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Cramér's estimate for stable processes with power drift

Christophe Profeta* Thomas Simon †

Abstract

We investigate the upper tail probabilities of the all-time maximum of a stable Lévy process with a power negative drift. The asymptotic behaviour is shown to be exponential in the spectrally negative case and polynomial otherwise, with explicit exponents and constants. Analogous results are obtained, at a less precise level, for the fractionally integrated stable Lévy process. We also study the lower tail probabilities of the integrated stable Lévy process in the presence of a power positive drift.

Keywords: Extremes ; Lower tail probabilities ; Power drift ; Stable process.
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1 Introduction and statement of the results

Let L be a real strictly α-stable Lévy process with characteristic exponent

$$\Psi(\lambda) \;=\; \log(\mathbb{E}[e^{\mathrm{i}\lambda L_1}]) \;=\; -\,(\mathrm{i}\lambda)^\alpha e^{-\mathrm{i}\pi\alpha\rho\,\mathrm{sgn}(\lambda)} \;=\; -\,|\lambda|^\alpha e^{\mathrm{i}\pi\alpha(1/2-\rho)\,\mathrm{sgn}(\lambda)}, \qquad \lambda\in\mathbb{R},$$

where $\alpha\in(0,2]$ is the scaling parameter and $\rho=\mathbb{P}[L_1>0]$ is the positivity parameter. Recall e.g. from Lemma 14.11 and Theorem 14.19 in [19] that $\rho\in[1-1/\alpha,1/\alpha]$ for $\alpha\in[1,2]$ and $\rho\in[0,1]$ for $\alpha\in(0,1)$, and that with this normalization, for $\alpha\in(0,2)$ the Lévy measure of L has density

$$\nu(x) \;=\; \frac{\Gamma(1+\alpha)}{\pi}\left(\frac{\sin(\pi\alpha(1-\rho))}{|x|^{1+\alpha}}\mathbf{1}_{\{x<0\}} \;+\; \frac{\sin(\pi\alpha\rho)}{x^{1+\alpha}}\mathbf{1}_{\{x>0\}}\right).$$

Throughout, we assume that L takes positive values i.e. $\rho\neq 0$, and we exclude the degenerate case $\alpha=\rho=1$ where L is a unit drift. With these restrictions, L has no positive jumps if and only if $\alpha>1$ and $\rho=1/\alpha$.

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Consider the positive random variable

$$\mathbf{M}_{\alpha,\rho,\gamma} \;=\; \sup_{t\geq 0}\{L_t - t^\gamma\}.$$

It is well-known from e.g. Proposition 48.10 in [19] that

$$\mathbb{P}[\mathbf{M}_{\alpha,\rho,\gamma} < \infty] \;=\; \begin{cases} 1 & \text{if } \gamma\alpha > 1 \\ 0 & \text{if } \gamma\alpha \leq 1. \end{cases}$$

In this paper, we are concerned with the asymptotic behaviour of

$$\mathbb{P}[\mathbf{M}_{\alpha,\rho,\gamma} \geq x], \qquad x \to \infty,$$

in the relevant case $\gamma\alpha > 1$. In the literature, the evaluation of such asymptotics having various applications in insurance is coined as Cramér's estimate. In the case of a linear drift $\gamma = 1$, we refer to (XI.6.16) and (XII.5.10) in [9] for random walks and to [6] and [14, Section 7.2] for Lévy processes having one-sided exponential moments. Applied to stable Lévy processes, the main result of [6] shows

$$\mathbb{P}[\mathbf{M}_{\alpha,1/\alpha,1} \geq x] \;\sim\; e^{-x}, \qquad x \to \infty, \tag{1.1}$$

for $\alpha > 1$, and it is well-known that the asymptotics is in fact an equality - see [20] or Corollary VII.2 in [4]. For more general power drifts and a class of Gaussian processes fulfilling a certain scaling property, we refer to [12] which, applied to the important case of Brownian motion with a parabolic drift, yields

$$\mathbb{P}[\mathbf{M}_{2,1/2,2} \geq x] \;\sim\; \frac{1}{\sqrt{3}}\,\exp\left\{-\frac{4}{3\sqrt{3}}\,x^{3/2}\right\}, \qquad x \to \infty. \tag{1.2}$$

Let us mention that this estimate has been refined in Theorem 2.1 of [11], where a complete asymptotic expansion at infinity is obtained - see also Lemma 2.1 and the references therein for closed expressions of the density of $\mathbf{M}_{2,1/2,2}$ in terms of the Airy function. The first result of the present paper is the following general estimate, extending (1.1) and (1.2).

Theorem A. Assume $\gamma\alpha > 1$.

(a) If L has positive jumps, one has

$$\mathbb{P}[\mathbf{M}_{\alpha,\rho,\gamma} \geq x] \;\sim\; \frac{\sin(\pi\alpha\rho)}{\pi}\,\Gamma(\alpha - 1/\gamma)\Gamma(1 + 1/\gamma)\,x^{\frac{1}{\gamma}-\alpha}.$$

(b) If L has no positive jumps, one has

$$\mathbb{P}[\mathbf{M}_{\alpha,1/\alpha,\gamma} \geq x] \;\sim\; \sqrt{\frac{\alpha-1}{\gamma\alpha-1}}\;\exp\left\{-(\alpha-1)\,\gamma^{\frac{\alpha}{\alpha-1}}\,(\gamma\alpha-1)^{\frac{1-\gamma\alpha}{\gamma(\alpha-1)}}\,x^{\frac{\gamma\alpha-1}{\gamma(\alpha-1)}}\right\}.$$

In the specific case $\alpha \in (1/2, 2]$ and $\gamma = 2$, these estimates are somehow reminiscent of those previously obtained in [5] in the framework of Burgers turbulence with stable noise initial data. See Remark 2.6 below for more detail. Our arguments, quite different from those of [5], rely on the compensation formula for the case with positive jumps and on some ad hoc and rather involved estimates combined with Laplace's method in the spectrally negative case.

In the second part of the paper we consider the Riemann-Liouville (or fractionally integrated) stable process with parameter $\beta > 0$, defined as

$$L_t^{(\beta)} \;=\; \int_0^t (t-s)^\beta dL_s \;=\; \beta \int_0^t (t-s)^{\beta-1} L_s\, ds, \qquad t \ge 0.$$

The process $\{L_t^{(\beta)},\, t \ge 0\}$ is stable in the broad sense of [18], and by Proposition 3.4.1 therein we have

$$L_1^{(\beta)} \;\stackrel{d}{=}\; (1+\alpha\beta)^{-1/\alpha} L_1. \tag{1.3}$$

Recall also that $\{L_t^{(\beta)},\, t \ge 0\}$ is self-similar with index $\beta + 1/\alpha$, non-Markovian, and that it has a.s. continuous sample paths. Consider the positive random variable

$$\mathbf{M}^{(\beta)}_{\alpha,\rho,\gamma} \;=\; \sup_{t\ge 0}\{L_t^{(\beta)} - t^{\beta+\gamma}\},$$

which can be viewed as an extension of $\mathbf{M}_{\alpha,\rho,\gamma}$. Observe also from Theorem 10.5.1 in [18] and self-similarity that

$$\mathbb{P}[\mathbf{M}^{(\beta)}_{\alpha,\rho,\gamma} < \infty] \;=\; \mathbb{P}[\mathbf{M}_{\alpha,\rho,\gamma} < \infty]$$

for every $\beta > 0$. It is also easy to check that $\mathbf{M}^{(\beta)}_{\alpha,\rho,\gamma} \xrightarrow{d} \mathbf{M}_{\alpha,\rho,\gamma}$ as $\beta \to 0$ when $\gamma\alpha > 1$. As a rule, the non-Markovian character of a given process makes its passage times across a level more difficult to investigate and our second main result has a less precise character. Here and throughout, we use the standard notation $f(x) \asymp g(x)$ to express the fact that there exist two positive finite constants κ_1, κ_2 such that $\kappa_1 f(x) \le g(x) \le \kappa_2 f(x)$ as $x \to \infty$ or as $x \to 0$, the nature of the limit being clear from the context.

Theorem B. Assume $\gamma\alpha > 1$.

(a) If L has positive jumps, one has

$$\mathbb{P}[\mathbf{M}^{(\beta)}_{\alpha,\rho,\gamma} \ge x] \;\asymp\; x^{\frac{1-\gamma\alpha}{\beta+\gamma}}.$$

(b) If L has no positive jumps, one has

$$\log \mathbb{P}[\mathbf{M}^{(\beta)}_{\alpha,1/\alpha,\gamma} \ge x] \;\sim\; -c_{\alpha,\beta,\gamma}\, x^{\frac{\gamma\alpha-1}{(\alpha-1)(\gamma+\beta)}}$$

with $c_{\alpha,\beta,\gamma} = (\alpha-1)\,(\gamma+\beta)^{\frac{\alpha}{\alpha-1}}\,(\alpha\beta+1)^{\frac{\gamma+\beta-1-\alpha\beta}{(\alpha-1)(\gamma+\beta)}}\,(\gamma\alpha-1)^{\frac{1-\gamma\alpha}{(\alpha-1)(\gamma+\beta)}} \;>\; 0.$

The method to get these estimates differs here for the lower bound and the upper bound. The former uses a simple scaling argument, inspired by that of [12], and amounts to a comparison with the upper tails of L_1. The latter relies on telescoping sums for the case with positive jumps, and on a simple yet powerful association lemma in the spectrally negative case - see Lemma 3.1.

In the last part of the paper, we study the lower tail problem for the integrated stable process with a power positive drift. In a Gaussian framework, lower tail probabilities have many applications described in [15]. In a self-similar framework they are connected to the persistence probabilities, whose applications are also manifold - see the recent survey [3]. We show the following.

Theorem C. Assume $\gamma\alpha > 1$ and $\rho \in (0,1)$. For every $\mu \ge 0$, one has

$$\mathbb{P}\left[\sup_{0\le t\le 1}\left\{L_t^{(1)} + \mu t^{1+\gamma}\right\} \le \varepsilon\right] \;\asymp\; \varepsilon^{\frac{\alpha\rho}{(\alpha+1)(\alpha(1-\rho)+1)}}.$$

Above, we have excluded the case $\rho = 1$, where the estimate amounts by monotonicity to the one-dimensional estimate $\mathbb{P}[L_1 + \mu \le \varepsilon]$, which is exponentially small for $\mu = 0$ - see e.g. (14.35) in [19] - and zero for $\mu > 0$. Theorem C is an extension of Theorem A in [16] which dealt with the case $\mu = 0$. In this respect, we should mention that the condition $\gamma\alpha > 1$ on the drift power is optimal: in the Cauchy case $\alpha = \gamma = 1$, the same Theorem A in [16] shows that the lower tail probability exponent depends on μ. Our argument here relies in an essential way on the strong Markov property of the bidimensional process $\{(L_t^{(1)}, L_t),\, t \ge 0\}$ and is hence specific to the case $\beta = 1$. The other cases are believed to be challenging. To give one example, for $\alpha = 2, \mu = 0$ and $\beta \ge 2$ an integer, finding the exact values of the exponents κ_n in the asymptotics

$$\mathbb{P}\left[\sup_{0\le t\le 1}\left\{L_t^{(n)}\right\} \le \varepsilon\right] \asymp \varepsilon^{\kappa_n}$$

as $\varepsilon \to 0$ is still an open problem on Brownian motion - see Section 3.3 in [3]. In our proof the aforementioned association Lemma 3.1 plays also a significant role. Unfortunately, its one-sided character prevents us from dealing with the case of a negative power drift. We leave this question, whose connection to Burgers turbulence with stable Lévy process initial data in the case $\alpha > 1$ and $\gamma = 1$ is precisely described in Section 4.1 of [3], to future research.

2 Proof of Theorem A

2.1 The case with positive jumps

We will use the standard notation

$$c_+ \;=\; \frac{\Gamma(1+\alpha)}{\pi}\sin(\pi\rho\alpha) \;>\; 0$$

for simplicity. Defining for every $x > 0$ the stopping time

$$T_x \;=\; \inf\{t \ge 0;\, L_t > t^\gamma + x\},$$

we have $\mathbb{P}[\mathbf{M}_{\alpha,\rho,\gamma} \ge x] \;=\; \mathbb{P}[T_x < \infty]$. We also set $K_x = L_{T_x} - T_x^\gamma - x$ for the overshoot at T_x. For every $f : \mathbb{R}^+ \to \mathbb{R}^+$ measurable and such that $f(0) = 0$, the compensation formula - see [4] p. 7 or Theorem 19.2 in [19] - implies

$$\begin{aligned}
\mathbb{E}\left[f(K_x)\,\mathbf{1}_{\{T_x<\infty\}}\right] &= \mathbb{E}\left[\sum_{t\ge 0} f\left(L_{t^-} + \Delta L_t - t^\gamma - x\right)\mathbf{1}_{\{L_u<u^\gamma+x\ \forall u<t,\, t^\gamma+x<L_{t^-}+\Delta L_t\}}\right]\\
&= c_+\mathbb{E}\left[\int_0^\infty dt\int_0^\infty f\left(L_t + s - t^\gamma - x\right)\mathbf{1}_{\{L_u<u^\gamma+x\ \forall u<t,\, t^\gamma+x<L_t+s\}}\, s^{-1-\alpha} ds\right]\\
&= c_+\int_0^\infty dt\int_0^\infty \mathbb{E}\left[f(z - t^\gamma - x)\mathbf{1}_{\{L_u<u^\gamma+x\ \forall u<t,\, t^\gamma+x<z\}}(z-L_t)^{-1-\alpha}\right] dz.
\end{aligned}$$

Taking $f(u) = \mathbf{1}_{\{u>0\}}$ and integrating in z, we obtain

$$\begin{aligned}
\mathbb{P}[K_x > 0,\, T_x < \infty] &= \frac{c_+}{\alpha}\int_0^\infty \mathbb{E}\left[(t^\gamma + x - L_t)^{-\alpha}\,\mathbf{1}_{\{L_u<u^\gamma+x\ \forall u<t\}}\right] dt\\
&= \frac{c_+}{\alpha}\left(\int_0^\infty \mathbb{E}\left[(s^\gamma + 1 - x^{\frac{1}{\gamma\alpha}-1}L_s)^{-\alpha}\,\mathbf{1}_{\{x^{\frac{1}{\gamma\alpha}-1}L_v<v^\gamma+1\ \forall v<s\}}\right] ds\right) x^{\frac{1}{\gamma}-\alpha}\\
&\sim \frac{c_+}{\alpha}\left(\int_0^\infty (s^\gamma+1)^{-\alpha}\, ds\right) x^{\frac{1}{\gamma}-\alpha}\\
&\sim \frac{\sin(\pi\alpha\rho)}{\pi}\,\Gamma(\alpha - 1/\gamma)\Gamma(1+1/\gamma)\, x^{\frac{1}{\gamma}-\alpha}
\end{aligned}$$

where the second equality follows by scaling, the convergence on the third line is obtained by bounded and monotone convergence (decomposing into $\{L_s < 0\}$ and $\{L_s \ge 0\}$ inside the expectation), and the evaluation of the integral on the fourth line is standard. To conclude the proof, it remains to show that L does not creep at T_x, in other words that

$$\mathbb{P}[K_x = 0,\, T_x < \infty] \;=\; 0. \tag{2.1}$$

The latter is in accordance with the well-known fact that L does not creep at a fixed level $x > 0$ - see Theorem VI.19 and Lemma VIII.1 in [4]. However, this result does not apply here since we consider the first passage time above a moving boundary. To show (2.1), fix $x > 0$ and decompose

$$\mathbb{P}[L_s \ge s^\gamma + x] \;=\; P_1(s) \,+\, P_2(s)$$

for every $s \ge 0$, with

$$\left\{\begin{array}{lcl} P_1(s) & = & \mathbb{P}[\widetilde{L}_{s-T_x} + T_x^\gamma \ge s^\gamma,\, K_x = 0,\, T_x < s] \\ P_2(s) & = & \mathbb{P}[\widetilde{L}_{s-T_x} + L_{T_x} \ge s^\gamma + x,\, K_x > 0,\, T_x < s], \end{array}\right.$$

where $\widetilde{L}$ is a copy of L which is independent of (T_x, L_{T_x}), by the strong Markov property. On the one hand, we see by scaling and e.g. Property 1.2.15 in [18] that

$$\mathbb{P}[L_s \ge s^\gamma + x] \;\sim\; \frac{c_+}{\alpha}\, s^{1-\gamma\alpha}.$$

On the other hand, we have

$$P_1(s) \;\ge\; \mathbb{P}[\widetilde{L}_{s-T_x} \ge s^\gamma,\, K_x = 0,\, T_x < s/2] \;\ge\; \mathbb{P}[\widetilde{L}_1 \ge 2^{\frac{1}{\alpha}} s^{\gamma - \frac{1}{\alpha}}]\, \mathbb{P}[K_x = 0,\, T_x < s/2]$$

and passing to the limit, we obtain

$$\liminf_{s\to\infty} s^{\gamma\alpha - 1}\, P_1(s) \;\ge\; \frac{c_+}{2\alpha}\, \mathbb{P}[K_x = 0,\, T_x < \infty].$$

Hence, we see that (2.1) is a consequence of

$$P_2(s) \;\sim\; \frac{c_+}{\alpha}\, s^{1-\gamma\alpha}. \tag{2.2}$$

Applying the compensation formula as above, we obtain

$$P_2(s) \;=\; c_+ \int_0^s dt \int_0^\infty \mathbb{P}[\widetilde{L}_{s-t} + L_t + z \ge s^\gamma + x,\, L_t + z > t^\gamma + x,\, L_u < u^\gamma + x \;\forall u < t]\, z^{-1-\alpha}\, dz.$$

Changing the variables $z = s^\gamma y$ and $t = su$, we see that $c_+^{-1} s^{\gamma\alpha - 1} P_2(s)$ equals

$$\int_0^1 du \int_0^\infty \mathbb{P}\left[s^{\frac{1}{\alpha}} \widetilde{L}_{1-u} + s^{\frac{1}{\alpha}} L_u + s^\gamma y \ge s^\gamma + x, \right.$$
$$\left. s^{\frac{1}{\alpha}} L_u > s^\gamma (u^\gamma - y) + x,\, s^{\frac{1}{\alpha}} L_u < s^\gamma\, u^\gamma + x \;\forall u < 1 \right] y^{-1-\alpha} dy,$$

which converges as $s \to \infty$ to

$$\int_0^1 du \int_1^\infty y^{-1-\alpha} dy \;=\; \frac{1}{\alpha}.$$

This shows (2.2), and completes the proof.

□

Remark 2.1. (a) Setting, here and throughout, $L_t^* = \sup\{L_s,\, s \in [0,t]\}$ for every $t > 0$, we have

$$\lim_{\gamma\to\infty} \mathbb{P}[\mathbf{M}_{\alpha,\rho,\gamma} \geq x] \;=\; \mathbb{P}[L_1^* \geq x]$$

for every $x \geq 0$. Passing formally to the limit $\gamma \to \infty$ in Theorem A (a), we can infer

$$\mathbb{P}[L_1^* \geq x] \;\sim\; \frac{\Gamma(\alpha)\sin(\pi\alpha\rho)}{\pi}\, x^{-\alpha} \tag{2.3}$$

which is a standard and rigorous estimate - see Theorem 10.5.1 in [18] and Proposition VIII.4 in [4].

(b) Taking $f(u) = \mathbf{1}_{\{u\geq rx\}}$ for some $r > 0$ and applying as above the compensation formula leads to the estimate

$$\mathbb{P}[K_x \geq rx,\, T_x < \infty] \;\sim\; \frac{c_+}{\alpha}\left(\int_0^\infty (r + u^\gamma + 1)^{-\alpha}\, du\right) x^{\frac{1}{\gamma}-\alpha} \;\sim\; (r+1)^{\frac{1}{\gamma}-\alpha}\mathbb{P}[T_x < \infty].$$

This implies the following limit theorem for the law of the renormalized overshoot:

Proposition 2.2.

$$\mathcal{L}\left(x^{-1}K_x \,\middle|\, T_x < +\infty\right) \;\to\; \mathrm{Pareto}\,(\alpha - 1/\gamma) \qquad \textit{as } x \to \infty.$$

Recall, indeed, that the standard Pareto distribution with parameter $\delta > 0$ has distribution function $1 - (r+1)^{-\delta}$ on $(0,\infty)$. This observation seems new even in the classical case of a linear drift $\gamma = 1$ with $\alpha > 1$. Notice that still in the case of a linear drift, the limit behaviour of the overshoot is very different for Lévy processes having finite exponential moments. If we consider for example the tempered stable subordinator with negative unit drift and Lévy measure having density

$$\nu(x) \;=\; \frac{\alpha\, e^{-cx}}{\Gamma(1-\alpha)x^{\alpha+1}}\, \mathbf{1}_{\{x>0\}}$$

for some $c \in (0,1)$, then we are in the framework of [6] with $\omega \in (0,1)$ and $\mu^* < \infty$ so that $C > 0$ in (5) therein. By Remark 2 of [6], this implies that K_x converges at infinity to some proper random variable - see also Theorem 4.2 in [13] for more general results.

(c) In the case $\alpha > 1, \rho = 1 - 1/\alpha$ and $\gamma = 1$, the Laplace transform of $\mathbf{M}_{\alpha,1-1/\alpha,1}$ can be computed with the help of Zolotarev's well-known general formula - see [20]: one finds

$$\mathbb{E}[e^{-\lambda \mathbf{M}_{\alpha,1-1/\alpha,1}}] \;=\; \frac{1}{1+\lambda^{\alpha-1}}\cdot$$

This Laplace transform can be easily inverted and yields the identity in law

$$\mathbf{M}_{\alpha,1-1/\alpha,1} \;\stackrel{d}{=}\; \mathbf{L}^{\frac{1}{\alpha-1}} \,\times\, \mathbf{Z}_{\alpha-1}$$

where $\mathbf{L} \sim \mathrm{Exp}(1)$ and $\mathbf{Z}_{\alpha-1}$ has a standard positive $(\alpha-1)-$stable law with Laplace transform

$$\mathbb{E}[e^{-\lambda \mathbf{Z}_{\alpha-1}}] \;=\; e^{-\lambda^{\alpha-1}},$$

both random variables being independent. This shows that the law of $\mathbf{M}_{\alpha,1-1/\alpha,1}$ is the so-called Mittag-Leffler distribution of parameter $\alpha - 1$ which is studied e.g. in Exercise 34.4 of [19] - see also the references therein. In particular, there exists a closed expression for the survival function of $\mathbf{M}_{\alpha,1-1/\alpha,1}$ in terms of the classical Mittag-Leffler function, which leads to a complete and simple asymptotic expansion at infinity: one has

$$\mathbb{P}[\mathbf{M}_{\alpha,1-1/\alpha,1} > x] \;=\; E_{\alpha-1}(-x^{\alpha-1}) \;\sim\; \sum_{n\geq 1} \frac{(-1)^{n-1}x^{-(\alpha-1)n}}{\Gamma(1-(\alpha-1)n)}$$

where we have used Formula 18.1(7) in [8] and the standard notation for asymptotic expansions given e.g. in Appendix C of [1]. Observe from the complement formula for the Gamma function that the first term matches the one that can be derived from Theorem A (a), in this specific case. Notice also the following closed formula for the distribution function, as a convergent series:

$$\mathbb{P}[\mathbf{M}_{\alpha,1-1/\alpha,1} \leq x] \;=\; 1 \,-\, E_{\alpha-1}(-x^{\alpha-1}) \;=\; \sum_{n\geq 1} \frac{(-1)^{n-1}x^{(\alpha-1)n}}{\Gamma(1+(\alpha-1)n)}.$$

Let us finally refer to [10] for related results in the presence of a compound Poisson process.

2.2 The case with no positive jumps

Applying the strong Markov property at T_x and using the absence of positive jumps, we get

$$\begin{aligned}\int_0^\infty (1-e^{-\lambda t})\,\mathbb{P}[L_t > t^\gamma + x]\,dt &= \mathbb{E}\left[\mathbf{1}_{\{T_x<\infty\}} \int_0^\infty (1-e^{-\lambda(T_x+t)})\,\mathbf{1}_{\{\widetilde{L}_t + T_x^\gamma > (t+T_x)^\gamma\}}\,dt\right] \\ &= \mathbb{E}\left[\mathbf{1}_{\{T_x<\infty\}} \int_0^\infty (1-e^{-\lambda(T_x+t)})\,\mathbf{1}_{\{t^{1/\alpha}\widetilde{L}_1^+ > (t+T_x)^\gamma - T_x^\gamma\}}\,dt\right]\end{aligned}$$

where we have set $a^+ = \max(a,0)$ and, on the right-hand side, $\widetilde{L}$ is an independent copy of L. Integrating both sides on $(0,\infty)$ with respect to $\lambda^{-\nu-1}d\lambda$ with $\nu\in(0,1)$, we deduce

$$\begin{aligned}\int_0^\infty t^\nu\,\mathbb{P}[L_t > t^\gamma + x]\,dt &= \mathbb{E}\left[\mathbf{1}_{\{T_x<\infty\}} \int_0^\infty (T_x+t)^\nu\,\mathbf{1}_{\{t^{1/\alpha}\widetilde{L}_1^+ > (t+T_x)^\gamma - T_x^\gamma\}}\,dt\right] \\ &= \mathbb{E}\left[\mathbf{1}_{\{T_x<\infty\}}\,T_x^{1+\nu} \int_0^\infty (1+t)^\nu\,\mathbf{1}_{\{\widetilde{L}_1^+ > T_x^{\gamma-1/\alpha}\,\varphi_{\alpha,\gamma}(t)\}}\,dt\right],\end{aligned}$$

where

$$\varphi_{\alpha,\gamma}(t) \;=\; \frac{(1+t)^\gamma - 1}{t^{1/\alpha}}$$

is an increasing homeomorphism from $(0,\infty)$ to $(0,\infty)$, because $\alpha\gamma > 1$ and $\alpha > 1$. This implies the identity

$$\begin{aligned}\int_0^\infty t^\nu\,\mathbb{P}[L_1^+ > t^{-1/\alpha}(t^\gamma + x)]\,dt &= \mathbb{E}\left[\mathbf{1}_{\{T_x<\infty\}}\,T_x^{1+\nu} \int_0^\infty (1+t)^\nu\,\mathbf{1}_{\{\varphi_{\alpha,\gamma}^{-1}(T_x^{1/\alpha-\gamma}\widetilde{L}_1^+) > t\}}\,dt\right] \\ &= \frac{1}{1+\nu}\,\mathbb{E}\left[\mathbf{1}_{\{T_x<\infty\}}\,T_x^{1+\nu}\left((1+\varphi_{\alpha,\gamma}^{-1}(T_x^{1/\alpha-\gamma}\widetilde{L}_1^+))^{1+\nu} - 1\right)\right]\end{aligned} \tag{2.4}$$

which extends to all $\nu > -1$ by analyticity, since L_1^+ has moments of every order. We will now study the asymptotic behaviour of both sides of (2.4), introducing the crucial parameter

$$\nu_0 \;=\; \frac{\alpha(\gamma-1)}{\alpha-1} \;>\; -1.$$

We begin with the left-hand side, which is easy.

Lemma 2.3. *One has*

$$\int_0^\infty t^\nu\,\mathbb{P}[L_1^+ > t^{-1/\alpha}(t^\gamma + x)]\,dt$$
$$\sim \frac{\gamma^{\frac{\alpha}{1-\alpha}}\left((\gamma\alpha-1)^{-1}x\right)^{\frac{\nu-\nu_0}{\gamma}}}{\sqrt{(\alpha-1)(\gamma\alpha-1)}}\,\exp\left\{-(\alpha-1)\,\gamma^{\frac{\alpha}{\alpha-1}}\,(\gamma\alpha-1)^{\frac{1-\gamma\alpha}{\gamma(\alpha-1)}}\,x^{\frac{\alpha\gamma-1}{\gamma(\alpha-1)}}\right\}.$$

Proof. By (14.35) in [19], we have the asymptotic behaviour

$$p_1(x)\;\sim\;\frac{\alpha^{-\frac{1}{2(\alpha-1)}}}{\sqrt{2\pi(\alpha-1)}}\,x^{\frac{2-\alpha}{2(\alpha-1)}}\exp\left\{-(\alpha-1)\,\alpha^{\frac{\alpha}{1-\alpha}}\,x^{\frac{\alpha}{\alpha-1}}\right\}$$

at infinity, where p_1 stands for the density of the random variable L_1. Making the change of variable $y = x^{\frac{\alpha}{\alpha-1}}$ and applying Watson's lemma - see e.g. Theorem C.3.1 in [1], this easily implies

$$\mathbb{P}[L_1^+ > x]\;\sim\;\frac{\alpha^{\frac{1}{2(\alpha-1)}}}{\sqrt{2\pi(\alpha-1)}}\,x^{-\frac{\alpha}{2(\alpha-1)}}\,\exp\left\{-(\alpha-1)\,\alpha^{\frac{\alpha}{1-\alpha}}\,x^{\frac{\alpha}{\alpha-1}}\right\}. \tag{2.5}$$

See also Theorem 2.5.3 in [21]. On the other hand, we can rewrite

$$\int_0^\infty t^\nu\,\mathbb{P}[L_1^+ > t^{-1/\alpha}(t^\gamma + x)]\,dt\;=\;x^{\frac{\nu+1}{\gamma}}\int_0^\infty s^\nu\,\mathbb{P}[L_1^+ > x^{\frac{\alpha\gamma-1}{\alpha\gamma}}\eta(s)]\,ds \tag{2.6}$$

where $\eta(s) = s^{-1/\alpha}(s^\gamma+1)$ reaches its global minimum on $(0,\infty)$ at $s_* = (\alpha\gamma-1)^{-1/\gamma}$, with

$$\eta(s_*)\;=\;\gamma\alpha(\gamma\alpha-1)^{\frac{1-\gamma\alpha}{\gamma\alpha}}\qquad\text{and}\qquad\eta''(s_*)\;=\;\frac{\gamma(\gamma\alpha-1)^{\frac{2\alpha+1}{\alpha\gamma}}}{\alpha}\cdot$$

Plugging (2.5) into the right-hand side of (2.6), we obtain

$$\int_0^\infty t^\nu\,\mathbb{P}[L_1^+ > t^{-1/\alpha}(t^\gamma + x)]\,dt$$
$$\sim\;\frac{\alpha^{\frac{1}{2(\alpha-1)}}}{\sqrt{2\pi(\alpha-1)}}\,x^{\frac{\nu+(1-\nu_0)/2}{\gamma}}\int_0^\infty s^\nu\,\eta(s)^{\frac{\alpha}{2(1-\alpha)}}\,\exp\left\{-(\alpha-1)\,\alpha^{\frac{\alpha}{1-\alpha}}\,\eta(s)^{\frac{\alpha}{\alpha-1}}x^{\frac{1+\nu_0}{\gamma}}\right\}ds,$$

which yields the required asymptotic behaviour, by Laplace's method. $\square$

We will now analyze the right-hand side of (2.4), which is more involved. Introducing the function

$$\Phi_{\alpha,\gamma,\nu}(x)\;=\;x^{-\frac{(1+\nu)\alpha}{\gamma\alpha-1}}\left((1+\varphi_{\alpha,\gamma}^{-1}(x))^{1+\nu}-1\right)$$

on $(0,\infty)$, we can rewrite (2.4) as

$$\int_0^\infty t^\nu\,\mathbb{P}[L_1^+ > t^{-1/\alpha}(t^\gamma + x)]\,dt\;=\;\frac{1}{1+\nu}\,\mathbb{E}\left[\mathbf{1}_{\{T_x<\infty\}}\left(\widetilde{L}_1^+\right)^{\frac{(1+\nu)\alpha}{\gamma\alpha-1}}\,\Phi_{\alpha,\gamma,\nu}(T_x^{1/\alpha-\gamma}\widetilde{L}_1^+)\right]. \tag{2.7}$$

Taking $\nu = \nu_0$ and observing that $\varphi_{\alpha,\gamma}^{-1}(t)\sim(t/\gamma)^{\frac{\alpha}{\alpha-1}}$ as $t\to 0$ and $\varphi_{\alpha,\gamma}^{-1}(t)\sim t^{\frac{\alpha}{\gamma\alpha-1}}$ as $t\to\infty$, we get

$$\lim_{x\to 0}\Phi_{\alpha,\gamma,\nu_0}(x)\;=\;(1+\nu_0)\gamma^{\frac{\alpha}{1-\alpha}}\;>\;0\qquad\text{and}\qquad\lim_{x\to\infty}\Phi_{\alpha,\gamma,\nu_0}(x)\;=\;1.$$

Therefore, since $\Phi_{\alpha,\gamma,\nu_0}$ is continuous and positive on $(0,\infty)$, we have

$$0 \;<\; \inf_{x>0}\{\Phi_{\alpha,\gamma,\nu_0}(x)\} \;<\; \sup_{x>0}\{\Phi_{\alpha,\gamma,\nu_0}(x)\} \;<\; \infty.$$

Going back to (2.7) and using the facts that $\widetilde{L}_1^+$ has positive moments of every order and is independent of T_x, we finally get from Lemma 2.3 the crude asymptotics

$$\mathbb{P}[T_x < \infty] \;\asymp\; \exp\left\{-(\alpha-1)\,\gamma^{\frac{\alpha}{\alpha-1}}\,(\gamma\alpha-1)^{\frac{1-\gamma\alpha}{\gamma(\alpha-1)}}\,x^{\frac{\alpha\gamma-1}{\gamma(\alpha-1)}}\right\}. \tag{2.8}$$

In order to obtain an exact asymptotics and finish the proof, we will need the following technical lemma.

Lemma 2.4. *For every $\nu \in (-1, \gamma - 1/\alpha - 1]$, the function $\Phi_{\alpha,\gamma,\nu}$ is an increasing homeomorphism from $(0,\infty)$ to $(0,1)$.*

Proof. First, it is easy to see from the aforementioned asymptotics of $\varphi_{\alpha,\gamma}$ at zero and infinity that

$$\lim_{x\to 0}\Phi_{\alpha,\gamma,\nu}(x) \;=\; 0 \qquad \text{and} \qquad \lim_{x\to\infty}\Phi_{\alpha,\gamma,\nu}(x) \;=\; 1$$

for $\nu \in (-1, \gamma - 1/\alpha - 1]$, and it is plain that $\Phi_{\alpha,\gamma,\nu}$ is continuous. Since $\varphi_{\alpha,\gamma}$ increases on $(0,\infty)$, we are reduced to show that

$$z \;\mapsto\; \Phi_{\alpha,\gamma,\nu}\left(\varphi_{\alpha,\gamma}(z)\right) \;=\; \frac{\left((1+z)^{1+\nu}-1\right)\, z^{\frac{(1+\nu)}{\gamma\alpha-1}}}{\left((1+z)^{\gamma}-1\right)^{\frac{(1+\nu)\alpha}{\gamma\alpha-1}}}$$

increases on $(0,\infty)$. Setting $y = (1+z)^\gamma - 1$ and $f_c(x) = (1+x)^c - x^c$, we obtain

$$\left(\Phi_{\alpha,\gamma,\nu}\left(\varphi_{\alpha,\gamma}(z)\right)\right)^{\frac{\gamma\alpha-1}{1+\nu}} \;=\; \left(f_{\frac{1+\nu}{\gamma}}(y^{-1})\right)^{\frac{\gamma\alpha-1}{1+\nu}}\, f_{\frac{1}{\gamma}}(y^{-1})$$

which, since f_c decreases for $c \in (0,1]$, shows that $\Phi_{\alpha,\gamma,\nu}$ increases for $\gamma \geq 1$ and $\nu \in (-1,\gamma-1]$. Assuming last $\gamma < 1$, we need to prove that

$$x \;\mapsto\; g_{\alpha,\gamma,\nu}(x) \;=\; \left(f_{\frac{1+\nu}{\gamma}}\,(x)\right)^{\alpha-\frac{1}{\gamma}}\left(f_{\frac{1}{\gamma}}\,(x)\right)^{\frac{1+\nu}{\gamma}}$$

decreases on $(0,\infty)$. Setting $c = \frac{1+\nu}{\gamma} \in (0,1)$, we compute

$$g'_{\alpha,\gamma,\nu}(x) = \frac{c\, g_{\alpha,\gamma,\nu}(x)}{\gamma(1+x)}\left(\alpha\gamma - (\alpha\gamma-1)\frac{x^{c-1}}{f_c(x)} - \frac{x^{\frac{1}{\gamma}-1}}{f_{\frac{1}{\gamma}}(x)}\right) < \frac{c\, g_{\alpha,\gamma,\nu}(x)}{\gamma(1+x)}\left(\alpha\gamma - (\alpha\gamma-1)\frac{x^{c-1}}{f_c(x)}\right).$$

It is easy to see that $x \mapsto x^{1-c} f_c(x)$ increases from $(0,+\infty)$ to $(0,c)$, and we finally obtain

$$g'_{\alpha,\gamma,\nu}(x) \;<\; \frac{((c-1)\gamma\alpha+1)g_{\alpha,\gamma,\nu}(x)}{\gamma(1+x)} \;\leq\; 0$$

as soon as $\nu \leq \gamma - 1/\alpha - 1$.

□

Corollary 2.5. *For every $A \geq 0$, one has*

$$\frac{\mathbb{P}[T_x \leq A]}{\mathbb{P}[T_x < +\infty]} \;\to\; 0$$

as $x \to \infty$.

Proof. Set $\nu = \varepsilon - 1$ with $\varepsilon > 0$ small enough for $\Phi_{\alpha,\gamma,\varepsilon-1}$ to increase on $(0,\infty)$. By (2.7) and the fact that $\widetilde{L}_1^+$ and T_x are independent, we have

$$\begin{aligned}\varepsilon \int_0^\infty t^{\varepsilon-1}\, \mathbb{P}[L_1^+ > t^{-1/\alpha}(t^\gamma + x)]\, dt &= \mathbb{E}\left[\mathbf{1}_{\{T_x<\infty\}} \left(\widetilde{L}_1^+\right)^{\frac{\alpha}{\gamma\alpha-1}} \Phi_{\alpha,\gamma,\varepsilon-1}(T_x^{1/\alpha-\gamma}\widetilde{L}_1^+)\right] \\ &\geq \mathbb{E}\left[\left(\widetilde{L}_1^+\right)^{\frac{\alpha}{\gamma\alpha-1}} \Phi_{\alpha,\gamma,\varepsilon-1}(A^{1/\alpha-\gamma}\widetilde{L}_1^+)\right] \mathbb{P}[T_x \leq A].\end{aligned}$$

Combining now the crude asymptotics (2.8) and Lemma 2.3, we deduce that there exists $K > 0$ such that

$$\frac{\mathbb{P}[T_x \leq A]}{\mathbb{P}[T_x < +\infty]} \leq K\, x^{\frac{\varepsilon-1-\nu_0}{\gamma}} \to 0$$

as $x \to \infty$, taking $\varepsilon > 0$ small enough.

□

We can now finish the proof. Taking $\nu = \nu_0$ in (2.7), we first decompose the quantity

$$\gamma^{\frac{\alpha}{\alpha-1}} \int_0^\infty t^{\nu_0}\, \mathbb{P}[L_1^+ > t^{-1/\alpha}(t^\gamma + x)]\, dt$$

into

$$\mathbb{E}[(\widetilde{L}_1^+)^{\frac{\alpha}{\alpha-1}}]\, \mathbb{P}[T_x < \infty] + \frac{1}{\Phi_{\alpha,\gamma,\nu_0}(0+)}\, \mathbb{E}\left[\mathbf{1}_{\{T_x<\infty\}}(\widetilde{L}_1^+)^{\frac{\alpha}{\alpha-1}} \left(\Phi_{\alpha,\gamma,\nu_0}(T_x^{1/\alpha-\gamma}\widetilde{L}_1^+) - \Phi_{\alpha,\gamma,\nu_0}(0+)\right)\right].$$

Applying Lemma 2.3 and the moment evaluation

$$\mathbb{E}[(\widetilde{L}_1^+)^{\frac{\alpha}{\alpha-1}}] = \frac{1}{\alpha-1}$$

which is e.g. a consequence of (2.6.20) in [21], we see that the proof will be complete as soon as

$$\mathbb{E}\left[\mathbf{1}_{\{T_x<\infty\}}(\widetilde{L}_1^+)^{\frac{\alpha}{\alpha-1}} \left(\Phi_{\alpha,\gamma,\nu_0}(T_x^{1/\alpha-\gamma}\widetilde{L}_1^+) - \Phi_{\alpha,\gamma,\nu_0}(0+)\right)\right] = o(\mathbb{P}[T_x < \infty]), \quad x \to \infty. \tag{2.9}$$

But, decomposing according to $\{T_x \leq A\}$ or $\{T_x > A\}$, the left-hand side of (2.9) is bounded by

$$\begin{aligned}&\frac{2}{\alpha-1} \sup_{z>0}\{\Phi_{\alpha,\gamma,\nu_0}(z)\}\, \mathbb{P}[T_x \leq A] \\ &\qquad + \mathbb{E}\left[(\widetilde{L}_1^+)^{\frac{\alpha}{\alpha-1}} \sup_{z\geq A}\left\{\left|\Phi_{\alpha,\gamma,\nu_0}(z^{1/\alpha-\gamma}\widetilde{L}_1^+) - \Phi_{\alpha,\gamma,\nu_0}(0+)\right|\right\}\right] \mathbb{P}[T_x < \infty]\end{aligned}$$

and (2.9) follows by Corollary 2.5, the continuity of $\Phi_{\alpha,\gamma,\nu_0}$ at zero, and dominated convergence.

□

Remark 2.6. As mentioned in the introduction, in the case $\gamma = 2$ our Theorem A echoes a large deviation estimate which had been previously obtained in [5]. More precisely, if we set

$$\underline{\mathbf{M}}^{[x]}_{\alpha,\rho,\gamma} = \sup_{t\in[0,x]}\{L_t - t^\gamma\} \qquad \text{and} \qquad \overline{\mathbf{M}}^{[x]}_{\alpha,\rho,\gamma} = \sup_{t\geq x}\{L_t - t^\gamma\},$$

then the main result of [5] states that

$$\mathbb{P}[\overline{\mathbf{M}}^{[x]}_{\alpha,\rho,2} \geq \underline{\mathbf{M}}^{[x]}_{\alpha,\rho,2}] \asymp x^{1-2\alpha}$$

if L has positive jumps, and that

$$\log \mathbb{P}[\overline{\mathbf{M}}^{[x]}_{\alpha,1/\alpha,2} \geq \underline{\mathbf{M}}^{[x]}_{\alpha,1/\alpha,2}] \;\sim\; -\kappa_\alpha \, x^{\frac{2\alpha-1}{\alpha-1}}$$

for some explicit $\kappa_\alpha \in (0,\infty)$ if L does not have positive jumps. Roughly speaking, when x is large the event

$$\left\{\overline{\mathbf{M}}^{[x]}_{\alpha,\rho,2} \geq \underline{\mathbf{M}}^{[x]}_{\alpha,\rho,2}\right\}$$

amounts to the fact that the translated process $L_t - L_x - (t-x)^2$ crosses a level of size x^2 for some $t \geq x$, which explains heuristically why the asymptotics of

$$\mathbb{P}[\overline{\mathbf{M}}^{[x]}_{\alpha,\rho,2} \geq \underline{\mathbf{M}}^{[x]}_{\alpha,\rho,2}] \qquad \text{and} \qquad \mathbb{P}[\mathbf{M}_{\alpha,\rho,2} \geq x^2]$$

are comparable. One might wonder if all our above arguments could not help refine the results of [5], but we have not investigated this question.

3 Proof of Theorem B

3.1 The lower bound

This part is easy and relies essentially on the identity (1.3). Introducing

$$T^{(\beta)}_x \;=\; \inf\{t \geq 0,\, L^{(\beta)}_t = t^{\gamma+\beta} + x\} \qquad \text{and} \qquad \widehat{T}^{(\beta)}_x \;=\; \inf\{t \geq 0,\, L^{(\beta)}_t = (t^{\gamma+\beta}+1)\, x^{\frac{\gamma\alpha-1}{\alpha(\gamma+\beta)}}\},$$

we see by the scaling property $\{L^{(\beta)}_{yt},\, t \geq 0\} \stackrel{d}{=} \{y^{\beta+1/\alpha} L^{(\beta)}_t,\, t \geq 0\}$ with $y = x^{\frac{1}{\gamma+\beta}}$ that

$$\mathbb{P}[\mathbf{M}^{(\beta)}_{\alpha,\rho,\gamma} \geq x] \;=\; \mathbb{P}[T^{(\beta)}_x < \infty] \;=\; \mathbb{P}[\widehat{T}^{(\beta)}_x < \infty]. \tag{3.1}$$

Setting

$$s_* \;=\; \arg\min\{s^{-\beta-1/\alpha}(s^{\gamma+\beta}+1)\} \;=\; \left(\frac{1+\alpha\beta}{\gamma\alpha-1}\right)^{\frac{1}{\gamma+\beta}}$$

and

$$m_* \;=\; \min_{s>0}\{s^{-\beta-1/\alpha}(s^{\gamma+\beta}+1),\, s>0\} \;=\; \alpha(\gamma+\beta)(\alpha\beta+1)^{-\frac{1+\alpha\beta}{\alpha(\gamma+\beta)}}(\gamma\alpha-1)^{\frac{1-\gamma\alpha}{\alpha(\gamma+\beta)}},$$

a further scaling argument implies

$$\mathbb{P}[\widehat{T}^{(\beta)}_x < \infty] \;\geq\; \mathbb{P}\left[L^{(\beta)}_{s_*} \geq (s_*^{\gamma+\beta}+1)\, x^{\frac{\gamma\alpha-1}{\alpha(\gamma+\beta)}}\right] \;=\; \mathbb{P}[L_1 \geq (1+\alpha\beta)^{1/\alpha} m_*\, x^{\frac{\gamma\alpha-1}{\alpha(\gamma+\beta)}}]. \tag{3.2}$$

When L has positive jumps, applying Property 1.2.15 in [18] to the stable random variable L_1 and using (3.1) and (3.2) yield the required lower bound

$$\mathbb{P}[\mathbf{M}^{(\beta)}_{\alpha,\rho,\gamma} \geq x] \;\geq\; \kappa\, x^{\frac{1-\gamma\alpha}{\gamma+\beta}}, \qquad x \to \infty,$$

for some $\kappa > 0$. When L has no positive jumps, we obtain from (2.5), (3.1) and (3.2) the required lower bound

$$\liminf_{x\to\infty} x^{\frac{1-\gamma\alpha}{(\alpha-1)(\gamma+\beta)}} \log \mathbb{P}[\mathbf{M}^{(\beta)}_{\alpha,1/\alpha,\gamma} \geq x] \;\geq\; -c_{\alpha,\beta,\gamma}.$$

3.2 The upper bound in the case with positive jumps

Introducing the parameter

$$\delta \;=\; \frac{\gamma\alpha-1}{\alpha(\gamma+\beta)} \;\in\; (0,1)$$

and fixing $\varepsilon > 0$ small enough such that $\eta = 2^\delta(1+\varepsilon)^{\delta-1} > 1$, define the stochastically increasing family of stopping times

$$\widehat{T}_x^{(\beta,k)} \;=\; \inf\{t \geq 0,\, L_t^{(\beta)} - (1+\varepsilon)^{-k} t^{\gamma+\beta} x^\delta \;=\; 2^k x^\delta\}, \qquad k \geq 0.$$

Since $\mathbb{P}[\widehat{T}_x^{(\beta,k)} < \infty] \to 0$ as $k \to \infty$, by (3.1) we have the telescoping decomposition

$$\mathbb{P}[\mathbf{M}_{\alpha,\rho,\gamma}^{(\beta)} \geq x] \;=\; \mathbb{P}[\widehat{T}_x^{(\beta,0)} < \infty] \;=\; \sum_{k\geq 0} \left(\mathbb{P}[\widehat{T}_x^{(\beta,k)} < \infty] \,-\, \mathbb{P}[\widehat{T}_x^{(\beta,k+1)} < \infty] \right).$$

We first consider the case $\gamma + \beta \geq 1$. Setting $r_k = (3 \times 2^k(1+\varepsilon)^k)^{\frac{1}{\gamma+\beta}}$, we can bound

$$\begin{aligned}
\mathbb{P}[\widehat{T}_x^{(\beta,k)} < \infty] \;&\leq\; \mathbb{P}\left[\sup_{t\in[0,r_k]}\{L_t^{(\beta)}\} \geq 2^k x^\delta\right] \;+\; \mathbb{P}\left[\sup_{t\geq r_k}\{L_t^{(\beta)} - (1+\varepsilon)^{-k} t^{\gamma+\beta} x^\delta\} \geq 2^k x^\delta\right] \\
&\leq\; \mathbb{P}\left[L_1^* \geq \eta^k 3^{\delta-1} x^\delta\right] \;+\; \mathbb{P}\left[\sup_{t\geq 0}\{L_{t+r_k}^{(\beta)} - (1+\varepsilon)^{-k} t^{\gamma+\beta} x^\delta\} \geq 2^{k+2} x^\delta\right],
\end{aligned}$$

where in the second line we have used the a.s. inequality $\sup_{t\in[0,1]}\{L_t^{(\beta)}\} \leq L_1^*$, which is obvious, and the equally obvious deterministic inequality

$$(t + r_k)^{\gamma+\beta} \;\geq\; t^{\gamma+\beta} \,+\, r_k^{\gamma+\beta} \tag{3.3}$$

for all $t \geq 0$, which follows from $\gamma + \beta \geq 1$. The next step is to write down the process decomposition

$$\begin{aligned}
L_{t+r_k}^{(\beta)} &= \left(\beta \int_0^{r_k} (t + r_k - u)^{\beta-1}\, L_u\, du \;+\; t^\beta\, L_{r_k}\right) \;+\; \beta \int_0^t (t-s)^{\beta-1}\,(L_{s+r_k} - L_{r_k})\, ds \\
&\stackrel{d}{=} \left(\beta \int_0^{r_k} (t + r_k - u)^{\beta-1}\, L_u\, du \;+\; t^\beta\, L_{r_k}\right) \;+\; \widetilde{L}_t^{(\beta)} \;\leq\; (t+r_k)^\beta L_{r_k}^* \;+\; \widetilde{L}_t^{(\beta)}
\end{aligned} \tag{3.4}$$

with $\{\widetilde{L}_t^{(\beta)},\, t \geq 0\}$ an independent copy of $\{L_t^{(\beta)},\, t \geq 0\}$, which implies

$$\begin{aligned}
&\mathbb{P}\left[\sup_{t\geq 0}\{L_{t+r_k}^{(\beta)} - (1+\varepsilon)^{-k} t^{\gamma+\beta} x^\delta\} \geq 2^{k+2} x^\delta\right] \\
&\quad\leq\; \mathbb{P}[\widehat{T}_x^{(\beta,k+1)} < \infty] \;+\; \mathbb{P}\left[\sup_{t\geq 0}\{L_{r_k}^*(t+r_k)^\beta - \varepsilon\,(1+\varepsilon)^{-k-1} t^{\gamma+\beta} x^\delta\} \geq 2^{k+1} x^\delta\right] \\
&\quad\leq\; \mathbb{P}[\widehat{T}_x^{(\beta,k+1)} < \infty] \;+\; \mathbb{P}\left[c_\beta\, r_k^\beta L_{r_k}^* \;+\; \sup_{t\geq 0}\{c_\beta\, L_{r_k}^* t^\beta - \varepsilon\,(1+\varepsilon)^{-k-1} t^{\gamma+\beta} x^\delta\} \geq 2^{k+1} x^\delta\right],
\end{aligned}$$

where $c_\beta = 2^{|\beta-1|}$ and we have used $(t+s)^\beta \leq c_\beta(t^\beta + s^\beta)$ for all $t, s \geq 0$. The second term on the right-hand side is bounded by

$$\begin{aligned}
&\mathbb{P}\left[L_1^* \geq \eta^k 3^{\delta-1} c_\beta^{-1} x^\delta\right] \;+\; \mathbb{P}\left[\sup_{t\geq 0}\{c_\beta\, L_{r_k}^* t^\beta - \varepsilon\,(1+\varepsilon)^{-k-1} t^{\gamma+\beta} x^\delta\} \geq 2^k x^\delta\right] \\
&\qquad\qquad = \;\mathbb{P}\left[L_1^* \geq \eta^k 3^{\delta-1} c_\beta^{-1} x^\delta\right] \;+\; \mathbb{P}\left[L_1^* \geq \eta^k \kappa\, x^\delta\right]
\end{aligned}$$

for some positive constant κ not depending on k, x. Setting $\hat{\kappa} = \min\{\kappa, 3^{\delta-1} c_\beta^{-1}\} > 0$, and putting everything together, we finally obtain

$$\mathbb{P}[\mathbf{M}_{\alpha,\rho,\gamma}^{(\beta)} \geq x] \;\leq\; 3 \sum_{k\geq 0} \mathbb{P}\left[L_1^* \geq \eta^k \hat{\kappa}\, x^\delta\right] \;\sim\; \frac{3\,\hat{\kappa}^{-\alpha}\Gamma(\alpha)\sin(\pi\alpha\rho)}{\pi(1-\eta^{-\alpha})}\, x^{\frac{1-\gamma\alpha}{\gamma+\beta}},$$

where the estimate follows at once from (2.3) and direct summation. This completes the proof for $\gamma + \beta \geq 1$. The case $\gamma + \beta < 1$ follows along the same lines, except that (3.3) is not true anymore. We hence set

$$\lambda \;=\; \frac{\varepsilon}{2(1+\varepsilon)} \in (0,1) \qquad \text{and} \qquad r_k \;=\; (3\lambda^{-1} \times 2^k (1+\varepsilon)^k)^{\frac{1}{\gamma+\beta}}, \quad k \geq 0.$$

Using the obvious inequality $(t + r_k)^{\gamma+\beta} \geq (1-\lambda)t^{\gamma+\beta} + \lambda r_k^{\gamma+\beta}$ leads first to

$$\mathbb{P}[\widehat{T}_x^{(\beta,k)} < \infty] \;\leq\; \mathbb{P}\left[L_1^* \geq \eta^k 3^{\delta-1} x^\delta\right] + \mathbb{P}\left[\sup_{t\geq 0}\{L_{t+r_k}^{(\beta)} - (1-\lambda)(1+\varepsilon)^{-k} t^{\gamma+\beta} x^\delta\} \geq 2^{k+2} x^\delta\right].$$

Then, we can bound

$$\begin{aligned} &\mathbb{P}\left[\sup_{t\geq 0}\{L_{t+r_k}^{(\beta)} - (1-\lambda)(1+\varepsilon)^{-k} t^{\gamma+\beta} x^\delta\} \geq 2^{k+2} x^\delta\right] \\ &\quad\leq\; \mathbb{P}[\widehat{T}_x^{(\beta,k+1)} < \infty] \;+\; \mathbb{P}\left[\sup_{t\geq 0}\{2L_{r_k}^*(t+r_k)^\beta - \varepsilon\,(1+\varepsilon)^{-k-1} t^{\gamma+\beta} x^\delta\} \geq 2^{k+2} x^\delta\right], \end{aligned}$$

and the proof is finished similarly.

□

3.3 The upper bound in the case without positive jumps

The argument relies on the following well-known association lemma, which will also be used during the proof of Theorem C.

Lemma 3.1. *Let F, G be two bounded functionals on the Skorokhod space $\mathcal{D}(\mathbb{R}^+, \mathbb{R})$ being both non-increasing or both non-decreasing. Then, one has*

$$\mathbb{E}\left[F(L_u,\, u\geq 0)\, G(L_u,\, u \geq 0)\right] \;\geq\; \mathbb{E}\left[F(L_u,\, u\geq 0)\right] \mathbb{E}\left[G(L_u,\, u\geq 0)\right].$$

Proof. By càd-làg approximation, it is enough to consider the case when F, G depend only on a finite number of points. With the notation of Chapter 4.6 in [18], we are hence reduced to show that the random vector $(L_{t_1}, L_{t_2}, \ldots, L_{t_n})$ is associated for every $n \geq 2$ and $0 < t_1 < \ldots < t_n$. By independence of the increments we have $(L_{t_1}, L_{t_2}, \ldots, L_{t_n}) = (X_1, X_1 + X_2, \ldots, X_1 + \ldots + X_n)$, where the X_i's are mutually independent real random variables, making the vector $X = (X_1, \ldots, X_n)$ trivially associated. We can then apply e.g. Exercise 4.25 p. 220 in [18].

□

Let us now finish the proof. For simplicity, we will set T_x for $T_x^{(\beta)}$. Let $\varepsilon > 0$ and fix δ small enough such that $\eta = 1 - (1-\varepsilon)(\delta+1)^{\beta+\gamma} > 0$. Using the absence of positive jumps, we obtain

$$\begin{aligned} \int_0^\infty \mathbb{P}[L_t^{(\beta)} \geq (1-\varepsilon)t^{\beta+\gamma} + x]\, dt &\geq \int_0^\infty \mathbb{P}\left[L_t^{(\beta)} - L_{T_x}^{(\beta)} \geq (1-\varepsilon)t^{\beta+\gamma} - T_x^{\beta+\gamma},\; T_x < +\infty\right] dt \\ &\geq \int_0^\delta \mathbb{P}\left[L_{T_x(t+1)}^{(\beta)} - L_{T_x}^{(\beta)} \geq -\eta\, T_x^{\beta+\gamma},\; 1 < T_x < +\infty\right] dt. \end{aligned} \tag{3.5}$$

By (1.3) and a change of variable, the left-hand side equals

$$\int_0^\infty \mathbb{P}[L_t^{(\beta)} \geq (1-\varepsilon)t^{\beta+\gamma} + x]\, dt \;=\; \kappa_\varepsilon \int_0^\infty t^{-\frac{\alpha\beta}{1+\alpha\beta}}\, \mathbb{P}[L_1 \geq (1+\alpha\beta)^{1/\alpha} t^{-1/\alpha}(t^{\frac{\gamma+\beta}{1+\alpha\beta}} + c_\varepsilon x)]\, dt$$

for some positive constants $\kappa_\varepsilon, c_\varepsilon$ such that $c_\varepsilon \to 1$ as $\varepsilon \to 0$ and, by Lemma 2.3, we first deduce

$$\log \int_0^\infty \mathbb{P}[L_t^{(\beta)} \geq (1-\varepsilon)t^{\beta+\gamma} + x]\, dt \;\sim\; -c_{\alpha,\beta,\gamma}(c_\varepsilon x)^{\frac{\gamma\alpha-1}{(\alpha-1)(\gamma+\beta)}}.$$

We shall now separate the proof according as $\beta \geq 1$ or $\beta < 1$.

Assume first $\beta \geq 1$. Bounding the right-hand side of (3.5) leads to

$$\int_0^\infty \mathbb{P}[L_t^{(\beta)} \geq (1-\varepsilon)t^{\beta+\gamma}+x]\, dt \geq \delta\, \mathbb{P}\left[\inf_{u\geq 1, t\leq\delta}\{u^{-\beta-\gamma}(L_{u(t+1)}^{(\beta)} - L_u^{(\beta)})\} \geq -\eta,\ 1 < T_x < \infty\right],$$

whence

$$\mathbb{P}\left[\inf_{u\geq 1, t\leq\delta}\{u^{-\beta-\gamma}(L_{u(t+1)}^{(\beta)} - L_u^{(\beta)})\} \geq -\eta,\ T_x < \infty\right] \leq\; \delta^{-1}\int_0^\infty \mathbb{P}[L_t^{(\beta)} \geq (1-\varepsilon)t^{\beta+\gamma} + x]\, dt \;+\; \mathbb{P}[T_x \leq 1]. \tag{3.6}$$

We next observe that the contribution of $\mathbb{P}[T_x \leq 1]$ in the right-hand side of (3.6) is negligible, using the obvious bound

$$\mathbb{P}[T_x \leq 1] \;\leq\; \mathbb{P}[\tau_x \leq 1]$$

with $\tau_x = \inf\{t \geq 0,\ L_t^{(\beta)} = x\}$, the crude estimates

$$\log \mathbb{P}[\tau_x \leq 1] \;\asymp\; \log \mathbb{P}[L_1 > x] \;\asymp\; -x^{\frac{\alpha}{\alpha-1}}$$

and the strict inequality

$$\frac{\alpha\gamma - 1}{(\alpha-1)(\beta+\gamma)} \;<\; \frac{\alpha}{\alpha-1}.$$

Above, the crude estimates are a consequence of (1.3), (2.5) and

$$\mathbb{P}[L_1^{(\beta)} > x] \;\leq\; \mathbb{P}[\tau_x \leq 1] \;\leq\; \mathbb{P}\left[\sup_{t\leq 1}\{L_t\} > x\right] \;=\; \alpha\, \mathbb{P}[L_1 > x],$$

the last equality being well-known as the reflection principle for spectrally negative stable Lévy processes - see e.g. Exercises 29.7 and 29.18 in [19]. Finally, we notice that

$$u^{-\beta-\gamma}(L_{u(t+1)}^{(\beta)} - L_u^{(\beta)}) \;=\; \beta\int_0^{1+t} \left((1+t-s)^{\beta-1} - (1-s)^{\beta-1}\mathbf{1}_{\{s\leq 1\}}\right)\, \frac{L_{us}}{u^\gamma}\, ds$$

is an increasing functional of $\{L_s,\ s \geq 0\}$, because $\beta \geq 1$. Since $\mathbf{1}_{\{T_x<\infty\}}$ is also an increasing functional of L, we deduce from Lemma 3.1 that

$$\mathbb{P}\left[\inf_{u\geq 1, t\leq\delta}\{u^{-\beta-\gamma}(L_{u(t+1)}^{(\beta)} - L_u^{(\beta)})\} \geq -\eta,\ T_x < \infty\right] \geq\; \mathbb{P}\left[\inf_{u\geq 1, t\leq\delta}\{u^{-\beta-\gamma}(L_{u(t+1)}^{(\beta)} - L_u^{(\beta)})\} \geq -\eta\right]\, \mathbb{P}\left[T_x < \infty\right] \;=\; \kappa\, \mathbb{P}\left[T_x < \infty\right]$$

for some $\kappa > 0$ not depending on x. Putting everything together, we get

$$\limsup_{x\to\infty} x^{\frac{1-\gamma\alpha}{(\alpha-1)(\gamma+\beta)}}\, \log \mathbb{P}[T_x < \infty] \;\leq\; -c_{\alpha,\beta,\gamma}\, c_\varepsilon^{\frac{\gamma\alpha-1}{(\alpha-1)(\gamma+\beta)}},$$

which, letting $\varepsilon \to 0$, completes the proof in the case $\beta \geq 1$.

Assume second $\beta < 1$. We set

$$\sigma_t = \beta \int_0^1 \left\{ (1-s)^{\beta-1} - (1+t-s)^{\beta-1} \right\} s^\gamma ds$$

which is a positive increasing function on $(0,\infty)$ such that $\sigma_t \to 0$ as $t \to 0$. Replacing $T_x^{\beta+\gamma}$ by

$$T_x^{\beta+\gamma} = \frac{\beta}{\sigma_t} \int_0^{T_x} \left\{ (T_x - s)^{\beta-1} - (T_x(1+t) - s)^{\beta-1} \right\} s^\gamma ds$$

we deduce using a change of variable that

$$L^{(\beta)}_{T_x(t+1)} - L^{(\beta)}_{T_x} + \frac{\eta}{2} T_x^{\beta+\gamma} \geq \beta T_x^\beta \int_1^{1+t} (1+t-s)^{\beta-1} L_{sT_x} ds - T_x^\beta h_\beta(t) \sup_{u\geq 0} \left\{ L_u - \frac{\eta}{2\sigma_t} u^\gamma \right\}$$

where $h_\beta(t) = 1 + t^\beta - (1+t)^\beta$ is increasing in t. Going back to (3.5), and taking $a < \delta$, the right-hand side is then greater than :

$$a\mathbb{P}\left[F_\delta(L) - \frac{h_\beta(\delta)}{\delta^\gamma x^{\frac{\gamma}{\beta+\gamma}}} \sup_{s\geq 0} \left\{ L_s - \frac{\eta}{2\sigma_a} s^\gamma \right\} \geq -\eta/2,\ \delta x^{\frac{1}{\beta+\gamma}} < T_x < \infty \right] \tag{3.7}$$

where

$$F_\delta(L) = \beta \inf_{t\leq\delta} \int_1^{1+t} (1+t-s)^{\beta-1} \inf_{u\geq 1} \frac{L_{su}}{u^\gamma}\, ds$$

is an increasing functional of L. We next observe that, cutting (3.7) in two as in (3.6), the second term will be negligible by taking δ small enough since

$$\mathbb{P}[T_x \leq \delta x^{\frac{1}{\gamma+\beta}}] \;\leq\; \mathbb{P}[\tau_x \leq \delta x^{\frac{1}{\gamma+\beta}}]$$

and

$$\log \mathbb{P}[\tau_x \leq \delta x^{\frac{1}{\gamma+\beta}}] \;\asymp\; \log \mathbb{P}[L_1 > \delta^{-\beta-\frac{1}{\alpha}} x^{\frac{\alpha\gamma-1}{\alpha(\gamma+\beta)}}] \;\asymp\; -\delta^{-\frac{\alpha\beta+1}{\alpha-1}} x^{\frac{\alpha\gamma-1}{(\alpha-1)(\gamma+\beta)}}.$$

Thus, it remains to deal with the term :

$$\mathbb{P}\left[F_\delta(L) \geq -\eta/4,\, T_x < \infty\right] - \mathbb{P}\left[h_\beta(\delta) \sup_{s\geq 0} \left\{ L_s - \frac{\eta}{2\sigma_a} s^\gamma \right\} \geq \frac{\eta}{4} \delta^\gamma x^{\frac{\gamma}{\beta+\gamma}} \right].$$

From Theorem A and using the scaling of L, the second term behaves as

$$\log \mathbb{P}\left[h_\beta(\delta)\, (\sigma_a)^{\frac{1}{\alpha\gamma-1}} \sup_{s\geq 0} \left\{ L_s - \frac{\eta}{2} s^\gamma \right\} \geq \frac{\eta}{4} \delta^\gamma x^{\frac{\gamma}{\beta+\gamma}} \right] \asymp -\, (\sigma_a)^{-\frac{1}{\gamma(\alpha-1)}}\, x^{\frac{\gamma\alpha-1}{(\alpha-1)(\gamma+\beta)}}$$

which is negligible by taking a small enough. The proof is then concluded as in the case $\beta \geq 1$ by applying Lemma 3.1 to the term $\mathbb{P}\left[F_\delta(L) \geq -\eta/4,\ T_x < \infty\right]$.

□

Remark 3.2. In the particular case $\beta = 1$ of the integrated stable process, we may proceed as in the proof of Theorem A, and obtain a more precise upper bound. The strong Markov property at T_x for the two-dimensional process

$$\{(L_t^{(1)}, L_t),\, t \geq 0\},$$

a scaling argument and (1.3) imply firstly

$$\int_0^\infty t^{\nu_0}\, \mathbb{P}[L_t^{(1)} > t^{1+\gamma} + x]\, dt$$
$$=\; \mathbb{E}\left[\mathbf{1}_{\{T_x<\infty\}}\, T_x^{1+\nu_0} \int_0^\infty (1+t)^{\nu_0}\, \mathbf{1}_{\{\widetilde{L}_1^+ + (tT_x)^{-1/\alpha}(L_{T_x} - (1+\gamma)T_x^\gamma) > T_x^{\gamma-1/\alpha}\, \psi_{\alpha,\gamma}(t)\}}\, dt \right]$$

where $\psi_{\alpha,\gamma}(t) = t^{-1-1/\alpha}((t+1)^{1+\gamma} - 1 - (1+\gamma)t)$ is again an homeomorphism from $(0,\infty)$ to $(0,\infty)$, and T_x and $\widetilde{L}_1^+$ are independent. We can then bound

$$\int_0^\infty t^{\nu_0}\,\mathbb{P}[L_t^{(1)} > t^{1+\gamma} + x]\,dt \;\geq\; \mathbb{E}\left[\mathbf{1}_{\{T_x<\infty\}}\,T_x^{1+\nu_0}\int_0^\infty (1+t)^{\nu_0}\,\mathbf{1}_{\{\widetilde{L}_1^+ > T_x^{\gamma-1/\alpha}\,\psi_{\alpha,\gamma}(t)\}}\,dt\right],$$

using the crucial fact that the derivative of $t \mapsto L_t^{(1)} - t^{1+\gamma}$ at T_x, which equals $L_{T_x} - (1+\gamma)T_x^\gamma$, is a.s. non-negative. This leads to

$$\int_0^\infty t^{\nu_0}\,\mathbb{P}[L_1^+ > t^{-1-1/\alpha}(t^{1+\gamma}+x)]\,dt \;\geq\; \frac{1}{1+\nu_0}\,\mathbb{E}\left[\mathbf{1}_{\{T_x<\infty\}}\left(\widetilde{L}_1^+\right)^{\frac{(1+\nu_0)\alpha}{\gamma\alpha-1}}\Psi_{\alpha,\gamma}(T_x^{1/\alpha-\gamma}\,\widetilde{L}_1^+)\right]$$

where

$$\Psi_{\alpha,\gamma}(x) \;=\; x^{-\frac{(1+\nu_0)\alpha}{\gamma\alpha-1}}\left((1+\psi_{\alpha,\gamma}^{-1}(x))^{1+\nu_0} - 1\right)$$

is again bounded away from zero and ∞, by the fateful choice of ν_0. We finally obtain

$$\mathbb{P}[T_x < \infty] \;\leq\; \kappa_+\int_0^\infty t^{\nu_0}\,\mathbb{P}[L_1^+ > t^{-1-1/\alpha}(t^{1+\gamma}+x)]\,dt$$

for some $\kappa_+ \in (0,\infty)$, and an appropriate modification of Lemma 1 yields

$$\mathbb{P}[\mathbf{M}^{(1)}_{\alpha,1/\alpha,\gamma} \geq x] \;\leq\; \hat{\kappa}_+\,\exp\left\{-c_{\alpha,1,\gamma}\,x^{\frac{\gamma\alpha-1}{(\alpha-1)(1+\gamma)}}\right\}$$

at infinity, for some other $\hat{\kappa}_+ \in (0,\infty)$. Unfortunately, the precise lower bound which can be derived from (2.5) is different: one gets

$$\mathbb{P}[\mathbf{M}^{(1)}_{\alpha,1/\alpha,\gamma} \geq x] \;\geq\; \hat{\kappa}_-\,x^{\frac{1-\gamma\alpha}{2(\alpha-1)(1+\gamma)}}\,\exp\left\{-c_{\alpha,1,\gamma}\,x^{\frac{\gamma\alpha-1}{(\alpha-1)(1+\gamma)}}\right\}$$

for some $\hat{\kappa}_- \in (0,\infty)$, and the exact polynomial speed before the exponential term remains unknown. We believe that this speed is given in the lower bound, and we refer to Remark 3.4 (c) below for a general conjecture.

3.4 A more precise estimate in the Brownian case

In this paragraph we specify the general results of [12] to the process $L^{(\beta)}$ in the case $\alpha = 2$, and we get a refinement of Theorem B (b). Observe that in this framework we can also consider the wider range $\beta > -1/2$. The following proposition is a consequence of Theorem 1 in [12] but the exact asymptotics does not seem to have been written down anywhere, and we hence provide a detailed proof. It turns out that a transition phenomenon occurs at $\beta = 1/2$.

Proposition 3.3. *Assume $\gamma > 1/2$.*

(a) *If $\beta \in (-1/2, 1/2)$, there exists $\kappa_{\beta,\gamma} > 0$ such that*

$$\mathbb{P}[\mathbf{M}^{(\beta)}_{2,1/2,\gamma} \geq x] \;\sim\; \kappa_{\beta,\gamma}\,x^{\frac{2\beta(1-2\gamma)}{(2\beta+1)(\gamma+\beta)}}\,\exp\left\{-c_{2,\beta,\gamma}\,x^{\frac{2\gamma-1}{\gamma+\beta}}\right\}.$$

(b) *If $\beta > 1/2$, there exists $\tilde{\kappa}_{\beta,\gamma} > 0$ such that*

$$\mathbb{P}[\mathbf{M}^{(\beta)}_{2,1/2,\gamma} \geq x] \;\sim\; \tilde{\kappa}_{\beta,\gamma}\,x^{\frac{1-2\gamma}{2(\gamma+\beta)}}\,\exp\left\{-c_{2,\beta,\gamma}\,x^{\frac{2\gamma-1}{\gamma+\beta}}\right\}.$$

Proof. With our normalization for the characteristic exponent, one has $L_1 \sim \mathcal{N}(0,2)$ and a scaling argument implies

$$\mathbf{M}^{(\beta)}_{2,1/2,\gamma} \;\stackrel{d}{=}\; \left(\frac{2}{2\beta+1}\right)^{\frac{\gamma+\beta}{2\gamma-1}}\,\widetilde{\mathbf{M}}_{\beta,\gamma}$$

where

$$\widetilde{\mathbf{M}}_{\beta,\gamma} \;=\; \sup_{t>0}\left\{\sqrt{2\beta+1}\int_0^t (t-s)^\beta\,dB_s \,-\, t^{\gamma+\beta}\right\}$$

and $\{B_t,\, t\ge 0\}$ is a standard linear Brownian motion. Setting $H=\beta+1/2$, the process

$$X_t \;=\; \sqrt{2\beta+1}\int_0^t (t-s)^\beta\,dB_s, \qquad t\ge 0,$$

is Gaussian with mean 0 and variance t^{2H}, and self-similar with index H. With the notation of Section 1 in [12], we have

$$s_0 \;=\; \left(\frac{2\beta+1}{2\gamma-1}\right)^{\frac{1}{\gamma+\beta}} \qquad \text{and} \qquad A \;=\; 2\sqrt{\frac{c_{2,\beta,\gamma}}{2\beta+1}}. \tag{3.8}$$

We now wish to apply Theorem 1 in [12], whose statement deals with case $H\in(0,1)$ but we can actually consider any $H>0$ by Remark 1 therein. Following (7) in [12], we need to evaluate the behaviour of $\mathbb{E}[(Y_t-Y_s)^2]$ as $t,s\to s_0$, having set $Y_t=t^{-H}X_t$ for all $t>0$. Because of the time normalization, we have not found any precise reference for this behaviour in the literature and so we give the details. For $0<s<t$, we compute

$$\mathbb{E}[(Y_t-Y_s)^2] \;=\; 2\,-\,I_\beta(x)$$

with $x=st^{-1}\in(0,1)$ and

$$I_\beta(x) \;=\; 2(2\beta+1)\sqrt{x}\int_0^1 (1-u)^\beta(1-xu)^\beta\,du.$$

We need to study the asymptotic behaviour of $I_\beta(x)$ as $y=1-x\to 0$. If $\beta>1/2$, rewriting

$$I_\beta(x) \;=\; 2(2\beta+1)\sqrt{1-y}\int_0^1 (1-u)^{2\beta}(1+yu(1-u)^{-1})^\beta\,du,$$

making a Taylor expansion of order 2 of both quantities in y and evaluating the two underlying Beta integrals leads to

$$I_\beta(x) \;=\; 2\,-\,\frac{(2\beta+1)y^2}{4(2\beta-1)}\,+\,o(y^2).$$

This shows that (7) in [12] holds with

$$\alpha \;=\; 2 \qquad \text{and} \qquad D \;=\; \frac{(2\beta+1)}{4(2\beta-1)s_0^2}. \tag{3.9}$$

If $\beta<1/2$, the argument does not apply because the second Beta integral diverges. We first rewrite

$$I_\beta(x) \;=\; \frac{2(2\beta+1)\sqrt{x}}{\beta+1}\,{}_2F_1\left[\begin{matrix}-\beta\;\;1\\ \beta+2\end{matrix};x\right] \;=\; \frac{2(2\beta+1)\sqrt{x}y^\beta}{\beta+1}\,{}_2F_1\left[\begin{matrix}-\beta\;\;\beta+1\\ \beta+2\end{matrix};-xy^{-1}\right],$$

where the first equality follows from Euler's integral representation and the second one from Pfaff's transformation for the hypergeometric function - see respectively 2.1.3(10) and 2.1.4(22) in [8]. Applying next the residue transformation 2.1.4(17) in [8], we obtain

$$\begin{aligned} I_\beta(x) \;&=\; 2x^{\beta+1/2}{}_2F_1\left[\begin{matrix}-\beta\;\;-1-2\beta\\ -2\beta\end{matrix};-yx^{-1}\right] \,-\, \frac{2\Gamma(\beta+1)\Gamma(-2\beta)}{\Gamma(-\beta)}\,x^{-\beta-1/2}y^{2\beta+1}\\ &=\; 2\,-\,\frac{\Gamma^2(\beta+1)}{\Gamma(2\beta+1)\,\cos(\pi\beta)}\,y^{2\beta+1}\,+\,O(y^2). \end{aligned}$$

This shows that (7) in [12] holds with

$$\alpha \;=\; 2\beta+1 \qquad \text{and} \qquad D \;=\; \frac{\Gamma^2(\beta+1)}{\Gamma(2\beta+1)\,\cos(\pi\beta)\,s_0^{2\beta+1}}. \tag{3.10}$$

Putting (3.8) and (3.9) resp. (3.10) together with (10) resp. (9) in [12], using the standard estimate $\sqrt{2\pi}\Psi(u) \sim u^{-1}e^{-u^2/2}$ for the tail of the unit normal distribution, and proceeding to the necessary simplifications, we obtain our required asymptotics with the two different regimes.

□

Remark 3.4. (a) For $\beta = 1/2$, the transformation 2.1.4(18) in [8] with $m = 2$ exhibits a logarithmic term: one has the non-trivial closed formula

$$I_{1/2}(x) \;\;=\;\; 2 \;+\; \frac{y^2}{2(1-y)}(\psi(3/2) \;-\; \psi(3) \;+\; \log(y) \;-\; \log(1-y))$$

where ψ is the digamma function. This implies

$$\mathbb{E}[(Y_t - Y_s)^2] \;\sim\; -\frac{(t-s)^2\log|t-s|}{2s_0^2}$$

as $t, s \to s_0$, and we cannot apply the results of [12]. We believe that

$$\mathbb{P}[\mathbf{M}^{(1/2)}_{2,1/2,\gamma} \geq x] \;\sim\; \kappa\,(\log x)^{\delta}\,x^{\frac{1-2\gamma}{2\gamma+1}}\;\exp\left\{-c_{2,1/2,\gamma}\,x^{\frac{2\gamma-1}{(\gamma+1/2)}}\right\}$$

for some $\kappa > 0$ and $\delta \neq 0$ to be determined, the logarithmic correction being heuristically due to the 1-self-similarity of

$$t \;\mapsto\; \int_0^t \sqrt{t-s}\,dB_s.$$

(b) The constants $\kappa_{\beta,\gamma}$ and $\tilde\kappa_{\beta,\gamma}$ can also be evaluated from Theorem 1 in [12], but they have a complicated form in general. For $\beta > 1/2$, one gets

$$\tilde\kappa_{\beta,\gamma} \;=\; \sqrt{\frac{2(2\beta\gamma-\beta-\gamma+1)}{\pi\,c_{2,\beta,\gamma}\,(2\beta-1)(2\gamma-1)}}.$$

For $\beta \in (-1/2, 1/2)$, one obtains

$$\kappa_{\beta,\gamma} \;=\; \frac{H_{2\beta+1}}{\sqrt{(2\beta+1)(2\gamma-1)}}(\gamma+\beta)^{-\frac{4\beta}{2\beta+1}}\left(\frac{2\gamma-1}{2\beta+1}\right)^{\frac{2\beta(2\gamma-1)}{(2\beta+1)(\gamma+\beta)}}\left(\frac{\Gamma^2(\beta+1)}{\Gamma(2\beta+1)\,\cos(\pi\beta)}\right)^{\frac{1}{2\beta+1}}$$

where

$$H_{2\beta+1} \;=\; \lim_{T\to\infty}\frac{1}{T}\,\mathbb{E}\left[\exp\left\{\max_{0\leq t\leq T}(\sqrt{2}B_H(t) - t^{2H})\right\}\right]$$

for $\{B_H(t),\, t \geq 0\}$ a standard fractional Brownian motion with Hurst parameter $H = \beta + 1/2$. In the literature on extreme values, the constant $H_{2\beta+1}$ is usually called the Pickands constant and we refer to [7] for more detail. It does not seem to the authors that the Pickands constant is explicit, save for $\beta = 0$ where the reflection principle and Laplace's method yield

$$\mathbb{E}\left[\exp\left\{\max_{0\leq t\leq T}(\sqrt{2}B_t - t)\right\}\right] \;=\; 1 \;+\; \frac{T^{3/2}}{2\sqrt{\pi}}\int_0^1 \sqrt{s}\left(\int_0^\infty x\,e^{-\frac{sT(x-1)^2}{4}}\,dx\right)ds \;\sim\; T,$$

so that $H_1 = 1$ and

$$\kappa_{0,\gamma} \;=\; \frac{1}{\sqrt{2\gamma-1}}$$

in accordance with Theorem A (b).

(c) From Proposition 3.3, it is plausible to conjecture that for $\alpha \in (1,2)$ one has

$$\mathbb{P}[\mathbf{M}^{(\beta)}_{\alpha,1/\alpha,\gamma} \geq x] \;\sim\; \kappa_{\alpha,\beta,\gamma}\, x^{\frac{\alpha\beta(1-\gamma\alpha)}{(\alpha-1+\alpha\beta)(\alpha-1)(\gamma+\beta)}}\; \exp\left\{-c_{\alpha,\beta,\gamma}\, x^{\frac{\gamma\alpha-1}{(\alpha-1)(\gamma+\beta)}}\right\}$$

if $\beta \in (0, 1-1/\alpha)$ and

$$\mathbb{P}[\mathbf{M}^{(\beta)}_{\alpha,1/\alpha,\gamma} \geq x] \;\sim\; \tilde{\kappa}_{\alpha,\beta,\gamma}\, x^{\frac{1-\gamma\alpha}{2(\alpha-1)(\gamma+\beta)}}\; \exp\left\{-c_{\alpha,\beta,\gamma}\, x^{\frac{\gamma\alpha-1}{(\alpha-1)(\gamma+\beta)}}\right\}$$

if $\beta > 1 - 1/\alpha$, where $\kappa_{\alpha,\beta,\gamma}$ and $\tilde{\kappa}_{\alpha,\beta,\gamma}$ are some positive and finite constants.

4 Proof of Theorem C

Following the notation of [16], we will set

$$\theta \;=\; \frac{\rho}{\alpha(1-\rho)+1}$$

once and for all. The upper bound follows easily from

$$\mathbb{P}\left[\sup_{0\leq t\leq 1}\left\{L^{(1)}_t + \mu t^{1+\gamma}\right\} \leq\, \varepsilon\right] \;\leq\; \mathbb{P}\left[\sup_{0\leq t\leq 1}\left\{L^{(1)}_t\right\} \leq \varepsilon\right] \;\leq\; \kappa\, \varepsilon^{\frac{\alpha\theta}{\alpha+1}}$$

for some $\kappa \in (0,\infty)$, where the first inequality follows from $\mu \geq 0$ and the second one from Theorem A in [16] and scaling.

The lower bound is more involved and we will need the strong Markovian character of the two-dimensional process $\{(L^{(1)}_t, L_t),\, t \geq 0\}$, setting by $\mathbb{P}_{(x,y)}$ for its law starting from $(x,y) \in \mathbb{R}^2$, with the convention $\mathbb{P} = \mathbb{P}_{(0,0)}$. Define the stopping time

$$R_\varepsilon \;=\; \inf\left\{t \geq 0,\, L^{(1)}_t + \mu\varepsilon^{\frac{\gamma\alpha-1}{\alpha+1}} t^{1+\gamma} = 0\right\}$$

and observe first that, by scaling and translation,

$$\mathbb{P}\left[\sup_{0\leq t\leq 1}\left\{L^{(1)}_t + \mu t^{1+\gamma}\right\} \leq\, \varepsilon\right] \;=\; \mathbb{P}_{(-1,0)}\left[R_\varepsilon \;\geq\; \varepsilon^{-\frac{\alpha}{\alpha+1}}\right].$$

Notice also that $\mathbb{P}_{(x,y)}\left[R_\varepsilon \leq R_0\right] = 1$ for every $x < 0$ and $y \in \mathbb{R}$, because $\mu \geq 0$. Applying the strong Markov property at R_ε, we obtain

$$\mathbb{P}_{(-1,0)}\left[R_0 \geq 2\varepsilon^{-\frac{\alpha}{\alpha+1}}\right] \;=\; \mathbb{E}_{(-1,0)}\left[\mathbb{P}_{\left(-\mu\varepsilon^{\frac{\gamma\alpha-1}{\alpha+1}} R_\varepsilon^{1+\gamma},\, L_{R_\varepsilon}\right)}\left[R_0 + x \geq 2\varepsilon^{-\frac{\alpha}{\alpha+1}}\right]_{\{x=R_\varepsilon\}}\right]$$

whose right-hand side is, by comparison, smaller than

$$\mathbb{P}_{(-1,0)}\left[R_\varepsilon \geq \varepsilon^{-\frac{\alpha}{\alpha+1}}\right] + \mathbb{E}_{(-1,0)}\left[\mathbf{1}_{\left\{R_\varepsilon \leq \varepsilon^{-\frac{\alpha}{\alpha+1}}\right\}}\, \mathbb{P}_{\left(-\mu\varepsilon^{\frac{\gamma\alpha-1}{\alpha+1}} R_\varepsilon^{1+\gamma},\, -\mu(1+\gamma)\varepsilon^{\frac{\gamma\alpha-1}{\alpha+1}} R_\varepsilon^{\gamma}\right)}\left[R_0 \geq \varepsilon^{-\frac{\alpha}{\alpha+1}}\right]\right].$$

Indeed, the derivative of $t \mapsto L^{(1)}_t + \mu\varepsilon^{\frac{\gamma\alpha-1}{\alpha+1}} t^{1+\gamma}$ at R_ε equals $L_{R_\varepsilon} + \mu(1+\gamma)\varepsilon^{\frac{\gamma\alpha-1}{\alpha+1}} R_\varepsilon^{\gamma}$ and is a.s. non-negative under $\mathbb{P}_{(-1,0)}$. On the other hand, a further scaling argument shows that

$$\mathbb{P}_{(-x,-y)}[R_0 \geq t] \;=\; \mathbb{P}_{\left(-1,-yx^{-\frac{1}{\alpha+1}}\right)}\left[x^{\frac{\alpha}{\alpha+1}} R_0 \geq t\right]$$

for every $x, y, t \geq 0$. If we now assume $\mu \leq 1$ this implies, again by comparison,

$$\begin{aligned}
&\mathbb{E}_{(-1,0)}\left[\mathbf{1}_{\{R_\varepsilon \leq \varepsilon^{-\frac{\alpha}{\alpha+1}}\}}\,\mathbb{P}_{(-\mu\varepsilon^{\frac{\gamma\alpha-1}{\alpha+1}}R_\varepsilon^{1+\gamma},-\mu(1+\gamma)\varepsilon^{\frac{\gamma\alpha-1}{\alpha+1}}R_\varepsilon^{\gamma})}\left[R_0 \geq \varepsilon^{-\frac{\alpha}{\alpha+1}}\right]\right]\\
&= \mathbb{E}_{(-1,0)}\left[\mathbf{1}_{\{R_\varepsilon \leq \varepsilon^{-\frac{\alpha}{\alpha+1}}\}}\,\mathbb{P}_{(-1,-\mu^{\frac{\alpha}{\alpha+1}}(1+\gamma)(\varepsilon^{\frac{\alpha}{\alpha+1}}R_\varepsilon)^{\frac{\gamma\alpha-1}{\alpha+1}})}\left[\left(\mu x^{1+\gamma}\,\varepsilon^{\frac{\gamma\alpha-1}{\alpha+1}}\right)^{\frac{\alpha}{\alpha+1}}R_0 \geq \varepsilon^{-\frac{\alpha}{\alpha+1}}\right]_{\{x=R_\varepsilon\}}\right]\\
&\qquad\qquad \leq \mathbb{E}_{(-1,0)}\left[\mathbb{P}_{(-1,-1-\gamma)}\left[x^{\frac{\alpha(\gamma+1)}{\alpha+1}}R_0 \;\geq \mu^{-\frac{\alpha}{\alpha+1}}\varepsilon^{-\frac{\alpha^2(\gamma+1)}{(\alpha+1)^2}}\right]_{\{x=R_\varepsilon\}}\right]\\
&\qquad\qquad\qquad\qquad \leq\; \mathbb{P}_{(-1,-1-\gamma)}\left[\widetilde{R}_0^{\frac{\alpha(\gamma+1)}{\alpha+1}}R_0 \;\geq \mu^{-\frac{\alpha}{\alpha+1}}\varepsilon^{-\frac{\alpha^2(\gamma+1)}{\alpha+1}}\right]
\end{aligned}$$

where $\widetilde{R}_0$ is an independent copy of R_0. Putting everything together, we obtain

$$\mathbb{P}_{(-1,0)}\left[R_\varepsilon \geq \varepsilon^{-\frac{\alpha}{\alpha+1}}\right] \geq \mathbb{P}_{(-1,0)}\left[R_0 \geq 2\varepsilon^{-\frac{\alpha}{\alpha+1}}\right] - \mathbb{P}_{(-1,-1-\gamma)}\left[\widetilde{R}_0^{\frac{\alpha(\gamma+1)}{\alpha+1}}R_0 \;\geq \mu^{-\frac{\alpha}{\alpha+1}}\varepsilon^{-\frac{\alpha^2(\gamma+1)}{\alpha+1}}\right].$$

Now by Theorem A in [16] we have

$$\mathbb{P}_{(-1,x)}\left[R_0 > t\right] \;\asymp\; t^{-\theta}$$

for every $x \in \mathbb{R}$ and since $\alpha(\gamma+1) > \alpha+1$, we can also infer from Lemma 2 in [17] that

$$\mathbb{P}_{(-1,-1-\gamma)}\left[\widetilde{R}_0^{\frac{\alpha(\gamma+1)}{\alpha+1}}R_0 \;\geq\; t\right] \;\asymp\; t^{\frac{\theta(\alpha+1)}{\alpha(\gamma+1)}}.$$

This implies that there exists two finite constants $\kappa_2 \geq \kappa_1 > 0$ independent of μ, ε such that

$$\mathbb{P}_{(-1,0)}\left[R_\varepsilon \geq \varepsilon^{-\frac{\alpha}{\alpha+1}}\right] \;\geq\; \kappa_1\,\varepsilon^{\frac{\alpha\theta}{\alpha+1}} \;-\; \kappa_2\,\mu^{\frac{\theta}{\gamma+1}}\varepsilon^{\frac{\alpha\theta}{\alpha+1}},$$

which completes the proof of the lower bound for $\mu \leq \mu_0$ with $\mu_0 = (\kappa_1/2\kappa_2)^{(\gamma+1)/\theta} > 0$.

Assuming finally $\mu > \mu_0$ and setting $\bar{\mu} = \mu^{\frac{\alpha}{\alpha\gamma-1}}$ and $\bar{\mu}_0 = \mu_0^{\frac{\alpha}{\alpha\gamma-1}}$ for simplicity, we have

$$\begin{aligned}
&\mathbb{P}\left[\sup_{0\leq t\leq 1}\left\{L_t^{(1)} + \mu t^{1+\gamma}\right\} \leq\, \varepsilon\right]\\
&\qquad = \mathbb{P}\left[\sup_{0\leq t\leq \bar{\mu}}\left\{L_t^{(1)} + t^{1+\gamma}\right\} \leq\, \varepsilon\bar{\mu}^{\frac{\alpha+1}{\alpha}}\right]\\
&\qquad \geq \mathbb{P}\left[\sup_{0\leq t\leq \bar{\mu}}\left\{L_t^{(1)} + t^{1+\gamma}\right\} \leq\, \varepsilon\bar{\mu}_0^{\frac{\alpha+1}{\alpha}}\right]\\
&\qquad \geq \mathbb{P}\left[\sup_{\bar{\mu}_0\leq t\leq \bar{\mu}}\left\{L_t^{(1)} + t^{1+\gamma}\right\} \leq\, 0,\; \sup_{0\leq t\leq \bar{\mu}_0}\left\{L_t^{(1)} + t^{1+\gamma}\right\} \leq\, \varepsilon\bar{\mu}_0^{\frac{\alpha+1}{\alpha}}\right]\\
&\qquad \geq \mathbb{P}\left[\sup_{\bar{\mu}_0\leq t\leq \bar{\mu}}\left\{L_t^{(1)} + t^{1+\gamma}\right\} \leq\, 0\right]\mathbb{P}\left[\sup_{0\leq t\leq \bar{\mu}_0}\left\{L_t^{(1)} + t^{1+\gamma}\right\} \leq\, \varepsilon\bar{\mu}_0^{\frac{\alpha+1}{\alpha}}\right]\\
&\qquad = \mathbb{P}\left[\sup_{\bar{\mu}_0\leq t\leq \bar{\mu}}\left\{L_t^{(1)} + t^{1+\gamma}\right\} \leq\, 0\right]\mathbb{P}\left[\sup_{0\leq t\leq 1}\left\{L_t^{(1)} + \mu_0 t^{1+\gamma}\right\} \leq\, \varepsilon\right]\\
&\qquad \geq \kappa\,\varepsilon^{\frac{\alpha\theta}{\alpha+1}}
\end{aligned}$$

for some $\kappa > 0$, where the first and fifth equalities are obtained by scaling, the fourth inequality follows from Lemma 3.1, and the last inequality is a consequence of the strict positivity of $\bar{\mu}_0$. This completes the proof.

$\square$

Remark 4.1. Using the same argument and Lemma VIII.4 in [4], one can show the following lower tail probabilities estimate for the Lévy stable process with a positive power drift. If $\alpha\gamma > 1$ and $\rho \in (0,1)$, then for every $\mu \geq 0$ one has

$$\mathbb{P}\left[\sup_{0\leq t\leq 1}\{L_t + \mu t^\gamma\} \leq \varepsilon\right] \asymp \varepsilon^{\alpha\rho}. \tag{4.1}$$

We leave the detail, which is simpler than the above, to the interested reader. This estimate for small values echoes the persistence result for large time obtained in Theorem 1 of [2], which reads

$$\mathbb{P}\left[\sup_{0\leq t\leq T}\{L_t + \mu t^\gamma\} \leq 1\right] = T^{-\rho+o(1)} \tag{4.2}$$

for every $\mu \geq 0$, with $\alpha\gamma < 1, \rho \in (0,1)$, and under the additional assumption $\alpha \in (0,1)$. Observe that in the absence of self-similarity, the estimates (4.1) and (4.2) are different ones and cannot be deduced from one another, save for $\mu = 0$.

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