has two vertices $x$ and $y$, then, again by Claim 21, $w^{*}(x), w^{*}(y) \leqslant 0$, and so $w^{*}(x)+$ $w^{*}(y) \leqslant 0$.

If $C$ is a bidirected path $[x, y, z]$, then, by Claim $16, x$ and $z$ have both neighbourhood valency at least 4 and so by Claim $21 w^{*}(x), w^{*}(z) \leqslant-\frac{1}{9}-\frac{\varepsilon}{6}$. Moreover, $y$ sends at least $\frac{2}{8}\left(-\frac{10}{3}+4-\varepsilon\right)=$ $\frac{1}{6}-\frac{\varepsilon}{4}$ to its neighbour out of $C$. Hence

$$
w^{*}(y) \leqslant w(y)-\left(\frac{1}{6}-\frac{\varepsilon}{4}\right) \leqslant \frac{1}{3}+\varepsilon-\frac{\delta}{3}-\frac{1}{6}+\frac{\varepsilon}{4}=\frac{1}{6}+\frac{5}{4} \varepsilon-\frac{\delta}{3} .
$$

Altogether, we get that

$$
w^{*}(x)+w^{*}(y)+w^{*}(z) \leqslant \frac{1}{6}+\frac{5}{4} \varepsilon-\frac{\delta}{3}+2\left(-\frac{1}{9}-\frac{\varepsilon}{6}\right)=-\frac{1}{18}+\frac{11}{12} \varepsilon-\frac{\delta}{3} \leqslant 0
$$

because $\delta \geqslant 6 \varepsilon$.
Finally, if $C$ is a bidirected star with centre $x$ and three other vertices $y, z, w$, then $w^{*}(x) \leqslant$ $w(x)=\frac{1}{3}+\varepsilon-\frac{\delta}{4}$. Moreover, each of $y, z, w$ has neighbourhood valency at least 4 by Claim 16 and so has final charge at most $-\frac{1}{9}+\frac{5}{3} \varepsilon-\frac{\delta}{4}$ by Claim 21. Hence

$$
w^{*}(x)+w^{*}(y)+w^{*}(z)+w^{*}(w) \leqslant \frac{1}{3}+\varepsilon-\frac{\delta}{4}+3\left(-\frac{1}{9}+\frac{5}{3} \varepsilon-\frac{\delta}{4}\right) \leqslant 6 \varepsilon-\delta \leqslant 0
$$

because $\delta \geqslant 6 \varepsilon$.
As a consequence of these last claims, we have $\rho(D) \leqslant \sum_{v \in V(D)} w(v)=\sum_{v \in V(D)} w^{*}(v) \leqslant$ $0 \leqslant 1$, a contradiction. This proves Theorem 8 .

## 4 An upper bound on $o_{k}(n)$

In this section, we show that, for every fixed $k$, there are infinitely many values of $n$ such that $o_{k}(n) \leqslant\left(2 k-\frac{7}{2}\right) n$. The proof is strongly based on the proof of [2], Theorem 4.4], which shows $o_{k}(n) \leqslant(2 k-3) n$ for every $k, n$ (with $n$ large enough). For $k=4$, the construction implies in particular that there is a 4 -dicritical oriented graph with 76 vertices and 330 arcs, and there are infinitely many 4 -dicritical oriented graphs with $m / n \leqslant 9 / 2$.

Proposition 26. Let $k \geqslant 3$ be an integer. For infinitely many values of $n \in \mathbb{N}$, there exists $a$ $k$-dicritical oriented graph $\vec{G}_{k}$ on $n$ vertices with at most $\left(2 k-\frac{7}{2}\right) n$ arcs.

Proof. Let us fix $n_{0} \in \mathbb{N}$. We will show, by induction on $k$, that there exists a $k$-dicritical oriented graph $\vec{G}_{k}$ on $n$ vertices with at most $\left(2 k-\frac{7}{2}\right) n$ arcs, such that $n \geqslant n_{0}$.

When $k=3$, the result is known ([2, Corollary 4.3]). We briefly describe the construction for completeness. Start from any orientation of an odd cycle on $2 n_{0}+1$ vertices. Then for each arc $x y$ in this orientation, add a directed triangle $\vec{C}_{3}$ and every arc from $y$ to $V\left(\vec{C}_{3}\right)$ and every arc from $V\left(\vec{C}_{3}\right)$ to $x$ (see Figure 5). This gadget forces $x$ and $y$ to have different colours in every 2 -dicolouring. Since we started from an orientation of an odd cycle, the result is a 3-dicritical oriented graph on $4\left(2 n_{0}+1\right)$ vertices and $10\left(2 n_{0}+1\right)$ arcs.

Let us fix $k \geqslant 4$ and assume that there exists such a $(k-1)$-dicritical oriented graph $\vec{G}_{k-1}$ on $n_{k-1} \geqslant n_{0}$ vertices with $m_{k-1} \leqslant\left(2(k-1)-\frac{7}{2}\right) n_{k-1}$ arcs. We start from any tournament $T$ on $k$ vertices. Then we add, for each arc $x y$ of $T$, a copy $\vec{G}_{k-1}^{x y}$ of $\vec{G}_{k-1}$, all arcs from $y$ to $\vec{G}_{k-1}^{x y}$ and all arcs from $\vec{G}_{k-1}^{x y}$ to $x$. Figure 6 illustrates a possible construction of $\vec{G}_{4}$, where $T$ is the transitive tournament on 4 vertices.

Let $\vec{G}_{k}$ be the resulting oriented graph. By construction, $n_{k}=\left|V\left(\vec{G}_{k}\right)\right|$ and $m_{k}=\left|A\left(\vec{G}_{k}\right)\right|$


Figure 5: A 3-dicritical oriented graph with $\frac{5}{2} n$ arcs.


Figure 6: A 4-dicritical oriented graph with at most $\frac{9}{2} n$ arcs.
satisfy:

$$
\begin{aligned}
n_{k} & =k+\binom{k}{2} n_{k-1} \\
m_{k} & =\binom{k}{2}+\binom{k}{2} \times 2 \times n_{k-1}+\binom{k}{2} \times m_{k-1} \\
& \leqslant\binom{ k}{2}+\binom{k}{2}\left(2+2(k-1)-\frac{7}{2}\right) n_{k-1} \\
& =\binom{k}{2}+\binom{k}{2}\left(2 k-\frac{7}{2}\right) n_{k-1} \\
& =\binom{k}{2}+\left(2 k-\frac{7}{2}\right)\left(n_{k}-k\right) \\
& \leqslant\left(2 k-\frac{7}{2}\right) n_{k}
\end{aligned}
$$

where in the last inequality we used $k\left(2 k-\frac{7}{2}\right) \geqslant\binom{ k}{2}$, which holds when $k \geqslant 2$. We will now prove that $\vec{G}_{k}$ is indeed $k$-dicritical.

We first prove that $\vec{\chi}\left(\vec{G}_{k}\right)=k$. Assume that there exists a $(k-1)$-dicolouring $\alpha$ of $\vec{G}_{k}$. Then there exist $x, y \in V(T)$ such that $\alpha(x)=\alpha(y)$. Since $\vec{\chi}\left(\vec{G}_{k-1}\right)=k-1$, there exists
$z \in V\left(\vec{G}_{k-1}^{x y}\right)$ such that $\alpha(z)=\alpha(x)$. But then $(x, y, z, x)$ is a monochromatic directed triangle in $\alpha$ : a contradiction.

Let us now prove that $\vec{\chi}\left(\vec{G}_{k} \backslash\{u v\}\right) \leqslant k-1$ for every arc $u v \in A\left(\vec{G}_{k}\right)$. This implies immediately that $\vec{\chi}\left(\vec{G}_{k}=k\right.$ and shows the result.

Consider first an arc $u v$ in $A(T)$. We colour each copy $\vec{G}_{k-1}^{x y}$ of $\vec{G}_{k-1}$ with a $(k-1)$-dicolouring of $\vec{G}_{k-1}$. We then choose a distinct colour for every vertex in $T$, except $u$ and $v$ which receive the same colour. This results in a $(k-1)$-dicolouring of $\vec{G}_{k} \backslash\{u v\}$.

Consider now an arc $u v$ of $\vec{G}_{k-1}^{x y}$ for some $x y \in A(T)$. Because $\vec{G}_{k-1}$ is $(k-1)$-dicritical, there exists a $(k-2)$-dicolouring $\xi$ of $\vec{G}_{k-1}^{x y} \backslash\{u v\}$. Hence we colour $\vec{G}_{k-1}^{x y} \backslash\{u v\}$ with $\xi$, every other copy of $\vec{G}_{k-1}$ a $(k-1)$-dicolouring of $\vec{G}_{k-1}$, and we choose a distinct colour for every vertex in $T$, except $x$ and $y$ which both receive colour $k-1$. This results in a $(k-1)$-dicolouring of $\vec{G}_{k} \backslash\{u v\}$.

Consider finally an arc $u v$ arc from $u \in V(T)$ to $v \in V\left(\vec{G}_{k-1}^{u y}\right)$ (the case of $u \in V\left(\vec{G}_{k-1}^{x v}\right)$ and $v \in V(T)$ being symmetric). Because $\vec{G}_{k-1}$ is dicritical, there exists a $(k-1)$-dicolouring $\gamma$ of $\vec{G}_{k-1}^{u y}$ in which $v$ is the only vertex coloured $k-1$. Hence, we colour $\vec{G}_{k-1}^{u y}$ with $\gamma$, every other copy of $\vec{G}_{k-1}$ with a $(k-1)$-dicolouring of $\vec{G}_{k-1}$, and we choose a distinct colour for every vertex in $T$, except $u$ and $y$ which both receive colour $k-1$. This results in a $(k-1)$-dicolouring of $\vec{G}_{k} \backslash\{u v\}$.

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