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On the minimum number of arcs in 4-dicritical oriented graphs *

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Abstract

The dichromatic number $\vec{\chi}(D)$ of a digraph D is the minimum number of colours needed to colour the vertices of a digraph such that each colour class induces an acyclic subdigraph. A digraph D is k -dicritical if $\vec{\chi}(D) = k$ and each proper subdigraph H of D satisfies $\vec{\chi}(H) < k$.

For integers k and n , we define $d_k(n)$ (respectively $o_k(n)$) as the minimum number of arcs possible in a k -dicritical digraph (respectively oriented graph). Kostochka and Stiebitz have shown [10] that $d_4(n) \geq \frac{10}{3}n - \frac{4}{3}$. They also conjectured that there is a constant c such that $o_k(n) \geq cd_k(n)$ for $k \geq 3$ and n large enough. This conjecture is known to be true for $k = 3$ (Aboulker et al. [2]).

In this work, we prove that every 4-dicritical oriented graph on n vertices has at least $(\frac{10}{3} + \frac{1}{51})n - 1$ arcs, showing the conjecture for $k = 4$. We also characterise exactly the k -dicritical digraphs on n vertices with exactly $\frac{10}{3}n - \frac{4}{3}$ arcs.

1 Introduction

Let G be a graph. We denote by $V(G)$ its vertex set and by $E(G)$ its edge set; we set $n(G) = |V(G)|$ and $m(G) = |E(G)|$. A **k -colouring** of G is a function $\varphi : V(G) \rightarrow [k]$. It is **proper** if for every edge $uv \in E(G)$, $\varphi(u) \neq \varphi(v)$. The smallest integer k such that G has a proper k -colouring is the **chromatic number**, and is denoted by $\chi(G)$. Since χ is non decreasing with respect to the subgraph relation, it is natural to consider the minimal graphs (for this relation) which are not $(k - 1)$ -colourable. Following this idea, Dirac defined **k -critical** graphs as the graphs G with $\chi(G) = k$ and $\chi(H) < k$ for every proper subgraph H of G . A first property of k -critical graph is

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that their minimum degree is at least $k - 1$. Indeed, if a vertex v has degree at most $k - 2$, then a $(k - 1)$ -colouring of $G - v$ can be easily extended to G , contradicting the fact that $\chi(G) = k$. As a consequence, the number of edges in a k -critical graph is at least $\frac{k-1}{2}n$. This bound is tight for complete graphs and odd cycles, but Dirac [4] proved an inequality of the form $m \geq \frac{k-1+\varepsilon_k}{2}n - c_k$ for every n -vertex k -critical graph with m edges, for some c_k and $\varepsilon_k > 0$. This shows that, for n sufficiently large, the average degree of a k -critical graph is at least $k - 1 + \varepsilon_k$. This initiated the quest after the best lower bound on the number of edges in n -vertex k -critical graphs. This problem was almost completely solved by Kostochka and Yancey in 2014 [11].

Theorem 1 (Kostochka and Yancey [11]).

Every k -critical graph on n vertices has at least $\frac{1}{2}(k - \frac{2}{k-1})n - \frac{k(k-3)}{2(k-1)}$ edges. For every k , this bound is tight for infinitely many values of n .

Kostochka and Yancey [12] also characterised k -critical graphs for which this inequality is an equality, and all of them contain a copy of K_{k-2} , the complete graph on $k - 2$ vertices. This motivated the following conjecture of Postle [13].

Conjecture 2 (Postle [13]). For every integer $k \geq 4$, there exists $\varepsilon_k > 0$ such that every k -critical K_{k-2} -free graph G on n vertices has at least $\frac{1}{2}(k - \frac{2}{k-1} + \varepsilon_k)n - \frac{k(k-3)}{2(k-1)}$ edges.

For $k = 4$, the conjecture trivially holds as there is no K_2 -free 4-critical graph. Moreover, this conjecture has been confirmed for $k = 5$ by Postle [13], for $k = 6$ by Gao and Postle [6], and for $k \geq 33$ by Gould, Larsen, and Postle [7].

Let D be a digraph. We denote by $V(D)$ its vertex set and by $A(D)$ its arc set; we set $n(D) = |V(D)|$ and $m(D) = |E(D)|$. A **k -colouring** of D is a function $\varphi : V(D) \rightarrow [k]$. It is a **k -dicolouring** if every directed cycle C in D is not monochromatic for φ (that is $|\varphi(V(C))| > 1$). Equivalently, it is a k -dicolouring if every colour class induces an acyclic subdigraph. The smallest integer k such that D has a k -dicolouring is the **dichromatic number** of D and is denoted by $\vec{\chi}(D)$.

A **digon** in D is a pair of opposite arcs between two vertices. Such a pair of arcs $\{uv, vu\}$ is denoted by $[u, v]$. We say that D is a **bidirected graph** if every pair of adjacent vertices forms a digon. In this case, D can be viewed as obtained from an undirected graph G by replacing each edge $\{u, v\}$ of G by the digon $[u, v]$. We say that D is a bidirected G , and we denote it by \overleftrightarrow{G} . Observe that $\chi(G) = \vec{\chi}(\overleftrightarrow{G})$. Thus every statement on proper colouring of undirected graphs can be seen as a statement on dicolouring of bidirected graphs.

Exactly as in the undirected case, one can define **k -dicritical** digraphs to be digraphs D with $\vec{\chi}(D) = k$ and $\vec{\chi}(H) < k$ for every proper subdigraph H of D . It is easy to check that if G is a k -critical graph, then \overleftrightarrow{G} is k -dicritical. Kostochka and Stiebitz [10] conjectured that the k -dicritical digraphs with the minimum number of arcs are bidirected graphs. Thus they conjectured the following generalisation of Theorem 1 to digraphs.

Conjecture 3 (Kostochka and Stiebitz [10]). Let $k \geq 2$. Every k -dicritical digraph on n vertices has at least $(k - \frac{2}{k-1})n - \frac{k(k-3)}{k-1}$ arcs. Moreover, equality holds only if D is bidirected.

In the case $k = 2$, this conjecture is easy and weak as it states that a 2-dicritical digraph on n vertices has at least two arcs, while, for all $n \geq 2$, the unique 2-dicritical digraph of order n is the directed n -cycle which has n arcs. The case $k = 3$ of the conjecture has been confirmed by Kostochka and Stiebitz [10]. Using a Brooks-type result for digraphs due to Harutyunyan and Mohar [8], they proved the following: if D is a 3-dicritical digraph of order $n \geq 3$, then $m(D) \geq 2n$ and equality holds if and only if n is odd and D is a bidirected odd cycle. The conjecture has also been proved for $k = 4$ by Kostochka and Stiebitz [10]. However, the conjecture is open for every $k \geq 5$. Recently, this problem has been investigated by Aboulker and Vermande [3] who proved the weaker bound $(k - \frac{1}{2} - \frac{2}{k-1})n - \frac{k(k-3)}{k-1}$ for the number of arcs in an n -vertex k -dicritical digraph.

For integers k and n , let $d_k(n)$ denote the minimum number of arcs in a k -dicritical digraph of order n . By the above observations, $d_2(n) = n$ for all $n \geq 2$, and $d_3(n) \geq 2n$ for all possible n , and equality holds if and only if n is odd and $n \geq 3$. Moreover, if n is even then $d_3(n) = 2n + 1$ (see [2]).

Kostochka and Stiebitz [9] showed that if a k -critical graph G is triangle-free (that is has no cycle of length 3), then $m(G)/n(G) \geq k - o(k)$ as $k \rightarrow +\infty$. Informally, this means that the minimum average degree of a k -critical triangle-free graph is (asymptotically) twice the minimum average degree of a k -critical graph. Similarly to this undirected case, it is expected that the minimum number of arcs in a k -dicritical digraph of order n is larger than $d_k(n)$ if we impose this digraph to have no short directed cycles, and in particular if the digraph is an **oriented graph**, that is a digraph with no digon. Let $o_k(n)$ denote the minimum number of arcs in a k -dicritical oriented graph of order n (with the convention $o_k(n) = +\infty$ if there is no k -dicritical oriented graph of order n). Clearly $o_k(n) \geq d_k(n)$.

Conjecture 4 (Kostochka and Stiebitz [10]). For any $k \geq 3$, there is a constant $\alpha_k > 0$ such that $o_k(n) > (1 + \alpha_k)d_k(n)$ for n sufficiently large.

For $k = 3$, this conjecture has been recently confirmed by Aboulker, Bellitto, Havet, and Rambaud [2] who proved that $o_3(n) \geq (2 + \frac{1}{3})n + \frac{2}{3}$.

In view of Conjecture 2, Conjecture 4 can be generalized to $\overleftrightarrow{K}_{k-2}$ -free digraphs.

Conjecture 5. For any $k \geq 4$, there is a constant $\beta_k > 0$ such that every k -dicritical $\overleftrightarrow{K}_{k-2}$ -free digraph D on n vertices has at least $(1 + \beta_k)d_k(n)$ arcs.

Together with Conjecture 3, this conjecture would imply the following generalisation of Conjecture 2.

Conjecture 6. For every integer $k \geq 4$, there exists $\varepsilon_k > 0$ such that every k -dicritical $\overleftrightarrow{K}_{k-2}$ -free digraph D on n vertices has at least $(k - \frac{2}{k-1} + \varepsilon_k)n - \frac{k(k-3)}{k-1}$ arcs.

A \overleftrightarrow{K}_2 -free digraph is an oriented graph, and there are infinitely many 4-dicritical oriented graphs. Thus, while Conjecture 2 holds vacuously for $k = 4$, this is not the case for Conjecture 6. In this paper, we prove that Conjectures 4, 5, and 6 hold for $k = 4$.

Theorem 7. *If \vec{G} is a 4-dicritical oriented graph, then*

$$m(\vec{G}) \geq \left(\frac{10}{3} + \frac{1}{51} \right) n(\vec{G}) - 1.$$

To prove Theorem 7, we use an approach similar to the proof of the case $k = 5$ of Conjecture 2 by Postle [13]. This proof is based on the potential method, which was first popularised by Kostochka and Yancey [11] when they proved Theorem 1. The idea is to prove a more general result on every 4-dicritical digraphs that takes into account the digons.

With a slight abuse, we call **digon** a subdigraph isomorphic to $\overleftrightarrow{K_2}$, the bidirected complete graph on two vertices. We also call **bidirected triangle** a subdigraph isomorphic to $\overleftrightarrow{K_3}$, the bidirected complete graph on three vertices. A **packing** of digons and bidirected triangles is a set of vertex-disjoint digons and bidirected triangles. To take into account the digons, we define a parameter $T(D)$ as follows.

$$T(D) = \max\{d + 2t \mid \text{there exists a packing of } d \text{ digons and } t \text{ bidirected triangles}\}$$

Clearly, $T(D) = 0$ if and only if D is an oriented graph.

Let ε, δ be fixed non-negative real numbers. We define the **potential** (with respect to ε and δ) of a digraph D to be

$$\rho(D) = \left(\frac{10}{3} + \varepsilon \right) n(D) - m(D) - \delta T(D).$$

Thus Theorem 7 can be rephrased as follows.

Theorem 7. *Set $\varepsilon = \frac{1}{51}$ and $\delta = 6\varepsilon = \frac{2}{17}$. If \vec{G} is a 4-dicritical oriented graph, then $\rho(\vec{G}) \leq 1$.*

In fact, we prove a more general statement which holds for every 4-dicritical digraph (with or without digons), except for some exceptions called the **4-Ore digraphs**. Those digraphs, which are formally defined in Section 2, are the bidirected graphs whose underlying graph is one of the 4-critical graphs reaching equality in Theorem 1. In particular, every 4-Ore digraph D has $\frac{10}{3}n(D) - \frac{4}{3}$ arcs. Moreover, the statement holds for all non-negative constants ε and δ satisfying the following inequalities:

- $\delta \geq 6\varepsilon$;
- $3\delta - \varepsilon \leq \frac{1}{3}$;

Theorem 8. *Let $\varepsilon, \delta \geq 0$ be constants satisfying the aforementioned inequalities. If D is a 4-dicritical digraph with n vertices, then*

- (i) $\rho(D) \leq \frac{4}{3} + \varepsilon n - \delta \frac{2(n-1)}{3}$ if D is 4-Ore, and
- (ii) $\rho(D) \leq 1$ otherwise.

In order to provide some intuition to the reader, let us briefly describe the main ideas of our proof. We will consider a minimum counterexample D to Theorem 8, and show that every subdigraph of D must have large potential. To do so, we need to construct some smaller 4-dicritical digraphs to leverage the minimality of D . These smaller 4-dicritical digraphs will be constructed by identifying some vertices of D . This is why, in the definition of the potential, we consider $T(D)$ instead of the number of digons: when identifying a set of vertices, the number of digons may be arbitrary larger in the resulting digraph, but $T(D)$ increases at most by 1. Using the fact that every subdigraph of D has large potential, we will prove that some subdigraphs are forbidden in D . Using this, we get the final contradiction by a discharging argument.

In addition to Theorem 7, Theorem 8 has also the following consequence when we take $\varepsilon = \delta = 0$.

Corollary 9. *If D is a 4-dicritical digraph, then $m(D) \geq \frac{10}{3}n(D) - \frac{4}{3}$. Moreover, equality holds if and only if D is 4-Ore, otherwise $m(D) \geq \frac{10}{3}n(D) - 1$.*

This is a slight improvement on a result of Kostochka and Stiebitz [10] who proved the inequality $m(D) \geq \frac{10}{3}n(D) - \frac{4}{3}$ without characterising the equality case.

Another interesting consequence of our result is the following bound on the number of vertices in a 4-dicritical oriented graph embedded on a fixed surface. Since a graph on n vertices embedded on a surface of Euler characteristic c has at most $3n - 3c$ edges, we immediately deduce the following from Theorem 7.

Corollary 10. *If \vec{G} is a 4-dicritical oriented graph embedded on a surface of Euler characteristic c , then $n(\vec{G}) \leq \frac{17}{6}(1 - 3c)$.*

The previous best upper bound was $n(\vec{G}) \leq 4 - 9c$ [10].

In Section 2 we prove some first preliminary results on 4-Ore digraphs, before proving Theorem 8 in Section 3. In Section 4, we show that $o_k(n) \leq (2k - \frac{7}{2})n$ for every fixed k and infinitely many values of n . The proof is strongly based on the proof of [2, Theorem 4.4], which shows $o_k(n) \leq (2k - 3)n$ for every fixed k, n (with n large enough). For $k = 4$, the construction implies in particular that there is a 4-dicritical oriented graph with 76 vertices and 330 arcs, and there are infinitely many 4-dicritical oriented graphs with $m/n \leq 9/2$.

2 The 4-Ore digraphs and their properties

We start with a few notations. We denote by $\llbracket x_1, \dots, x_n \rrbracket$ the bidirected path with vertex set $\{x_1, \dots, x_n\}$ in this order. If $x_1 = x_n$, $\llbracket x_1, \dots, x_n \rrbracket$ denotes the bidirected cycle of order n with cyclic order x_1, \dots, x_n . If D is a digraph, for any $X \subseteq V(D)$, $D - X$ is the subdigraph induced by $V(D) \setminus X$. We abbreviate $D - \{x\}$ into $D - x$. Moreover, for any $F \subseteq V(D) \times V(D)$, $D \setminus F$ is the subdigraph $(V(D), A(D) \setminus F)$ and $D \cup F$ is the digraph $(V(D), A(D) \cup F)$.

Let D_1, D_2 be two bidirected graphs, $[x, y] \subseteq A(D_1)$, and $z \in V(D_2)$. An **Ore-composition** D of D_1 and D_2 with **replaced digon** $[x, y]$ and **split vertex** z is a digraph obtained by removing $[x, y]$ of D_1 and z of D_2 , and adding the set of arcs $\{xz_1 \mid zz_1 \in A(D_2) \text{ and } z_1 \in Z_1\}, \{z_1x \mid$

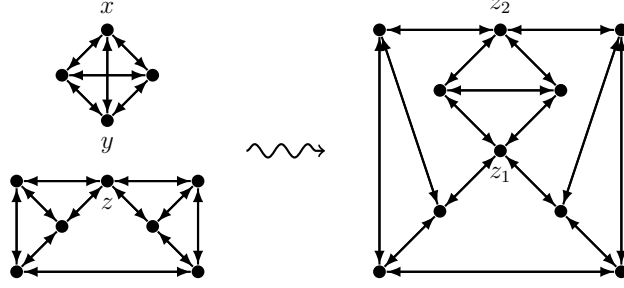


Figure 1: An example of a 4-Ore digraph obtained by an Ore-composition of two smaller 4-Ore digraphs, with replaced digon $[x, y]$ and split vertex z .

$z_1 z \in A(D_2)$ and $z_1 \in Z_1$, $\{yz_2 \mid zz_2 \in A(D_2) \text{ and } z_2 \in Z_2\}$, $\{z_2 y \mid z_2 z \in A(D_2) \text{ and } z_2 \in Z_2\}$, where (Z_1, Z_2) is a partition of $N_{D_2}(z)$ into non-empty sets. We call D_1 the **digon side** and D_2 the **split side** of the Ore-composition. The class of the **4-Ore digraphs** is the smallest class containing \overleftrightarrow{K}_4 which is stable under Ore-composition. See Figure 1 for an example of a 4-Ore digraph. Observe that all the 4-Ore-digraphs are bidirected.

Proposition 11 (Dirac [5], see also [12]). *4-Ore digraphs are 4-dicritical.*

Proof. One can easily show that a bidirected digraph is 4-dicritical if and only if its undirected underlying graph is 4-critical. Then the result follows from the undirected analogous proved by [5]. \square

Lemma 12. *Let D be a 4-dicritical bidirected digraph and $v \in V(D)$. Let (N_1^+, N_2^+) and (N_1^-, N_2^-) be two partitions of $N(v)$. Consider D' the digraph with vertex set $V(D) \setminus \{v\} \cup \{v_1, v_2\}$ with $N^+(v_i) = N_i^+$, $N^-(v_i) = N_i^-$ for $i = 1, 2$ and $D' \langle V(D) \setminus \{v\} \rangle = D - v$. Then D' has a 3-dicolouring with v_1 and v_2 coloured the same except if $N_1^+ = N_1^-$ (that is D' is bidirected).*

Proof. Suppose that D' is not bidirected. Consider a vertex $u \in N_D(v)$ such that $v_1 u, uv_2 \in A(D')$ or $v_2 u, uv_1 \in A(D')$. Without loss of generality, suppose $v_1 u, uv_2 \in A(D')$. As D is 4-dicritical, $D \setminus [u, v]$ has a proper 3-dicolouring φ . We set $\varphi(v_1) = \varphi(v_2) = \varphi(v)$ and claim that φ is a 3-dicolouring of D' . To show that, observe that φ is a proper 3-colouring of the underlying undirected graph of $D' \setminus \{v_1 u, uv_2\}$, and so φ is a 3-dicolouring of D' as wanted. \square

Lemma 13. *Let D be a digraph. If v is a vertex of D , then $T(D - v) \geq T(D) - 1$.*

Proof. Let M be a packing of d digons and t bidirected triangles in H such that $d + 2t = T(D)$. If v belongs to a digon $[u, v]$ in M , then $M \setminus \{[u, v]\}$ witnesses the fact that $T(D - v) \geq T(D) - 1$. If v belongs to a bidirected triangle $\llbracket u, v, w, u \rrbracket$, then $M \setminus \{\llbracket u, v, w, u \rrbracket\} \cup [u, v]$ witnesses the fact that $T(D - v) \geq T(D) - 2 + 1$. Otherwise $T(D - v) \geq T(D)$. \square

Lemma 14. *If D_1, D_2 are two digraphs, and D is an Ore-composition of D_1 and D_2 , then $T(D) \geq T(D_1) + T(D_2) - 2$. Moreover, if D_1 or D_2 is isomorphic to \overleftrightarrow{K}_4 , then $T(D) \geq T(D_1) + T(D_2) - 1$.*

Proof. Let D be the Ore-composition of D_1 (the digon side with replaced digon $[x, y]$) and D_2 (the split side with split vertex z). One can easily see that $T(D) \geq T(D_1 - x) + T(D - z) \geq T(D_1) + T(D_2) - 2$ by Lemma 13. Moreover, if D_1 (resp. D_2) is a copy of \overleftrightarrow{K}_4 , then $T(D_1 - x) = 2 = T(D_1)$ (resp. $T(D_2 - z) = 2 = T(D_2)$) and therefore $T(D) \geq T(D_1) + T(D_2) - 1$. \square

Lemma 15. *If D is 4-Ore, then $T(D) \geq \frac{2}{3}(n(D) - 1)$.*

Proof. If D is \overleftrightarrow{K}_4 , then the result is clear. Suppose now that D is an Ore-composition of D_1 and D_2 . Then $n(D) = n(D_1) + n(D_2) - 1$ and, by Lemma 14, $T(D) \geq T(D_1) + T(D_2) - 2$. By induction, $T(D_1) \geq \frac{2}{3}(n(D_1) - 1)$ and $T(D_2) \geq \frac{2}{3}(n(D_2) - 1)$, and so $T(D) \geq \frac{2}{3}(n(D_1) + n(D_2) - 1 - 1) = \frac{2}{3}(n(D) - 1)$. \square

Let D be a digraph. A **diamond** in D is a subdigraph isomorphic to \overleftrightarrow{K}_4 minus a digon $[u, v]$, with vertices different from u and v having degree 6 in D . An **emerald** in D is a subdigraph isomorphic to \overleftrightarrow{K}_3 whose vertices have degree 6 in D .

Let R be an induced subdigraph of D with $n(R) < n(D)$. The **boundary** of R in D , denoted by $\partial_D(R)$, or simply $\partial(R)$ when D is clear from the context, is the set of vertices of R having a neighbour in $V(D) \setminus R$. We say that R is **Ore-collapsible** if the boundary of R contains exactly two vertices u and v and $R \cup [u, v]$ is 4-Ore.

Lemma 16. *If D is 4-Ore and $v \in V(D)$, then there exists either an Ore-collapsible subdigraph of D disjoint from v or an emerald of D disjoint from v .*

Proof. If D is a copy of \overleftrightarrow{K}_4 , then $D - v$ is an emerald. Otherwise, D is the Ore-composition of two 4-Ore digraphs: D_1 the digon side with replaced digon $[x, y]$, and D_2 the split side with split vertex z . If $v \in V(D_2 - z)$, then $D_1 \setminus \{x, y\}$ is an Ore-collapsible subdigraph with boundary $\{y, z\}$. Otherwise $v \in V(D_1)$ and we apply induction on D_2 to find an emerald or an Ore-collapsible subdigraph in D_2 disjoint from z . \square

Lemma 17. *If $D \neq \overleftrightarrow{K}_4$ is 4-Ore and T is a copy of \overleftrightarrow{K}_3 in D , then there exists either an Ore-collapsible subdigraph of D disjoint from T or an emerald of D disjoint from T .*

Proof. As D is not \overleftrightarrow{K}_4 , it is an Ore-composition of two 4-Ore digraphs: D_1 the digon side with replaced digon $[x, y]$, and D_2 the split side with split vertex z . As x and y are non adjacent, we have either $T \subseteq D_1$, $T \subseteq D_2 - z$, or T contains a vertex $w \in \{x, y\}$ and two vertices in $V(D_2 - z)$.

If $T \subseteq D_1$, then by Lemma 16, in D_2 there exists either an Ore-collapsible subdigraph O or an emerald E disjoint from z . In the former case O is an Ore-collapsible subdigraph of D disjoint from T , and in the later one E is an emerald in D disjoint from T .

If $T \subseteq D_2 - z$, then $D_1 \setminus \{x, y\}$ is an Ore-collapsible subdigraph disjoint from T .

Assume now that T contains a vertex $w \in \{x, y\}$ and two vertices in $V(D_2 - z)$. Without loss of generality, we may assume that $y \notin T$. Let z_1 and z_2 be the two vertices of T disjoint from w . Then $\{z, z_1, z_2\}$ induces a bidirected triangle T' in D_2 . If $D_2 \neq \overleftrightarrow{K}_4$, then by induction in D_2 , there exists either an Ore-collapsible subdigraph O or an emerald E disjoint from T' . In the former case

O is an Ore-collapsible subdigraph of D disjoint from T , and in the later one E is an emerald in D disjoint from T .

Henceforth we may assume that $D_2 = \overleftrightarrow{K}_4$. This implies that y has exactly one neighbour in $D_2 - z$ and so its degree is the same in D_1 and D . By Lemma 16, in D_1 there exists either an Ore-collapsible subdigraph O or an emerald E disjoint from x . In the former case O is an Ore-collapsible subdigraph of D disjoint from T , and in the later one E is an emerald in D disjoint from T even if $y \in V(E)$ because y has the same degree in D_1 and D . \square

Lemma 18. *If R is an Ore-collapsible induced subdigraph of a 4-Ore digraph D , then there exists a diamond or an emerald of D whose vertices lie in $V(R)$.*

Proof. Let D be a digraph. Let R be a minimal counterexample to this lemma, and let $\partial(R) = \{u, v\}$ and $H = D\langle R \rangle \cup [u, v]$. If $H = \overleftrightarrow{K}_4$, then R is a diamond in D . Suppose now that H is the Ore-composition of two 4-Ore digraphs H_1 (the digon side with replaced digon $[x, y]$) and H_2 (the split side with split vertex z). If $\{u, v\} \not\subset V(H_2)$, then by Lemma 16 there exists an Ore-collapsible subdigraph in H_2 disjoint from z . As it is smaller than H , it contains an emerald or a diamond as desired, a contradiction.

Now assume that $\{u, v\} \subset V(H_2)$, then H_1 is an Ore-collapsible subdigraph of D smaller than H , and by induction, H_1 contains a diamond or an emerald in D . \square

Lemma 19. *If D is a 4-Ore digraph and v is a vertex in D , then D contains a diamond or an emerald disjoint from v .*

Proof. Follows from Lemmas 16 and 18. \square

Lemma 20. *If D is a 4-Ore digraph and T is a bidirected triangle in D , then either $D = \overleftrightarrow{K}_4$ or D contains a diamond or an emerald disjoint from T .*

Proof. Follows from Lemmas 17 and 18. \square

The following theorem was formulated for undirected graphs, but by replacing every edge by a digon, it can be restated as follows:

Theorem 21 (Kostochka and Yancey [12], Theorem 6). *Let D be a 4-dicritical bidirected digraph. If $\frac{10}{3}n(D) - m(D) > 1$, then D is 4-Ore and $\frac{10}{3}n(D) - m(D) = \frac{4}{3}$.*

Lemma 22. *If D is a 4-Ore digraph with n vertices, then $\rho(D) \leq \frac{4}{3} + \varepsilon n - \delta \frac{2(n-1)}{3}$.*

Proof. Follows from Theorem 21 and Lemma 15. \square

Lemma 23 (Kostochka and Yancey [12], Claim 16). *Let D be a 4-Ore digraph. If $R \subseteq D$ and $0 < n(R) < n(D)$, then $\frac{10}{3}n(R) - m(R) \geq \frac{10}{3}$.*

Lemma 24. *Let D be a 4-Ore digraph obtained from a copy J of \overleftrightarrow{K}_4 by successive Ore-compositions with 4-Ore digraphs, vertices and digons in J being always on the digon side. Let $[u, v]$ be a digon in $D\langle V(J) \rangle$. For every 3-dicolouring φ of $D \setminus [u, v]$, vertices in $V(J)$ receive distinct colours except u and v .*

Proof. We proceed by induction on $n(D)$, the result holding trivially when D is \overleftrightarrow{K}_4 . Now assume that D is the Ore-composition of D_1 , the digon side containing J , and D_2 , with D_1 and D_2 being 4-Ore digraphs. Let $[x, y] \subseteq A(D_1)$ be the replaced digon in this Ore-composition, and let $z \in V(D_2)$ be the split vertex. Let φ be a 3-dicolouring of $D \setminus [u, v]$. Then φ induces a 3-dicolouring of $D \setminus (V(D_2 - z) \cup \{x, y\})$. Necessarily $\varphi(x) \neq \varphi(y)$, for otherwise φ_2 defined by $\varphi_2(v) = \varphi(v)$ if $v \in V(D_2 - z)$ and $\varphi_2(z) = \varphi(x)$ is a 3-dicolouring of D_2 , contradicting the fact that 4-Ore digraphs have dichromatic number 4 by Lemma 11. Hence φ induces a 3-dicolouring of $D_1 \setminus [u, v]$. So, by the induction hypothesis, vertices in $V(J)$ have distinct colours in φ , except u and v . \square

Lemma 25. *Let D be a 4-Ore digraph obtained from a copy J of \overleftrightarrow{K}_4 by successive Ore-compositions with 4-Ore digraphs, vertices and digons in J being always on the digon side. Let v be a vertex in $V(J)$. For every 3-dicolouring φ of $D - v$, vertices in J receive distinct colours.*

Proof. We proceed by induction on $n(D)$, the result holding trivially when D is \overleftrightarrow{K}_4 . Now assume that D is the Ore-composition of D_1 , the digon side containing J , and D_2 , with D_1 and D_2 being 4-Ore digraphs. Let $[x, y] \subseteq A(D_1)$ be the replaced digon in this Ore-composition, and let $z \in V(D_2)$ be the split vertex. Let φ be a 3-dicolouring of $D - v$. If $v \in \{x, y\}$, then φ is a 3-dicolouring of $D_1 - v$ and the result follows by induction. Now assume $v \notin \{x, y\}$. Then φ induces a 3-dicolouring of $D \setminus (V(D_2 - z) \cup \{x, y\})$. Necessarily $\varphi(x) \neq \varphi(y)$, for otherwise φ_2 defined by $\varphi_2(v) = \varphi(v)$ if $v \in V(D_2 - z)$ and $\varphi_2(z) = \varphi(x)$ is a 3-dicolouring of D_2 , contradicting the fact that 4-Ore digraphs have dichromatic number 4 by Lemma 11. Hence φ induces a 3-dicolouring of $D_1 - v$. So, by the induction hypothesis, vertices in $V(J)$ have distinct colours in φ . \square

3 Proof of Theorem 8

Let D be a 4-dicritical digraph, R be an induced subdigraph of D with $4 \leq n(R) < n(D)$ and φ a 3-dicolouring of R . The φ -**identification** of R in D , denoted by $D_\varphi(R)$ is the digraph obtained from D by identifying for each $i \in [3]$ the vertices coloured i in $V(R)$ to a vertex x_i , adding the digons $[x_i, x_j]$ for all $1 \leq i < j \leq 3$. Observe that $D_\varphi(R)$ is not 3-dicolourable. Indeed, assume for a contradiction that $D_\varphi(R)$ has a 3-dicolouring φ' . Since $\{x_1, x_2, x_3\}$ induces a \overleftrightarrow{K}_3 , we may assume without loss of generality that $\varphi'(x_i) = i$ for $i \in [3]$. Consider the 3-colouring φ'' of D defined by $\varphi''(v) = \varphi'(v)$ if $v \notin R$ and $\varphi''(v) = \varphi(v)$ if $v \in R$. One easily checks that φ'' is a 3-dicolouring of D , a contradiction to the fact that $\chi(D) = 4$.

Now let W be a 4-dicritical subdigraph of $D_\varphi(R)$ and $X = \{x_1, x_2, x_3\}$. Then we say that $R' = D \setminus ((V(W) \setminus X) \cup R)$ is the **dicritical extension** of R with **extender** W . We call $X_W = X \cap V(W)$ the **core** of the extension. Note that X_W is not empty, because W is not a subdigraph of D . Thus $1 \leq |X_W| \leq 3$. See Figure 2 for an example of a φ -identification and a dicritical extension.

Let D be a counterexample to Theorem 8 with minimum number of vertices. By Lemma 22, D is not 4-Ore. Thus $\rho(D) > 1$.

Claim 1. *If \tilde{D} is a 4-dicritical digraph with $n(\tilde{D}) < n(D)$, then $\rho(\tilde{D}) \leq \frac{4}{3} + 4\varepsilon - 2\delta$.*

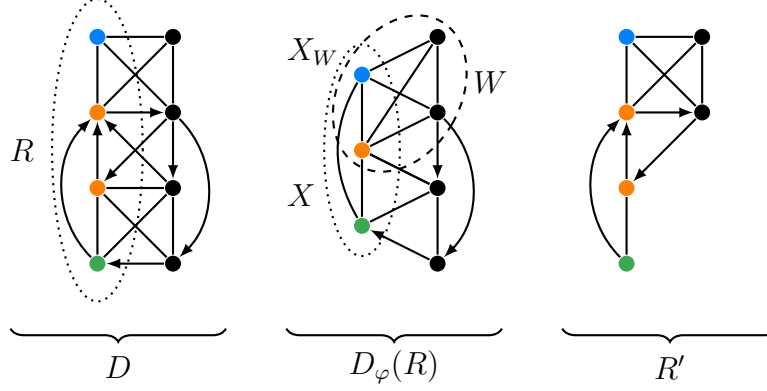


Figure 2: A 4-dicritical digraph D together with an induced subdigraph R of D and φ a 3-dicolouring of R , the φ -identification $D_\varphi(R)$ of R in D and the dicritical extension R' of R with extender W and core X_W . For clarity, the digons are represented by undirected edges.

Proof of claim. If \tilde{D} is not 4-Ore, then $\rho(\tilde{D}) \leq 1$ by minimality of D . Thus $\rho(\tilde{D}) \leq \frac{4}{3} + 4\varepsilon - 2\delta$ because $4\varepsilon - 2\delta \geq \frac{-1}{3}$. Otherwise, by Lemma 22, $\rho(\tilde{D}) \leq \frac{4}{3} + \varepsilon n(\tilde{D}) - \delta \frac{2(n(\tilde{D})-1)}{3} \leq \frac{4}{3} + 4\varepsilon - 2\delta$ because $\delta \geq \frac{3}{2}\varepsilon$ and $n(\tilde{D}) \geq 4$. \diamond

Claim 2. Let R be a subdigraph of D with $4 \leq n(R) < n(D)$. If R' is a dicritical extension of R with extender W and core X_W , then

$$\rho(R') \leq \rho(W) + \rho(R) - \left(\rho(\overleftarrow{K}_{|X_W|}) + \delta \cdot T(\overleftarrow{K}_{|X_W|}) \right) + \delta \cdot (T(W) - T(W - X_W))$$

and in particular

$$\rho(R') \leq \rho(W) + \rho(R) - \frac{10}{3} - \varepsilon + \delta.$$

Proof of claim. We have

- $n(R') = n(W) - |X_W| + n(R)$,
- $m(R') \geq m(W) + m(R) - m(\overleftarrow{K}_{|X_W|})$,
- $T(R') \geq T(W - X_W) + T(R)$

and by summing these inequalities, we get the first result.

Now observe that $T(W) - T(W - X_W) \leq |X_W|$ by Lemma 13, and that the maximum of $-\left(\rho(\overleftarrow{K}_{|X_W|}) + \delta T(\overleftarrow{K}_{|X_W|}) \right) + \delta |X_W|$ is reached when $|X_W| = 1$, in which case it is equal to $-\frac{10}{3} - \varepsilon + \delta$. The second inequality follows. \diamond

Claim 3. If R is a subdigraph of D with $4 \leq n(R) < n(D)$, then $\rho(R) \geq \rho(D) + 2 - 3\varepsilon + \delta > 3 - 3\varepsilon + \delta$.

Proof of claim. We proceed by induction on $n - n(R)$. Let R' be a dicritical extension of R with extender W and core X_W . By Claim 2, we have

$$\rho(R') \leq \rho(W) + \rho(R) - \frac{10}{3} - \varepsilon + \delta.$$

Either $V(R') = V(D)$ and so $\rho(R') \geq \rho(D)$ or $V(R')$ is a proper subset of $V(D)$ and, since R is a proper subdigraph of R' , by induction $\rho(R') \geq \rho(D) + 2 - 3\varepsilon + \delta \geq \rho(D)$. In both cases, $\rho(R') \geq \rho(D)$. Now W is smaller than D so $\rho(W) \leq \frac{4}{3} + 4\varepsilon - 2\delta$ by Claim 1. Thus

$$\rho(D) \leq \rho(R') \leq \frac{4}{3} + 4\varepsilon - 2\delta + \rho(R) - \frac{10}{3} - \varepsilon + \delta.$$

This gives $\rho(R) \geq \rho(D) + 2 - 3\varepsilon + \delta > 3 - 3\varepsilon + \delta$, because $\rho(D) > 1$. \diamond

As a consequence of Claim 3, any subdigraph (proper or not) of size at least 4 has potential at least $\rho(D)$.

We say that an induced subdigraph R of D is **collapsible** if, for every 3-dicolouring φ of R , its dicritical extension R' (with extender W and core X_W) is D , has core of size 1 (i.e. $|X_W| = 1$), and the border $\partial_D(R)$ of R is monochromatic in φ .

Claim 4. *Let R be an induced subdigraph of D and φ a 3-dicolouring of R such that $\partial(R)$ is not monochromatic in φ . If D is a dicritical extension of R dicoloured by φ with extender W and core X_W with $|X_W| = 1$, then*

$$\rho(R) \geq \rho(D) + 3 - 3\varepsilon + \delta.$$

Proof of claim. Assume D is a dicritical extension of R dicoloured by φ with extender W and core X_W with $|X_W| = 1$. Observe that each of the following inequalities holds:

- $n(D) = n(W) - |X_W| + n(R) = n(X) + n(R) - 1$,
- $m(D) \geq m(W) + m(R) - m(\overleftrightarrow{K_{|X_W|}}) + 1 = m(W) + m(R) + 1$ because $\partial_D(R)$ is not monochromatic in φ , and
- $T(D) \geq T(W - X_W) + T(R) \geq T(W) + T(R) - 1$ by Lemma 13.

By Claim 1, we have

$$\rho(D) \leq \rho(W) + \rho(R) - \left(\frac{10}{3} + \varepsilon\right) - 1 + \delta \leq \left(\frac{4}{3} + 4\varepsilon - 2\delta\right) + \rho(R) - \frac{13}{3} - \varepsilon + \delta$$

and so $\rho(R) \geq \rho(D) + 3 - 3\varepsilon + \delta$. \diamond

Claim 5. *If R is a subdigraph of D with $4 \leq n(R) < n(D)$ and R is not collapsible, then $\rho(R) \geq \rho(D) + \frac{8}{3} - \varepsilon - \delta > \frac{11}{3} - \varepsilon - \delta$.*

Proof of claim. Let R' be a dicritical extension of R dicoloured by φ with extender W and core X_W .

(i) If R' is not D , then it has a dicritical extension R'' with extender W' . By (the consequence of) Claim 3, we have $\rho(D) \leq \rho(R'')$; by Claim 2 (applied twice), $\rho(R'') \leq \rho(R) + \rho(W') + \rho(W) + 2(-\frac{10}{3} - \varepsilon + \delta)$; both W and W' are smaller than D , so, by Claim 1, $\rho(W), \rho(W') \leq \frac{4}{3} + 4\varepsilon - 2\delta$. Those three inequalities imply

$$\rho(D) \leq \rho(R'') \leq \rho(R) + 2\left(\frac{4}{3} + 4\varepsilon - 2\delta\right) + 2\left(-\frac{10}{3} - \varepsilon + \delta\right) = \rho(R) - 4 + 6\varepsilon - 2\delta$$

$$\text{and so } \rho(R) \geq \rho(D) + 4 - 6\varepsilon + 2\delta \geq \rho(D) + \frac{8}{3} - \varepsilon - \delta.$$

(ii) If $R' = D$ and $|X_W| = 2$, then $\rho(\overleftarrow{K_{|X_W|}}) + \delta T(\overleftarrow{K_{|X_W|}}) = \frac{14}{3} + 2\varepsilon$, and, by Lemma 13, $T(W) - T(W - X_W) \leq |X_W| = 2$. Thus, by Claim 2,

$$\rho(D) \leq \rho(W) + \rho(R) - \frac{14}{3} - 2\varepsilon + 2\delta$$

Now, since W is smaller than D , $\rho(W) \leq \frac{4}{3} + 4\varepsilon - 2\delta$ by Claim 1. Thus

$$\rho(D) \leq \rho(R) + \frac{4}{3} + 4\varepsilon - 2\delta - \frac{14}{3} - 2\varepsilon + 2\delta = \rho(R) - \frac{10}{3} + 2\varepsilon$$

$$\text{and so } \rho(R) \geq \rho(D) + \frac{10}{3} - 2\varepsilon \geq \rho(D) + \frac{8}{3} - \varepsilon - \delta.$$

(iii) If $R' = D$ and $|X_W| = 3$, then $\rho(\overleftarrow{K_{|X_W|}}) + \delta T(\overleftarrow{K_{|X_W|}}) = 4 + 3\varepsilon$, and, by Lemma 13, $T(W) - T(W - X_W) \leq |X_W| = 3$. Thus, by Claim 2,

$$\rho(D) \leq \rho(W) + \rho(R) - 4 - 3\varepsilon + 3\delta.$$

Now, since W is smaller than D , $\rho(W) \leq \frac{4}{3} + 4\varepsilon - 2\delta$ by Claim 1. Thus

$$\rho(D) \leq \rho(R) + \frac{4}{3} + 4\varepsilon - 2\delta - 4 - 3\varepsilon + 3\delta = \rho(R) - \frac{8}{3} + \varepsilon + \delta$$

$$\text{and so } \rho(R) \geq \rho(D) + \frac{8}{3} - \varepsilon - \delta.$$

(iv) If $R' = D$, $|X_W| = 1$ and $\partial(R)$ is not monochromatic in φ , then, by Claim 4, we have $\rho(R) \geq \rho(D) + 3 - 3\varepsilon + \delta \geq \rho(D) + \frac{8}{3} - \varepsilon - \delta$.

If R is not collapsible, then, by definition, it has a dicritical extension R' satisfying the hypothesis of one of the cases (i)–(iv). In any case, $\rho(R) \geq \rho(D) + \frac{8}{3} - \varepsilon - \delta$. \diamond

Recall that a **k -cutset** in a graph G is a set S of k vertices such that $G - S$ is not connected. A graph is **k -connected** if it has more than k vertices and has no $(k - 1)$ -cutset. A **k -cutset** in a digraph is a k -cutset in its underlying graph, and a digraph is **k -connected** if its underlying graph is k -connected.

Claim 6. D is 2-connected.

Proof of claim. Suppose for contradiction that $\{x\}$ is a 1-cutset in D . Let (A_0, B_0) be a partition of $V(D - x)$ into non-empty sets such that there is no edge between A_0 and B_0 , and set $A = A_0 \cup \{x\}$ and $B = B_0 \cup \{x\}$.

Since D is 4-dicritical, there exist a 3-dicolouring φ_A of $D\langle A \rangle$ and a 3-dicolouring φ_B of $D\langle B \rangle$. Free to swap the colours, we may assume $\varphi_A(x) = \varphi_B(x)$. Let φ be defined by $\varphi(v) = \varphi_A(v)$ if $v \in A$ and $\varphi(v) = \varphi_B(v)$ if $v \in B$. Since $\vec{\chi}(D) = 4$, D , coloured with φ , must contain a monochromatic directed cycle. Such a directed cycle must be contained in $D\langle A \rangle$ or $D\langle B \rangle$, a contradiction. \diamond

Claim 7. D is 3-connected. In particular, D contains no diamond.

Proof of claim. Suppose for contradiction that $\{x, y\}$ is a 2-cutset of D . Let (A_0, B_0) be a partition of $V(D) \setminus \{x, y\}$ into non-empty sets such that there is no edge between A_0 and B_0 , and set $A = A_0 \cup \{x, y\}$ and $B = B_0 \cup \{x, y\}$.

Assume for a contradiction that there exists a 3-dicolouring φ_A of $D\langle A \rangle$ and a 3-dicolouring φ_B of $D\langle B \rangle$ such that $\varphi_A(x) \neq \varphi_A(y)$ and $\varphi_B(x) \neq \varphi_B(y)$. Free to swap the colours, we may assume $\varphi_A(x) = \varphi_B(x)$ and $\varphi_A(y) = \varphi_B(y)$. Let φ be defined by $\varphi(v) = \varphi_A(v)$ if $v \in A$ and $\varphi(v) = \varphi_B(v)$ if $v \in B$. Every directed cycle either is in $D\langle A \rangle$, or is in $D\langle B \rangle$ or contains both x and y . Therefore it cannot be monochromatic with φ because φ_A and φ_B are 3-dicolourings of $D\langle A \rangle$ and $D\langle B \rangle$ respectively, and $\varphi(x) \neq \varphi(y)$. Thus φ is a 3-dicolouring of D , a contradiction. Henceforth either $D\langle A \rangle$ or $D\langle B \rangle$ has no 3-dicolouring φ such that $\varphi(x) \neq \varphi(y)$. Suppose without loss of generality that it is $D\langle A \rangle$.

Let $D_A = D\langle A \rangle \cup [x, y]$. D_A is not 3-dicolourable because in every 3-dicolouring of $D\langle A \rangle$, x and y are coloured the same. Let D_B be the digraph obtained from $D\langle B \rangle$ by identifying x and y into a vertex z . Assume for a contradiction that D_B has a 3-dicolouring ψ_B . Set $\psi(x) = \psi(y) = \psi_B(z)$, and $\psi(u) = \psi_B(u)$ for every $u \in B \setminus \{x, y\}$. Then consider a 3-dicolouring ψ_A of $D\langle A \rangle$ such that $\psi_A(x) = \psi(x) = \psi_A(y) = \psi(y)$ (such a colouring exists because A is a proper subdigraph of D) and we set $\psi(u) = \psi_A(u)$ for every $u \in V(A) \setminus \{x, y\}$. As D is not 3-dicolourable, it contains a monochromatic directed cycle C (with respect to ψ). The cycle C is not included in $D\langle A \rangle$ nor in D_B . As a consequence, there is a monochromatic directed path from $\{x, y\}$ to $\{x, y\}$ in B , and so there is a monochromatic directed cycle in D_B for ψ_B , a contradiction. Therefore D_B is not 3-dicolourable.

Now D_A has a 4-dicritical subdigraph W_A which necessarily contains $\{x, y\}$, and D_B has a 4-dicritical subdigraph W_B which necessarily contains z . As W_A and W_B are 4-dicritical digraphs smaller than D , we have $\rho(W_A), \rho(W_B) \leq \frac{4}{3} + 4\varepsilon - 2\delta$ by Claim 1. Let H be the subdigraph of D induced by $V(W_A) \cup V(W_B - z)$.

Note that $n(H) = n(W_A) + n(W_B) - 1$ and $m(H) \geq m(W_A) + m(W_B) - 2$. Moreover $T(H) \geq T(W_A - x) + T(W_B - z) \geq T(W_A) + T(W_B) - 2$, by Lemma 13. Hence we have

$$\begin{aligned}
\rho(H) &\leq \rho(W_A) + \rho(W_B) - \left(\frac{10}{3} + \varepsilon\right) + (m(W_A) + m(W_B) - m(H)) + 2\delta \\
&\leq \rho(W_A) + \rho(W_B) - \frac{10}{3} - \varepsilon + 2 + 2\delta \\
&= \rho(W_A) + \rho(W_B) - \frac{4}{3} - \varepsilon + 2\delta \\
&\leq 2\left(\frac{4}{3} + 4\varepsilon - 2\delta\right) - \frac{4}{3} - \varepsilon + 2\delta \\
&= \frac{4}{3} + 7\varepsilon - 2\delta
\end{aligned} \tag{1}$$

By Claim 3, if $n(H) < n(D)$ then $\rho(H) > 3 - 3\varepsilon + \delta$. As $10\varepsilon - 3\delta \leq \frac{5}{3}$, we deduce that $H = D$. Hence $1 < \rho(D) = \rho(H) \leq \frac{4}{3} + 7\varepsilon - 2\delta + (m(W_A) + m(W_B) - m(H) - 2)$ and so $m(H) = m(W_A) + m(W_B) - 2$ because $2\delta - 7\varepsilon \leq \frac{2}{3}$. In particular, there is no arc between x and y in D . Moreover, no arc was suppressed when identifying x and y into z to obtain D_B , so x and y have no common out-neighbour (resp. in-neighbour) in B_0 .

We first show that either W_A or W_B is not 4-Ore. Assume for contradiction that both W_A and W_B are 4-Ore. If $H = D$ is not bidirected, then by Lemma 12, $D\langle B \rangle$ admits a 3-dicolouring φ_B such that $\varphi_B(x) = \varphi_B(y)$. Now let φ_A be a 3-dicolouring of $D\langle A \rangle$. We have $\varphi_A(x) = \varphi_A(y)$. Free to exchange colour we may assume, $\varphi_A(x) = \varphi_A(y) = \varphi_B(x) = \varphi_B(y)$. Hence we can define the 3-colouring φ of D by $\varphi(v) = \varphi_A(v)$ if $v \in A$, and $\varphi(v) = \varphi_B(v)$ if $v \in B$. Observe that, since A is bidirected, all neighbours of x and y in $D\langle A \rangle$ have a colour distinct from $\varphi(x)$. Therefore there is no monochromatic directed cycle in D coloured by φ . Thus φ is a 3-dicolouring of D , a contradiction. Therefore, $H = D$ is bidirected, and so H is an Ore-composition of W_A and W_B (because D is 2-connected by Claim 6), and so D is 4-Ore, a contradiction. Henceforth, we may assume that either W_A or W_B is not 4-Ore.

If none of W_A and W_B are a 4-Ore, then by minimality of D , $\rho(W_A) \leq 1$ and $\rho(W_B) \leq 1$. Together with Equation (1), this yields

$$\rho(H) \leq \frac{2}{3} - \varepsilon + 2\delta \leq 1$$

because $2\delta - \varepsilon \leq \frac{1}{3}$, a contradiction.

If none of W_A and W_B is \overleftrightarrow{K}_4 , then $\rho(W_A) + \rho(W_B) \leq 1 + (\frac{4}{3} + 7\varepsilon - 4\delta)$ (recall that if a digraph is 4-Ore but not \overleftrightarrow{K}_4 , then it has potential at most $\frac{4}{3} + 7\varepsilon - 4\delta$ by Lemma 22). Thus, with Equation (1), we get

$$\rho(H) \leq 1 + \left(\frac{4}{3} + 7\varepsilon - 4\delta\right) - \frac{4}{3} - \varepsilon + 2\delta = 1 + 6\varepsilon - 2\delta \leq 1$$

because $\delta \geq 3\varepsilon$.

Finally, if exactly one of W_A or W_B is isomorphic to $\overleftrightarrow{K_4}$, then either $T(W_A - x) = T(W_A) = 2$ (if $W_A = \overleftrightarrow{K_4}$) or $T(W_B - z) = T(W_B) = 2$ (if $W_B = \overleftrightarrow{K_4}$). Therefore $T(H) \geq T(W_A - x) + T(W_B - z) \geq T(W_A) + T(W_B) - 1$ by Lemma 13, and so

$$\rho(H) \leq \rho(W_A) + \rho(W_B) - \left(\frac{10}{3} + \varepsilon\right) + 2 + \delta.$$

Now the non 4-Ore digraph among W_A, W_B has potential at most 1 and the other has potential $\rho(\overleftrightarrow{K_4}) = \frac{4}{3} + 4\varepsilon - 2\delta$. Thus

$$\rho(H) \leq 1 + \left(\frac{4}{3} + 4\varepsilon - 2\delta\right) - \left(\frac{10}{3} + \varepsilon\right) + 2 + \delta = 1 + 3\varepsilon - \delta \leq 1$$

because $\delta \geq 3\varepsilon$.

In all three cases, $\rho(D) = \rho(H) \leq 1$, which is a contradiction. Hence D is 3-connected. \diamond

Claim 8. *If R is a collapsible subdigraph of D , u, v are in the boundary of R and $D \setminus \langle R \rangle \cup [u, v]$ is 4-Ore, then there exists $R' \subseteq R$ such that*

- (i) *either R' is an Ore-collapsible subdigraph of D , or*
- (ii) *R' is an induced subdigraph of R , $n(R') < n(R)$, and there exist u', v' in $\partial_D(R')$ such that $R' \cup [u', v']$ is 4-Ore.*

Proof of claim. If $\partial(R) = \{u, v\}$, then R is Ore-collapsible and we are done. Suppose now that there exists $w \in \partial(R)$ distinct from u and v . Let $H = D \setminus \langle R \rangle \cup [u, v]$. Observe that $H \neq \overleftrightarrow{K_4}$ as u, v and w receive the same colour in any 3-dicolouring of $D \setminus \langle R \rangle$ because R is collapsible. Hence H is the Ore-composition of two 4-Ore digraphs H_1 (the digon side with replaced digon $[x, y]$) and H_2 (the split side with split vertex z).

If u or v is in $V(H_2)$, then $R' = D \setminus \langle V(H_1) \rangle$ with $u' = x, v' = y$ satisfies (ii). Now we assume that $u, v \in V(H_1) \setminus V(H_2)$. By repeating this argument successively on H_1 , and then on the digon-side of H_1 , etc, either we find a subdigraph R' satisfying (ii) or u and v are in a copy J of $\overleftrightarrow{K_4}$ such that H is obtained by Ore-compositions between J and some 4-Ore digraphs with J being always in the digon side.

Observe that $w \notin V(J)$ because in any 3-dicolouring of $H \setminus [u, v]$, vertices in J receive different colours by Lemma 24, except u and v . Hence at one step in the succession of Ore-compositions, w was in the split-side S when a digon e in J has been replaced. However $e \neq [u, v]$, so either u or v is not in e . Suppose without loss of generality that e is not incident to v .

We claim that $H' = R - v \cup [u, w]$ is not 3-dicolourable. Otherwise, let φ be a 3-dicolouring of H' . Then φ is a 3-dicolouring of $H - v$ with H 4-Ore, so vertices in $J - v$ must receive pairwise different colours by Lemma 25. Let φ' be a 3-dicolouring of R . Without loss of generality, we may assume that $\varphi(x) = \varphi'(x)$ for every $x \in V(J - v)$. If $y \in S$, let $\varphi''(y) = \varphi(y)$, and let $\varphi''(y) = \varphi'(y)$ if $y \notin S$. Then φ'' is a 3-dicolouring of R but with $\varphi(u) \neq \varphi(w)$, contradicting the fact that R is collapsible. This shows that $H' = R - v \cup [u, w]$ is not 3-dicolourable.

Hence $R - v \cup [u, w]$ contains a 4-dicritical digraph K . By Lemma 23, $R' = D\langle V(K) \rangle$, as a subdigraph of H which is a 4-Ore, satisfies $\frac{10}{3}n(R') - m(R') \geq \frac{10}{3}$. This implies that $\frac{10}{3}n(K) - m(K) \geq \frac{4}{3}$. Note also that K is bidirected because $R - v$ is bidirected. Thus, by Theorem 21, K is 4-Ore. Hence R' with u, w satisfies (ii). \diamond

Claim 9. *If R is a subdigraph of D with $n(R) < n(D)$ and $u, v \in V(R)$, then $R \cup [u, v]$ is 3-dicolourable. As a consequence, there is no collapsible subdigraph in D .*

Proof of claim. Assume for a contradiction that the statement is false. Consider a smallest induced subdigraph R for which the statement does not hold. Then $K = R \cup [u, v]$ is 4-vertex-dicritical, that is for every vertex $v \in V(K)$, $\bar{\chi}(K - v) < 4 = \bar{\chi}(K)$. Note that 4-vertex-dicritical digraphs smaller than D satisfy the outcome of Theorem 8 since adding arcs does not increase the potential. Note that $\rho(R) \leq \rho(K) + 2 + \delta$.

If R is not collapsible, then, by Claim 5, $\rho(R) \geq \rho(D) + \frac{8}{3} - \varepsilon - \delta > \frac{11}{3} - \varepsilon - \delta$. But we also have $\rho(R) \leq \rho(K) + 2 + \delta \leq \frac{10}{3} + 4\varepsilon - \delta$ by Claim 1, which is a contradiction because $5\varepsilon \leq \frac{1}{3}$. Hence R is collapsible.

Let φ be a 3-dicolouring of R . Observe that $\varphi(u) = \varphi(v)$ for otherwise $R \cup [u, v]$ would be 3-dicolourable. Let R' be the dicritical extension of R with extender W and core X_W . We have $R' = D$ and $|X_W| = 1$. Since R is collapsible, for every two vertices u', v' on the boundary of R , $R \cup [u', v']$ is not 3-dicolourable. Hence, free to consider u', v' instead of u, v , we can suppose that u and v are on the boundary of R . If K is 4-Ore, then, by Claim 8 and by minimality of R , we have that R is Ore-collapsible, and so has boundary of size 2. This contradicts the fact that D is 3-connected. Hence K is not 4-Ore.

By Claim 2, we have

$$\begin{aligned} 1 < \rho(D) = \rho(R') &\leq \rho(W) + \rho(R) - \frac{10}{3} - \varepsilon + \delta \\ &\leq \rho(W) + (\rho(K) + 2 + \delta) - \frac{10}{3} - \varepsilon + \delta \end{aligned}$$

and as $\rho(K) \leq 1$ (because it is not 4-Ore and by minimality of D) we get

$$1 < 1 + \rho(W) - \left(\frac{4}{3} + \varepsilon - 2\delta \right)$$

that is $\rho(W) > \frac{4}{3} + \varepsilon - 2\delta$. But as W is smaller than D , it satisfies Theorem 8. Thus, since $\varepsilon - 2\delta \geq \frac{-1}{3}$, W must be 4-Ore. Moreover, W must be isomorphic to \overleftrightarrow{K}_4 , for otherwise $\rho(W)$ would be at most $\frac{4}{3} + 7\varepsilon - 4\delta$, and $\frac{4}{3} + 7\varepsilon - 4\delta \geq \rho(W) > \frac{4}{3} + \varepsilon - 2\delta$ would contradict $\delta \geq 3\varepsilon$. Hence $\rho(W) = \rho(\overleftrightarrow{K}_4) = \frac{4}{3} + 4\varepsilon - 2\delta$ and $T(W - X_W) = 2 = T(W)$. Thus, by Claim 2 and because $\delta \geq 3\varepsilon$, we have

$$1 < \rho(D) \leq \rho(W) + \rho(K) + 2 + \delta - \frac{10}{3} - \varepsilon \leq \rho(K) + 3\varepsilon - \delta \leq \rho(K) \leq 1,$$

a contradiction.

This implies that D does not contain any collapsible subdigraph. Indeed, assume for a contradiction that D contains a collapsible subdigraph R , and let u, v be two vertices in its boundary. Then there exists a 3-dicolouring φ of $R \cup [u, v]$, for which $\partial(R)$ is not monochromatic, a contradiction. \diamond

Claim 10. *If R is a subdigraph of D with $n(R) < n(D)$ and $u, v, u', v' \in R$, then $R \cup \{uv, u'v'\}$ is 3-dicolourable. In particular, D contains no copy of $\overleftrightarrow{K_4}$ minus two arcs.*

Proof of claim. Assume for a contradiction that the statement is false. Consider a smallest subdigraph R for which the statement does not hold. Then $K = R \cup \{uv, u'v'\}$ is 4-dicritical and smaller than D , so $\rho(K) \leq \frac{4}{3} + 4\varepsilon - 2\delta$ by Claim 1. By Claim 9, R is not collapsible, so, by Claim 5, we have $\rho(R) \geq \rho(D) + \frac{8}{3} - \varepsilon - \delta > \frac{11}{3} - \varepsilon - \delta$. But $\rho(R) \leq \rho(K) + 2 + 2\delta \leq \frac{10}{3} + 4\varepsilon$, which is a contradiction as $5\varepsilon + \delta \leq \frac{1}{3}$. \diamond

For any $v \in V(D)$, we denote by $n(v)$ its number of neighbours, that is $n(v) = |N^+(v) \cup N^-(v)|$, and by $d(v)$ its number of incident arcs, that is $d(v) = d^+(v) + d^-(v)$.

Claim 11. *Vertices of degree 6 in D have either three or six neighbours.*

Proof of claim. Let x be a vertex of degree 6.

If $n(x) = 4$, then let a, b, c, d be its neighbours such that $N^+(x) = \{a, b, c\}$ and $N^-(x) = \{a, b, d\}$. Consider $D' = D - x \cup dc$. By Claim 10, D' has a 3-dicolouring φ . If $|\varphi(N^-(x))| < 3$, then choosing $\varphi(x)$ in $\{1, 2, 3\} \setminus \varphi(N^-(x))$, we obtain a 3-dicolouring of D , a contradiction. Hence $\varphi(N^-(x)) = \{1, 2, 3\}$. We set $\varphi(x) = \varphi(d)$. As D is not 3-dicolourable, D contains a monochromatic directed cycle C . This cycle C must contain the arc dx , and an out-neighbour z of x . Since $\varphi(a), \varphi(b)$ and $\varphi(d)$ are all distinct, necessarily $z = c$. But then $C - x \cup dc$ is a monochromatic directed cycle in D' , a contradiction.

Similarly, if $n(x) = 5$, let $N^+(x) = \{a, b, c\}$ and $N^-(x) = \{a, d, e\}$, and consider $D' = D - x \cup db \cup dc$. By Claim 10, D' has a 3-dicolouring φ . If $|\varphi(N^-(x))| < 3$, then choosing $\varphi(x)$ in $\{1, 2, 3\} \setminus \varphi(N^-(x))$, we obtain a 3-dicolouring of D , a contradiction. Hence $\varphi(N^-(x)) = \{1, 2, 3\}$. We set $\varphi(x) = \varphi(d)$. As D is not 3-dicolourable, there is a monochromatic directed cycle C , which must contain the arc dx and an out-neighbour z of x . Note that z must be b or c because $\varphi(a) \neq \varphi(d)$. Then $C - x \cup dz$ is a monochromatic directed cycle in D' , a contradiction. \diamond

Claim 12. *There is no bidirected triangle containing two vertices of degree 6. In particular, D contains no emerald.*

Proof of claim. Suppose that $D\langle\{x, y, z\}\rangle = \overleftrightarrow{K_3}$ and $d(x) = d(y) = 6$. By Claim 11, x and y have exactly three neighbours, and $N[x] \neq N[y]$ because D contains no copy of $\overleftrightarrow{K_4}$ minus two arcs by Claim 10. Let u (resp. v) be the unique neighbour of x distinct from y and z (resp. x and z). Consider $D' = D - \{x, y\} \cup [u, v]$. By Claim 9, D' has a 3-dicolouring φ . Without loss of generality, suppose that $\varphi(u) = 1$ and $\varphi(v) = 2$. If $\varphi(z) = 1$ (resp. $\varphi(z) = 2, \varphi(z) = 3$), we set

$\varphi(x) = 2$ and $\varphi(y) = 3$ (resp. $\varphi(x) = 3$ and $\varphi(y) = 1$, $\varphi(x) = 2$ and $\varphi(y) = 1$). In each case, this yields a 3-dicolouring of D , a contradiction. \diamond

So now we know that D contains no emerald, and no diamond by Claim 7.

Claim 13. *If R is an induced subdigraph of D with $4 \leq n(R) < n(D)$, then $\rho(R) \geq \rho(D) + 3 + 3\varepsilon - 3\delta$, except if $D - R$ contains a single vertex which has degree 6 in D .*

Proof of claim. Let R be an induced subdigraph of D with $4 \leq n(R) < n(D)$. By Claim 9, R is not collapsible. Let φ be a 3-dicolouring of R , R' be a dicritical extension of R with extender W and core X_W (with respect to φ). By (the consequence of) Claim 3, we know that $\rho(R') \geq \rho(D)$.

Assume first that $R' \neq D$. Then, by Claims 3 and 2,

$$\rho(D) + 2 - 3\varepsilon + \delta \leq \rho(R') \leq \rho(W) + \rho(R) - \frac{10}{3} - \varepsilon + \delta.$$

Since $\rho(W) \leq \frac{4}{3} + 4\varepsilon - 2\delta$ by Claim 1, we have $\rho(R) \geq \rho(D) + 4 - 6\varepsilon + 2\delta \geq \rho(D) + 3 + 3\varepsilon - 3\delta$, because $1 \geq 9\varepsilon - 5\delta$. In the following we suppose that $R' = D$. We distinguish three cases depending on the cardinality of $|X_W|$.

- Assume first that $|X_W| = 2$. Then, by Claim 2 and Lemma 13,

$$\rho(D) \leq \rho(R') \leq \rho(W) + \rho(R) - \frac{20}{3} - 2\varepsilon + 2 + 2\delta$$

and, as $\rho(W) \leq \frac{4}{3} + 4\varepsilon - 2\delta$ by Claim 1, we have $\rho(R) \geq \rho(D) + \frac{10}{3} - 2\varepsilon \geq \rho(D) + 3 + 3\varepsilon - 3\delta$ because $5\varepsilon - 3\delta \leq \frac{1}{3}$.

- Assume now that $|X_W| = 3$. If there is a vertex $v \in V(D - R)$ with two out-neighbours (resp. two in-neighbours) in $V(R)$ with the same colour for φ , then

- $n(R') = n(W) - |X_W| + n(R)$,
- $m(R') \geq m(W) + m(R) - m(\overleftarrow{K_{|X_W|}}) + 1$ because v has two in- or out-neighbour in $V(R)$ with the same colour for φ ,
- $T(R') \geq T(W - X_W) + T(R)$.

It follows that

$$\rho(D) \leq \rho(R') \leq \rho(W) + \rho(R) - (10 + 3\varepsilon - 6) + 3\delta - 1$$

and so $\rho(R) \geq \rho(D) - \frac{4}{3} - 4\varepsilon + 2\delta + 5 + 3\varepsilon - 3\delta \geq \rho(D) + \frac{11}{3} - \varepsilon - \delta \geq \rho(D) + 3 + 3\varepsilon - 3\delta$ because $4\varepsilon - 2\delta \leq \frac{2}{3}$. Now we assume that there is no vertex with two out-neighbours (resp. two in-neighbours) in R with the same colour for φ . In other words, the in-degrees and out-degrees of vertices in $D - R$ are the same in D and in W .

If W is not 4-Ore, then by Claim 2

$$\rho(D) \leq \rho(R') \leq \rho(W) + \rho(R) - (10 + 3\varepsilon - 6) + 3\delta$$

and, as $\rho(W) \leq 1$, we have $\rho(R) \geq \rho(D) + 3 + 3\varepsilon - 3\delta$.

Now suppose W is 4-Ore. If $W \neq \overleftrightarrow{K_4}$, then, by Lemma 20, W contains a diamond or an emerald disjoint from X , and this gives a diamond or an emerald in D because the degrees of vertices in $D - R$ are the same in D and in W , which is a contradiction.

Now suppose that $W = \overleftrightarrow{K_4}$. Then $D - R$ has a single vertex of degree 6 in D .

- Assume finally that $|X_W| = 1$. Since R is not collapsible by Claim 9, φ may have been chosen so that $\partial(R)$ is not monochromatic in φ . Then, by Claim 4, $\rho(R) \geq \rho(D) + 3 - 3\varepsilon + \delta \geq \rho(D) + 3 + 3\varepsilon - 3\delta$, because $6\varepsilon - 4\delta \leq 0$.

◇

In D , we say that a vertex v is a **simple in-neighbour** (resp. **simple out-neighbour**) if v is a in-neighbour (resp. out-neighbour) of u and $[u, v]$ is not a digon in D . If v is a simple in-neighbour or simple out-neighbour of u , we simply say that v is a **simple neighbour** of u .

Claim 14. *Vertices of degree 7 have seven neighbours. In other words, every vertex of degree 7 has only simple neighbours.*

Proof of claim. Let x be a vertex of degree 7. We suppose, without loss of generality, that $d^-(x) = 3$ and $d^+(x) = 4$.

If $n(x) = 4$, then x has a unique simple out-neighbour a . As D is 4-dicritical, $D \setminus xa$ has a 3-dicolouring φ . But then every directed cycle is either in $D \setminus xa$ or it contains xa and thus an in-neighbour t of x . In the first case, it is not monochromatic because φ is a 3-dicolouring of $D \setminus xa$, and in the second case, it is not monochromatic because $[t, x]$ is a digon and so $\varphi(t) \neq \varphi(x)$. Hence φ is a 3-dicolouring of D , a contradiction.

If $n(x) = 5$, let $N^-(x) = \{a, b, c\}$ and $N^+(x) = \{a, b, d, e\}$. By Claim 10, $D' = D - x \cup \{cd, ce\}$ has a 3-dicolouring φ . If $|\varphi(N^-(x))| < 3$, then choosing $\varphi(x)$ in $\{1, 2, 3\} \setminus \varphi(N^-(x))$ gives a 3-dicolouring of D , a contradiction. If $|\varphi(N^-(x))| = 3$, then we set $\varphi(x) = \varphi(c)$. Suppose for a contradiction that there is a monochromatic directed cycle C in D (with φ). Necessarily C contains x (since φ is a 3-dicolouring of $D - x$) and so it must contain c and one vertex y in $\{d, e\}$ because $\varphi(a)$, $\varphi(b)$, and $\varphi(c)$ are all distinct. Then $C - x \cup cy$ is a monochromatic directed cycle in D' , a contradiction. Therefore φ is a 3-dicolouring of D , a contradiction.

If $n(x) = 6$, let $N^-(x) = \{a, b, c\}$ and $N^+(x) = \{a, d, e, f\}$. Consider $D' = D - x \cup \{bd, be, bf\}$.

We first show that D' is not 3-dicolourable. Assume for a contradiction that there is a 3-dicolouring φ of D' . If $|\varphi(N^-(x))| < 3$, then choosing $\varphi(x)$ in $\{1, 2, 3\} \setminus \varphi(N^-(x))$ gives a 3-dicolouring of D , a contradiction. Hence $|\varphi(N^-(x))| = 3$. We set $\varphi(x) = \varphi(b)$. Since D is not 3-dicolourable, there exists a monochromatic directed cycle C in D (with φ). Necessarily

C contains x (since φ is a 3-dicolouring of $D - x$) and so it must contain b and one vertex y in $\{d, e, f\}$ because $\varphi(a)$, $\varphi(b)$, and $\varphi(c)$ are all distinct. Then $C - x \cup by$ is a monochromatic directed cycle in D' , a contradiction. This gives a 3-dicolouring of D , a contradiction.

Henceforth D' is not 3-dicolourable, and so it contains a 4-dicritical digraph \tilde{D} , smaller than D . If \tilde{D} does not contain the three arcs bd, be, bf , then it can be obtained from a proper induced subdigraph of D by adding at most two arcs, and so it is 3-dicolourable by Claim 10, a contradiction.

Hence $\{b, d, e, f\} \subseteq V(\tilde{D})$. Now consider $U = D \langle V(\tilde{D}) \cup \{x\} \rangle$.

• Assume first that $a \notin V(U)$ or $c \notin V(U)$. Then we have

- $n(U) = n(\tilde{D}) + 1$,
- $m(U) \geq m(\tilde{D}) + 1$ and
- $T(U) \geq T(\tilde{D} - b) \geq T(\tilde{D}) - 1$ by Lemma 13.

Hence

$$\begin{aligned}
 \rho(U) &\leq \rho(\tilde{D}) + \frac{10}{3} + \varepsilon - 1 + \delta \\
 &\leq \frac{4}{3} + 4\varepsilon - 2\delta + \frac{10}{3} + \varepsilon - 1 + \delta \quad \text{by Claim 1,} \\
 &= 1 + \frac{8}{3} + 5\varepsilon - \delta \\
 &< \rho(D) + \frac{8}{3} + 5\varepsilon - \delta \\
 &\leq \rho(D) + 3 + 3\varepsilon - 3\delta \quad \text{because } \frac{1}{3} \geq 2\delta + 2\varepsilon.
 \end{aligned}$$

Hence by Claim 13, $D - U$ has a single vertex of degree 6 (in D), which must be either a or c . Then we have

- $n(D) = n(\tilde{D}) + 2$,
- $m(D) \geq m(\tilde{D}) - 3 + 11$ and
- $T(D) \geq T(\tilde{D} - b) \geq T(\tilde{D}) - 1$.

Thus

$$\begin{aligned}
 \rho(D) &\leq \rho(\tilde{D}) + 2 \left(\frac{10}{3} + \varepsilon \right) - 8 + \delta \\
 &\leq \left(\frac{4}{3} + 4\varepsilon - 2\delta \right) - \frac{4}{3} + 2\varepsilon + \delta \quad \text{by Claim 1,} \\
 &\leq 1 \quad \text{because } 6\varepsilon - \delta \leq 1.
 \end{aligned}$$

This is a contradiction.

• Assume now that $a, c \in V(U)$, then we have

- $n(U) = n(\tilde{D}) + 1$,
- $m(U) \geq m(\tilde{D}) + 4$ and
- $T(U) \geq T(\tilde{D} - b) \geq T(\tilde{D}) - 1$ by Lemma 13.

Thus

$$\begin{aligned}
\rho(U) &\leq \rho(\tilde{D}) + \frac{10}{3} + \varepsilon - 4 + \delta \\
&\leq \left(\frac{4}{3} + 4\varepsilon - 2\delta \right) + \frac{10}{3} + \varepsilon - 4 + \delta \quad \text{by Claim 1,} \\
&\leq 1 \quad \text{because } 5\varepsilon - \delta \leq \frac{1}{3}.
\end{aligned}$$

Together with the consequence of Claim 3, we get that $\rho(D) \leq \rho(U) \leq 1$, a contradiction. \diamond

The 8^+ -**valency** of a vertex v , denoted by $\nu(v)$, is the number of arcs incident to v and a vertex of degree at least 8.

Let D_6 be the subdigraph of D induced by the vertices of degree 6 incident to digons. Let us describe the connected components of D_6 and their neighbourhoods. Remember that vertices of degree 7 are incident to no digon by Claim 14, and so they do not have neighbours in $V(D_6)$. If v is a vertex in D_6 , we define its **neighbourhood valency** to be the sum of the 8^+ -valency of its neighbours of degree at least 8. We denote the neighbourhood valency of v by $\nu_N(v)$.

Claim 15. *If $[x, y]$ is a digon and both x and y have degree 6, then either*

- (i) *the two neighbours of y distinct from x have degree at least 8, or*
- (ii) *the two neighbours of x distinct from y have degree at least 8 and $\nu_N(x) \geq 4$.*

Proof of claim. Let $[x, y]$ be a digon in D with $d(x) = d(y) = 6$. By Claim 11 $n(x) = n(y) = 3$. Let u and v be the two neighbours of x different from y . By Claim 14, u and v have degree 6 or at least 8.

If u and v are linked by a digon, then by Claim 12, u and v do not have degree 6, so they have degree 8. Moreover $\nu(u) \geq 2$ and $\nu(v) \geq 2$. Thus $\nu_N(x) = \nu(u) + \nu(v) \geq 4$ and (ii) holds. Henceforth, we may assume that u and v are not linked by a digon.

Let D' the digraph obtained by removing x and y and identifying u and v into a single vertex $u \star v$. We claim that D' is not 3-dicolourable. To see that, suppose for contradiction that there exists a 3-dicolouring φ of D' . Then set $\varphi(u) = \varphi(v) = \varphi(u \star v)$, choose $\varphi(y)$ in $\{1, 2, 3\} \setminus \varphi(N(y) \setminus \{x\})$, and finally choose $\varphi(x)$ in $\{1, 2, 3\} \setminus \{\varphi(u \star v), \varphi(y)\}$. One can easily see that φ is now a 3-dicolouring of D , a contradiction. This proves that D' is not 3-dicolourable and so it contains a 4-dicritical digraph \tilde{D} , which must contain $u \star v$ because every subdigraph of D is 3-dicolourable. Let R be the subdigraph of D induced by $(V(\tilde{D}) \setminus \{u \star v\}) \cup \{u, v, x\}$. We have

- $n(R) = n(\tilde{D}) + 2$,
- $m(R) \geq m(\tilde{D}) + 4$ and

- $T(R) \geq T(\tilde{D} - u \star v) + 1 \geq T(\tilde{D})$ because $[x, u]$ is a digon, and by Lemma 13.

If \tilde{D} is not 4-Ore, then $\rho(\tilde{D}) \leq 1$ by minimality of D , and so

$$\begin{aligned} \rho(R) &\leq \rho(\tilde{D}) + 2 \left(\frac{10}{3} + \varepsilon \right) - 4 \\ &\leq 1 + \frac{8}{3} + 2\varepsilon \\ &< \rho(D) + 3 + 3\varepsilon - 3\delta \quad \text{because } \varepsilon - 3\delta \geq -\frac{1}{3}. \end{aligned}$$

Similarly, if \tilde{D} is 4-Ore but not \overleftrightarrow{K}_4 , then

$$\begin{aligned} \rho(R) &\leq \rho(\tilde{D}) + 2 \left(\frac{10}{3} + \varepsilon \right) - 4 \\ &\leq \left(\frac{4}{3} + 7\varepsilon - 4\delta \right) + \frac{8}{3} + 2\varepsilon \quad \text{by Lemma 22,} \\ &= 1 + 3 + 9\varepsilon - 4\delta \\ &< \rho(D) + 3 + 9\varepsilon - 4\delta \\ &\leq \rho(D) + 3 + 3\varepsilon - 3\delta \quad \text{because } \delta \geq 6\varepsilon. \end{aligned}$$

In both cases (that is when \tilde{D} is not \overleftrightarrow{K}_4), by Claim 13, $D - R$ is a single vertex of degree 6, namely y . Then every neighbour w of y different from x has degree at least 6 in \tilde{D} (because \tilde{D} is 3-dicritical) and so has degree at least 8 in D and (i) holds.

Assume now that \tilde{D} is a copy of \overleftrightarrow{K}_4 . Let us denote by a, b, c the vertices of \tilde{D} different from $u \star v$. Suppose for a contradiction that u has degree 6. Then u has exactly three neighbours by Claim 11. If $|N(u) \cap \{a, b, c\}| = 2$, then $D\langle\{u, a, b, c\}\rangle$ is a copy of \overleftrightarrow{K}_4 minus a digon, contradicting Claim 9. If $|N(u) \cap \{a, b, c\}| \leq 1$, then v must be adjacent to at least two vertices of $\{a, b, c\}$ with a digon, and so $D\langle\{v, a, b, c\}\rangle$ contains a copy of \overleftrightarrow{K}_4 minus a digon, contradicting Claim 9. Hence u has degree at least 8, and by symmetry so does v . Moreover $D\langle\{a, b, c\}\rangle$ is a bidirected triangle, and so by Claim 12, at least two of these vertices have degree at least 8 (remember that vertices of degree 7 are in no digon by Claim 14). Hence at least four arcs between $\{u, v\}$ and $\{a, b, c\}$ are incident to two vertices of degree at least 8. In other word, $\nu_N(x) = \nu(u) + \nu(v) \geq 4$, so (ii) holds. \diamond

Claim 16. *Let C be a connected component of D_6 . Then C is one of the following (see Figure 3):*

- (i) *a single vertex, or*
- (ii) *a bidirected path on two vertices, or*

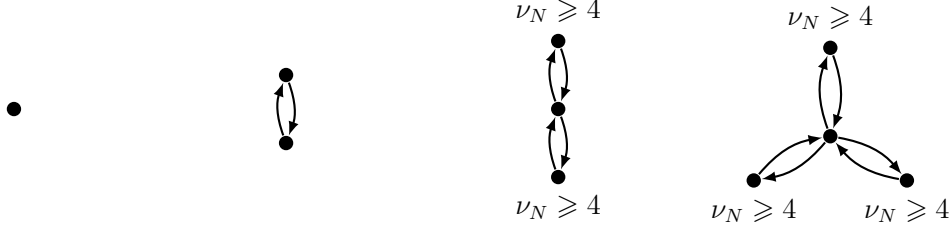


Figure 3: The possible connected components of D_6 .

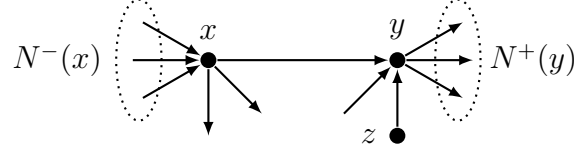


Figure 4: An example of an out-chelou arc xy .

(iii) a bidirected path on three vertices, whose extremities have neighbourhood valency at least 4,
or

(iv) a star on four vertices, whose non-central vertices have neighbourhood valency at least 4.

Proof of claim. First observe that C does not contain a bidirected path $\llbracket x, y, z, w \rrbracket$ on four vertices, because otherwise, by Claim 15 applied on $[y, z]$, either y or z has two neighbours of degree at least 8, a contradiction. Observe also that C contains no bidirected triangle by Claim 12.

Moreover, if $\llbracket x, y, z \rrbracket$ is a bidirected path in C on three vertices, then by Claim 15 applied both on $[y, z]$ and $[z, y]$, x and z have both neighbourhood valency at least 4. The statement of the claim follows. \diamond

An arc xy is said to be **out-chelou** if

- (i) $yx \notin A(D)$, and
- (ii) $d^+(x) = 3$, and
- (iii) $d^-(y) = 3$, and
- (iv) there exists $z \in N^-(y) \setminus N^+(y)$ distinct from x .

Symmetrically, we say that an arc xy is **in-chelou** if yx is out-chelou in the digraph obtained from D by reversing every arc. See Figure 4 for an example of an out-chelou arc.

Claim 17. *There is no out-chelou arc and no in-chelou arc in D .*

Proof of claim. By directional duality, it suffices to prove that D has no out-chelou arcs.

Let xy be an out-chelou arc with $z \in N^-(y) \setminus (N^+(y) \cup \{x\})$. Consider $D' = D - \{x, y\} \cup \{zz' \mid z' \in N^+(y) \setminus N^-(y)\}$. We claim that D' is not 3-dicolourable. To see that, suppose

for contradiction that there is a 3-dicolouring φ of D' . As $d^+(x) = 3$, we can choose $\varphi(x)$ in $\{1, 2, 3\} \setminus \varphi(N^+(x) \setminus \{y\})$ to obtain a 3-dicolouring of $D - y$. If $|\varphi(N^-(y))| < 3$, then choosing $\varphi(y)$ in $\{1, 2, 3\} \setminus \varphi(N^-(y))$ gives a 3-dicolouring of D , a contradiction. Hence $|\varphi(N^-(y))| = 3$. Set $\varphi(x) = \varphi(z)$. Suppose there is a monochromatic directed cycle C in D . It must contain y and thus z , its unique in-neighbour with its colour. Let z' be the out-neighbour of y in C . It must be in $N^+(y) \setminus N^-(y)$, so zz' is an arc in D' . Thus $C - y \cup zz'$ is a monochromatic directed cycle in D' , a contradiction. Therefore φ is a 3-dicolouring of D , a contradiction. Hence D' is not 3-dicolourable.

Consequently, D' contains a 3-dicritical digraph \tilde{D} , which is smaller than D and contains z , for otherwise \tilde{D} would be a subdigraph of D . Consider $U = D \setminus (V(\tilde{D}) \cup \{y\})$. We have

- $n(U) = n(\tilde{D}) + 1$,
- $m(U) \geq m(\tilde{D}) + 1$ and
- $T(U) \geq T(\tilde{D} - z) \geq T(\tilde{D}) - 1$ by Lemma 13.

First if \tilde{D} is not 4-Ore, then by minimality of D we have $\rho(\tilde{D}) \leq 1$, so

$$\rho(U) \leq \rho(\tilde{D}) + \frac{10}{3} + \varepsilon - 1 + \delta \leq \frac{10}{3} + \varepsilon + \delta \leq \frac{11}{3} - \varepsilon - \delta$$

$$2\varepsilon + 2\delta \leq \frac{1}{3}.$$

Next if \tilde{D} is 4-Ore, but not isomorphic to \overleftrightarrow{K}_4 , then $\rho(\tilde{D}) \leq \frac{4}{3} + 7\varepsilon - 4\delta$ by Lemma 22, and

$$\rho(U) \leq \rho(\tilde{D}) + \frac{10}{3} + \varepsilon - 1 + \delta \leq \frac{11}{3} + 8\varepsilon - 3\delta \leq \frac{11}{3} - \varepsilon - \delta$$

because $9\varepsilon - 2\delta \leq 0$.

Finally if \tilde{D} is isomorphic to \overleftrightarrow{K}_4 , then we have $T(U) \geq T(\tilde{D} - z) \geq T(\tilde{D})$ and $\rho(\tilde{D}) = \frac{4}{3} + 4\varepsilon - 2\delta$. So the same computation yields

$$\rho(U) \leq \rho(\tilde{D}) + \frac{10}{3} + \varepsilon - 1 \leq \frac{11}{3} + 5\varepsilon - 2\delta \leq \frac{11}{3} - \varepsilon - \delta$$

because $6\varepsilon - \delta \leq 0$.

In all cases, we have $\rho(U) \leq \frac{11}{3} - \varepsilon - \delta$. This contradicts Claim 5 because U is not collapsible by Claim 9. \diamond

We now use the discharging method. For every vertex v , let $\sigma(v) = \frac{\delta}{|C|}$ if v has degree 6 and is in a component C of D_6 of size at least 2, and $\sigma(v) = 0$ otherwise. Clearly $T(D)$ is at least the number of connected components of size at least 2 of D_6 so $\sum_{v \in V(D)} \sigma(v) \leq \delta T(D)$. We define the **initial charge** of v to be $w(v) = \frac{10}{3} + \varepsilon - \frac{d(v)}{2} - \sigma(v)$. We have

$$\rho(D) \leq \sum_{v \in V(D)} w(v).$$

We now redistribute this total charge according to the following rules:

- (R1) A vertex of degree 6 incident to no digon sends $\frac{1}{12} - \frac{\varepsilon}{8}$ to each of its neighbours.
- (R2) A vertex of degree 6 incident to digons sends $\frac{2}{d(v)-\nu(v)}(-\frac{10}{3} + \frac{d(v)}{2} - \varepsilon)$ to each neighbour v of degree at least 8 (so $\frac{1}{d(v)-\nu(v)}(-\frac{10}{3} + \frac{d(v)}{2} - \varepsilon)$ via each arc of the digon).
- (R3) A vertex of degree 7 with $d^-(v) = 3$ (resp. $d^+(v) = 3$) sends $\frac{1}{12} - \frac{\varepsilon}{8}$ to each of its in-neighbours (resp. out-neighbours).

For every vertex v , let $w^*(v)$ be the final charge of v .

Claim 18. *If v has degree at least 8, then $w^*(v) \leq 0$.*

Proof of claim. Let v be a vertex of degree at least 8. If v is not adjacent to a vertex of degree at most 7, then $w^*(v) = w(v) = \frac{10}{3} + \varepsilon - \frac{d(v)}{2} \leq 0$ (because $\varepsilon \leq \frac{2}{3}$). Otherwise, $d(v) - \nu(v) \geq 1$ and

$$\begin{aligned} \frac{1}{d(v) - \nu(v)} \left(-\frac{10}{3} + \frac{d(v)}{2} - \varepsilon \right) &\geq \frac{1}{d(v)} \left(-\frac{10}{3} + \frac{d(v)}{2} - \varepsilon \right) \\ &\geq \frac{1}{12} - \frac{\varepsilon}{8}. \end{aligned}$$

Thus v receives at most $\frac{1}{d(v)-\nu(v)}(-\frac{10}{3} + \frac{d(v)}{2} - \varepsilon)$ per arc incident with a vertex of degree 6 or 7. Since there are $d(v) - \nu(v)$ such arcs, $w^*(v) \leq w(v) - \frac{10}{3} - \varepsilon + \frac{d(v)}{2} = 0$. \diamond

Claim 19. *If v has degree 7, then $w^*(v) \leq 0$.*

Proof of claim. By Claim 14, v has seven neighbours. Without loss of generality, let us suppose that $d^-(v) = 3$ and $d^+(v) = 4$. By Claim 17, the in-neighbours of v can not have out-degree 3. In particular, they do not have degree 6, and if they have degree 7, they do not send anything to v by Rule (R3). Hence v receives at most four times the charge $\frac{1}{12} - \frac{\varepsilon}{8}$ by (R1) or (R3), and it sends three times this charge by (R3). Hence

$$\begin{aligned} w^*(v) &\leq w(v) + \frac{1}{12} - \frac{\varepsilon}{8} \\ &= -\frac{1}{12} + \frac{7}{8}\varepsilon \end{aligned}$$

and the result comes because $\varepsilon \leq \frac{2}{21}$. \diamond

Claim 20. *If v is a vertex of degree 6 incident to no digon, then $w^*(v) \leq 0$.*

Proof of claim. The vertex v sends $\frac{1}{12} - \frac{\varepsilon}{8}$ to each of its neighbours, and it receives no charge as all its in-neighbours (resp. out-neighbours) have out-degree (resp. in-degree) at least 4, by Claim 17. As a consequence,

$$w^*(v) = w(v) - 6 \left(\frac{1}{12} - \frac{\varepsilon}{8} \right) = -\frac{1}{6} + \frac{7\varepsilon}{4}$$

and the result comes because $\varepsilon \leq \frac{2}{21}$. \diamond

Claim 21. *Let v be a vertex in D_6 having at least two neighbours of degree at least 8. Then $w^*(v) \leq 0$. Moreover, if v is not an isolated vertex in D_6 and $\nu_N(v) \geq 4$, then $w^*(v) \leq -\frac{1}{9} + \frac{5}{3}\varepsilon - \frac{\delta}{4}$.*

Proof of claim. Observe that v receives no charge and sends the following charge to each of its neighbour u with degree at least 8:

$$\begin{aligned} \frac{2}{d(u) - \nu(u)} \left(-\frac{10}{3} - \varepsilon + \frac{d(u)}{2} \right) &\geq \frac{2}{d(u)} \left(-\frac{10}{3} - \varepsilon + \frac{d(u)}{2} \right) \\ &= 1 - \frac{2}{d(u)} \left(\frac{10}{3} + \varepsilon \right) \\ &\geq \frac{2}{8} \left(-\frac{10}{3} - \varepsilon + 4 \right) \\ &= \frac{1}{6} - \frac{\varepsilon}{4}. \end{aligned}$$

Assume first that v is isolated in D_6 . By Claim 14, its three neighbours do not have degree 7, and so have degree at least 8. Thus v sends three times at least $\frac{1}{6} - \frac{\varepsilon}{4}$, and so

$$w^*(v) \leq w(v) - 3 \left(\frac{1}{6} - \frac{\varepsilon}{4} \right) = -\frac{1}{6} + \frac{7}{4}\varepsilon$$

and the result comes because $\varepsilon \leq \frac{2}{21}$.

Assume now that v is in a connected component C of D_6 of size at least 2. By Claim 16, $\sigma(v) \geq \frac{\delta}{4}$, so $w(v) \leq \frac{1}{3} + \varepsilon - \frac{\delta}{4}$. Moreover it sends two times at least $\frac{1}{6} - \frac{\varepsilon}{4}$. Hence

$$w^*(v) \leq \left(\frac{1}{3} + \varepsilon - \frac{\delta}{4} \right) - 2 \left(\frac{1}{6} - \frac{\varepsilon}{4} \right) = \frac{3}{2}\varepsilon - \frac{\delta}{4}$$

and the result comes because $\delta \geq 6\varepsilon$. This shows the first part of the statement.

We will now prove the second part of the statement. Assume that v is not an isolated vertex in D_6 and $\nu_N(v) \geq 4$. Let u_1 and u_2 be the two neighbours of v with degree at least 8. For every $i \in \{1, 2\}$ we have

$$\frac{2}{d(u_i) - \nu(u_i)} \left(-\frac{10}{3} - \varepsilon + \frac{d(u_i)}{2} \right) = 1 - \frac{1}{d(u_i) - \nu(u_i)} \left(\frac{20}{3} + 2\varepsilon - \nu(u_i) \right)$$

Case 1: $\nu(u_i) \geq 7$ for some $i \in \{1, 2\}$. Without loss of generality suppose $i = 1$. Then we have

$$1 - \frac{1}{d(u_1) - \nu(u_1)} \left(\frac{20}{3} + 2\varepsilon - \nu(u_1) \right) \geq 1$$

because $\nu(u_1) \geq 7 \geq \frac{20}{3} + 2\varepsilon$ as $\varepsilon \leq \frac{1}{6}$. Then the total charge sent by v is at least 1, and thus

$$w^*(v) \leq w(v) - 1 \leq \left(\frac{1}{3} + \varepsilon - \frac{\delta}{4} \right) - 1 = -\frac{2}{3} + \varepsilon - \frac{\delta}{4}$$

Thus, we have $w^*(v) \leq -\frac{1}{9} + \frac{5}{3}\varepsilon - \frac{\delta}{4}$ because $\varepsilon, \delta \geq 0$.

Case 2: $\nu(u_1), \nu(u_2) \leq 6$. Let $f : [0, 6] \rightarrow \mathbb{R}$ be the function defined by

$$f(x) = \frac{2}{8-x} \left(-\frac{10}{3} - \varepsilon + \frac{8}{2} \right) = 1 - \frac{1}{8-x} \left(\frac{20}{3} - 2\varepsilon - x \right)$$

for every $x \in [0, 6]$. Observe that f is non decreasing and convex on $[0, 6]$ because $-\frac{10}{3} - \varepsilon + \frac{8}{2} \geq 0$. For $i = 1, 2$, we have

$$\frac{2}{d(u_i) - \nu(u_i)} \left(-\frac{10}{3} - \varepsilon + \frac{d(u_i)}{2} \right) \geq f(\nu(u_i))$$

because the function $d \mapsto 1 - \frac{1}{d - \nu(u_i)} \left(\frac{20}{3} + 2\varepsilon - \nu(u_i) \right)$ is non decreasing on $[8, +\infty[$ as $\nu(u_i) \leq 6 \leq \frac{20}{3} + 2\varepsilon$. Hence the charge sent by v to u_i is at least $f(\nu(u_i))$. By hypothesis we have $\nu_N(v) = \nu(u_1) + \nu(u_2) \geq 4$. It follows that the total charge sent by v is at least

$$\begin{aligned} f(\nu(u_1)) + f(\nu(u_2)) &\geq 2f\left(\frac{\nu(u_1) + \nu(u_2)}{2}\right) \quad \text{by convexity of } f \\ &\geq 2f(2) \quad \text{because } f \text{ is non decreasing} \\ &= \frac{4}{9} - \frac{2}{3}\varepsilon. \end{aligned}$$

Hence

$$w^*(v) \leq w(v) - \left(\frac{4}{9} - \frac{2}{3}\varepsilon \right) \leq \left(\frac{1}{3} + \varepsilon - \frac{\delta}{4} \right) - \frac{4}{9} + \frac{2}{3}\varepsilon = -\frac{1}{9} + \frac{5}{3}\varepsilon - \frac{\delta}{4}.$$

showing the second part of the statement. \diamond

Claim 22. *If C is a connected component of D_6 , then $\sum_{v \in V(C)} w^*(v) \leq 0$.*

Proof of claim. If C has a unique vertex v , then, by Claim 21, we have $w^*(v) \leq 0$ as wanted.

If C has two vertices x and y , then, again by Claim 21, $w^*(x), w^*(y) \leq 0$, and so $w^*(x) + w^*(y) \leq 0$.

If C is a bidirected path $[x, y, z]$, then, by Claim 16, x and z have both neighbourhood valency at least 4 and so by Claim 21 $w^*(x), w^*(z) \leq -\frac{1}{9} - \frac{\varepsilon}{6}$. Moreover, y sends at least $\frac{2}{8}(-\frac{10}{3} + 4 - \varepsilon) = \frac{1}{6} - \frac{\varepsilon}{4}$ to its neighbour out of C . Hence

$$w^*(y) \leq w(y) - \left(\frac{1}{6} - \frac{\varepsilon}{4} \right) \leq \frac{1}{3} + \varepsilon - \frac{\delta}{3} - \frac{1}{6} + \frac{\varepsilon}{4} = \frac{1}{6} + \frac{5}{4}\varepsilon - \frac{\delta}{3}.$$

Altogether, we get that

$$w^*(x) + w^*(y) + w^*(z) \leq \frac{1}{6} + \frac{5}{4}\varepsilon - \frac{\delta}{3} + 2 \left(-\frac{1}{9} - \frac{\varepsilon}{6} \right) = -\frac{1}{18} + \frac{11}{12}\varepsilon - \frac{\delta}{3} \leq 0$$

because $\delta \geq 6\varepsilon$.

Finally, if C is a bidirected star with centre x and three other vertices y, z, w , then $w^*(x) \leq w(x) = \frac{1}{3} + \varepsilon - \frac{\delta}{4}$. Moreover, each of y, z, w has neighbourhood valency at least 4 by Claim 16 and so has final charge at most $-\frac{1}{9} + \frac{5}{3}\varepsilon - \frac{\delta}{4}$ by Claim 21. Hence

$$w^*(x) + w^*(y) + w^*(z) + w^*(w) \leq \frac{1}{3} + \varepsilon - \frac{\delta}{4} + 3 \left(-\frac{1}{9} + \frac{5}{3}\varepsilon - \frac{\delta}{4} \right) \leq 6\varepsilon - \delta \leq 0$$

because $\delta \geq 6\varepsilon$. ◇

As a consequence of these last claims, we have $\rho(D) \leq \sum_{v \in V(D)} w(v) = \sum_{v \in V(D)} w^*(v) \leq 0 \leq 1$, a contradiction. This proves Theorem 8. □

4 An upper bound on $o_k(n)$

In this section, we show that, for every fixed k , there are infinitely many values of n such that $o_k(n) \leq (2k - \frac{7}{2})n$. The proof is strongly based on the proof of [2, Theorem 4.4], which shows $o_k(n) \leq (2k - 3)n$ for every k, n (with n large enough). For $k = 4$, the construction implies in particular that there is a 4-dicritical oriented graph with 76 vertices and 330 arcs, and there are infinitely many 4-dicritical oriented graphs with $m/n \leq 9/2$.

Proposition 26. *Let $k \geq 3$ be an integer. For infinitely many values of $n \in \mathbb{N}$, there exists a k -dicritical oriented graph \vec{G}_k on n vertices with at most $(2k - \frac{7}{2})n$ arcs.*

Proof. Let us fix $n_0 \in \mathbb{N}$. We will show, by induction on k , that there exists a k -dicritical oriented graph \vec{G}_k on n vertices with at most $(2k - \frac{7}{2})n$ arcs, such that $n \geq n_0$.

When $k = 3$, the result is known ([2, Corollary 4.3]). We briefly describe the construction for completeness. Start from any orientation of an odd cycle on $2n_0 + 1$ vertices. Then for each arc xy in this orientation, add a directed triangle \vec{C}_3 and every arc from y to $V(\vec{C}_3)$ and every arc from $V(\vec{C}_3)$ to x (see Figure 5). This gadget forces x and y to have different colours in every 2-dicolouring. Since we started from an orientation of an odd cycle, the result is a 3-dicritical oriented graph on $4(2n_0 + 1)$ vertices and $10(2n_0 + 1)$ arcs.

Let us fix $k \geq 4$ and assume that there exists such a $(k - 1)$ -dicritical oriented graph \vec{G}_{k-1} on $n_{k-1} \geq n_0$ vertices with $m_{k-1} \leq (2(k - 1) - \frac{7}{2})n_{k-1}$ arcs. We start from any tournament T on k vertices. Then we add, for each arc xy of T , a copy \vec{G}_{k-1}^{xy} of \vec{G}_{k-1} , all arcs from y to \vec{G}_{k-1}^{xy} and all arcs from \vec{G}_{k-1}^{xy} to x . Figure 6 illustrates a possible construction of \vec{G}_4 , where T is the transitive tournament on 4 vertices.

Let \vec{G}_k be the resulting oriented graph. By construction, $n_k = |V(\vec{G}_k)|$ and $m_k = |A(\vec{G}_k)|$

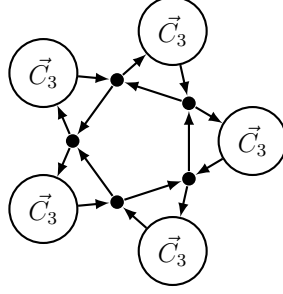


Figure 5: A 3-dicritical oriented graph with $\frac{5}{2}n$ arcs.

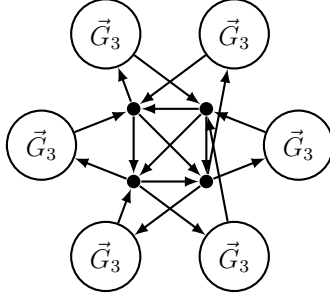


Figure 6: A 4-dicritical oriented graph with at most $\frac{9}{2}n$ arcs.

satisfy:

$$\begin{aligned}
 n_k &= k + \binom{k}{2} n_{k-1} \\
 m_k &= \binom{k}{2} + \binom{k}{2} \times 2 \times n_{k-1} + \binom{k}{2} \times m_{k-1} \\
 &\leq \binom{k}{2} + \binom{k}{2} \left(2 + 2(k-1) - \frac{7}{2} \right) n_{k-1} \\
 &= \binom{k}{2} + \binom{k}{2} \left(2k - \frac{7}{2} \right) n_{k-1} \\
 &= \binom{k}{2} + \left(2k - \frac{7}{2} \right) (n_k - k) \\
 &\leq \left(2k - \frac{7}{2} \right) n_k
 \end{aligned}$$

where in the last inequality we used $k \left(2k - \frac{7}{2} \right) \geq \binom{k}{2}$, which holds when $k \geq 2$. We will now prove that \vec{G}_k is indeed k -dicritical.

We first prove that $\vec{\chi}(\vec{G}_k) = k$. Assume that there exists a $(k-1)$ -dicolouring α of \vec{G}_k . Then there exist $x, y \in V(T)$ such that $\alpha(x) = \alpha(y)$. Since $\vec{\chi}(\vec{G}_{k-1}) = k-1$, there exists

$z \in V(\vec{G}_{k-1}^{xy})$ such that $\alpha(z) = \alpha(x)$. But then (x, y, z, x) is a monochromatic directed triangle in α : a contradiction.

Let us now prove that $\vec{\chi}(\vec{G}_k \setminus \{uv\}) \leq k - 1$ for every arc $uv \in A(\vec{G}_k)$. This implies immediately that $\vec{\chi}(\vec{G}_k) = k$ and shows the result.

Consider first an arc uv in $A(T)$. We colour each copy \vec{G}_{k-1}^{xy} of \vec{G}_{k-1} with a $(k-1)$ -dicolouring of \vec{G}_{k-1} . We then choose a distinct colour for every vertex in T , except u and v which receive the same colour. This results in a $(k-1)$ -dicolouring of $\vec{G}_k \setminus \{uv\}$.

Consider now an arc uv of \vec{G}_{k-1}^{xy} for some $xy \in A(T)$. Because \vec{G}_{k-1} is $(k-1)$ -dicritical, there exists a $(k-2)$ -dicolouring ξ of $\vec{G}_{k-1}^{xy} \setminus \{uv\}$. Hence we colour $\vec{G}_{k-1}^{xy} \setminus \{uv\}$ with ξ , every other copy of \vec{G}_{k-1} a $(k-1)$ -dicolouring of \vec{G}_{k-1} , and we choose a distinct colour for every vertex in T , except x and y which both receive colour $k-1$. This results in a $(k-1)$ -dicolouring of $\vec{G}_k \setminus \{uv\}$.

Consider finally an arc uv arc from $u \in V(T)$ to $v \in V(\vec{G}_{k-1}^{xy})$ (the case of $u \in V(\vec{G}_{k-1}^{xy})$ and $v \in V(T)$ being symmetric). Because \vec{G}_{k-1} is dicritical, there exists a $(k-1)$ -dicolouring γ of \vec{G}_{k-1}^{xy} in which v is the only vertex coloured $k-1$. Hence, we colour \vec{G}_{k-1}^{xy} with γ , every other copy of \vec{G}_{k-1} with a $(k-1)$ -dicolouring of \vec{G}_{k-1} , and we choose a distinct colour for every vertex in T , except u and y which both receive colour $k-1$. This results in a $(k-1)$ -dicolouring of $\vec{G}_k \setminus \{uv\}$. \square

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