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Geodesic regression on $SE(3)$ and application to estimation of positions

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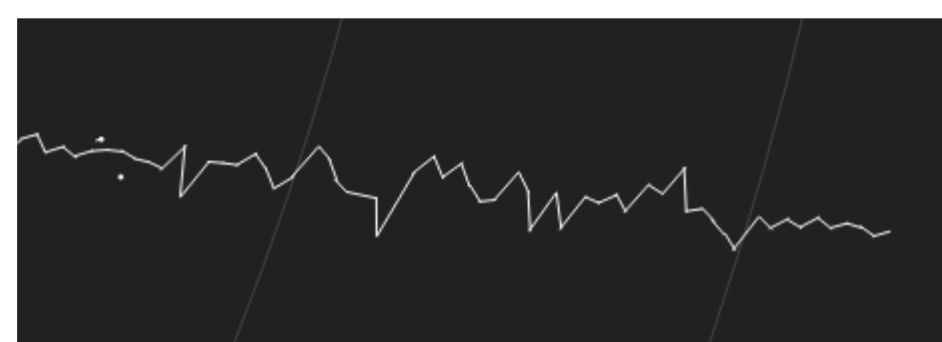
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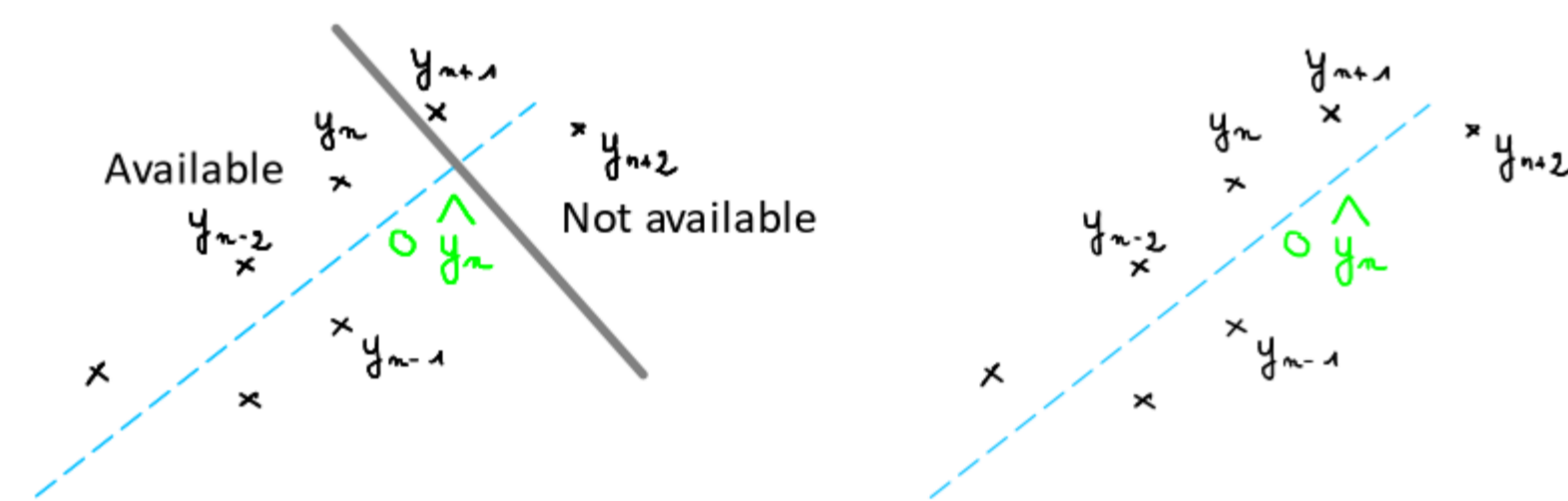
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Our application

- Estimating the true position of a mobile object (aircraft, drone, ...) from measured (therefore noisy) raw positions



- Two cases :
 - From past positions: prediction : frequently solved with linear / extended / unscented Kalman filters, or particle filters [1]
 - From past and future positions: smoothing



- This may be difficult with a highly maneuvering target [6]

Our framework

- Instead of considering the trajectory in the state space (\mathbb{R}^3), the movement of the mobile object is considered as a sequence of displacements: rotations and translations in \mathbb{R}^3
- This set is called the Special Euclidean group $SE(3)$
- $SE(3)$ is a Lie group, which is both:
 - a differentiable manifold
 - a group in which both the product and inverse maps are smooth

Underlying Lie group: $SE(3)$

- An element of $SE(3)$ can be represented by a 4-by-4 matrix

$$\begin{pmatrix} R & T \\ 0 & 1 \end{pmatrix}, R \in SO(3), T \in \mathbb{R}^3 :$$

- An element of its Lie algebra $\mathfrak{se}(3)$ can be represented by a 4-by-4 matrix:

$$\begin{pmatrix} \Omega & U \\ 0 & 0 \end{pmatrix}, \Omega \text{ skew-symmetric}, U \in \mathbb{R}^3$$

Our method

- A minimizing geodesic between two points g_0, g_1 with respect to $\langle \cdot, \cdot \rangle_g$ is a curve $\gamma: [0, 1] \rightarrow \mathcal{G}$, such that $\gamma(0) = g_0, \gamma(1) = g_1$ minimizing the functional:

$$L(\gamma) = \frac{1}{2} \int_0^1 \langle \dot{\gamma}(t), \dot{\gamma}(t) \rangle_g dt \quad (1)$$

- Left invariance of the metric implies it is enough to seek after a solution of the reduced problem:

$$L(\gamma) = \frac{1}{2} \int_0^1 \langle \dot{\xi}(t), \dot{\xi}(t) \rangle dt \quad (2)$$

where ξ is a curve in \mathfrak{g} .

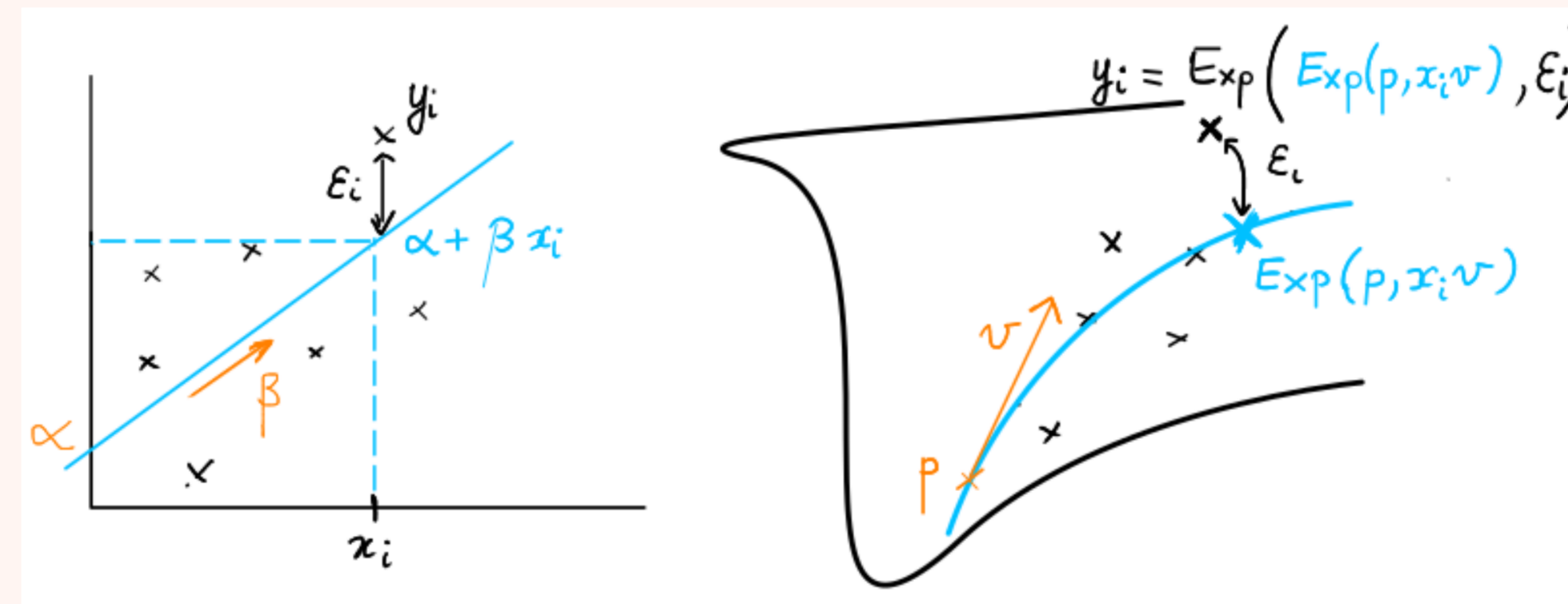
- ξ satisfies the so-called Euler-Arnold equation [4]:

$$\dot{\xi}(t) = \text{ad}_{\xi(t)}^\dagger \xi(t) \quad (3)$$

- An extended notion of parametric linear regression on Riemannian manifolds, called intrinsic geodesic regression, has been independently developed in [2] and [5]
- Let y_1, \dots, y_N be a set of points representing the trajectory of the mobile on a Riemannian manifold (\mathcal{M}, g) : the idea is to find a geodesic curve $\gamma(t)$ on the manifold that best fits those points in a "linear" manner at known times t_1, \dots, t_N
- The geodesic regression model is expressed as follows in [2]

$$Y = \text{Exp}(\text{Exp}(p, tv), \varepsilon)$$

where $\gamma(t) = \text{Exp}(p, tv)$ is the geodesic curve given by the initial conditions $\gamma(0) = p$ and $\dot{\gamma}(0) = v$, and ε is a Gaussian random variable taking values in the tangent space at $\gamma(t)$.



- In [3], this geodesic regression model is generalized to a polynomial one where the curve $\gamma(t)$ is a Riemannian polynomial of order k , as defined by
- $$\nabla_{\dot{\gamma}(t)} (\nabla_{\dot{\gamma}(t)} \dots (\nabla_{\dot{\gamma}(t)} \dot{\gamma}(t))) = \nabla_{\dot{\gamma}(t)}^k \dot{\gamma}(t) = 0, \text{ where } \nabla \text{ denotes the Levi - Civita connection}$$
- The estimate is found by minimizing a least squares criterion based on a Riemannian distance between the model $\gamma(t)$ and the data

$$E(\gamma) = \frac{1}{N} \sum_{j=1}^N d(\gamma(t_j), y_j)^2$$

under the constraint that γ shall be a geodesic

- One can prove, by taking the differential of E with respect to the initial conditions that it is equivalent to cancelling out the Riemannian gradient

$$\frac{2}{N} \sum_{j=1}^N \partial_p \gamma(t_j, p)^\dagger \text{Log}(\gamma(t_j, p), y_j)$$

Results: some examples work, some do not

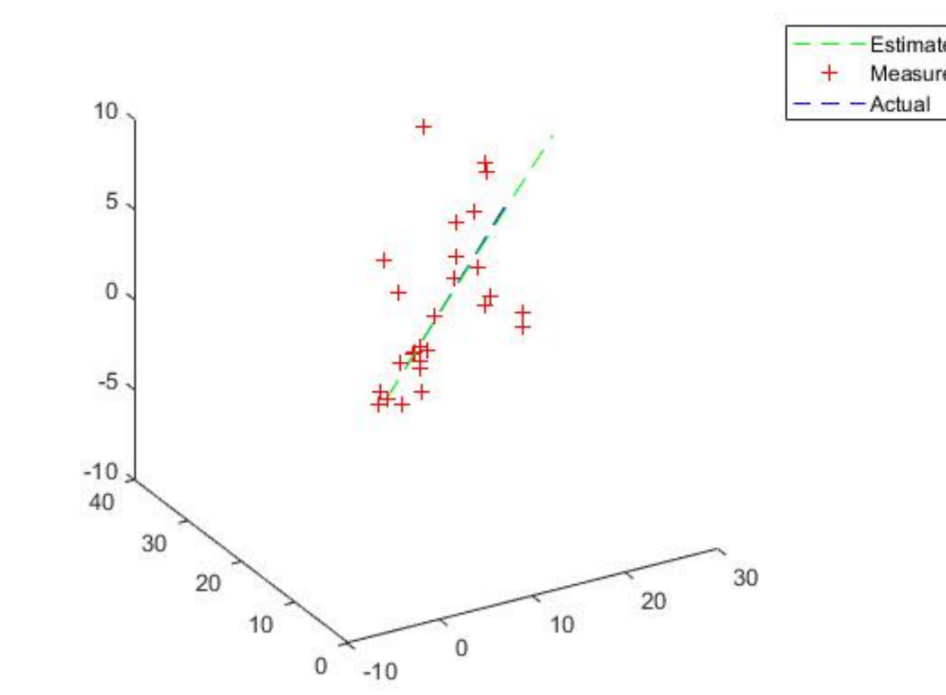


Figure 1. A straight line

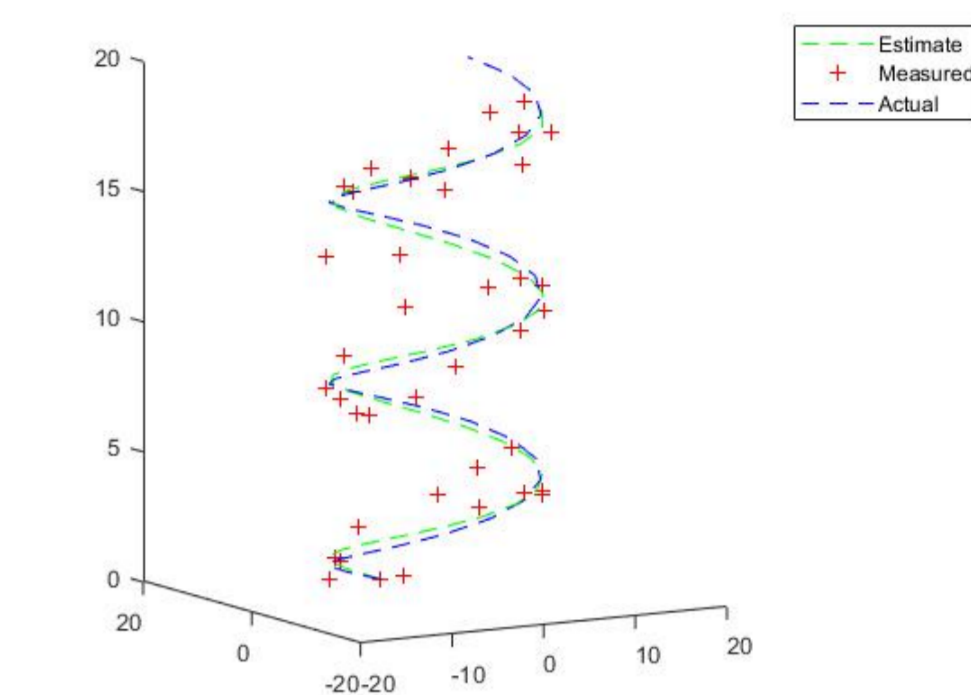


Figure 2. An helix

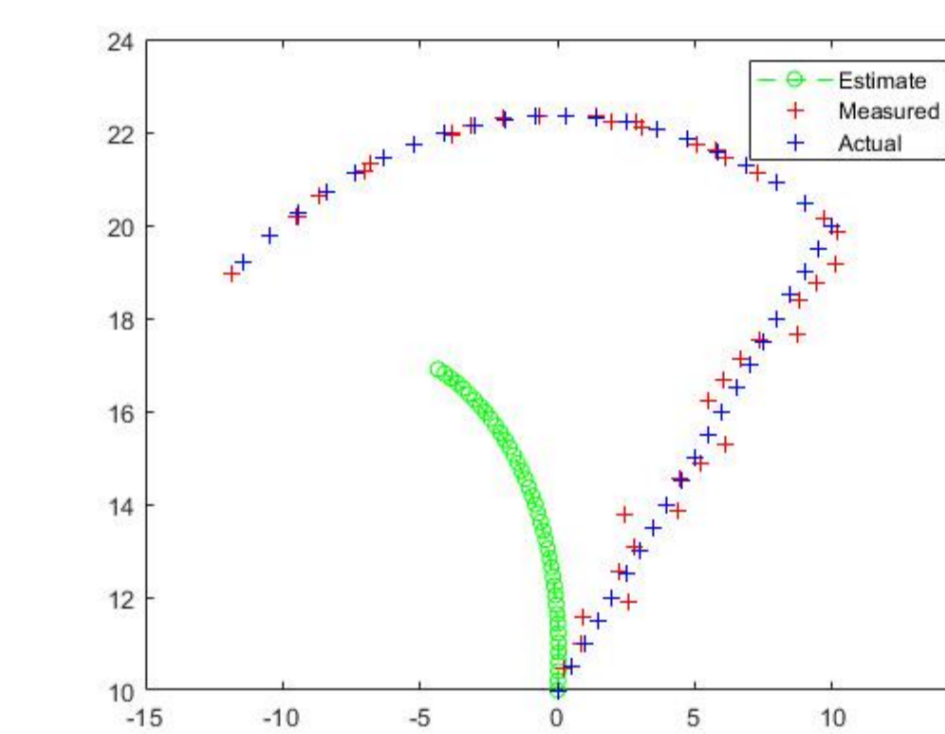


Figure 3. A straight line followed by a turn

Now what's next?

- Quantifying the goodness of the fit of the model to the data
- Local polynomials
- Describing the noise in $se(3)$: the Gaussian hypothesis holds only if the noise is "small"

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