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# About the Determinant of Complete non-ambiguous Trees

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Complete non-ambiguous trees (CNATs) are combinatorial objects which appear in various contexts. Recently, Chen and Ohlig studied the notion of leaf permutation on these objects, and proposed a series of nice conjectures. Most of them were proved by Selig and Zhu, through a connexion with the abelian sandpile model. But one conjecture remained open, about the distribution of a natural statistic named determinant. We prove this conjecture, in a bijective way.

## 1 Introduction

Non-ambiguous trees (NATs) were first defined in [1] and may be seen as a proper way to draw a binary tree on a square grid (see Definition 2.1). They were put to light as a special case of tree-like tableaux, which have been found to have applications in the PASEP model of statistical mechanics [5, 3]. The initial study of NATs revealed nice properties, mostly in an enumerative context [1, 2]. This includes enumeration formulas with respect to fixed constraints (hook formula), and new bijective proofs of combinatorial identities. When the underlying binary tree is *complete*, we are led to complete non-ambiguous trees (CNATs). These objects were first considered in [1], where it was proved that their enumerating sequence is related to the formal power series of the logarithm of the Bessel function of order 0. To end this early study of NATs, an extension to higher dimension was proposed in [9].

Recent works have revealed new facets of these objects. In [6], some striking mathematical cross-connections were obtained, such as a bijection between CNATs and fully-tiered trees of weight 0. In [7], CNATs were linked to the abelian sandpile model. In the same article, it was noticed that if we restrict a CNAT to its leaf dots, we obtain a permutation. This link was investigated in [4], where nice properties were derived, and several conjectures proposed. By using the connection with the abelian sandpile model, a large number of conjectures were proved very recently [10]. But a conjecture remained open. It asserts that when considering the set of CNATs of a fixed odd size, the number of them with an underlying permutation with even and odd determinant (signature) are equal. We give a bijective proof of this (Theorem 2.7), and include the case of the even size, which was suggested in [4].

**In this extended abstract, some technical proofs are omitted.**

## 2 Definitions and statement of the result

We first recall the definition of (complete) non-ambiguous trees, as in [1].

**Definition 2.1.** A *non-ambiguous tree* (NAT)  $T$  is a filling of an  $m \times n$  rectangular grid, where each cell is either dotted or not, satisfying the following conditions:

**(Existence of a root)** The top-left cell is dotted; we call it the *root* of  $T$ .

**(Non ambiguity)** Aside from the root, every dotted cell of  $T$  has either a dotted cell above it in the same column, or a dotted cell to its left in the same row, but not both.

**(Minimality)** Every row and every column of  $T$  contains at least one dotted cell.

**Remark 2.2.** The use of the word *tree* to describe these objects comes from the following observation. Given a NAT  $T$ , we connect every dot  $d$  different from the root to its *parent* dot  $p(d)$ , which is the dot immediately above it in the same column, or to its left in the same row (because of the condition of non ambiguity, exactly one of these must exist).

A NAT is said to be complete if the underlying tree is complete, *i.e.* every internal dot has exactly two children.

**Definition 2.3.** A *complete non-ambiguous tree* (CNAT) is a NAT in which every dot has either both a dot below it in the same column and a dot to its right in the same row (in which case the dot is said to be an *internal dot*), or neither of these (in which case the dot is said to be a *leaf*).

The *size* of a CNAT is its number of leaf dots, or equivalently one more than its number of internal dots.

We denote by  $\mathcal{T}_n$  the set of CNATs of size  $n$  and  $T_n = |\mathcal{T}_n|$ .

Figure 1 gives an example of this notion.

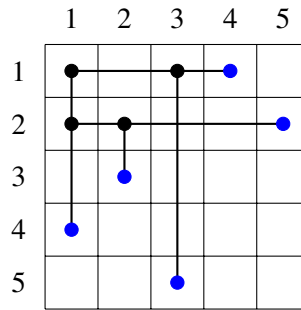


Figure 1: A CNAT of size 5. Leaf dots are represented in blue, and internal dots in black.

As in this figure, it will be convenient to label by integers the rows and columns respectively from top to bottom and from left to right (in such a way that the root appears in the cell  $(1, 1)$ ). Moreover, given a dot  $d$  in a CNAT, we denote by  $c(d)$  and  $r(d)$  the (label of) its column and row. For a given internal dot, its child in the same row is called its *right child* and its child in the same column is called its *left child*.

**Remark 2.4.** We may observe that any right leaf  $l$  in a CNAT  $T$  is the only dot in its column: there is no dot above  $l$  because this would contradict the minimality condition of Definition 2.3, and there is no dot below  $l$  because  $l$  is a leaf. In the same way, any left leaf  $l$  in  $T$  is the only dot in its row.

We may see a CNAT  $T$  as a matrix  $M(T)$  where dotted cells are 1's and undotted cells are 0's. For example, the CNAT of Figure 1 is encoded matricially as

$$\begin{pmatrix} 1 & 0 & 1 & 1 & 0 \\ 1 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{pmatrix}.$$

The numbers  $T_n$  of CNATs of size  $n$  appear as the series A002190 in [11]. As proved in [1], these numbers give a combinatorial interpretation for the development of the Bessel function  $J_0$ .

Let us now introduce the notion of permutation associated to a CNAT.

**Definition 2.5.** Let  $T$  be a CNAT of size  $n$ . It is clear that in any column of  $T$  the bottom-most dot of is a leaf, as well as the right-most dot of any row. Thus every row and every column must have exactly one leaf dot. As such, the set of leaf dots of a CNAT  $T$  of size  $n$  forms the graphical representation of an  $n$ -permutation  $\pi(T)$ . We say that  $\pi(T)$  is the permutation associated with the CNAT  $T$ .

For example, the CNAT of Figure 1 has associated permutation  $\pi(T) = 45312$ .

A careful study of permutations associated to CNATs was initiated in [4], where the following proposition is proved.

**Proposition 2.6.** *Let  $T$  be a CNAT. We have:*

$$\det M(T) = \operatorname{sgn} \pi(T).$$

Let us denote by  $T(n; \varepsilon)$  the number of CNATs of size  $n$  with determinant equal to  $\varepsilon$ . We are now in a position to state the main result of this article.

**Theorem 2.7.** *If  $n > 1$  is odd:*

$$T(n; +1) = T(n; -1) = \frac{T_n}{2}. \quad (1)$$

*If  $n$  is even (let us set  $n = 2p$ ):*

$$T(2p; +1) = \frac{T_{2p} + (-1)^p T_p}{2} \text{ and } T(2p; -1) = \frac{T_{2p} - (-1)^p T_p}{2}. \quad (2)$$

The odd case corresponds to Conjecture 2.6 in [4], the even case to Remark 2.7 in the same paper.

### 3 A bijective proof of Theorem 2.7

This section is devoted to proving our main result. This proof is bijective. More precisely, we shall:

1. introduce a subset  $\mathcal{A}_{2p} \subset \mathcal{T}_{2p}$  of CNATs of even size, with  $\mathcal{A}_{2p} = T_p$ , and such that for any  $T \in \mathcal{A}_{2p}$  we know that  $\operatorname{sgn} \pi(T) = (-1)^p$ ;
2. construct an *involution*  $\Phi$  on the set of CNATs such that if  $T$  is not in any of the sets  $\mathcal{A}_{2p}$  we have:

$$\operatorname{sgn} \pi(\Phi(T)) = -\operatorname{sgn} \pi(T).$$

We first introduce a useful notion on the leaves of CNATs.

**Definition 3.1.** A leaf in a CNAT is said to be *short* if its parent is in a cell adjacent to it. Otherwise the leaf is said to be *long*. Moreover, we denote by  $\mathcal{A}_n$  the set of CNATs of size  $n$  with only short leaves.

Figure 2 illustrates this notion.

The elements of  $\mathcal{A}_p$  are designed to be the fixed points of our involution  $\Phi$ . We treat their case with two propositions, whose proofs are omitted.

**Proposition 3.2.** *When the size  $n = 2p + 1$  is odd, the set  $\mathcal{A}_{2p+1}$  is empty. When the size  $n = 2p$  is even, the set  $\mathcal{A}_{2p}$  is in bijection with  $\mathcal{T}_p$ .*

**Proposition 3.3.** *Let  $T$  be an element of  $\mathcal{A}_{2p}$ , its determinant is given by:*

$$\operatorname{sgn} \pi(T) = (-1)^p.$$

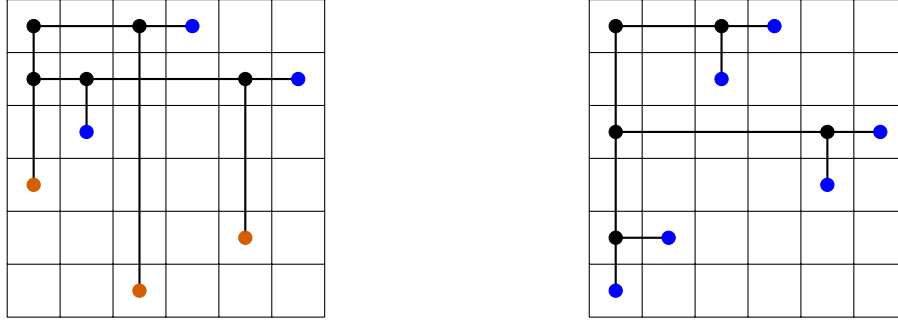


Figure 2: (Left) A CNAT with short leaves in blue and long leaves in red.  
(Right) An element of  $\mathcal{A}_6$ .

We now come to the definition of a function  $\Phi$  on  $\mathcal{T}_n$ , which is the key construction of this work. We first introduce the following notion.

**Definition 3.4.** Let  $T$  be a CNAT, and  $l_1$  and  $l_2$  two leaves in  $T$  with respective parent  $p_1$  and  $p_2$ . If  $l_1$  and  $l_2$  are both left leaves, they are said to be *interacting* if

$$r(p_1) < r(l_2) < r(l_1) \text{ or } r(p_2) < r(l_1) < r(l_2).$$

The definition is similar for right leaves.

This notion is illustrated by Figure 3.

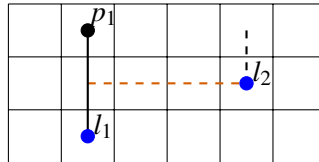


Figure 3: Interacting leaves.

The interest of this notion of interacting leaves is put to light by the following operation.

**Definition 3.5.** For two interacting left leaves  $l_1$  and  $l_2$  in a CNAT  $T$ , we define the *switch* of these two leaves as the exchange of the row labels of  $l_1$  and  $l_2$ . More precisely:

- we erase  $l_1$  and we put a new leaf  $l'_1$  in the same column, in row  $r(l_2)$ ;
- we erase  $l_2$  and we put a new leaf  $l'_2$  in the same column, in row  $r(l_1)$ .

By doing this, we obtain an object  $T' = S(T, l_1, l_2)$ .

We have the same operation for right (interacting) leaves.

This notion is illustrated by Figure 4

**Proposition 3.6.** Let  $T$  be in  $\mathcal{T}_n$ . For two interacting leaves  $l_1$  and  $l_2$  in  $T$ ,  $S(T, l_1, l_2)$  is in  $\mathcal{T}_n$ .

*Proof.* The only condition in Definition 2.1 that is not trivially satisfied in non ambiguity. This is a direct consequence of Remark 2.4.  $\square$



Figure 4: Switching interacting leaves.

The technical part of the construction of  $\Phi$  now relies on the following lemma, whose proof is omitted.

**Lemma 3.7.** *Any CNAT with a long leaf has at least two interacting leaves.*

Let us now precise the construction of  $\Phi : \mathcal{T}_n \rightarrow \mathcal{T}_n$ . First of all, we set that for any  $T \in \mathcal{A}_n$ ,  $\Phi(T) = T$ . This case done, we are reduced to the case where  $T$  has at least one long leaf. By Lemma 3.7,  $T$  contains interacting leaves. To define  $\Phi$ , we want to *choose* a pair of interacting leaves. Since the set of interacting leaves may change when we switch leaves, we have to choose in a way such that we create an involution. If  $T$  contains left interacting leaves, we consider the (non-empty) set  $\{(r(l_1), r(l_2)); l_1 \text{ and } l_2 \text{ interacting}\}$  and choose  $l_1$  and  $l_2$  which correspond to the lexicographical maximum of this set. Let us call these interacting leaves *active*. This done, we set  $\Phi(T) = S(T, l_1, l_2)$ . And if  $T$  contains only right interacting leaves, we consider the lexicographical maximum of the set  $\{(c(l_1), c(l_2)); l_1 \text{ and } l_2 \text{ interacting}\}$  to choose the pair of active leaves.

**Proposition 3.8.** *The function  $\Phi$  is an involution on  $\mathcal{T}_n$ . Moreover, if  $\mathcal{T} \notin \mathcal{A}_n$  then*

$$\text{sgn } \pi(\Phi(T)) = -\text{sgn } \pi(T).$$

*Proof.* Omitted. □

Figure 5 shows an example of the application of  $\Phi$ .

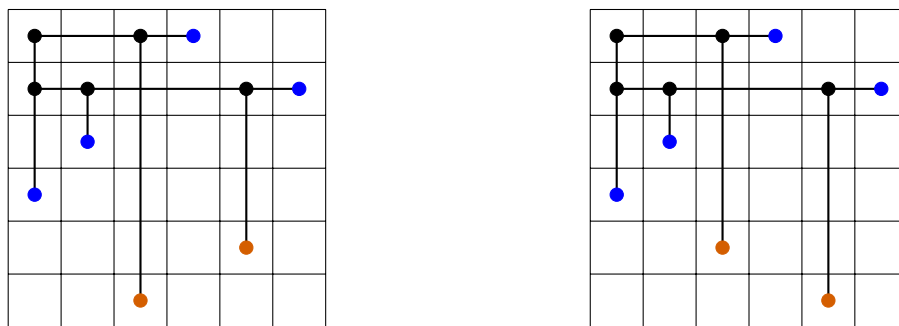


Figure 5: A CNAT and its image under  $\Phi$ . Active leaves appear in red.

We can now conclude the proof of our main result.

*Proof of Theorem 2.7.* It is a consequence of Propositions 3.2, 3.3 and 3.8. □

## References

- [1] Jean-Christophe Aval, Adrien Boussicault, Mathilde Bouvel & Matteo Silimbani (2014): *Combinatorics of non-ambiguous trees*. *Advances in Applied Mathematics* 56, pp. 78–108, doi:10.1016/j.aam.2013.11.004.
- [2] Jean-Christophe Aval, Adrien Boussicault, Bérénice Delcroix-Oger, Florent Hivert & Patxi Laborde-Zubieta (2021): *Non-ambiguous trees: new results and generalization*. *European Journal of Combinatorics* 95, p. 103331, doi:10.1016/j.ejc.2021.103331.
- [3] Jean-Christophe Aval, Adrien Boussicault & Philippe Nadeau (2011): *Tree-like Tableaux*. *FPSAC'11 - 23rd International Conference on Formal Power Series and Algebraic Combinatorics* 20, doi:10.37236/3440.
- [4] Daniel Chen & Sebastian Ohlig (2023): *Associated Permutations of Complete Non-Ambiguous Trees*, doi:10.48550/arXiv.2210.11117. arXiv:2210.11117.
- [5] Sylvie Corteel & Lauren Williams (2006): *A Markov Chain on Permutations which Projects to the PASEP*. *International Mathematics Research Notices* 17, doi:10.1093/imrn/rnm055.
- [6] William Dugan, Sam Glennon, Paul E. Gunnells & Einar Steingrímsson (2019): *Tiered trees, weights, and  $q$ -Eulerian numbers*. *Journal of Combinatorial Theory, Series A* 164, pp. 24–49, doi:10.1016/j.jcta.2018.12.002.
- [7] Mark Dukes, Thomas Selig, Jason P. Smith & Einar Steingrímsson (2019): *Permutation graphs and the abelian sandpile model, tiered trees and non-ambiguous binary trees*. *Electronic Journal of Combinatorics* 26(3), p. p3.29, doi:10.37236/8225.
- [8] Patxi Laborde Zubieta (2015): *Occupied corners in tree-like tableaux*. *Seminaire Lotharingien de Combinatoire* 74, p. 74b, doi:10.48550/arXiv.1505.06098.
- [9] Patxi Laborde-Zubieta (2017): *Combinatoire de l'ASEP, arbres non-ambigus et polyminos parallélogrammes périodiques*. Ph.D. thesis, Université de Bordeaux.
- [10] Thomas Selig & Haoyue Zhu (2023): *Complete non-ambiguous trees and associated permutations: connections through the Abelian sandpile model*. *arXiv e-prints:arXiv:2303.15756*, doi:10.48550/arXiv.2303.15756. arXiv:2303.15756.
- [11] Neil J. A. Sloane & The OEIS Foundation Inc. (2022): *The on-line encyclopedia of integer sequences*. <http://oeis.org/?language=english>.