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Acceleration or finite speed propagation in weakly monostable reaction-diffusion equations

Emeric Bouin ^{*} Jérôme Coville [†] Xi Zhang [‡]

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Abstract

This paper focuses on propagation phenomena in reaction-diffusion equations with a weakly monostable nonlinearity. The reaction term can be seen as an intermediate between the classical logistic one (or Fisher-KPP) and the standard weak Allee effect one. We investigate the effect of the decay rate of the initial data on the propagation rate. When the right tail of the initial data is sub-exponential, finite speed propagation and acceleration may happen and we derive the exact separation between the two situations. When the initial data is sub-exponentially unbounded, acceleration unconditionally occurs. Estimates for the locations of the level sets are expressed in terms of the decay of the initial data. In addition, sharp exponents of acceleration for initial data with sub-exponential and algebraic tails are given. Numerical simulations are presented to illustrate the above findings.

Keywords: reaction-diffusion equations, propagation phenomena, weakly monostable equations

1 Introduction

In this paper, we study rates of invasion in the following one-dimensional reaction-diffusion equations

$$\begin{cases} u_t(t, x) = u_{xx}(t, x) + f(u(t, x)), & t > 0, x \in \mathbb{R}, \\ u(0, x) = u_0(x) \geq 0, & x \in \mathbb{R}. \end{cases} \quad (1.1)$$

Hypothesis 1.1. *The non-linearity $f \in C^1([0, 1], \mathbb{R})$ is of the weakly monostable type, in the sense that*

$$f(0) = f(1) = 0, \quad f(s) > 0 \text{ for any } s \in (0, 1), \quad f'(1) < 0,$$

and there exists $s_0 \in (0, 1)$, $K \geq 0$, $\alpha > 0$ and $r > 0$ such that

$$f(s) \leq r \frac{s}{(1 + |\ln s|)^\alpha} \quad \text{for all } s \in (0, 1), \quad (1.2)$$

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and

$$f(s) \geq r \frac{s}{(1 + |\ln s|)^\alpha} (1 - Ks) \quad \text{for all } s \in (0, s_0]. \quad (1.3)$$

After Kolmogorov, Petrovskii and Piskunov [24], and Fisher [16], the classical monostable equation is equation (1.1) with Fisher-KPP type nonlinearity, that is,

$$f(0) = f(1) = 0 \text{ and } 0 < f(s) \leq f'(0)s \text{ for all } s \in (0, 1). \quad (1.4)$$

In population dynamics, this type of non-linearity is commonly used to model the situation where growth per capita is maximal at low densities. The decay rate of the initial data at infinity is crucially important for the propagation problem. For the Fisher-KPP equation with front-like initial data, initial data u_0 with exponentially bounded decay, that is,

$$\lim_{x \rightarrow +\infty} u_0(x) e^{\varepsilon x} < \infty \quad \text{for some } \varepsilon, \quad (1.5)$$

lead to finite propagation speed [12,27]. On the other hand, for an exponentially unbounded initial data, meaning that condition (1.5) is not met, or

$$\lim_{x \rightarrow +\infty} u_0(x) e^{\varepsilon x} > +\infty \quad \text{for any } \varepsilon, \quad (1.6)$$

Hamel and Roques [20] have presented evidence of acceleration of the solution to the Fisher-KPP equation. They also provided an expression of the locations of level sets based on the decay of the initial data. We refer to references [3, 7–9, 13, 15, 17, 21, 22, 25] for the further results about propagation in KPP equations.

When an Allee effect occurs, meaning that the per capita growth is no longer maximal at low densities, the KPP assumption (1.4) becomes unrealistic. Hence, incorporating the Allee effect into models becomes necessary. An acceleration phenomenon may take place in the degenerate situation $f'(0) = 0$. Indeed, when the initial data is front-like and the nonlinearity $f(s) \sim rs^{\alpha+1}$ with $\alpha > 0$ as $s \rightarrow 0^+$, Alfaro [2] has studied the balance between the decay rate of the initial data at infinity and the weak Allee effect and found that for exponentially unbounded tails but lighter than algebraic acceleration does not occur in the presence of the Allee effect, which is in contrast with the KPP equation. Similarly to the KPP situation, the initial data with exponentially bounded decay lead to a finite propagation speed [23, 28]. On the other hand, algebraic decay leads to acceleration despite the Allee effect and the position of the level sets of $u(t, \cdot)$ as $t \rightarrow \infty$ propagates polynomially fast [2, 26]. We refer to references [1, 6, 14, 19] for other kinds of Allee effect.

It is worth mentioning that these results about propagation phenomena in degenerate monostable equations are based on the assumption $f(s) \sim rs^{\alpha+1}$ with some $\alpha > 0$ and $r > 0$ as $s \rightarrow 0^+$. This assumption is used to quantify the degeneracy. In this paper, we also take into account that the growth per capita is small at small densities, but we quantify the degeneracy by a weakly monostable type nonlinearity f satisfying $f(s) \sim r \frac{s}{|\ln s|^\alpha}$ with $\alpha > 0$ and $r > 0$ as $s \rightarrow 0^+$, like $f(s) = r \frac{s}{(1 + |\ln s|)^\alpha} (1 - s)$ for $s \in (0, 1)$. Notice that such nonlinearity is between the KPP type and the Allee effect type near the right side of zero point, see Figure 1. Thus, this type of nonlinear term fill an existing gap between two classical nonlinearities.

To describe the propagation speed, we introduce three notations. For any $\lambda \in (0, 1)$, the (upper) level set of $u(t, x)$ is defined by

$$E_\lambda(t) := \{x \in \mathbb{R} : u(t, x) \geq \lambda\}.$$

Let $x_\lambda(t)$ be the largest element of level set of $u(t, x)$ defined by

$$x_\lambda(t) := \sup E_\lambda(t).$$

For any subset $\Lambda \subset (0, 1]$, we set

$$u_0^{-1}\{\Lambda\} := \{x \in \mathbb{R} : u_0(x) \in \Lambda\}$$

the inverse image of Λ by u_0 .

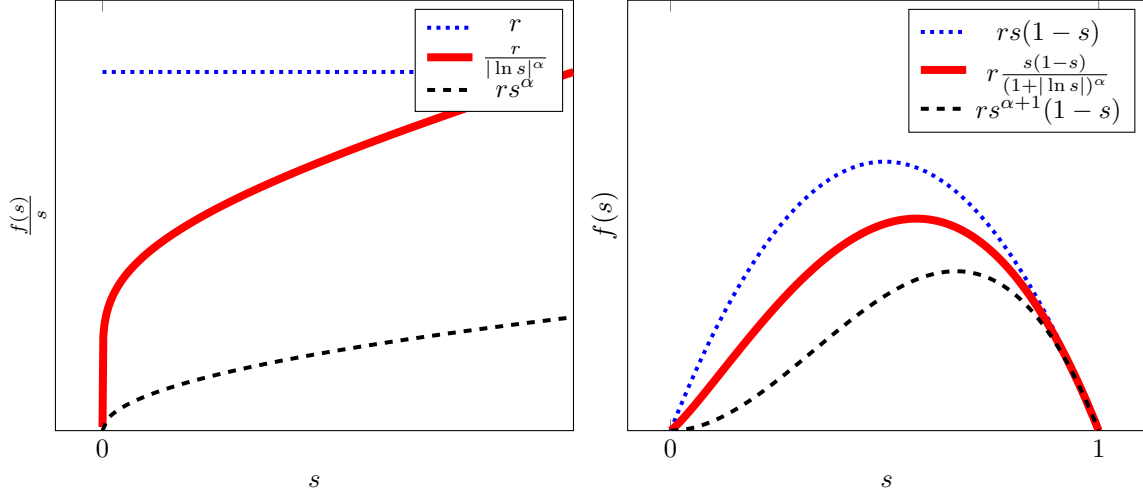


Figure 1: Comparison of the size of three kinds of nonlinearities near zero, where the parameter r and α are positive.

Hypothesis 1.2. *The initial data $u_0 : \mathbb{R} \rightarrow [0, 1]$ is uniformly continuous and asymptotically front-like, in the sense that*

$$u_0 > 0 \text{ in } \mathbb{R}, \quad \liminf_{x \rightarrow -\infty} u_0 > 0, \quad \lim_{x \rightarrow +\infty} u_0 = 0.$$

In this paper, we always denote by $u(t, x)$ the solution to (1.1) with initial data u_0 . We mainly consider the following types of initial data:

- Sub-exponentially bounded for large x , that is, there exist $x_0 > 0$ such that, for any $x > x_0$,

$$u_0(x) \lesssim e^{-\mu x^\beta},$$

with $\beta \in (0, 1)$ and $\mu > 0$.

- Sub-exponential decay for large x , that is, there exist $x_0 > 0$ such that, for any $x > x_0$,

$$u_0(x) \asymp e^{-\mu x^\beta},$$

with $\beta \in (0, 1)$ and $\mu > 0$ ¹.

- Algebraic decay for large x , that is, there exists $x_0 > 1$ such that, for any $x > x_0$,

$$u_0(x) \asymp \frac{1}{x^\beta},$$

with $\beta > 0$.

- Initial data u_0 that decay as a negative power of $\ln x$ for large x , that is, there exists $x_0 > e$ such that, for any $x > x_0$,

$$u_0(x) \asymp (\ln x)^{-\beta},$$

with $\beta > 0$.

Our first result shows that for sub-exponentially bounded initial data, acceleration does not happen.

¹The notation $a \asymp b$ means that there exists a constant C such that $Cb \leq a \leq C^{-1}b$.

Theorem 1.3. *Let $\alpha > 0$ and $\beta > 0$ be such that*

$$\beta \geq \frac{1}{\alpha + 1}.$$

Assume that the non-linearity f and the initial data u_0 satisfy Hypotheses 1.1 and 1.2, respectively. Assume that there exist $x_0 > 0$ and $\mu > 0$ such that

$$u_0(x) \lesssim e^{-\mu x^\beta} \quad \text{for any } x \geq x_0. \quad (1.7)$$

Then, for any $\lambda \in (0, 1)$, there exist some positive constants c and a time T_λ such that

$$\Gamma < \frac{x_\lambda(t)}{t} < c \quad \text{for any } t > T_\lambda. \quad (1.8)$$

Now, we turn to cases where it is assumed that the initial data u_0 decay more slowly than $e^{-\varepsilon x^{\frac{1}{\alpha+1}}}$ as $x \rightarrow +\infty$ for any $\varepsilon > 0$, that is,

$$\forall \varepsilon > 0, \quad \exists x_\varepsilon \in \mathbb{R}, \quad u_0(x) \geq e^{-\varepsilon x^{\frac{1}{\alpha+1}}} \text{ in } [x_\varepsilon, +\infty). \quad (1.9)$$

Let us denote

$$\varphi_0(x) := -\ln u_0(x) \geq 0. \quad (1.10)$$

Notice that if u_0 is C^2 , then we can get

$$\varphi_0'(x) = -\frac{u_0'}{u_0}(x) \quad \text{and} \quad \varphi_0''(x) = -\left(\frac{u_0'}{u_0}\right)'(x).$$

Observe that if we assume that $\varphi_0'(x) = o(\varphi_0^{-\alpha}(x))$ as $x \rightarrow +\infty$, then condition (1.9) is fulfilled.

For such initial data, we have the following result.

Lemma 1.4. *Assume that the non-linearity f and the initial data u_0 satisfy Hypotheses 1.1 and 1.2, respectively. Assume that u_0 is of class C^2 and non-increasing on $[\xi_0, +\infty)$ for some $\xi_0 > 0$, and*

$$\varphi_0'(x) = o(\varphi_0^{-\alpha}(x)) \text{ and } \varphi_0''(x) = o(\varphi_0'(x)) \quad \text{as } x \rightarrow +\infty. \quad (1.11)$$

Then, for any fixed $\lambda \in (0, 1)$ and small $\varepsilon > 0$, there is a time $T_{\lambda, \varepsilon}$ such that

$$E_\lambda(t) \subset u_0^{-1} \left\{ \left[e^{-[(r+\varepsilon)(\alpha+1)t]^{\frac{1}{\alpha+1}}}, e^{-[(r-\varepsilon)(\alpha+1)t]^{\frac{1}{\alpha+1}}} \right] \right\} \quad \text{for any } t > T_{\lambda, \varepsilon}.$$

It is easy to check that initial data $u_0(x) \asymp e^{-\mu x^\beta}$ satisfy (1.11) in the regime $\beta < \frac{1}{\alpha+1}$. Thus, according to the above lemma, we obtain the following theorem.

Theorem 1.5. *Let $\alpha > 0$ and $\beta > 0$ be such that*

$$\beta < \frac{1}{\alpha + 1}.$$

Assume that the nonlinearity f and the initial data u_0 satisfy Hypotheses 1.1 and 1.2, respectively. Assume that there exists $x_0 > 0$ and $\mu > 0$ such that

$$u_0(x) \asymp e^{-\mu x^\beta} \quad \text{for any } x \geq x_0.$$

Then, for any $\lambda \in (0, 1)$ and $\varepsilon > 0$, there exists a time $T'_{\lambda, \varepsilon}$ such that ²

$$x_\lambda(t) \asymp_{\lambda, \varepsilon, \mu} t^{\frac{1}{\beta(\alpha+1)}} \quad \text{for any } t > T'_{\lambda, \varepsilon}.$$

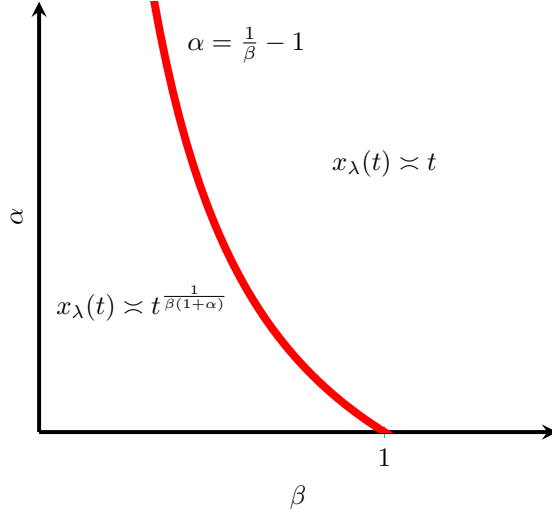


Figure 2: The separation for sub-exponential decay initial data case.

For initial data with algebraic tails, that is $u_0(x) \asymp x^{-\beta}$ for $\beta > 0$, by Lemma 1.4, we just obtain a rough estimate:

$$C_1 e^{\frac{1}{\beta}((r-\varepsilon)(\alpha+1)t)^{\frac{1}{\alpha+1}}} \leq x_\lambda(t) \leq C_2 e^{\frac{1}{\beta}((r+\varepsilon)(\alpha+1)t)^{\frac{1}{\alpha+1}}}$$

for some constants C_1 and C_2 . Notice that the position of the level set depends strongly on the constant ε . Hence the estimate is not enough for such initial data.

To get an exact estimate of the position of the level sets, we add a concavity assumption, that we believe not to be a huge restriction given that classical sub-exponentials are usually log-concave functions.

Lemma 1.6. *Assume that the nonlinearity f and the initial data satisfy Hypotheses 1.1 and 1.2, respectively. Assume that u_0 is of class C^2 and nonincreasing on $[\xi_0, +\infty)$ for some $\xi_0 > 0$, and*

$$\varphi_0'(x) = o(\varphi_0^{-\alpha}(x)) \text{ and } \varphi_0''(x) = o(\varphi_0'(x)) \text{ as } x \rightarrow +\infty.$$

Assume that

$$\varphi_0''(x) \leq 0 \text{ for large } x. \quad (1.12)$$

Then, for any $\lambda \in (0, 1)$, there are two constants $\underline{C}_\lambda > 0$ and $\bar{C}_\lambda > 0$ and a time T_λ such that

$$E_\lambda(t) \subset u_0^{-1} \left\{ \left[\bar{C}_\lambda e^{-[r(\alpha+1)t]^{\frac{1}{\alpha+1}}}, \underline{C}_\lambda e^{-[r(\alpha+1)t]^{\frac{1}{\alpha+1}}} \right] \right\} \text{ for any } t > T_\lambda. \quad (1.13)$$

We point out that (1.12) is used only in the proof of the lower bound. Our approach can be used to prove the exact result in the KPP situation of [20].

Equipped with the above lemma, we can get exact estimates for the level sets of the solution to equation (1.1) with the algebraic decay initial data. We check the assumptions in Lemma 1.6 and obtain the following theorem.

Theorem 1.7. *Assume that the nonlinearity f and the initial data satisfy Hypotheses 1.1 and 1.2, respectively. Assume that there exist $x_0 > 1$ and $\beta > 0$ such that*

$$u_0(x) \asymp \frac{1}{x^\beta} \text{ for any } x \geq x_0.$$

²The notation $a \asymp_{\Lambda_1, \Lambda_2, \dots} b$ means that there exists a constant $C_{\Lambda_1, \Lambda_2, \dots}$, depending on some constants $\Lambda_1, \Lambda_2, \dots$, such that $C_{\Lambda_1, \Lambda_2, \dots} b \leq a \leq C_{\Lambda_1, \Lambda_2, \dots}^{-1} b$.

Then, for any $\lambda \in (0, 1)$, there exists a time T'_λ such that

$$x_\lambda(t) \asymp_\lambda e^{\frac{1}{\beta}[r(\alpha+1)t]^{\frac{1}{\alpha+1}}} \quad \text{for any } t > T'_\lambda.$$

Observe that when $\alpha = 0$, one recovers the rate of the KPP situation [20, 22]. For the degenerate monostable case, Alfaro [2] shows that if $f(s) = s^{1+\alpha}(1-s)$ then

$$x_\lambda(t) \asymp t^{\frac{1}{\alpha\beta}} \quad \text{for } t \text{ large enough,}$$

where $0 < \alpha < \frac{1}{\beta}$.

Thanks to Lemma 1.6, we can also get the following theorem for the initial data $u_0(x) \asymp (\ln x)^{-\beta}$.

Theorem 1.8. *Assume that the nonlinearity f and the initial data u_0 satisfy Hypotheses 1.1 and 1.2, respectively. Assume that there exists $x_0 > e$ and $\beta > 0$ such that*

$$u_0(x) \asymp (\ln x)^{-\beta} \quad \text{for any } x \geq x_0. \quad (1.14)$$

Then, for any $x(t) \in E_\lambda(t)$, there exists a time T''_λ , such that

$$\ln x_\lambda(t) \asymp_\lambda e^{\frac{1}{\beta}[r(\alpha+1)t]^{\frac{1}{\alpha+1}}} \quad \text{for any } t > T''_\lambda.$$

Remark 1.9. *One can obtain Lemma 1.4 and Theorem 1.5 under the weaker hypothesis*

$$f(s) \sim r \frac{s}{(1 + |\ln s|)^\alpha} \quad \text{as } s \rightarrow 0^+,$$

for some $r > 0$ and $\alpha > 0$. However, Lemma 1.6 and Theorems 1.7 and 1.8 need crucially Hypothesis 1.1 with both precise bounds (1.2) and (1.3) for f . An insight can be easily seen in the proofs in Sections 3 and 4.

Nevertheless, Lemmas 1.4 and 1.6 and Theorems 1.5, 1.7 and 1.8, are true under Hypothesis 1.1 where (1.3) is replaced by

$$f(s) \geq r \frac{s}{(1 + |\ln s|)^\alpha} (1 - Ks^\delta) \quad \text{for any } s \in (0, s_0],$$

for some $\delta > 0$, $s_0 \in (0, 1)$ and $K \geq 0$. Their proofs are similar to the one we provide in Sections 3 and 4 but a bit messier so we have chosen to stick to $\delta = 1$ for the sake of readability.

The rest of this paper is organized as follows. In Section 2, we shall prove that the solution to equation (1.1), starting from an exponentially unbounded initial data, propagates at constant speed. In Section 3 and Section 4, we provide the proof of the main results, Lemma 1.4 and Lemma 1.6, respectively. In Section 5, some numerical simulations shall be given to illustrate our main results.

This paper is the first part of our work on weakly monostable equations; a companion paper [11] with non-local dispersal follows. In this latter paper, we have proved the existence and nonexistence of traveling waves, and studied the effect of the tails of the dispersal kernel on the propagation rate. Exact rates of invasion have been provided for the sub-exponential and algebraic tails.

2 Finite speed propagation: Proof of Theorem 1.3

In this section, we prove Theorem 1.3: the level sets of the solution to (1.1) moves at a constant speed.

As in [2, Theorem 2.3], we can also obtain that, for any $\lambda \in (0, 1)$, there is a time $t_\lambda > 0$ and $\Gamma > 0$ such that

$$\emptyset \neq E_\lambda(t) \subset (\Gamma t, +\infty) \quad \text{for any } t > t_\lambda. \quad (2.1)$$

Indeed, we consider the equation

$$\begin{cases} v_t - v_{xx} - r_1 v^2(1-v) = 0, & t > 0, x \in \mathbb{R}, \\ v(0, x) = v_0 \geq 0, \end{cases} \quad (2.2)$$

where the initial data $v_0(x) = \inf_{y \leq 0} u_0(y) \mathbb{1}_{(-\infty, 0)}(x)$ and $r_1 > 0$ small enough so that $r_1 s^2(1-s) \leq f(s)$ for all $s \in (0, 1)$. According to [29], the solution $v(t, x)$ to (2.2) satisfies $\lim_{t \rightarrow \infty} \inf_{x \leq \Gamma t} v(t, x) = 1$ for some $\Gamma > 0$. It follows from the comparison principle that propagation of $u(t, x)$ is at least linear, that is,

$$\liminf_{t \rightarrow \infty} \inf_{x \leq \Gamma t} u(t, x) = 1. \quad (2.3)$$

On the other hand, we can reproduce the proof of [20, Theorem 1.1 part a], which does not require the KPP assumption, and get

$$\lim_{x \rightarrow +\infty} u(t, x) = 0 \quad \text{for any } t \geq 0. \quad (2.4)$$

Thus, combining (2.3) and (2.4), we can conclude (2.1).

Inspired by [2], for the initial data with sub-exponential decay, we use a suitable shifted profile which construction now follows. Take $\alpha > 0$ and $\beta > 0$ such that

$$\beta \geq \frac{1}{\alpha + 1}.$$

Let us define

$$w(z) := M e^{-\mu z^p} (\leq 1) \quad \text{for } z \geq z_0 := \left(\frac{\ln M}{\mu} \right)^{\frac{1}{p}}, \quad (2.5)$$

where $p := \frac{1}{\alpha+1} < 1$ and $M > e$.

Lemma 2.1. *Assume that f satisfies Hypothesis 1.1. Then, for any $M > e$, there is $c > 0$ such that*

$$w''(z) + c w'(z) + f(w(z)) \leq 0, \quad \forall z \geq z_0.$$

Proof. By definition of w , we have, for $z \geq z_0$,

$$w'(z) = -\mu p z^{p-1} w(z) \quad \text{and} \quad w'' = \left(\mu p(1-p) z^{p-2} + \mu^2 p^2 z^{2(p-1)} \right) w(z).$$

Since $1 - \ln M + \mu z^p \geq \frac{\mu}{2} z^p$ for any $z \geq \left(\frac{2(\ln M - 1)}{\mu} \right)^{\frac{1}{p}}$, then we have, for all $z \geq z_1 := \max\{z_0, \left(\frac{2(\ln M - 1)}{\mu} \right)^{\frac{1}{p}}\}$,

$$\begin{aligned} w''(z) + c w'(z) + f(w(z)) &\leq \mu w(z) \left(\frac{\mu p^2}{z^{2(1-p)}} + \frac{p(1-p)}{z^{2-p}} - \frac{cp}{z^{1-p}} + \frac{r}{\mu(1 - \ln M + \mu z^p)^\alpha} \right) \\ &\leq \mu w(z) \left(\frac{\mu p^2}{z^{2(1-p)}} + \frac{p(1-p)}{z^{2-p}} - \frac{cp - \frac{2^\alpha r}{\mu^{\alpha+1}}}{z^{1-p}} \right). \end{aligned}$$

Choosing $c > \frac{2^\alpha r}{\mu^{\alpha+1}}$, the above is nonpositive for z large enough, say $z \geq z_2$. On the other hand, for the remaining region $z_0 \leq z \leq z_2$, we have

$$\begin{aligned} w''(z) + c w'(z) + f(w(z)) &\leq \mu w(z) \left(\frac{\mu p^2}{z^{2(1-p)}} + \frac{p(1-p)}{z^{2-p}} - \frac{cp}{z^{1-p}} + \frac{r}{\mu(1 - \ln M + \mu z^p)^\alpha} \right) \\ &\leq \mu w(z) \left(\frac{\mu p^2}{z_0^{2(1-p)}} + \frac{p(1-p)}{z_0^{2-p}} - \frac{cp}{z_2^{1-p}} + \frac{r}{\mu(1 - \ln M + \mu z_0^p)^\alpha} \right), \end{aligned}$$

by taking c large enough so that the above is nonpositive. \square

Equipped with the above lemma, we can construct a supersolution to (1.1). Let $M = \max\{e, e^{x_0^\beta} \|u_0\|_\infty\}$ and define

$$v(t, x) := \begin{cases} w(x - x_0 - ct), & x > ct + x_0 + z_0, \\ 1, & x \leq ct + x_0 + z_0, \end{cases}$$

where c and x_0 is from the above lemma and (1.7) respectively. We claim that $v(t, x)$ is a supersolution for (1.1) for any $x \in \mathbb{R}$ and $t > 0$. Indeed, it is enough to check it when $v(t, x) < 1$, that is, $x > ct + x_0 + z_0$. It follows from the above lemma that

$$v_t - v_{xx} - f(v) \geq -(cw' - w'' + r \frac{w}{(1 - \ln w)^\alpha}) \geq 0.$$

For $x > x_0 + z_0$, since $p = \frac{1}{\alpha+1} \leq \beta$ and (1.7), then we have

$$v(0, x) = M e^{-\mu(x-x_0)^p} \geq e^{x_0^\beta} \|u_0\|_\infty e^{-\mu x^\beta} \geq u_0(x).$$

On the other hand, since $u_0 \leq 1$, for $x \leq x_0 + z_0$, we have $v(0, x) = 1 \geq u_0(x)$.

In the regime $\beta \geq \frac{1}{\alpha+1}$, the comparison principle then implies that for all $t > 0$ and $x \in \mathbb{R}$, we have

$$u(t, x) \leq v(t, x) \leq w(x - x_0 - ct).$$

Therefore, for any $\lambda \in (0, 1)$, there is T_λ large enough such that for all $t > T_\lambda$, we have

$$x_\lambda(t) \leq x_0 + \left(\frac{1}{\mu} \ln \frac{M}{\lambda}\right)^{\frac{1}{p}} + ct \leq (c+1)t,$$

which gives the upper bound of (1.8). Together with (2.1), the proof of Theorem 1.3 is complete.

3 The acceleration regime: Proof of Lemma 1.4

In this section, we prove Lemma 1.4: the level sets of solution to the equation (1.1) with front-like initial data that is sub-exponentially unbounded move by accelerating, and the locations of the level sets are expressed in terms of the decay of the initial data.

The long-time behaviour of the solution to the Cauchy problem (1.1) is captured approximately by the ODE

$$\begin{cases} w_t = \rho \frac{w}{(1 - \ln w)^\alpha}, & t > 0, x \in \mathbb{R}, \\ w(0, x) = u_0(x) \geq 0, & x \in \mathbb{R}, \end{cases} \quad (3.1)$$

where $\rho > 0$ is to be determined. We solve the above ODE and obtain

$$w(t, x) = \exp \left\{ 1 - \left[(1 + \varphi_0(x))^{\alpha+1} - \rho(\alpha+1)t \right]^{\frac{1}{\alpha+1}} \right\}, \quad (3.2)$$

where φ_0 is defined by (1.10). Notice that $w(t, x) \geq w(0, x) = u_0(x)$ since $w(x, \cdot)$ is increasing for each $x \in \mathbb{R}$. Let us define

$$x_0(t) := \sup \left\{ x \in \mathbb{R} : u_0(x) = \exp \left(1 - (\rho(\alpha+1)t + 1)^{\frac{1}{\alpha+1}} \right) \right\}. \quad (3.3)$$

Observe that $w(t, x_0(t)) = 1$ and $0 < w(t, x) \leq 1$ for $x \geq x_0(t)$. For any $x \geq x_0(t)$ and $t > 0$, we have

$$w_x = -\frac{w}{(1 - \ln w)^\alpha} \varphi_0'(1 + \varphi_0)^\alpha, \quad (3.4)$$

and

$$w_{xx} = \frac{w}{(1 - \ln w)^\alpha} \left\{ (\varphi_0')^2 (1 + \varphi_0)^{2\alpha} \left((1 - \ln w)^{-\alpha} + \alpha(1 - \ln w)^{-(\alpha+1)} - \alpha(1 + \varphi_0)^{-(\alpha+1)} \right) - \varphi_0'' (1 + \varphi_0)^\alpha \right\}. \quad (3.5)$$

For w_{xx} , we have the following estimate.

Lemma 3.1. *Let u_0 such that $\varphi_0 = -\ln u_0$ satisfies $\varphi_0' = o(\varphi_0^{-\alpha})$ and $\varphi_0'' = o(\varphi_0')$ as $x \rightarrow +\infty$. Then, for any small $\varepsilon > 0$, there exists $t^\# > 0$, depending on ε , such that*

$$|w_{xx}| < \varepsilon \frac{w}{(1 - \ln w)^\alpha} \quad \text{for any } x \geq x_0(t) \text{ and } t \geq t^\#. \quad (3.6)$$

Proof. Since $0 < u_0(x) \leq w(t, x) \leq 1$, we have

$$0 < (1 + \varphi_0)^{-(\alpha+1)} \leq (1 - \ln w)^{-(\alpha+1)} \leq 1.$$

It follows from $0 < w \leq 1$ for all $x \geq x_0(t)$ and $t > 0$ that, for any $x \geq x_0(t)$ and $t > 0$, we have

$$0 < (1 - \ln w)^{-\alpha} + \alpha(1 - \ln w)^{-(\alpha+1)} - \alpha(1 + \varphi_0)^{-(\alpha+1)} < 2. \quad (3.7)$$

In view of the definition (3.3) of $x_0(t)$, since u_0 is nonincreasing and $\lim_{x \rightarrow +\infty} u_0 = 0$, we have $x_0(t) \rightarrow +\infty$ as $t \rightarrow \infty$. For any small $\varepsilon > 0$, it follows from the assumption on $\varphi_0'(x)$ that there exists $t^\# > 0$ such that for $x > x_0(t)$ and $t \geq t^\#$, we have

$$|\varphi_0'(x)(1 + \varphi_0(x))^\alpha| < \sqrt{\frac{\varepsilon}{4}}.$$

On the other hand, it follows from $\varphi_0''(x) = o(\varphi_0'(x))$ that there exists X' such that for $x > x_0(t) \geq X'$ and $t \geq t^\#$, up to enlarge $t^\#$ if necessary, we have

$$|\varphi_0''(x)(1 + \varphi_0(x))^\alpha| < \varepsilon/2.$$

Therefore, by collecting the above estimates, we have, for any $x \geq x_0(t)$ and $t \geq t^\#$,

$$\begin{aligned} |w_{xx}(t, x)| &\leq \frac{w(t, x)}{(1 - \ln w(t, x))^\alpha} \left\{ (\varphi_0'(x)(1 + \varphi_0(x))^\alpha)^2 (1 - \ln w(t, x))^{-\alpha} \right. \\ &\quad \left. + \alpha(1 - \ln w(t, x))^{-(\alpha+1)} - \alpha(1 + \varphi_0(x))^{-(\alpha+1)} + |\varphi_0''(x)(1 + \varphi_0(x))^\alpha| \right\} \\ &\leq \frac{w(t, x)}{(1 - \ln w(t, x))^\alpha} \left(\frac{\varepsilon}{4} \times 2 + \frac{\varepsilon}{2} \right) = \varepsilon \frac{w(t, x)}{(1 - \ln w(t, x))^\alpha}, \end{aligned}$$

which gives the estimate (3.6). This completes the proof. \square

Here we present a lemma, which will play a key role in the proof of Lemma 1.6.

Lemma 3.2. *Let u_0 such that $\varphi_0 = -\ln u_0$ satisfies $\varphi_0' = o(\varphi_0^{-\alpha})$ and $\varphi_0'' = o(\varphi_0')$ as $x \rightarrow +\infty$. Then there is $t^1 > 0$ such that, for $x \geq x_0(t)$ and $t \geq t^1$, we have*

$$w_x + w_{xx} \leq 0. \quad (3.8)$$

Proof. By the assumptions $\varphi_0'(x) = o(\varphi_0^{-\alpha}(x))$ and $\varphi_0''(x) = o(\varphi_0'(x))$, there exists X_0 such that for all $x \geq X_0$, we have

$$\varphi_0'(1 + \varphi_0)^\alpha \leq \frac{1}{4} \quad \text{and} \quad |\varphi_0''| \leq \frac{1}{2} \varphi_0'.$$

Since $x_0(t) \rightarrow +\infty$ as $t \rightarrow \infty$, there is $t^1 > 0$ such that $x_0(t) > X_0$ for all $t \geq t^1$. In view of the definition (1.10) of φ , since u_0 is a nonincreasing function, then $\varphi' \geq 0$. It then follows from (3.4), (3.5) and (3.7) that we have, for all $x \geq x_0(t)$ and $t \geq t^1$,

$$\begin{aligned} w_x + w_{xx} &\leq \frac{w}{(1 - \ln w)^\alpha} (1 + \varphi_0)^\alpha \left(\varphi_0'(-1 + 2\varphi_0'(1 + \varphi_0)^\alpha) + |\varphi_0''| \right) \\ &\leq \frac{w}{(1 - \ln w)^\alpha} (1 + \varphi_0)^\alpha \left(-\frac{1}{2} \varphi_0' + \frac{1}{2} \varphi_0' \right) = 0. \end{aligned}$$

This completes the proof. \square

3.1 The upper bound

In this subsection, we prove the upper bound of the level sets in Lemma 1.4 by constructing an accurate supersolution.

We define

$$m(t, x) = \begin{cases} w(t + t^\#, x), & x \geq x_0(t + t^\#), \\ 1, & x < x_0(t + t^\#), \end{cases}$$

where $t^\#$ is defined in Lemma 3.1. Observe that $m(t, x)$ is well defined for all $t \geq 0$ and all $x \in \mathbb{R}$, and $0 < m(t, x) \leq 1$.

Let $\varepsilon > 0$ be given and define

$$\rho = r + \frac{\varepsilon}{2}. \quad (3.9)$$

Now, we prove that m is a supersolution of equation (1.1).

Lemma 3.3. *Let u_0 such that $\varphi_0 = -\ln u_0$ satisfies $\varphi_0' = o(\varphi_0^{-\alpha})$ and $\varphi_0'' = o(\varphi_0')$ as $x \rightarrow +\infty$. Then $m(t, x)$ is a supersolution to equation (1.1) for all $t > 0$ and $x \in \mathbb{R}$.*

Proof. To prove m is a supersolution, we need to check that $m_t - m_{xx} - f(m) \geq 0$ for all $t > 0$ and $x \in \mathbb{R}$. For $x < x_0(t + t^\#)$ and $t > 0$, since $m_t = m_{xx} = f(m) = 0$, we have

$$m_t(t, x) - m_{xx}(t, x) - f(m(t, x)) = 0.$$

On the other hand, for all $x \geq x_0(t + t^\#)$ and $t > 0$, by the definitions of m and w , we have

$$m_t(t, x) = w_t(t + t^\#, x) = \rho \frac{w(t + t^\#, x)}{(1 - \ln w(t + t^\#, x))^\alpha} = \left(r + \frac{\varepsilon}{2}\right) \frac{w(t + t^\#, x)}{(1 - \ln w(t + t^\#, x))^\alpha}.$$

Thus, by Lemma 3.1 and Hypothesis 1.1, for all $x \geq x_0(t + t^\#) > x_0(t)$ and $t > 0$, we obtain

$$\begin{aligned} m_t(t, x) - m_{xx}(t, x) - f(m(t, x)) &= w_t(t + t^\#, x) - w_{xx}(t + t^\#, x) - f(w(t + t^\#, x)) \\ &\geq \left(r + \frac{\varepsilon}{2}\right) \frac{w(t + t^\#, x)}{(1 - \ln w(t + t^\#, x))^\alpha} - \frac{\varepsilon}{2} \frac{w(t + t^\#, x)}{(1 - \ln w(t + t^\#, x))^\alpha} - r \frac{w(t + t^\#, x)}{(1 - \ln w(t + t^\#, x))^\alpha} = 0. \end{aligned}$$

This completes the proof. \square

In view of the definition of m , for $x < x_0(t + t^\#)$, since $u_0 \leq 1$, we have $m(0, x) = 1 \geq u_0(x)$. For $x \geq x_0(t + t^\#)$, since $w(\cdot, x)$ is nondecreasing for each $x \in \mathbb{R}$, we have $m(0, x) = w(t^\#, x) \geq w(0, x) = u_0(x)$. Thus, $m(0, x) \geq u_0(x) = u(0, x)$ for all $x \in \mathbb{R}$. Equipped with Lemma 3.3, it then follows from the comparison principle that

$$m(t, x) \geq u(t, x) \quad \text{for all } t > 0 \text{ and } x \in \mathbb{R}. \quad (3.10)$$

Now, we prove the upper bound in Lemma 1.4.

The proof of the upper bound. We need to prove that, for any $\lambda \in (0, 1)$ and large time t , we have

$$E_\lambda(t) \subset u_0^{-1} \left\{ \left[e^{-[(r+\varepsilon)(\alpha+1)t]^{\frac{1}{\alpha+1}}}, 1 \right] \right\}. \quad (3.11)$$

By (3.10) and the definition of m , we have

$$u(t, x) \leq m(t, x) \leq w(t + t^\#, x) \quad \text{for all } t > 0 \text{ and } x \in \mathbb{R}.$$

Let us pick a $y \in E_\lambda(t)$, then $w(t + t^\#, y) \geq \lambda$. It follows that, by the definitions of w and φ_0 , we have

$$w(t + t^\#, y) = \exp \left\{ 1 - [(1 - \ln u_0(y))^{\alpha+1} - \rho(\alpha+1)(t + t^\#)]^{\frac{1}{\alpha+1}} \right\} \geq \lambda,$$

whence

$$u_0(y) \geq \exp \left\{ 1 - \left[\rho(\alpha + 1)(t + t^\#) + (1 - \ln \lambda)^{\alpha+1} \right]^{\frac{1}{\alpha+1}} \right\}.$$

Since $\rho = r + \frac{\varepsilon}{2}$, there is a time $\bar{t}_{\lambda, \varepsilon} > 0$ such that

$$u_0(y) \geq e^{-[(r+\varepsilon)(\alpha+1)t]^{\frac{1}{\alpha+1}}} \quad \text{for any } t \geq \bar{t}_{\lambda, \varepsilon}, \quad (3.12)$$

which gives (3.11). This completes the proof. \square

3.2 The lower bound

In this subsection, we explore the lower bound of the level sets of a solution to (1.1) by constructing an adequate subsolution.

Let $\varepsilon > 0$ be given. We take

$$\max \left\{ r - \frac{\varepsilon}{2}, \frac{3}{4}r \right\} < \rho < r. \quad (3.13)$$

Let us define the function $g(y) := y(1 - My)$ with $M > 0$. Notice that

$$0 \leq g(y) \leq g\left(\frac{1}{2M}\right) = \frac{1}{4M}, \quad \forall y \in \left[0, \frac{1}{2M}\right].$$

We define

$$x_M(t) := \sup \left\{ x \in \mathbb{R} : u_0(x) = \exp \left\{ 1 - \left((1 + \ln(2M))^{\alpha+1} + \rho(\alpha + 1)t \right)^{\frac{1}{\alpha+1}} \right\} \right\} \geq x_0(t),$$

where $x_0(t)$ is defined by (3.3). Observe that $w(t, x_M(t)) = \frac{1}{2M}$ and $w(t, x) < \frac{1}{2M}$ for all $x > x_M(t)$. Let us define

$$\zeta := \inf_{x \in (-\infty, \xi_1)} u_0(x),$$

where $\xi_1 := \max\{\xi_0, x_0(0)\}$. Notice that $\zeta \in (0, 1]$ according to Hypothesis 1.2, and that u_0 is non-increasing on $[\xi_1, +\infty)$. We select large enough $M > 0$ so that

$$M \geq M_0 := \max \left\{ \frac{1}{2\zeta}, \frac{1}{4s_0} \right\}.$$

Then, by $u_0(x_M(0)) = \frac{1}{2M} < \zeta \leq u_0(\xi_1)$, we have $x_M(0) > \xi_1$.

Let us define

$$v(t, x) := \begin{cases} \frac{1}{4M}, & x \leq x_M(t), \\ g(w(t, x)), & x > x_M(t). \end{cases} \quad (3.14)$$

Since $M \geq \frac{1}{4s_0}$, then we have $0 < v(t, x) \leq s_0$ for all $t \geq 0$ and $x \in \mathbb{R}$.

Lemma 3.4. *Let u_0 such that $\varphi_0 = -\ln u_0$ satisfies $\varphi_0' = o(\varphi_0^{-\alpha})$ and $\varphi_0'' = o(\varphi_0')$ as $x \rightarrow +\infty$. Then there exists large enough $M > 0$ such that $v(t, x)$ is a subsolution to equation (1.1) for all $t > 0$ and $x \in \mathbb{R}$.*

Proof. In view of the definition (3.14) of v , we obtain

$$v_t(t, x) = \rho \frac{w(t, x)}{(1 - \ln w(t, x))^\alpha} \left(1 - 2Mw(t, x) \right)^+.$$

It then follows that

$$v_t(t, x) \leq \begin{cases} 0, & x \leq x_M(t), \\ \rho \frac{w(t, x)}{(1 - \ln w(t, x))^\alpha} \left(1 - 2Mw(t, x) \right), & x > x_M(t). \end{cases} \quad (3.15)$$

Since $\frac{1}{(1+y)^\alpha} \geq 1 - \alpha y$ for any $y \geq 0$, then, for any $x > x_M(t)$, we have

$$1 \geq \left(\frac{1 - \ln w}{1 - \ln v} \right)^\alpha = \left(\frac{1}{1 - \frac{\ln(1-Mw)}{1 - \ln w}} \right)^\alpha \geq 1 + \alpha \frac{\ln(1-Mw)}{1 - \ln w}.$$

It follows from the inequality $\ln(1-y) \geq -\tilde{c}y$ for any $y \in (0, \frac{1}{2})$, where $\tilde{c} = 2 \ln 2$, that, for any $x > x_M(t)$, we have

$$1 \geq \left(\frac{1 - \ln w}{1 - \ln v} \right)^\alpha \geq 1 - \alpha \tilde{c} M \frac{w}{1 - \ln w},$$

from $0 < Mw < \frac{1}{2}$. Thus, since $f(s) \geq r \frac{s}{(1-\ln s)^\alpha} (1 - Ks)$ for $s \in (0, s_0]$ and $0 < v(t, x) \leq s_0$ for all $t \geq 0$ and $x \in \mathbb{R}$, when $x > x_M(t)$, we have

$$\begin{aligned} f(v(t, x)) &\geq r \frac{v(t, x)}{(1 - \ln v(t, x))^\alpha} (1 - Kv(t, x)) \\ &\geq r \frac{w(t, x)(1 - Mw(t, x))}{(1 - \ln w(t, x))^\alpha} \left(\frac{1 - \ln w(t, x)}{1 - \ln v(t, x)} \right)^\alpha (1 - Kw(t, x)) \\ &\geq r \frac{w(t, x)}{(1 - \ln w(t, x))^\alpha} \left(1 - (M + K)w(t, x) - \alpha \tilde{c} M \frac{w(t, x)}{1 - \ln w(t, x)} - \alpha \tilde{c} K M^2 \frac{w^3(t, x)}{1 - \ln w(t, x)} \right). \end{aligned} \quad (3.16)$$

On the other hand, when $x \leq x_M(t)$, we have

$$f(v(t, x)) \geq 0, \quad (3.17)$$

thanks to $v \in (0, 1)$.

Now, let us estimate the value v_{xx} . In view of the definition (3.14) of v , we have $v_{xx}(t, x) = 0$ for $x \leq x_M(t)$ and

$$v_{xx}(t, x) = (1 - 2Mw(t, x)) w_{xx}(t, x) - 2Mw_x^2(t, x) \quad \text{for } x > x_M(t).$$

In view of (3.4) and (3.5), we get

$$v_{xx}(t, x) \geq -\varphi_0''(x)(1 + \varphi_0(x))^\alpha (1 - 2Mw(t, x)) \frac{w(t, x)}{(1 - \ln w(t, x))^\alpha} - 2M(\varphi_0'(x)(1 + \varphi_0(x))^\alpha)^2 \frac{w^2(t, x)}{(1 - \ln w(t, x))^{2\alpha}}.$$

Since $\varphi_0''(x) = o(\varphi_0'(x))$ and $\varphi_0'(x) = o(\varphi_0^{-\alpha}(x))$ as $x \rightarrow +\infty$, there exists $X_1 > \xi_0$ such that

$$\varphi_0''(x)(1 + \varphi_0(x))^\alpha \leq \frac{r - \rho}{4} \quad \text{and} \quad \varphi_0'(x)(1 + \varphi_0(x))^\alpha \leq \frac{\sqrt{r - \rho}}{2}, \quad (3.18)$$

as $x \rightarrow +\infty$. In view of the definition of x_M , we take M large enough, say $M > M_1 \geq M_0$, such that $x_M(t) > X_1$ for all $t > 0$. Thus, for $x > x_M(t)$, we have

$$\begin{aligned} v_{xx} &\geq -\frac{r - \rho}{4} \frac{w}{(1 - \ln w)^\alpha} - \frac{r - \rho}{2} M \frac{w^2}{(1 - \ln w)^{2\alpha}} \\ &\geq -\frac{r - \rho}{2} \frac{w}{(1 - \ln w)^\alpha}, \end{aligned} \quad (3.19)$$

thanks to $0 < w \leq \frac{1}{2M}$.

Collecting (3.15), (3.16), (3.17) and (3.19), for $x \leq x_M(t)$, we obtain

$$(v_t - v_{xx} - f(v))(t, x) \leq 0, \quad (3.20)$$

whereas, for $x > x_M(t)$,

$$\begin{aligned} (v_t - v_{xx} - f(v))(t, x) &\leq \frac{w(t, x)}{(1 - \ln w(t, x))^\alpha} \left(-\frac{1}{2}(r - \rho) + \left((-2\rho + r)M + rK \right) w(t, x) \right. \\ &\quad \left. + r\alpha \tilde{c} M \frac{w(t, x)}{1 - \ln w(t, x)} + \alpha \tilde{c} K M^2 \frac{w^3(t, x)}{1 - \ln w(t, x)} \right). \end{aligned}$$

For $0 < w \leq \frac{1}{2M}$, when $M \geq M_2 := \max \left\{ \frac{1}{2} \exp \left(2\alpha\tilde{c} \left(1 + \frac{1}{r} \right) - 1 \right), \frac{K}{4} \right\}$, we have

$$\frac{r + KMw^2}{1 - \ln w} \leq \frac{r + 1}{1 + \ln(2M)} \leq \frac{r}{2\alpha\tilde{c}}.$$

Thus, for any $x > x_M(t)$, we take M large enough, say

$$M \geq \tilde{M} := \max \left\{ \frac{rK}{2\rho - \frac{3}{2}r}, M_1, M_2 \right\},$$

so that

$$r\alpha\tilde{c}M \frac{w(t, x)}{1 - \ln w(t, x)} + \alpha\tilde{c}KM^2 \frac{w^3(t, x)}{1 - \ln w(t, x)} = \alpha\tilde{c}M \frac{r + KMw^2(t, x)}{1 - \ln w(t, x)} w(t, x) \leq \frac{1}{2}rMw(t, x),$$

whence

$$(v_t - v_{xx} - f(v))(t, x) \leq \frac{w^2(t, x)}{(1 - \ln w(t, x))^\alpha} \left((-2\rho + \frac{3}{2}r)M + rK \right) \leq 0, \quad (3.21)$$

thanks to $\rho > \frac{3}{4}r$. This completes the construction of the subsolution $v(t, x)$. \square

In view of (3.14), we notice that:

- when $x > x_M(0)$, we have $v(0, x) = g(w(0, x)) \leq w(0, x) = u_0(x)$;
- when $\xi_1 \leq x \leq x_M(0)$, since u_0 is nonincreasing on $[\xi_1, +\infty)$, we have $v(0, x) = \frac{1}{4M} < \frac{1}{2M} \leq u_0(x)$;
- when $x < \xi_1$, we have $v(0, x) = \frac{1}{4M} < \frac{1}{2M} < \zeta \leq u_0(x)$.

Thus, we obtain $v(0, x) \leq u_0(x) = u(0, x)$ for all $x \in \mathbb{R}$. As a consequence, the maximum principle yields

$$v(t, x) \leq u(t, x) \quad \text{for all } t \geq 0 \text{ and } x \in \mathbb{R}. \quad (3.22)$$

Now, we prove the lower bound in Lemma 1.5.

The proof of the lower bound. Firstly, we prove it for small θ . Let us fix

$$0 < \theta < \frac{1}{4M}.$$

We define the level set of $w(t, x)$ as

$$F_\theta(t) := \{x \in \mathbb{R}, w(t, x) = \theta\}.$$

Recall that $E_\theta(t)$ is not empty for $t > t_\theta$. It follows from Hypothesis 1.2 that there exists a time $t'_\theta > t_\theta$ such that, for any $t > t'_\theta$, the closed set $F_\theta(t)$ is nonempty. For any $t \geq t'_\theta$, denote

$$y_\theta(t) := \min F_\theta(t).$$

Then the function $y_\theta : [t'_\theta, +\infty) \rightarrow \mathbb{R}$ is nondecreasing and left-continuous. In addition, since u_0 is nonincreasing, for all points $t \geq t'_\theta$ where the function y_θ is discontinuous, there exist $a < b$ such that

$$u_0 = \exp \left\{ 1 - [\rho(\alpha + 1)t + (1 - \ln \theta)^{\alpha+1}]^{\frac{1}{\alpha+1}} \right\} \quad \text{on } [a, b];$$

if $[a, b]$ denotes the largest such interval, then $a = y_\theta(t)$ and $b = y_\theta(t^+) = \lim_{s \rightarrow t, s > t} y_\theta(s)$.

We claim that

$$\inf_{\Omega} u > 0,$$

where Ω is an open set defined by

$$\Omega := \{(t, x), t > 0, x < y_\theta(t)\}.$$

Let us evaluate $u(t, x)$ on the boundary $\partial\Omega$. $\partial\Omega$ consists in two parts:

- (1) $\{t'_\theta\} \times (-\infty, y_\theta(t'_\theta^+)]$;
- (2) $\{(t, x) | t > t'_\theta \text{ and } x \in [y_\theta(t), y_\theta(t^+)]\}$.

For the first part, $u(t'_\theta, \cdot)$ is continuous, positive, and $\liminf_{x \rightarrow -\infty} u(t'_\theta, x) > 0$. Thus, we have

$$\inf_{x \in (-\infty, y_\theta(t'_\theta^+)]} u(t'_\theta, x) > 0.$$

For the second part, if $t > t'_\theta$ and $x \in [y_\theta(t), y_\theta(t^+)]$, then $w(t, x) = \theta$, whence

$$u(t, x) \geq \theta - M\theta^2 > 0.$$

As a consequence, $\Theta := \inf_{\partial\Omega} u > 0$. Since $\Theta > 0$ is a subsolution of equation (1.1), the comparison principle yields

$$u(t, x) \geq \Theta \quad \text{for all } x \in \bar{\Omega}. \quad (3.23)$$

Let us pick any $x \in E_\lambda(t)$ for any $\lambda \in (0, \Theta)$. Then

$$x > y_\theta(t^+) \geq y_\theta(t) \quad \text{for any } t \geq t'_\theta.$$

Since $\rho > r - \frac{\varepsilon}{2}$, then there exists a time $t'_{\lambda, \varepsilon} > t'_\theta$ such that

$$u_0(x) \leq u_0(y_\theta(t)) = \exp \left\{ 1 - [\rho(\alpha + 1)t + (1 - \ln \theta)^{\alpha+1}]^{\frac{1}{\alpha+1}} \right\} \leq e^{-[(r-\varepsilon)(\alpha+1)t]^{\frac{1}{\alpha+1}}} \quad (3.24)$$

for any $t > t'_{\lambda, \varepsilon}$, which gives the lower bound for small λ .

Let us prove the lower bound for any $\lambda \in (0, 1)$. Let $\lambda \in [\Theta, 1)$ be given. Denote by u_θ the solution to 1.1 with initial data

$$u_{\theta,0} := \begin{cases} \Theta, & x \geq -1, \\ -\Theta x, & -1 < x < 0, \\ 0, & x \geq 0. \end{cases} \quad (3.25)$$

In view of (2.3), we can also obtain that, for some $\gamma_1 > 0$, we have $\lim_{t \rightarrow \infty} \inf_{x \leq \gamma_1 t} u_\theta(t, x) = 1$.

There exists a time $t''_\lambda > 0$ such that

$$u_\theta(t''_\lambda, x) > \lambda \quad \text{for all } x \leq 0. \quad (3.26)$$

Furthermore, by (3.23) and (3.25), we have

$$u(t, x) \geq u_{\theta,0}(x - y_\theta(T)) \quad \text{for any } x \in \mathbb{R} \text{ and } T \geq 0.$$

It follows from the comparison principle that

$$u(T + t, x) \geq u_\theta(t, x - y_\theta(T)).$$

By (3.26), we obtain

$$u(T + t''_\lambda, x) > \lambda, \quad \text{for all } x \leq y_\theta(T) \text{ and } T \geq 0.$$

Therefore there exists a time $t_{\lambda, \varepsilon} > \max(t'_{\lambda, \varepsilon}, t''_\lambda)$ such that

$$\begin{aligned} u_0(x_\lambda(t)) &< u_0(y_\theta(t - t''_\lambda)) \\ &= \exp \left\{ 1 - [\rho(\alpha + 1)(t - t''_\lambda) + (1 - \ln \lambda)^{\alpha+1}]^{\frac{1}{\alpha+1}} \right\} \\ &< e^{-[(r-\varepsilon)(\alpha+1)t]^{\frac{1}{\alpha+1}}} \end{aligned} \quad (3.27)$$

for any $t > t_{\lambda, \varepsilon}$, which gives the lower bound. This completes the proof. \square

Let $T_{\lambda, \varepsilon} = \max\{\bar{t}_{\lambda, \varepsilon}, t_{\lambda, \varepsilon}\}$. Thus, combining (3.12), (3.24) and (3.27), the proof of Lemma 1.4 is complete.

4 A more precise bound: Proof of Lemma 1.6

In this section, we give a more precise bound for the level sets of the solution to the equation (1.1).

4.1 The upper bound

We derive a more precise upper bound by translating the spatial variables in w , so that the supersolution can approximate the solution to (1.1) more accurately.

Let us define

$$m(t, x) = \begin{cases} w(t + t^1, x - t), & x \geq x_0(t + t^1) + t, \\ 1, & x < x_0(t + t^1) + t, \end{cases}$$

where $w(t, x)$ is defined by (3.2) with $\rho = r$ and t^1 is from (3.8).

We claim that m is a supersolution to equation 1.1 for any $t > 0$ and $x \in \mathbb{R}$.

To prove m is a supersolution, we need to check that $m_t - m_{xx} - f(m) \geq 0$ for all $t > 0$ and $x \in \mathbb{R}$. For $x < x_0(t + t^1) + t$ and $t > 0$, since $m_t(t, x) = m_{xx}(t, x) = f(m(t, x)) = 0$, we have

$$m_t(t, x) - m_{xx}(t, x) - f(m(t, x)) = 0.$$

For $x \geq x_0(t + t^1) + t$ and $t > 0$, by the definitions of m and w and Hypothesis 1.1, we have

$$m_t(t, x) - f(m(t, x)) \geq r \frac{w(t + t^1, x - t)}{(1 - \ln w(t + t^1, x - t))^\alpha} - w_x(t + t^1, x - t) - r \frac{w(t + t^1, x - t)}{(1 - \ln w(t + t^1, x - t))^\alpha} = -w_x(t + t^1, x - t).$$

Therefore, for all $x \geq x_0(t + t^1) + t$ and $t > 0$, since $x - t \geq x_0(t + t^1) > x_0(t)$ and $t + t^1 > t^1$ for $t > 0$, by Lemma 3.2, we have

$$m_t(t, x) - m_{xx}(t, x) - f(m(t, x)) \geq -w_x(t + t^1, x - t) - w_{xx}(t + t^1, x - t) \geq 0.$$

When $t = 0$, since $u_0 \leq 1$ and $w(\cdot, x)$ is nondecreasing for each $x \in \mathbb{R}$, we have that $m(0, x) = 1 \geq u_0(x) = u(0, x)$ for $x < x_0(t^1)$, and that $m(0, x) = w(t^1, x) \geq w(0, x) = u_0(x) = u(0, x)$ for $x \geq x_0(t^1)$. The comparison principle then yields that

$$u(t, x) \leq m(t, x) \quad \text{for all } t > 0 \text{ and } x \in \mathbb{R}. \quad (4.1)$$

The proof of the upper bound. Now, we prove that, for any $\lambda \in (0, 1)$ and large time t , there are a constant $\bar{C}_\lambda > 0$ we have

$$E_\lambda(t) \subset u_0^{-1} \left\{ [\bar{C}_\lambda e^{-(r(\alpha+1)t)^{\frac{1}{\alpha+1}}}, 1] \right\}. \quad (4.2)$$

It follows from (4.1) that

$$u(t, x) \leq m(t, x) \leq w(t + t^1, x - t) \quad \text{for all } t > 0 \text{ and } x \in \mathbb{R}.$$

If we pick $y \in E_\lambda(t)$, then $w(t + t^1, y - t) \geq \lambda$. Thus, by the definitions of w and φ_0 , we have

$$w(t + t^1, y - t) = \exp \left\{ 1 - [(1 - \ln u_0(y - t))^{\alpha+1} - \rho(\alpha+1)(t + t^1)]^{\frac{1}{\alpha+1}} \right\} \geq \lambda,$$

whence

$$u_0(y - t) \geq \exp \left\{ 1 - (r(\alpha+1)(t + t^1) + (1 - \ln \lambda)^{\alpha+1})^{\frac{1}{\alpha+1}} \right\}.$$

By the assumption $\varphi'_0(x) = o(\varphi_0^{-\alpha}(x))$ as $x \rightarrow +\infty$, we have, for some $C > 0$,

$$\frac{U_0(Ce^{-(r(\alpha+1)t)^{\frac{1}{\alpha+1}}})}{t} \rightarrow +\infty \quad \text{as } t \rightarrow \infty,$$

where $U_0(z) := \sup\{x \in \mathbb{R}, u_0(x) = z\}$. Therefore, there is a time \bar{t}_λ and a constant \bar{C}_λ such that

$$E_\lambda(t) \subset u_0^{-1}\left\{\left[\bar{C}_\lambda e^{-r(\alpha+1)t^{\frac{1}{\alpha+1}}}, 1\right]\right\} \quad \forall t \geq \bar{t}_\lambda, \quad (4.3)$$

proving (4.2). □

4.2 The lower bound

Assuming additionally that $\varphi_0''(x) \leq 0$ for large x , we derive a more precise lower bound. To do so, we take $\rho = r$ and recall

$$v(t, x) = \begin{cases} \frac{1}{4M}, & x \leq x_M(t), \\ g(w(t, x)), & x > x_M(t), \end{cases} \quad (4.4)$$

where $M \geq M_0 = \max\left\{\frac{1}{2\zeta}, \frac{1}{4s_0}\right\}$. We claim that

$$E_\lambda(t) \subset u_0^{-1}\left\{\left(0, \underline{C}e^{-[r(\alpha+1)t]^{\frac{1}{\alpha+1}}}\right)\right\} \quad \text{for } t \text{ large enough.} \quad (4.5)$$

In view of (3.5), since $\varphi_0''(x) \leq 0$ for large x , we obtain $w_{xx} \geq 0$ for any $x \geq x_M(t)$ with large enough M , say $M > M^* \geq M_0$. For $x > x_M(t)$, it follows from the assumption $\varphi_0'(x) = o(\varphi_0^{-\alpha}(x))$ as $x \rightarrow +\infty$, the definition (4.4) of v and (3.4) that we have for $M > M^*$, up to enlarge M^* ,

$$\begin{aligned} v_{xx}(t, x) &= (1 - 2Mw)w_{xx} - 2Mw_x^2 \\ &\geq -2M(\varphi_0'(1 + \varphi_0)^\alpha)^2 \frac{w^2}{(1 - \ln w)^{2\alpha}} \\ &\geq -\frac{1}{4}rM \frac{w^2}{(1 - \ln w)^{2\alpha}}. \end{aligned}$$

On the other hand, for $x \leq x_M(t)$, we have $v_{xx}(t, x) = 0$.

In view of (3.16) and (3.17), similar to (3.20), we obtain, for $x \leq x_M(t)$,

$$(v_t - v_{xx} - f(v))(t, x) \leq 0,$$

whereas, for $x > x_M(t)$, we have

$$(v_t - v_{xx} - f(v))(t, x) \leq \frac{w(t, x)}{(1 - \ln w(t, x))^\alpha} \left(\left(-\frac{3}{4}rM + rK\right)w(t, x) + \alpha\bar{c}M \frac{w(t, x)}{1 - \ln w(t, x)} + \alpha\bar{c}KM^2 \frac{w^3}{1 - \ln w(t, x)} \right).$$

For $0 < w \leq \frac{1}{2M}$, when $M \geq M^{**} := \max\left\{\frac{1}{2} \exp\left(4\alpha\bar{c}\left(1 + \frac{1}{r}\right) - 1\right), \frac{K}{4}\right\}$, we have

$$\frac{r + KMw^2}{1 - \ln w} \leq \frac{r + 1}{1 + \ln(2M)} \leq \frac{r}{4\alpha\bar{c}}.$$

Thus, for any $x > x_M(t)$, we take M large enough, say

$$M \geq \bar{M} := \max\left\{2rK, M^*, M^{**}\right\},$$

so that

$$r\alpha\bar{c}M \frac{w(t, x)}{1 - \ln w(t, x)} + \alpha\bar{c}KM^2 \frac{w^3(t, x)}{1 - \ln w(t, x)} = \alpha\bar{c}M \frac{r + KMw^2(t, x)}{1 - \ln w(t, x)} w(t, x) \leq \frac{1}{4}rMw(t, x),$$

whence

$$(v_t - v_{xx} - f(v))(t, x) \leq \frac{w^2(t, x)}{(1 - \ln w(t, x))^\alpha} \left(-\frac{1}{2}M + rK \right) \leq 0.$$

In view of (4.4), we notice that:

- when $x > x_M(0)$, we have $v(0, x) = g(w(0, x)) \leq w(0, x) = u_0(x)$;
- when $\xi_1 \leq x \leq x_M(0)$, since u_0 is nonincreasing on $[\xi_1, +\infty)$, we have $v(0, x) = \frac{1}{4M} < \frac{1}{2M} \leq u_0(x)$;
- when $x < \xi_1$, since $M > \frac{1}{2\zeta}$, we have $v(0, x) = \frac{1}{4M} \leq \frac{1}{2M} < \zeta \leq u_0(x)$.

Thus, we obtain $v(0, x) \leq u_0(x) = u(0, x)$ for all $x \in \mathbb{R}$. Therefore, the comparison principle implies that

$$v(t, x) \leq u(t, x) \quad \text{for all } x \in \mathbb{R}, t > 0.$$

It follows from the proof of the lower bound of Lemma 1.4 that, for $\lambda \in (0, \Theta)$, if $x \in E_\lambda(t)$, then there exists a time $t'_\lambda > t'_\theta$ such that

$$u_0(x) \leq u_0(y_\theta(t)) = \exp \left\{ 1 - [r(\alpha + 1)t + (1 - \ln \theta)^{\alpha+1}]^{\frac{1}{\alpha+1}} \right\} \leq \underline{C}_1 e^{-[r(\alpha+1)t]^{\frac{1}{\alpha+1}}} \quad \text{for any } t > t'_\lambda, \quad (4.6)$$

and, for $\lambda \in (\Theta, 1)$, there exists a time $\underline{t}_\lambda > \max\{t'_\lambda, \underline{t}''_\lambda\}$ such that

$$u_0(x) < u_0(y_\theta(t - \underline{t}''_\lambda)) = \exp \left\{ 1 - [r(\alpha + 1)(t - \underline{t}''_\lambda) + (1 - \ln \lambda)^{\alpha+1}]^{\frac{1}{\alpha+1}} \right\} < \underline{C}_2 e^{-[r(\alpha+1)t]^{\frac{1}{\alpha+1}}} \quad (4.7)$$

for any $t > \underline{t}_\lambda$ which gives (4.5). Let $T_\lambda = \max\{\underline{t}_\lambda, \bar{t}_\lambda\}$ and $\underline{C}_\lambda = \max\{\underline{C}_1, \underline{C}_2\}$. Thus, combining (4.3) (4.6) and (4.7), the proof of Lemma 1.6 is complete.

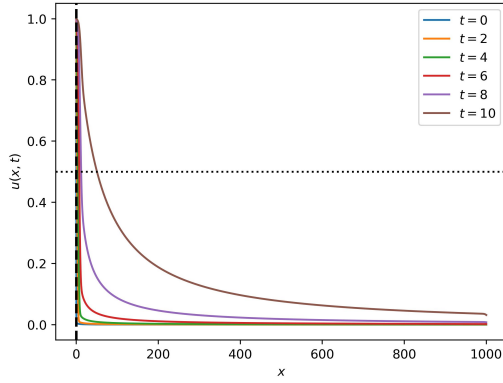
5 Numerical simulations

In this section, we provide some numerical simulations to illustrate the previous results.

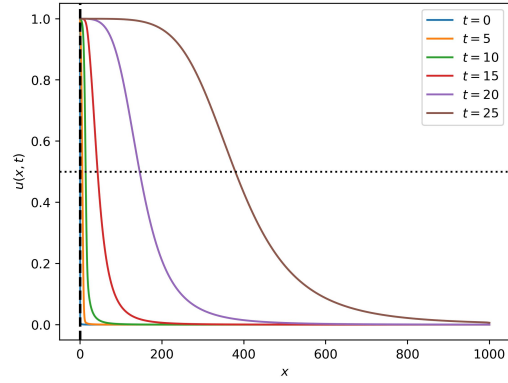
To get an approximate solution for equation (1.1), we discretize the equation in space by the finite difference method and then use *Implicit-Explicit* scheme (IMEX) [4, 5] to integrate it in time, where the implicit scheme handles the diffusion term while the explicit handles the reaction term. The influence of the initial data u_0 on the propagation speed is illustrated under the some fixed α in Figure 3-8. We mainly consider the initial data with two kinds of decay: sub-exponential decay and algebraic decay. In the following simulations, we all take $f(u) = \frac{u}{(1-\ln u)^\alpha} (1-u)$ for $u \in (0, 1)$. For the initial data with sub-exponential decay, we take the initial data to be $u_0 = \min\{e^{-5x^\beta}, 1\}$ and $\alpha = 0.2, 0.4, 0.6$. Figure 3, Figure 4 and Figure 5 show that the acceleration can be observed over a small time range when β is small. This is consistent with our theoretical results, that is, $x_\lambda(t) \asymp t^{\frac{1}{\beta(1+\alpha)}}$ tends to infinite as $\beta \rightarrow 0^+$. When β is large enough, as we show in Theorem 1.3, for initial data $u_0 \lesssim e^{-\mu x^\beta}$ with $\beta > \frac{1}{\alpha+1}$ and $\mu > 0$, the solution propagates at a finite rate. Notice that the width of the solution becomes larger and larger as β gets smaller and smaller. This is because the flattening effect [10, 18].

For initial data with algebraic decay, we take $u_0 = \min\{\frac{1}{1+100x^\beta}, 1\}$ and $\alpha = 0.2, 0.4, 0.6$. In Figure 6, Figure 7, and Figure 8, we observe that decreasing the parameter β leads to an increase of the propagation speed. Our theoretical findings support this observation, as demonstrated by the fact that $x_\lambda(t) \asymp \exp\left(\frac{r(\alpha+1)t}{\beta}\right)^{\frac{1}{\alpha+1}}$ tends to infinity as $\beta \rightarrow 0^+$. We can also observe the flattening effect. Therefore, the decay of the initial data is the key to the propagation of solution to equation (1.1). When the initial data increases, meaning β decreases, the propagation speed also increases.

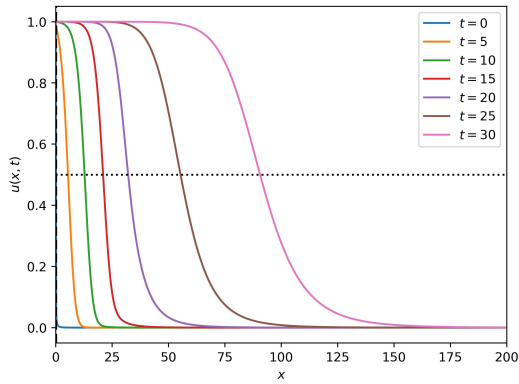
In Figure 9, we provide a comparison between the largest element $x_\lambda(t)$ of level sets $E_\lambda(t)$ of the solution with three different types of initial data. Observe that the slope of curve for the algebraic decay case is maximum, followed by the sub-exponential decay, and the sub-exponentially bounded case show a straight line. This is consistent with our theoretical results. We fit the corresponding theoretical results for each cases, as shown in the thick continuous curves in the figure 9. Notice that in each pair of curves when time t is large enough, our



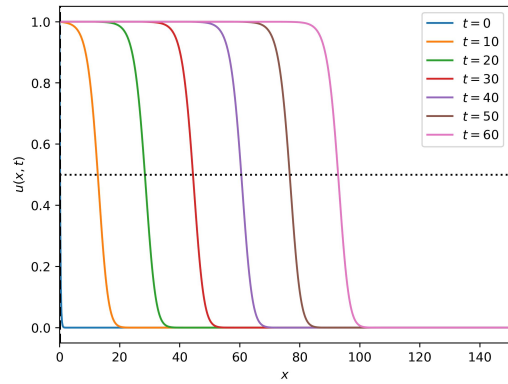
(a) $\beta = 0.1$



(b) $\beta = 0.2$

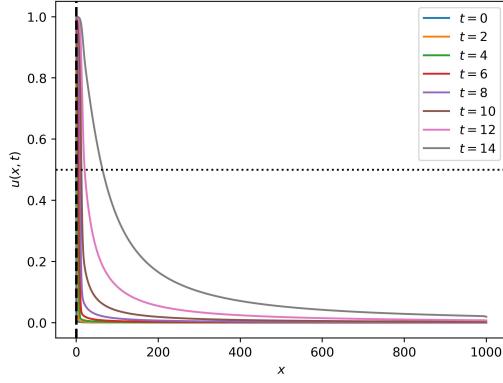


(c) $\beta = 0.3$

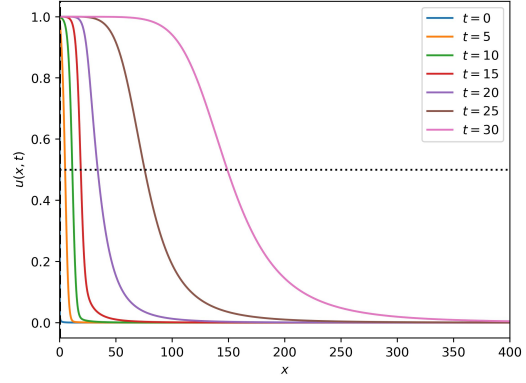


(d) $\beta = 1.0$

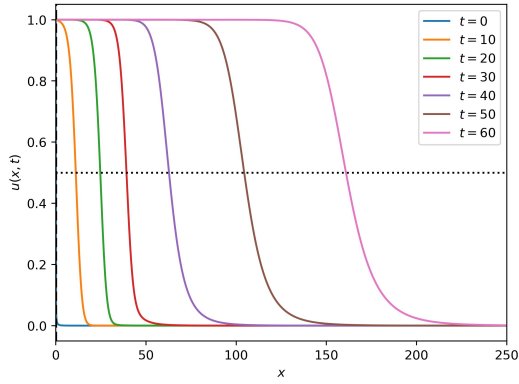
Figure 3: Numerical approximations of the solution to (1.1) with the initial data $u_0(x) = \min\{e^{-5x^\beta}, 1\}$ at different times for $\alpha = 0.2$ and different values of β . The threshold for acceleration is $\beta = \frac{1}{\alpha+1} = \frac{5}{6}$.



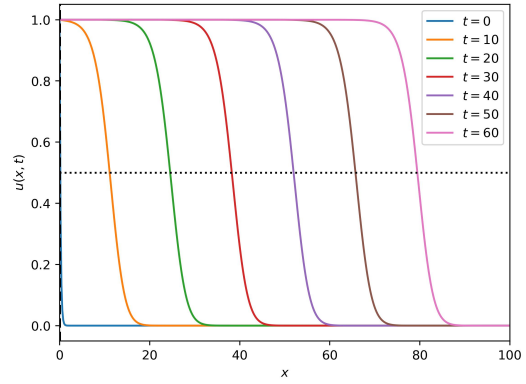
(a) $\beta = 0.1$



(b) $\beta = 0.2$

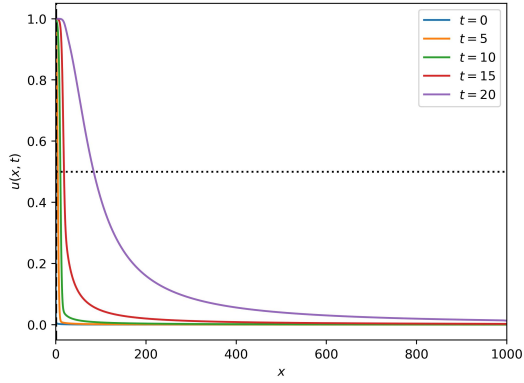


(c) $\beta = 0.3$

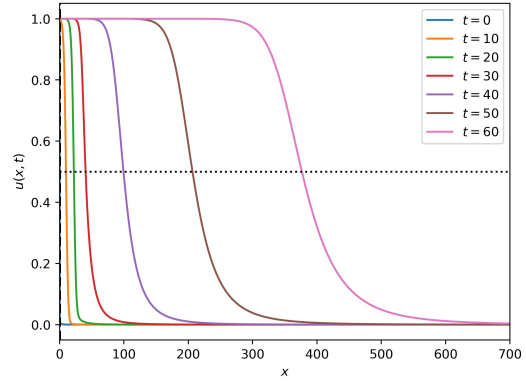


(d) $\beta = 1.0$

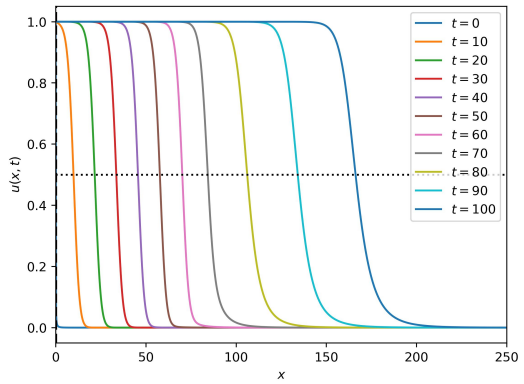
Figure 4: Numerical approximations of the solution to (1.1) with the initial data $u_0(x) = \min\{e^{-5x^\beta}, 1\}$ at different times for $\alpha = 0.4$ and different values of β . The threshold for acceleration is $\beta = \frac{1}{\alpha+1} = \frac{5}{7}$.



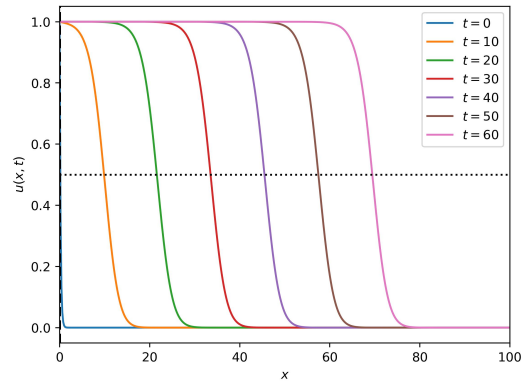
(a) $\beta = 0.1$



(b) $\beta = 0.2$



(c) $\beta = 0.3$



(d) $\beta = 1.0$

Figure 5: Numerical approximations of the solution to (1.1) with the initial data $u_0(x) = \min\{e^{-5x^\beta}, 1\}$ at different times for $\alpha = 0.6$ and different values of β . The threshold for acceleration is $\beta = \frac{1}{\alpha+1} = \frac{5}{8}$.

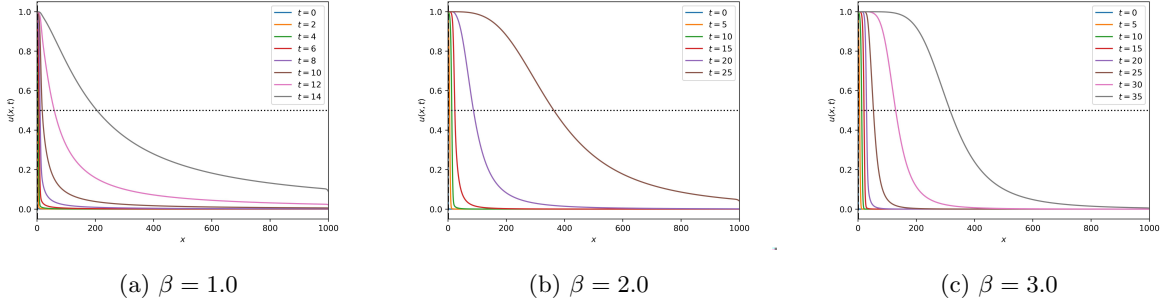


Figure 6: Numerical approximations of the solution to (1.1) with the initial data $u_0(x) = \min\{\frac{1}{1+100x^\beta}, 1\}$ at different times for $\alpha = 0.2$ and different values of β .

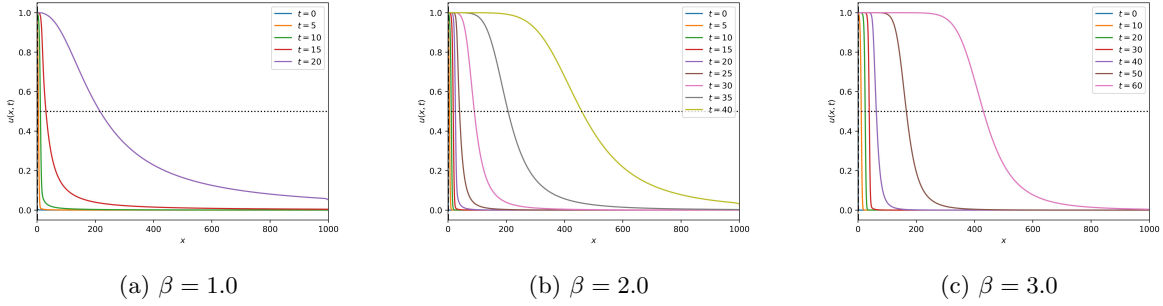


Figure 7: Numerical approximations of the solution to (1.1) with the initial data $u_0(x) = \min\{\frac{1}{1+100x^\beta}, 1\}$ at different times for $\alpha = 0.4$ and different values of β .

experimental results are consistent with the theoretical results. The curves we choose with theoretical rates are $x_\lambda(t) = 1.9t - 4.0$, $x_\lambda(t) = 0.0013t^{\frac{1}{0.28}} + 40.0$ and $x_\lambda(t) = 0.0236e^{(1.4t)^{\frac{1}{1.4}}} + 10$ respectively. Here, in order to better observe the trend of each pair of curves, we make a small downward translation for the thick continuous curves.

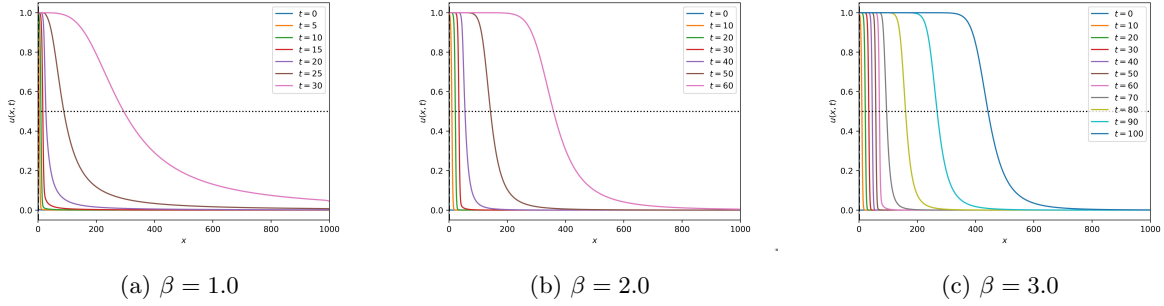


Figure 8: Numerical approximations of the solution to (1.1) with the initial data $u_0(x) = \min\{\frac{1}{1+100x^\beta}, 1\}$ at different times for $\alpha = 0.6$ and different values of β .

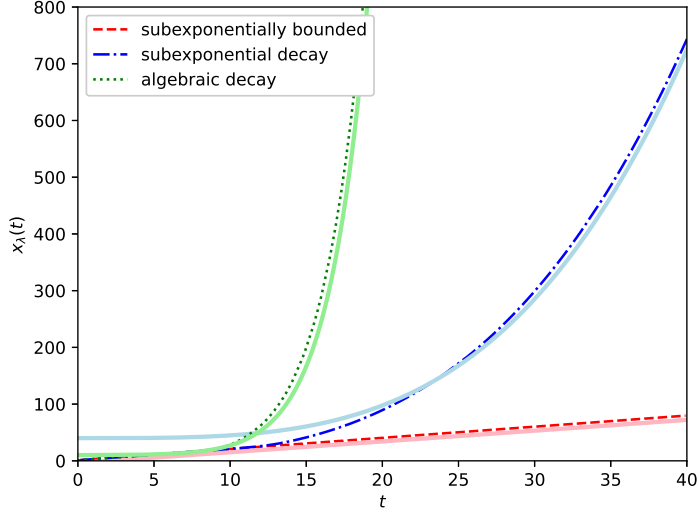


Figure 9: Comparison between the largest element $x_\lambda(t)$ of level sets $E_\lambda(t)$ of the solution starting from three types of initial data: sub-exponentially bounded $u_0(x) = \min\{e^{-5x}, 1\}$, sub-exponential decay $u_0(x) = \min\{e^{-5x^{0.2}}, 1\}$ and algebraic decay $u_0(x) = \min\{\frac{1}{1+100x}, 1\}$. In this figure, the thick continuous curves are theoretical results. Here, we choose $\alpha = 0.4$ and $\lambda = \frac{1}{2}$.

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