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# Structure-preserving Observers for port-Hamiltonian systems via contraction analysis

Mario Spirito

**Abstract**—We address the design of structure preserving observers for port-Hamiltonian systems. Despite what is done in the literature on nonlinear observers for this class of systems, we consider a Luenberger-like identity observer dynamics with an output injection term. We provide an analysis on the passivity properties of the observer dynamics and of the system-observer cascade. We then consider the convergence properties of the observer scheme by exploiting the contraction theory (or convergent dynamics) approach, and we show the exponential convergence rate of the state reconstruction. We conclude the work by applying the proposed technique to a velocity observation of a PMSM system with only current measurements.

**Index Terms**—Structure-preserving Observer, port-Hamiltonian systems, gradient-based observability/detectability

## I. INTRODUCTION

The port-Hamiltonian (pH) approach to modeling and controlling complex physical systems constitutes a well-established framework that originated with the seminal work by van der Schaft and Maschke [1], [2]. For a comprehensive overview of this field, including control techniques, one can refer to [3], [4], [5]. Port-Hamiltonian systems have the particular feature of describing all the main physical properties of the system under consideration, such as energy dissipation, passivity, and power conservation laws. Because of this feature, they have been identified as a powerful framework for the treatment of complex physical systems.

The literature on pH systems contains numerous contributions for the design of controllers, see, e.g., [6] and [4]. In contrast, the design of observers for port-Hamiltonian systems has received rather limited attention.

In particular, for linear pH systems, state reconstruction can be naturally addressed with a standard Luenberger observer [7], thus making the error dynamics Hurwitz. However, exploiting the predefined structure of pH systems gives additional insights into observer dynamics and its convergence properties (as it will be clear at the end of Section VI) also for linear pH systems. For this class of systems, a compensator based on a dual observer has been proposed in [8]. While in [9], the authors address the combined input-state reconstruction problem for linear port-Hamiltonian systems. The work [10] addresses the design of passive observers for linear port-Hamiltonian systems based on the LMI approach proposed in [11] and developed for stabilization purposes.

For nonlinear pH systems, there exist also several observer design methods. When referring to nonlinear pH systems, we can generally differentiate between two kinds of nonlinearities, viz. (a) nonlinearities in the interconnection structure and (b) nonlinearities in the gradient of the storage/Hamiltonian function. The former are characterized by state-dependent matrices of the pH systems; the latter are characterized by possibly non-quadratic Hamiltonians. It is obvious that the combination of both the nonlinearities can take place into account in the same dynamics and makes it simply more complicated to analyze.

The first work addressing the design of observers for nonlinear port-Hamiltonian systems was [12]. However, the proposed observer is shown to be only asymptotically convergent if the system reaches a steady state. Later, in [13], the authors present a passivity-based, globally exponentially convergent observer for pH systems with both nonlinear interconnection structure and Hamiltonian gradient. The proposed observer dynamics exploit additional measurements that are not conjugate to the system input and are available for feedback. Its design requires the solution of a set of algebraic equations and partial differential equations (PDEs).

Yaghmaei and Yazdanpanah in [14] propose an observer design for PHSs with constant system matrices and nonquadratic storages with an approach similar to the well-known interconnection and damping assignment passivity-based control (IDA-PBC), developed in [6]. It is worth noticing that the structure-preserving description in [14] is different from the one we consider in this work, since we provide an analysis on the observer dynamics with an explicit port-Hamiltonian structure (involving an explicit Hamiltonian function, an interconnection and a resistance matrices), from which we deduce the passivity properties between the observer input signals and their conjugate outputs.

The observer structure for nonlinear pH system (with linear Hamiltonian gradient) has also been augmented with an integral action to improve the robustness of the proportional passivity-based observer in [15]. Recently, in [16], the authors propose the design of a particular class of nonlinear port-Hamiltonian systems where the Hamiltonian function has a particular shape (quadratic form with some nonlinearity depending on part of the system state). Along with this particular Hamiltonian shape, the authors consider exploiting, for the output injection term in the observer dynamics, some measurements that are not conjugated to the original system input. Among these measurements a part of the system state is directly available, that is the critical state part involved in the Hamiltonian nonquadratic form part, which simplifies a lot

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the scenario.

In [17], the authors propose an observer-based controller describing the passivity property of the system-observer cascade. They then couple the observer dynamics with a Model Predictive Control-like technique to obtain a controller design for port-Hamiltonian systems.

Despite all the proposed techniques, it can be seen that the observers from [12] and [13] are the only two approaches that are applicable to port-Hamiltonian systems with both nonlinearities in the interconnection structure and in the Hamiltonian gradient. However, the observer approach in [12] has not been proved to be asymptotically convergent in general, and the observer design in [13] is delicate (or weak) as it relies on the solution of a set of algebraic equations and PDEs.

In this paper, we propose a complete analysis regarding the passivity of the observer dynamics along with its subtleties in reconstructing the system state. In particular, it is not generally guaranteed that the observer states are a passive output for the overall observer dynamics. On the contrary, the gradient of the observer Hamiltonian is always a passive output with respect to the output injection term. This article is motivated by and concurrently paves the way for the design of an observer-based controller, which is currently under development and will be the topic of future work. In this work, we consider an observer design based on the convergence properties of nonlinear systems as introduced in [18], [19] and re-proposed in [20]. By properly exploiting this contraction/convergent<sup>1</sup> properties of nonlinear systems, we are able to show exponential convergence of the observer and directly tune its convergence rate for some classes of port-Hamiltonian systems that are not restrictive in practical applications. This convergence property is achieved, for the class of systems under consideration, without involving any algebraic and/or partial differential equations.

The paper is structured as follows. In Section II, we introduce some preliminary concepts that are useful for the successive development. We then provide some motivations for the work towards observer-based controller, providing new, *gradient-based*, notions of controllability and stabilizability in Section III. We then discuss the structure-preserving functional observer dynamics in Section IV and study the passivity of the corresponding error dynamics in Section V. The theoretical part of the paper concludes with Section VI, in which we describe the design techniques for two classes of port-Hamiltonian systems whose nonlinearities are first in the system dynamics matrices and then in the gradient of the associated Hamiltonian function. We then merge the two results into a summarizing theorem that applies to general nonlinear port-Hamiltonian systems, whose input matrix is constant. We then provide some numerical examples to show the effectiveness of the proposed methods in Section VII, and we give some conclusions in Section VIII.

## Nomenclature

To apply a chain rule to a matrix  $J(x)$  times a vector  $Qx$  multiplication, we consider the following notation

$$\frac{\partial J(x)Qx}{\partial x} = \nabla J(x, Qx) + J(x)Q$$

where  $\nabla J(x, \cdot)$  represents the tensor (3D- matrix) obtained by involving the gradient of all the elements of  $J(x)$ .

## II. PRELIMINARIES

In this section, we introduce some basic concepts that are useful to the successive sections.

### A. Linear Luenberger Observers

We recall the Luenberger observer approach for an observable linear systems, with dynamics

$$\begin{aligned} \dot{x} &= Ax + Bu, & x(0) &= x_0 \\ y &= Cx \end{aligned} \quad (1)$$

where  $x \in \mathbb{R}^n$  is defined for all  $t \geq 0$ , from some unknown initial condition  $x_0 \in \mathbb{R}^n$ ,  $u$  and  $y$  are respectively the input and output signals that are available from feedback, while  $A \in \mathbb{R}^{n \times n}$ ,  $B \in \mathbb{R}^{n \times m}$  and  $C \in \mathbb{R}^{p \times n}$ .

Thus, for system (1), we define a Luenberger observer with dynamics

$$\dot{\hat{x}} = A\hat{x} + Bu + L(y - C\hat{x}), \quad \hat{x}(0) = \hat{x}_0 \quad (2)$$

where  $\hat{x} \in \mathbb{R}^n$  is the estimation of  $x$ , initialized at some known initial condition  $\hat{x}_0$  and the matrix  $L$  is designed such that the observation error  $\tilde{x} = \hat{x} - x$  has a asymptotically stable dynamics

$$\dot{\tilde{x}} = (A - LC)\tilde{x}, \quad \tilde{x}(0) = \hat{x}_0 - x_0 \quad (3)$$

or equivalently,  $L$  is designed such that  $A - LC$  is Hurwitz.

### B. Port-Hamiltonian Systems

Port-Hamiltonian systems can be formalized as systems generated by an energy function  $H$  with related dynamics

$$\begin{aligned} \dot{x} &= (J(x) - R(x)) \nabla H(x) + g(x)u \\ y &= g(x)^\top \nabla H(x) \end{aligned} \quad (4)$$

where  $x \in X \subset \mathbb{R}^n$ ,  $u \in \mathbb{R}^m$ ,  $y \in \mathbb{R}^m$  and the matrices of opportune dimensions, where  $J$  is skew-symmetric, while  $R \geq 0$  is symmetric matrices, while  $g(x) \in \mathbb{R}^{n \times m}$  plays the role of the input matrix. Moreover, having  $R(x)$  positive semidefinite on  $X$ , guarantees the passivity property with storage function  $H(x)$ , whenever it is lower bounded. Indeed, the energy balance equation gives us

$$\dot{H}(x) = y^\top u - \nabla H(x)^\top R(x) \nabla H(x) \leq y^\top u$$

thus, providing passivity with storage function  $H$ .

For the present developments, we consider the following Assumption.

<sup>1</sup>From the stability point of view these notions are synonymous.

**Assumption 1.** *The Hessian of the Hamiltonian is every bounded and positive definite, i.e., there exist positive real  $h_1$  and  $h_2$  such that for all  $x \in \mathbb{R}^n$*

$$h_1 I \leq \nabla^2 H(x) \leq h_2 I.$$

The above assumption implies also that  $H$  is lower bounded and convex, and it can moreover play the role of a Lyapunov function.

In the following, we consider the design of an observer dynamics that preserve the port-Hamiltonian structure (4) and preserve passivity. In particular, when applying the Luenberger observer approach to Port-Hamiltonian systems, one has to consider an additional desired property, along with guaranteeing the observer stability and the passivity characteristics of the original system, i.e., structure preservation. That is, in the Port-Hamiltonian formalism, it is crucial to have explicit Hamiltonian/energy terms involved in the system dynamics (describing the effort of the system), then a symplectic structure for the energy conservation, a positive semidefinite matrix describing possible dissipative elements in the system, and a particular relationship between the input and the output matrices so to guarantee the passivity property of the system. Hence, the resulting *structure-preserving* observer must have all the features listed above in order to fit into the Port-Hamiltonian formalism.

**Remark 1.** *Contrary to what has been addressed in [14], we highlight the difficulties of preserving the port-Hamiltonian structure for the observer dynamics, and along with its passivity property, in Section IV. The structure preservation addressed in [14] is then different to the one addressed here. Since they consider, as structure-preserving observer, a system whose dynamics has the characteristics of stability, and no passivity property is taken into account. Their only focus is on the observer error origin stability. In the next section, we discuss about the passivity properties of a functional observer in the port-Hamiltonian formalism and the related subtleties.*

### C. Contraction theory

Another fundamental concept we need to introduce is that of convergent/contracting dynamics.

Contraction theory in the context of nonlinear control is nowadays a counterpart for the Lyapunov approach, allowing to guarantee stability of the origin and convergence of the system trajectories.

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The approach involves a positive definite matrix, that plays the role of a metric for what is called the variational system.

This approach has been introduced in works such as [18], [19], for the case of constant metric, and re-proposed recently in [20], [21], [22], while their extension to the more general case of nonlinear Riemannian metrics can be found in [23] and [24]. See also [25] for the case of non-Euclidean  $L_1$  and  $L_\infty$  metrics.

In the following, we will use the standard convergence result in [20] that exploits the standard metric  $P$  to describe a convergence dynamics property. In order to do so, we first

introduce the definition of contractive systems.

Consider the system

$$\dot{x} = f(x, t) \quad (5)$$

where  $x \in X \subseteq \mathbb{R}^n$  and  $f : X \times \mathbb{R} \rightarrow \mathbb{R}^n$  is continuously differentiable in  $x$  and Lypscitz continuous in  $t$ . Let  $\phi(x_0, t)$  denote the trajectory of (5) at time  $t$  originated at  $x_0$  at time  $t = 0$ . We then have the following definition.

**Definition II.1.** *We say that system (5) defines a contraction if there exist real positive  $\alpha, \kappa > 0$  such that*

$$\|\phi(x_1, t) - \phi(x_2, t)\| \leq \kappa \exp(-\alpha t) \|x_1 - x_2\|$$

for all initial conditions  $x_1, x_2 \in X \subseteq \mathbb{R}^n$  and for all  $t \geq 0$ .

A sufficient condition to determine whether system (5) defines a contraction is the existence of a constant metric  $P$  for which the distance with this metric between any two system trajectories is monotonically decreasing in time. We thus have the following well-known result, also known as Demidovic condition in [20].

**Lemma 1.** *System (5) is a contraction on  $X$  if there exists a constant  $P \in \mathbb{R}^{n \times n}$  symmetric and positive definite, such that*

$$P \frac{df}{dx}(x, t) + \frac{df}{dx}^\top(x, t) P < 0 \quad (6)$$

for all  $x \in X \subset \mathbb{R}^n$ .

## III. SOME MOTIVATIONS

In general observer can be used for the purpose of reconstructing some unmeasurable state variables  $x$  or some desired state functional  $f(x)$ , whenever an output signal,  $y$ , is available from the system under consideration  $\dot{x} = F(x, u)$ ,  $y = h(x)$ . For this reason, they are very often used, in control applications, in dynamic output controller design to reconstruct the ideal state feedback control law

$$u = \alpha(x) \quad (7)$$

where  $\alpha : X \rightarrow \mathbb{R}^m$ , and it is such that the closed loop  $\dot{x} = F(x, \alpha(x))$  has some desired properties, such as stability of the origin or asymptotic tracking of a certain trajectory. Thus, because the system state is often unavailable for feedback, the control designer has to rely on a state estimation by exploiting an observer dynamics, e.g., (2) for linear systems. Hence, in this framework the designed controller is augmented by a copy of the system dynamics that is properly modified in order to achieve, possibly fast, the system state trajectory and exploit it for reconstructing the ideal state feedback control law (7), via the following controller dynamics

$$\begin{aligned} \dot{\hat{x}} &= f(\hat{x}, u) + L(\hat{x})(y - h(\hat{x})) \\ u &= \alpha(\hat{x}) \end{aligned} \quad (8)$$

### A. The port-Hamiltonian point of view

In the port-Hamiltonian formalism, the control law (7) should preserve the port-Hamiltonian structure in closed loop, i.e., the closed loop should always explicitly have a  $J_d$  and an  $R_d$  matrices and they have to multiply the gradient of a

Hamiltonian function  $\nabla H_d$ . Control laws with this closed loop property can be thought as being state-based, like the IDA-PBC technique [6] or the LMI approaches developed in [11], in which the state feedback control law reads as (7) with a particular pre-defined shape of the function  $\alpha$ . In particular, for the IDA-PBC approach we have

$$\alpha(x) = g^\dagger(x)((J_d(x) - R_d(x))\nabla H_d(x, t) - (J(x) - R(x))\nabla H(x))$$

that allows us to obtain the desired closed loop dynamics

$$\dot{x} = (J_d(x) - R_d(x))\nabla H_d(x, t)$$

if the following matching equation hold

$$g_\perp(x)((J_d(x) - R_d(x))\nabla H_d(x, t)) = g_\perp(x)((J(x) - R(x))\nabla H(x))$$

where  $g^\dagger$  is the right inverse of  $g(x)$  and  $g^\dagger$  is a the full rank left annihilator of  $g$ .

We refer to this kind of controller design as *state-based* controller. A particular case of the above controllers <sup>2</sup> is the case of what we call *gradient-based* controllers, that is, the control law explicitly exploits the gradient of the Hamiltonian function (or effort/co-state variable). For this kind of framework the control laws  $\alpha(x)$  in (7) reads as  $\alpha(x) = K(x)\nabla H(x)$ , for some mapping  $K : \mathbb{R}^n \rightarrow \mathbb{R}^{n \times n}$ .

According to the gradient-based control laws just introduced, we define for port-Hamiltonian systems the related controllability and stabilizability properties.

**Definition III.1** (G-B controllability). *A system (4) is to be Gradient-Based controllable if for any desired  $J_d(x)$  and  $R_d(x)$  there exists a  $K(x)$  defined for all  $x \in \mathbb{R}^n$ , such that*

$$J_d(x) - R_d(x) = J(x) - R(x) - g(x)K(x).$$

A weaker notion of Gradient-Based Controllability is given by the following definition.

**Definition III.2** (G-B controllable dissipativity). *A system (4) is to have a controllable dissipation if for any desired  $R_d(x)$  there exists a  $K(x)$ , defined for all  $x \in \mathbb{R}^n$ , such that*

$$R_d(x) = R(x) + \text{sym}(g(x)K(x)).$$

We then have the notion of Gradient-Based stabilizability.

**Definition III.3** (G-B stabilizability). *A system (4) is to be Gradient-Based stabilizable if there exists a  $K(x)$ , defined for all  $x \in \mathbb{R}^n$ , such that the closed loop*

$$\dot{x} = (J(x) - R(x) - g(x)K(x))\nabla H(x)$$

is globally asymptotically stable, i.e.,  $x \rightarrow 0$  as  $t \rightarrow \infty$ .

Sufficient conditions for G-B controllability for Port-Hamiltonian systems is a particular case of the IDA-PBC matching equations, i.e.,

$$g_\perp(x)(J_d(x) - R_d(x)) = g_\perp(x)(J(x) - R(x))$$

<sup>2</sup>That for us seems to be more relevant than the IDA-PBC approach because the the energy function plays the role of natural/explicit Lyapunov function and viceversa.

Sufficient conditions for G-B controllable dissipativity for Port-Hamiltonian systems, is now a particular case of the last matching equations, i.e.,

$$g_\perp(x)R_d(x) = g_\perp(x)R(x)$$

If one is only interested in obtaining a closed loop  $R_d$  that is positive *definite*,  $R_d > 0$ , we recall the following Lemma from [16, Lemma 4] that provides necessary and sufficient conditions to obtain a positive definite closed loop resistive matrix if the original system is stable

**Lemma 2.** *Consider two matrices  $R(s) \in \mathbb{R}^{n \times n}$  and  $g(s) \in \mathbb{R}^{n \times m}$  depending on some parameters  $s \in \mathbb{S} \subseteq \mathbb{R}^n$ , where  $R$  is positive semidefinite for all  $s \in \mathbb{S}$ . Then there exists a matrix  $K(s) \in \mathbb{R}^{m \times n}$  such that*

$$R(s) + \text{sym}(g(s)K(s)) > 0, \quad \forall s \in \mathbb{S} \quad (9)$$

if and only if (9) is satisfied for  $K(s) = g^\top(s)$ .

Note that however, this is only a particular case of G-B controllable dissipativity, and it is not directly G-B stabilizability, because we can have asymptotic stability of the origin even if  $R_d$  is only positive semidefinite, as recalled in the following Proposition from [11, Proposition 6].

**Lemma 3.** *Suppose that in the linear closed loop dynamics  $\dot{x} = (J_d - R_d)Q_d x$ ,  $R_d \geq 0$  and its kernel is spanned by  $\{r_1, \dots, r_k\}$ . Then the closed loop system is asymptotically stable if and only if*

$$[sI - Q_d J_d] [r_1 | \dots | r_k]$$

has rank  $k$  for every  $s = j\omega$ ,  $\omega \in \mathbb{R}$ .

Thus the above lemma provides sufficient and necessary conditions for gradient-based stabilizability for linear Port-Hamiltonian systems.

The observability properties are the dual definitions, in which we simply substitute  $g(x)K(x)$  in the above definitions with  $L^\top(x)g^\top(x)$ , as it will be clearer why from Section VI.

#### IV. STRUCTURE-PRESERVING OBSERVER FOR PORT-HAMILTONIAN SYSTEMS

We present in this section the structure-preserving functional observer dynamics of port-Hamiltonian system. We first introduce the following passivity assumption

**Assumption 2.** *For any given functional of interest  $f(x) : \mathbb{R}^n \rightarrow \mathbb{R}^p$ , there exists a locally invertible matrix valued function  $\psi : \mathbb{R}^n \rightarrow \mathbb{R}^{n \times p}$  such that  $f(x) = \psi^\top(x)\nabla H(x)$ .*

Then a structure-preserving, and thus passive, port-Hamiltonian Functional Observer for the functional  $f(x)$  reconstruction reads as

$$\begin{aligned} u_f &= \psi(\hat{x})^\dagger L(\hat{x})^\top y \\ \dot{\hat{x}} &= [(J(\hat{x}) - R(\hat{x})) - L(\hat{x})^\top g(\hat{x})^\top] \nabla H(\hat{x}) + \\ &\quad g(\hat{x})u + \psi(\hat{x})u_f \\ \hat{y} &= g(\hat{x})^\top \nabla H(\hat{x}) \\ f(\hat{x}) &= \psi(\hat{x})^\top \nabla H(\hat{x}) \end{aligned} \quad (10)$$

where  $\psi(\hat{x})^\dagger$  is the local inverse of  $\psi(x)$ , existing by assumption, and the signals  $(\hat{y}, f(\hat{x}))$  are the collocated outputs of the inputs  $(u, u_f)$ , that is, if  $R(\hat{x}) + \text{sym}(g(\hat{x})L(\hat{x})) \geq 0$  for all  $x$ , that the observer is passive with respect to the inputs  $(u, u_f)$  and their conjugated outputs  $(\hat{y}, f(\hat{x}))$ . It is also easy to notice that, if  $\hat{x} \rightarrow x$  also  $f(\hat{x}) \rightarrow f(x)$ , thus the full order passive observer (10) can reconstruct the functional of interest if  $\hat{x} - x \rightarrow 0$  for any trajectory  $x(t)$  originated at  $x_0$  at time  $t = 0$  of system (4).

**Remark 2.** *However, from an implementation point of view, there is no need to consider the existence of the matrix values function  $\psi$ , because we can obtain an estimate of  $f(x)$  by simply replacing  $\hat{x}$  into its argument, i.e., directly getting  $f(\hat{x})$  from the available state evolution. Moreover, substituting the definition of  $u_f$  into the observer dynamics hides the presence of the local inverse of  $\psi$ .*

**Remark 3.** *The presence of such matrix  $\psi$  has been introduced only to preserve the structure of an explicit passive system, thus making the observer dynamics structure preserving port-Hamiltonian with inputs  $(u, u_f)$  and outputs  $(\hat{y}, f(\hat{x}))$ . As a consequence, if such a matrix valued function  $\psi$  exists, although possibly unknown and unemployed, the observer dynamics is intrinsically passive from the inputs  $(u, u_f)$ .*

We now consider two particular cases, i.e., passive state and passive gradient reconstruction/observation.

#### A. Passive state observation

In this particular case, to obtain a *passive* state reconstruction, we need to assume the injectivity of the Hamiltonian gradient  $\nabla H(x)$ , i.e., there exists a locally invertible  $\psi$  such that  $x = \psi(x)\nabla H(x)$ . Thus the Observer dynamics reads as

$$\begin{aligned} u_f &= \psi(\hat{x})^\dagger L(\hat{x})^\top y \\ \dot{\hat{x}} &= [(J(\hat{x}) - R(\hat{x})) - L(\hat{x})^\top g(\hat{x})^\top] \nabla H(\hat{x}) + \\ &\quad g(\hat{x})u + \psi(\hat{x})u_f \\ \hat{y} &= g(\hat{x})^\top \nabla H(\hat{x}) \\ \hat{x} &= \psi(\hat{x})^\top \nabla H(\hat{x}). \end{aligned} \quad (11)$$

Again the knowledge of the matrix  $\psi$ , neither its locally invertibility property, is not necessary from the implementation point of view, but rather it plays an important role for a passivity analysis of the observer interconnections<sup>3</sup> when we use pure state-feedback techniques such as the IDA-PBC approach or any other control law of the  $\alpha(\hat{x})$ . Obtaining  $\psi(\hat{x})$  for the linear case, whenever  $Q = Q^\top > 0$  in  $\nabla H(x) = Qx$ , is easy since  $\psi(\hat{x})^\top = Q^{-1}$ . In the nonlinear context, i.e.,  $\nabla H(x) \neq Qx$ , we cannot provide with any general solution for  $\psi$ , and the problem should be addressed case by case.

#### B. Passive gradient observation

In this particular case, when we are interested in reconstruction the gradient of the system Hamiltonian, the matrix valued

<sup>3</sup>For example the design can be interested in some passivity properties of the system-observer cascade or closed loop.

function  $\psi$  always exists and it is trivially the identity matrix, thus the observer trivially reads as

$$\begin{aligned} u_f &= L(\hat{x})^\top y \\ \dot{\hat{x}} &= [(J(\hat{x}) - R(\hat{x})) - L(\hat{x})^\top g(\hat{x})^\top] \nabla H(\hat{x}) + \\ &\quad g(\hat{x})u + Iu_f \\ \hat{y} &= g(\hat{x})^\top \nabla H(\hat{x}) \\ \nabla H(\hat{x}) &= I \nabla H(\hat{x}). \end{aligned} \quad (12)$$

Reconstructing/Observing the Hamiltonian gradient is by far the simplest case of *passive* functional observation, this provides an additional motivation to explore more the gradient-based controller approach.

**Remark 4.** *It is clear from the above analysis that getting a **passive observer** might be a hard task, although it might not be needed to construct a passivity-preserving controller.*

As already touched before, any functional  $f$  can be reconstructed by the full order observer, whenever the states of the observer  $\hat{x}$  converge to the system state evolution  $x$ . Hence, from a convergence point of view, it is enough to study the deviation evolution between the observer's and the system's state, as proposed in the following section.

For the sake of exposition, from the following subsection on, we use the notation  $M = M(x)$ ,  $\hat{M} = M(\hat{x})$ , and  $\tilde{M} = \hat{M} - M$ , for  $M \in \{J, R, g, H\}$ , while  $\nabla \tilde{H} = \nabla \hat{H} - \nabla H = \nabla H(\hat{x}) - \nabla H(x)$ .

#### C. System-Observer Cascade Passivity

We consider now the augmented system given by the cascade of the system and the observer dynamics. For this augmented system, due to the dynamics interconnection, the passivity conditions has to be considered only between the input signals  $(u, u)$  and the conjugated outputs  $(y, \hat{y})$ . In particular, we have that the augmented system has again a port-Hamiltonian structure, i.e., defining  $H_a(x, \hat{x}) = H(x) + H(\hat{x})$  we have

$$\begin{aligned} \begin{pmatrix} \dot{x} \\ \dot{\hat{x}} \end{pmatrix} &= \begin{bmatrix} J - R & 0 \\ \hat{L}^\top \hat{g}^\top & \hat{J} - \hat{R} - \hat{L}^\top \hat{g}^\top \end{bmatrix} \begin{pmatrix} \nabla_x H_a(x, \hat{x}) \\ \nabla_{\hat{x}} H_a(x, \hat{x}) \end{pmatrix} \\ &\quad + \begin{bmatrix} g & 0 \\ 0 & \hat{g} \end{bmatrix} \begin{pmatrix} u \\ u \end{pmatrix} \\ \begin{pmatrix} y \\ \hat{y} \end{pmatrix} &= \begin{bmatrix} g & 0 \\ 0 & \hat{g} \end{bmatrix}^\top \begin{pmatrix} \nabla_x H_a(x, \hat{x}) \\ \nabla_{\hat{x}} H_a(x, \hat{x}) \end{pmatrix}. \end{aligned} \quad (13)$$

And thus, for storage function  $H_a$ , the passivity condition is satisfied is the cascade resistive matrix is positive semidefinite, i.e.

$$\begin{bmatrix} R & -\frac{1}{2}\hat{g}\hat{L} \\ -\frac{1}{2}\hat{L}^\top\hat{g}^\top & \hat{R} + \text{sym}(\hat{L}^\top\hat{g}^\top) \end{bmatrix} \geq 0 \quad (14)$$

and this condition can be satisfied, in general, under strong assumptions on  $L$ , despite what is reported in the analysis by [17, eq. (6)]. Thus in general, the cascade System-Observer dynamics is not passive from the inputs  $(u, u)$  to outputs  $(y, \hat{y})$ . This lack of passivity of the system-observer cascade is due to the fact the  $y(t)$  plays the role of an additional input to

the observer dynamics which does not have a corresponding passive output, we consider in the cascade dynamics. This conjugate output should look like  $y_y = L(\hat{x})\nabla H(\hat{x})$ , where if  $L = K$ , with  $K$  being a gradient-based controller gain matrix, then the conjugate passive output already provides the passivity-preserving control law. This argument is part of a current work under development on observer-based controller for port-Hamiltonian systems, and discussing it will be out of the scope of this paper.

For the sake of exposition, in the remaining part of the paper, we consider the observer state dynamics as

$$\dot{\hat{x}} = (\hat{J} - \hat{R} - \hat{L}^\top \hat{g}^\top) \nabla \hat{H} + \hat{g}u(t) + \hat{L}^\top y(t). \quad (15)$$

## V. OBSERVER ERROR PASSIVITY ANALYSIS

In order to analyze the convergence of the observer state  $\hat{x}$  on the system state trajectory  $x$ , we introduce the error coordinate  $\tilde{x} = \hat{x} - x$ . By considering the evolution in the observer dynamics (15) and in the system dynamics (4), we can obtain again a port Hamiltonian  $\tilde{x}$  dynamics, as reported in the following Proposition/Lemma

**Lemma 4.** *Given the observer dynamics (15) with state  $\hat{x}$  and in the system dynamics (4) with state  $x$ , the error dynamics reads as*

$$\begin{aligned} \dot{\tilde{x}} &= (\hat{J} - \hat{R} - \hat{L}^\top \hat{g}^\top) \nabla \tilde{H} + \tilde{g}u + (\tilde{J} - \tilde{R} - \hat{L}^\top \tilde{g}^\top) \nabla H \\ \tilde{y} &= \tilde{g}^\top \nabla \tilde{H} \end{aligned} \quad (16)$$

where  $\tilde{y}$  are the conjugate outputs to the inputs  $u$ .

*Proof.* The proof comes from substituting the observer's and system's dynamics into the time derivative of  $\tilde{x}$ , i.e.,  $\dot{\tilde{x}} = \dot{\hat{x}} - \dot{x}$ . Using the notation introduced in the lemma and substituting the definition of  $u_f$  in the observer dynamics, we have

$$\begin{aligned} \dot{\tilde{x}} &= (\hat{J} - \hat{R}) \nabla \hat{H} + \hat{L}^\top (g^\top \nabla H - \hat{g}^\top \nabla H + \hat{g}^\top \nabla H - \hat{g} \nabla \hat{H}) \\ &\quad + (\hat{g} - g)u - (J - R) \nabla H \\ &= (\hat{J} - \hat{R}) \nabla \tilde{H} - (J - R - (\hat{J} - \hat{R})) \nabla H + \tilde{g} \\ &\quad - \hat{L}^\top (\hat{g}^\top \nabla \tilde{H} + \tilde{g}^\top \nabla H) \\ &= (\hat{J} - \hat{R} - \hat{L}^\top \hat{g}^\top) \nabla \tilde{H} + \tilde{g}u + (\tilde{J} - \tilde{R} - \hat{L}^\top \tilde{g}^\top) \nabla H \end{aligned}$$

and thus we obtained an error dynamics that has a port-Hamiltonian structure, although the error Hamiltonian function  $\tilde{H}$  is not explicitly defined.  $\square$

In general  $\nabla \tilde{H}$  is not only a function of  $\tilde{x}$  and thus the corresponding Hamiltonian function  $\tilde{H}$ , obtained by integration in the  $\tilde{x}$  usually depends also on the system state variable  $x$ . However, by the involving the mean value theorem, as usually done in contraction analysis [20], we can obtain an explicit Hamiltonian function, this time time-varying, i.e.,

$$\begin{aligned} \nabla \tilde{H} &= \nabla H(\tilde{x} + x) - \nabla H(x) \\ &= \nabla^2 H((1-c)x + c(\tilde{x} + x)) \cdot (\tilde{x} + x - x) \\ &= \nabla^2 H(\bar{x}(t)) \tilde{x} = \nabla^2 \bar{H}(t) \tilde{x} \end{aligned}$$

where  $\bar{x} = x + c\tilde{x}$ , with a possibly time varying  $c \in [0, 1]$ . We can then conclude the following passivity result.

**Theorem V.1.** *The cascade (4)-(16), is passive with respect to the inputs  $(u, u)$  and outputs  $(y, \tilde{y})$ , with storage function*

$$\tilde{H}(x, \tilde{x}, t) = \frac{1}{2} \tilde{x}^\top \nabla^2 \bar{H}(t) \tilde{x} + H(x) \quad (17)$$

if the following passivity condition holds

$$\begin{bmatrix} \frac{d(\nabla^2 \bar{H}(t))^{-1}}{dt} - 2(\hat{R} + \text{sym}(\hat{L}^\top \hat{g}^\top)) & \star \\ (\tilde{J} - \tilde{R} - \hat{L}^\top \tilde{g}^\top)^\top & -2R \end{bmatrix} \leq 0. \quad (18)$$

*Proof.* With the definition of  $\tilde{H}(x, \tilde{x}, t)$  in (17), the system-Observer cascade dynamics, (4)-(16), reads as

$$\begin{aligned} \begin{pmatrix} \dot{x} \\ \dot{\tilde{x}} \end{pmatrix} &= \begin{bmatrix} J - R & 0 \\ \tilde{J} - \tilde{R} - \hat{L}^\top \tilde{g}^\top & \hat{J} - \hat{R} - \hat{L}^\top \hat{g}^\top \end{bmatrix} \begin{pmatrix} \nabla_x \tilde{H}(x, \tilde{x}, t) \\ \nabla_{\tilde{x}} \tilde{H}(x, \tilde{x}, t) \end{pmatrix} \\ &\quad + \begin{bmatrix} g & 0 \\ 0 & \tilde{g} \end{bmatrix} \begin{pmatrix} u \\ u \end{pmatrix} \\ \begin{pmatrix} y \\ \tilde{y} \end{pmatrix} &= \begin{bmatrix} g & 0 \\ 0 & \tilde{g} \end{bmatrix}^\top \begin{pmatrix} \nabla_x \tilde{H}(x, \tilde{x}, t) \\ \nabla_{\tilde{x}} \tilde{H}(x, \tilde{x}, t) \end{pmatrix}. \end{aligned} \quad (19)$$

Hence, along the dynamics (19), the time derivative of  $\tilde{H}(x, \tilde{x}, t)$ , reads as

$$\begin{aligned} \dot{\tilde{H}} &= \frac{1}{2} \tilde{x}^\top \frac{d}{dt} \nabla^2 \bar{H}(t) \tilde{x} + \tilde{x}^\top \nabla^2 \bar{H} \tilde{g}u \\ &\quad - \tilde{x}^\top \nabla^2 \bar{H}(t) (\hat{R} + \text{sym}(\hat{L}^\top \hat{g}^\top)) \nabla^2 \bar{H}(t) \tilde{x} \\ &\quad + \tilde{x}^\top \nabla^2 \bar{H}(t) [\tilde{J} - \tilde{R} - \hat{L}^\top \tilde{g}^\top] \nabla H \\ &\quad - \nabla H^\top R \nabla H + \nabla H^\top g u \\ &= \frac{1}{2} \tilde{x}^\top \nabla^2 \bar{H} [\nabla^2 \bar{H}^{-1} \frac{d}{dt} (\nabla^2 \bar{H}(t)) \nabla^2 \bar{H}^{-1}] \nabla^2 \bar{H} \tilde{x} \\ &\quad + \tilde{y}^\top u - \tilde{x}^\top \nabla^2 \bar{H}(t) (\hat{R} + \text{sym}(\hat{L}^\top \hat{g}^\top)) \nabla^2 \bar{H}(t) \tilde{x} \\ &\quad + \tilde{x}^\top \nabla^2 \bar{H}(t) [\tilde{J} - \tilde{R} - \hat{L}^\top \tilde{g}^\top] \nabla H \\ &\quad - \nabla H^\top R \nabla H + y^\top u. \end{aligned}$$

By noticing that

$$\nabla^2 \bar{H}^{-1} \frac{d}{dt} (\nabla^2 \bar{H}(t)) \nabla^2 \bar{H}^{-1} = \frac{d(\frac{d}{dt} \nabla^2 \bar{H}(t))^{-1}}{dt}$$

under condition (18) we obtain passivity, i.e.,

$$\dot{\tilde{H}}(x, \tilde{x}, t) \leq \tilde{y}^\top u + y^\top u.$$

Thus proof the theorem.  $\square$

However, if we additionally assume that there exists a function  $\tilde{H}(\tilde{x}, x)$  (time independent) such that

$$\frac{\partial \tilde{H}(\tilde{x}, x)}{\partial \tilde{x}} = \frac{\partial H}{\partial x}(\tilde{x} + x) - \frac{\partial H}{\partial x}(x), \quad \frac{\partial \tilde{H}(\tilde{x}, x)}{\partial x} = \frac{\partial H}{\partial x}(x), \quad (20)$$

then passivity conditions simplifies for the system cascade, as detailed in the following corollary.

**Corollary V.1.** *Assume there exists a Hamiltonian function  $\tilde{H}(x, \tilde{x})$  with property (20). Then the cascade (4)-(16), is passive with respect to the inputs  $(u, u)$  and outputs  $(y, \tilde{y})$ , with storage function  $\tilde{H}(x, \tilde{x})$ , if the following passivity condition holds*

$$\begin{bmatrix} -2(\hat{R} + \text{sym}(\hat{L}^\top \hat{g}^\top)) & (\tilde{J} - \tilde{R} - \hat{L}^\top \tilde{g}^\top) \\ (\tilde{J} - \tilde{R} - \hat{L}^\top \tilde{g}^\top)^\top & -2R \end{bmatrix} \leq 0. \quad (21)$$

The proof of the Corollary simply comes as an application of the previous theorem, for which the passivity condition (18) simplifies thanks to the time invariant Hamiltonian assumption  $\tilde{H} = \tilde{H}(x, \tilde{x})$ . Thus it is omitted.

A particular case happens, when the Hamiltonian function of the system  $H(x)$  is a quadratic form, i.e.,  $H(x) = x^\top Qx$ , with associate positive definite matrix,  $Q = Q^\top > 0$ . In particular, in this situation the Hamiltonian functions  $\tilde{H}(x, \tilde{x}, t)$  and  $\tilde{H}(x, \tilde{x})$  coincide having  $\tilde{H}(x, \tilde{x}) = \tilde{x}^\top Q\tilde{x} + H(x)$ .

This case is also particularly relevant for the convergence analysis of the observer dynamics, as considered in the following section.

## VI. OBSERVER DESIGN VIA CONTRACTION ANALYSIS

In this section, we present some results on the design of a convergent<sup>4</sup> observer dynamics (15), reported here for completeness,

$$\dot{\hat{x}} = (\hat{J} - \hat{R} - \hat{L}^\top \hat{g}^\top) \nabla \hat{H} + \hat{g}u(t) + \hat{L}^\top y(t) \quad (22)$$

for two classes of port-Hamiltonian systems with constant input matrix  $g(x) = g$ , i.e., for systems with quadratic form Hamiltonian function  $H$  and for systems with constant parameters  $J$  and  $R$ , but with non quadratic form Hamiltonian.

The key idea behind the following results is that the observer dynamics is the system dynamics initialized on a different initial condition as shown below.

In order to proceed with the analysis, we first introduce a modification of the system dynamics (4), that however does not change its state trajectory. In particular, we consider adding the terms  $L^\top(y - y)$  into the  $x$  dynamics, so that the state dynamics of (4) now reads as

$$\dot{x} = (J - R - L^\top g^\top) \nabla H + gu(t) + L^\top y(t) \quad (23)$$

where we consider the system output  $y$  as an additional 'fake input'. With this reformulation, we see that both the observer and system dynamics have the same structure with augmented input signals  $(u, y)$ . Moreover it is clear in this context that the observer dynamics is the system dynamics initialized on a different initial conditions. This reformulation allows us to exploit the contraction property, i.e., Lemma 1, on the original system (23), in order to obtain the observer convergence on a certain domain of interest.

In the following cases, we focus our attention on the output injection matrix  $L$  by considering it constant, along with the input matrix  $g$ .

Because the observer dynamics can suffer from the peaking problem, we consider the observer state space  $\hat{X}$  to be at least as large as the system state space  $X$ , i.e.,  $X \subseteq \hat{X}$ .

### A. Observer design with constant Hessian

In this subsection, we consider the case in which the system (23) has a quadratic form Hamiltonian, i.e.,  $H(x) = \frac{1}{2}x^\top Qx$ , with  $Q = Q^\top$  positive definite, so that  $\nabla H(x) = Qx$ , and we have the following

<sup>4</sup>By convergent observer dynamics we mean that  $\hat{x} \rightarrow x$  as  $t \rightarrow \infty$ .

**Theorem VI.1.** *Given system (23), with  $x \in X$ , with  $g(x) = g$  constant, and  $\nabla H(x) = Qx$ , with  $Q$  positive definite. The observer dynamics (15), with  $\hat{x} \in \hat{X}$ , is convergent on  $\hat{X}$ , i.e.,  $\hat{x} \xrightarrow[t \rightarrow \infty]{} x$ , if there exist a constant  $L$  such that for any  $x \in \hat{X}$*

$$R'(x) + \text{sym}(gL) > 0, \quad (24)$$

where

$$R'(x) = R - \text{sym}(\nabla J(x, Qx)Q^{-1} - \nabla R(x, Qx)Q^{-1}). \quad (25)$$

Moreover, for some positive  $\alpha$ , we can explicitly determine the exponential convergence of the observation error as

$$|\tilde{x}(t)| \leq \mu(Q) \exp(-\alpha t) |\tilde{x}(0)|. \quad (26)$$

*Proof.* To have a convergent observer, due to  $X \subset \hat{X}$ , we need to guarantee the convergence property of (23) for any  $x \in \hat{X}$ . That is, according to Lemma 1 and considering in (23),  $\dot{x} = F(x, t)$ , we analyze its gradient

$$\begin{aligned} \frac{\partial F}{\partial x}(x, t) &= \left( \frac{\partial J(x)}{\partial x} - \frac{\partial R(x)}{\partial x} \right) Qx \\ &\quad + (J(x) - R(x) - L^\top g^\top) Q \\ &= [\nabla J(x, Qx)Q^{-1} - \nabla R(x, Qx)Q^{-1} \\ &\quad + (J(x) - R(x) - L^\top g^\top)] Q \\ &= (J'(x) - R'(x) - L^\top g^\top) Q \end{aligned}$$

Then the convergence property can be studied with respect to the metric  $Q$ , i.e., defining  $V = \tilde{x}^\top Q\tilde{x}$  and taking its time derivative, we have by mean value theorem

$$\dot{V} = 2\tilde{x}^\top Q(F(\hat{x}, t) - F(x, t)) = 2\tilde{x}^\top Q \frac{\partial F}{\partial x}(\bar{x}(t), t)\tilde{x}$$

where  $\bar{x} = x + c(\hat{x} - x)$  for some possibly time varying  $c \in [0, 1]$ . We can hence write

$$\dot{V} = -\tilde{x}^\top Q(R'(\bar{x}(t)) + \text{sym}(gL))Q\tilde{x} < 0$$

Then, by standard arguments, since  $R' + \text{sym}(gL)$  is uniformly positive, there exists a positive  $\alpha$  such that for every  $x \in \hat{X}$

$$-R'(x) - \text{sym}(gL) \leq -\alpha Q,$$

so that we have (26) by applying the comparison lemma [26, Lemma 3.4] to the Lyapunov function.  $\square$

**Remark 5.** *This result is the dual of the one shown in [14] in which the authors assume that the system matrix are constants and the only nonlinearity is due to the gradient of the Hamiltonian  $\nabla H(\hat{x})$ .*

The latter case has been considered in the following subsection in which we provide a counterpart result exploiting the convergence/contractive analysis as stated in [20].

### B. Observer design with constant system parameters

In this subsection, we consider a system dynamics (23) whose matrices  $J, R, g$  are constant and that the only nonlinearity comes from the gradient of the Hamiltonian function  $\nabla H(x)$ . We additionally assume that the Hessian of  $H$  is diagonalizable by means of a constant orthogonal matrix  $T_H$ , i.e.,  $\nabla^2 H(x) = T_H^\top \Lambda(\nabla^2 H(x)) T_H$ , where  $\Lambda(\nabla^2 H)$  is the matrix with the eigenvalues of  $\nabla^2 H(x)$  on the main diagonal. For this class of the systems we have the following result

**Theorem VI.2.** *Consider (23) with constant matrices  $J, R, g$ , and consider  $\underline{Q} = \nabla^2 H(x_{\min})$  where  $x_{\min}$  is that, for all  $x \in \hat{X}$ ,  $\nabla^2 H(x) \geq \nabla^2 H(x_{\min})$ . Moreover, assume that there exists a constant orthogonal matrix  $T_H$  such that  $\nabla^2 H(x) = T_H^\top \Lambda(\nabla^2 H)(x) T_H$  for all  $x \in \hat{X}$ . Then the observer dynamics (15) is convergent on  $\hat{X}$  with metric  $\underline{Q}$  if there exists a constant  $L$  such that*

$$R(x) + \text{sym}(gL) > 0, \quad (27)$$

for all  $x \in \hat{X}$ . Moreover, for some positive  $\alpha$ , we can explicit determine the exponential convergence of the observation error as

$$|\tilde{x}(t)| \leq \mu(\underline{Q}) \exp(-\alpha t) |\tilde{x}(0)|. \quad (28)$$

*Proof.* The proof of this theorem leverages on the orthogonal transformation  $T_H$ . In particular, by definition of  $\underline{Q}$ , we first introduce  $\Delta Q(x) := \nabla^2 H(x) - \underline{Q} \geq 0$  for all  $x \in \hat{X}$ , that is the deviation matrix between  $\nabla^2 H(x)$  and  $\underline{Q}$  is always positive semidefinite on the domain of interest  $\hat{X}$ . Then, due to  $T_H$  is a constant transformation, it diagonalizes the matrix  $\underline{Q}$  and hence, as consequence, also the matrix  $\Delta Q(x)$ .

We now show by contradiction that the matrix

$$\underline{Q}(J - R - L^\top g^\top) \Delta Q(x)$$

is negative semidefinite for all  $x \in \hat{X}$ . Thus, assume it is positive semidefinite, then we have

$$\begin{aligned} \underline{Q}(J - R - L^\top g^\top) \Delta Q(x) &= \\ &= T_H^\top \Lambda(\underline{Q}) T_H (J - R - L^\top g^\top) T_H^\top \Lambda(\Delta Q(x)) T_H \\ &\leq \lambda_{\max}(\underline{Q})(J - R - L^\top g^\top) \lambda_{\max}(\Delta Q(x)) \leq 0 \end{aligned}$$

by the property (27) of  $L$ . Hence, we have a contradiction. And in particular, we have

$$\begin{aligned} T_H^\top \Lambda(\underline{Q}) T_H (J - R - L^\top g^\top) T_H^\top \Lambda(\Delta Q(x)) T_H \\ \leq \lambda_{\min}(\underline{Q})(J - R - L^\top g^\top) \lambda_{\min}(\Delta Q(x)) = 0 \end{aligned}$$

because  $\Delta Q(x)$  is positive semidefinite for all  $x \in \hat{X}$ .

We can now proceed with the convergence proof considering as Lyapunov function  $V = \tilde{x}^\top \underline{Q} \tilde{x}$ . Taking the time derivative of  $V$  we have

$$\begin{aligned} \dot{V} &= 2\tilde{x}^\top \underline{Q} (F(\hat{x}, t) - F(x, t)) \\ &= 2\tilde{x}^\top \underline{Q} \frac{\partial F}{\partial x}(\bar{x}, t) \tilde{x} \\ &= 2\tilde{x}^\top \underline{Q} [(J - R - L^\top g^\top) \nabla^2 H(\bar{x})] \tilde{x} \\ &= 2\tilde{x}^\top \underline{Q} [(J - R - L^\top g^\top) (\underline{Q} + \Delta Q(x))] \tilde{x} \\ &\leq 2\tilde{x}^\top \underline{Q} [(J - R - L^\top g^\top) \underline{Q}] \tilde{x} < 0 \end{aligned}$$

Then, by standard arguments, since  $R + \text{sym}(gL)$  is uniformly positive, there exists a positive  $\alpha$  such that

$$-R - \text{sym}(gL) \leq -\alpha \underline{Q},$$

so that we have (28) by applying the comparison lemma [26, Lemma 3.4] to the Lyapunov function.  $\square$

Note that our result, compared to [14], is more powerful since it does not involve finding the solution of a Algebraic Riccati Inequality (or ARE with a arbitrary small  $\epsilon$ , as presented in their paper). But on the other hand, it assumes the existence of a constant transformation matrix  $T_H$  to diagonalize the Hessian of  $H$ .

### C. Observer design: a general case

This result can be easily extended and merged with the results presented in the previous subsection (Sec.VI-A), under the existence assumption of such a transformation matrix  $T_H$ , if (24) can be satisfied for all  $x \in \hat{X}$  by a constant  $L$ .

**Theorem VI.3.** *Consider (23) with constant matrix  $g$ , and consider  $\underline{Q}$  and  $T_H$  defined as in Theorem VI.2. Then the observer dynamics (15) is convergent on  $\hat{X}$  with metric  $\underline{Q}$  if there exists a constant  $L$  such that (24) for all  $x \in \hat{X}$ . Moreover, for some positive  $\alpha$ , we can explicit determine the exponential convergence of the observation error as in (28)*

The proof is simply an application of the previous Theorems proofs and thus it is omitted.

**Remark 6.** *The results shown in this section hold for any value of  $u$  and  $y$ , and allows us to determine an exponential convergence of the observe dynamics with convergence rate  $\alpha$ . Such a value only depends on the choice of  $L$  and, because  $\underline{Q}$  is a fixed metric independent of  $L$ , we can possibly make  $\alpha$  arbitrary large by properly defining  $L$ . This property paves the way to the design of a Observer-based controller that leverages on the Separation Principle.*

### D. Discussion on the choice of $L$

The existence of such a matrix  $L$  can be determine through the sufficient condition introduced in [16, Lemma 4]. In particular, the authors show that there exists a matrix  $L$ , such that for all  $x \in \hat{X}$ , (24) holds with  $R'(x) \geq 0$ , if and only if (24) is satisfied by  $L = g^\top$ . Thus, for  $R'(x) \geq 0$ , this 'observability' condition can be easily check by first setting  $L = g^\top$ . We can then set  $L = \gamma g^\top$ , with positive scalar  $\gamma$  chosen sufficient high, to determine the desired convergence rate  $\alpha$ .

For  $R'(x)$  which is not sign defined, but bounded from below we can still get a similar property as detailed in the following generalization of Lemma 4 in [16]

**Lemma 5.** *Define  $\underline{R}'$  as the minimum value for  $R'$  in (25), i.e., for all  $x \in \hat{X}$ ,  $R'(x) \geq \underline{R}'$ , and assume it to take finite values on  $\hat{X}$ . If there exists a  $\Gamma = \Gamma^\top$  such that  $\underline{R}' + g\Gamma g^\top \geq 0$ , then there exists an  $L$  such that (24) if and only if (24) is satisfied by  $L = (\Gamma + I)g^\top$ .*

The proof reads as in [16] when considering  $R(x) = \underline{R}' + g\Gamma g^\top \geq 0$ , and thus it is omitted.

In this case, to obtain the observer convergence rate and satisfy (24), we can define  $L = (\Gamma + \gamma I)g^\top$ , with  $\Gamma$  as defined above, and set  $\gamma > 0$  according with the desired observer convergence rate.

Note that however, condition (24) is only sufficient and can be restrictive for proving the convergence of the observer dynamics. In particular, it is still possible to obtain the attractiveness of the origin for  $\tilde{x}$  with  $R'(x) + \text{sym}(gL) \geq 0$  for all  $x \in \hat{X}$ , if an additional condition on the error dynamics is fulfilled. For linear system, this condition is easy to find, as show in [11, Proposition 6] and exploited in the linear case below, and it is related to kernel space of  $R + \text{sym}(gL)$ .

The extension to the nonlinear case is left for a future work and it is currently under development. Moreover, we believe that this type of approach can lead to a separation principle also in the nonlinear context for the class of port-Hamiltonian systems with constant input matrix, because the exponential rate of convergence of the observer dynamics is not affected by the input signal.

#### E. The linear case

A particular case of the scenarios considered above is the linear case, for which we can give a larger characterizations of the observer convergence. In particular, it is well-known that for linear systems, the observation error dynamics is autonomous, i.e., (16) is independent from the  $u(t)$  and  $\nabla H$  terms. Thus in this scenario, a sufficient condition for the observer convergence is that (27) is satisfied. However, we can think of the matrix  $L$  to be parameterized into a symplectic  $J_L$ , positive definite  $R_L$  and the  $g$  matrices, i.e., we write

$$L = (J_L + R_L)g^\top.$$

With this parameterization of  $L$ , the observer error dynamics (16) reads as

$$\dot{\tilde{x}} = (J + gJ_Lg^\top - (R + gR_Lg^\top))Q\tilde{x} = (J_o - R_o)Q\tilde{x} \quad (29)$$

and if  $R_o \geq 0$  we can still have asymptotic stability of the origin for  $\tilde{x}$ , as described in the following corollary.

**Proposition 1.** *Denote by  $k$  the dimension of the kernel of  $R_o \geq 0$ , and assume it is spanned by  $\{r_1, \dots, r_k\}$ . Then (29) is asymptotically stable if and only if the matrix*

$$(sI - QJ_o) [r_1 | r_2 | \dots | r_k]$$

has rank  $k$ , for every  $s = j\omega$ ,  $\omega \in \mathbb{R}$ .

The proof of this proposition follows the same steps of that for [11, Proposition 6], and it is thus omitted.

We can then characterize the stability of an autonomous linear system, rather than placing its eigenvalues on the open left hand side, by exploiting the positive semidefiniteness of its equivalent resistive part along with some geometrical conditions related to its kernel space.

<sup>5</sup>Previously, we considered  $R_L = \Gamma + \gamma I$ .

## VII. EXAMPLE

*Linear gradient: sensorless velocity measurement of PMSM system [17]*

We consider here the case of a 3-phase Permanent Magnetic Synchronous Motor (PMSM) in the  $d - q$  coordinates, with unmeasured shaft velocity. Its dynamics reads as

$$\begin{pmatrix} \dot{\varphi}_d \\ \dot{\varphi}_q \\ \dot{p} \end{pmatrix} = \left( \begin{bmatrix} 0 & 0 & \varphi_q \\ 0 & 0 & -\varphi_d - \Phi \\ -\varphi_q & \varphi_d + \Phi & 0 \end{bmatrix} - \begin{bmatrix} r & 0 & 0 \\ 0 & r & 0 \\ 0 & 0 & \beta \end{bmatrix} \right) \begin{pmatrix} \frac{\varphi_d}{L} \\ \frac{\varphi_q}{L} \\ \frac{p}{J_m} \end{pmatrix} \\ + \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} u \\ y = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{pmatrix} \frac{\varphi_d}{L} \\ \frac{\varphi_q}{L} \\ \frac{p}{J_m} \end{pmatrix}$$

where  $\varphi_d, \varphi_q$  are the stator magnetic fluxes along the  $d$  and  $q$  axes, respectively, while  $\Phi$  is the constant rotor magnetic flux,  $L$  is the phase inductance,  $r$  the phase resistance,  $p$  is the shaft momentum,  $J_m$  its inertia, and  $\beta$  is the viscous friction coefficient. The control signals  $u$  are the two stator phase voltages  $v_d, v_q$ , and the available outputs the corresponding phase currents  $i_d, i_q$ . The associated Hamiltonian function is then

$$H = \frac{\varphi_d^2}{2L} + \frac{\varphi_q^2}{2L} + \frac{p^2}{2J_m}$$

This model has constant input matrix and linear Hamiltonian gradient, with a diagonal  $Q$  matrix, thus it fits precisely into the framework described in Section VI-A. We thus first have to investigate the partial derivative of  $J(x)Qx$  with respect to  $x$ , i.e.,

$$\frac{\partial J(x)Qx}{\partial x} = \frac{\partial}{\partial x} \begin{pmatrix} \varphi_q \frac{p}{J_m} \\ -(\varphi_d + \Phi) \frac{p}{J_m} \\ \Phi \frac{\varphi_q}{L} \end{pmatrix} = \begin{bmatrix} 0 & \frac{p}{J_m} & \frac{\varphi_q}{J_m} \\ -\frac{p}{J_m} & 0 & -\frac{\varphi_d + \Phi}{J_m} \\ 0 & \frac{\Phi}{L} & 0 \end{bmatrix}$$

We are then interested in the symmetric part of  $\frac{\partial J(x)Qx}{\partial x}Q^{-1}$ , that is

$$\text{sym} \left( \frac{\partial J(x)Qx}{\partial x}Q^{-1} \right) = \begin{bmatrix} 0 & 0 & \frac{1}{2}\varphi_q \\ 0 & 0 & -\frac{1}{2}\varphi_d \\ \frac{1}{2}\varphi_q & -\frac{1}{2}\varphi_d & 0 \end{bmatrix}$$

so that the resulting equivalent resistive matrix  $R'(x)$  in (25) is then given by

$$R - \text{sym} \left( \frac{\partial J(x)Qx}{\partial x}Q^{-1} \right) = \begin{bmatrix} r & 0 & \frac{1}{2}\varphi_q \\ 0 & r & -\frac{1}{2}\varphi_d \\ \frac{1}{2}\varphi_q & -\frac{1}{2}\varphi_d & \beta \end{bmatrix}.$$

In general, this  $R'(x)$  is not sign definite, since its principal minors have no positive sign, i.e.,

$$r > 0; \quad r^2 > 0; \quad r^2\beta - r\frac{1}{4}(\varphi_d^2 + \varphi_q^2) \stackrel{?}{>} 0.$$

Last inequality can be analyzed by considering the fact that the supplied current norm  $\sqrt{i_d^2 + i_q^2}$  saturates in any practical

TABLE I  
 PMSM SYSTEM PARAMETERS

$J_m$	0.012	$[Kg\ m^2]$	$L$	$3.8 \cdot 10^{-3}$	$[H]$
$\beta$	0.0026	$[Nms/rad]$	$r$	0.225	$[\Omega]$
$\Phi$	0.17	$[Wb]$	$i_{\max}$	15	$[A]$

application <sup>6</sup>. Then, we can write an upper bound for  $\varphi_d^2 + \varphi_q^2$ , that is valid for any working configuration of the motor, i.e.,

$$\varphi_d^2 + \varphi_q^2 = (i_d^2 + i_q^2)L^2 = i_{\max}^2 L^2. \quad (30)$$

Although a lower bound for the determinant of  $R'(x)$  can be found, there is no guarantee that the resistance  $r$  is sufficiently large to overcome the norm of the magnetic flux,  $\varphi_d^2 + \varphi_q^2$ , in any configuration and for any friction coefficient  $\beta$ . We thus consider as output injection matrix  $L^\top = \gamma G$ , where  $\gamma$  is a positive parameter to be tuned. We this choice the equivalent resistance matrix is

$$R'(x) + \gamma G G^\top = \begin{bmatrix} r + \gamma & 0 & \frac{1}{2}\varphi_q \\ 0 & r + \gamma & -\frac{1}{2}\varphi_d \\ \frac{1}{2}\varphi_q & -\frac{1}{2}\varphi_d & \beta \end{bmatrix}, \quad (31)$$

whose principal minors are given by

$$r + \gamma > 0; \quad (r + \gamma)^2 > 0; \quad (r + \gamma)^2 \beta - (r + \gamma) \frac{1}{4}(\varphi_d^2 + \varphi_q^2).$$

To guarantee that (31) is positive definite, it is easy to see that

$$\gamma \geq \max \left\{ \frac{i_{\max}^2 L^2 - r, 0 \right\}.$$

For the numerical results we set  $\gamma = 10$ , and we initialize

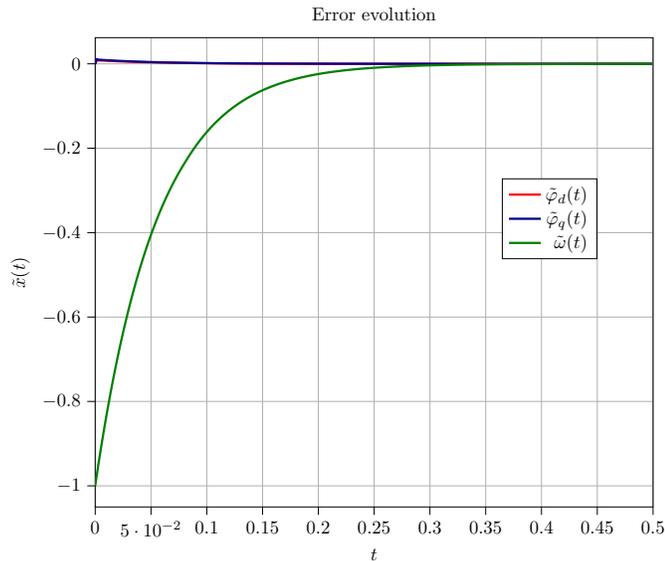


Fig. 1. Error evolution with  $u = (v_d, v_q)$  constant  $(-4V, 3V)$ .

system dynamics at  $(\varphi_d(0), \varphi_q(0), \omega(0)) = (0, 0.001, 1)$ , and the observer at the origin,  $\hat{x}(0) = 0$ . We then test the observer dynamics in two different scenarios. We first set the input voltage to a constant value  $(v_d, v_q) = (4V, 3V)$  and

<sup>6</sup>Moreover, the magnetic fluxes usually saturates even at lower current values.

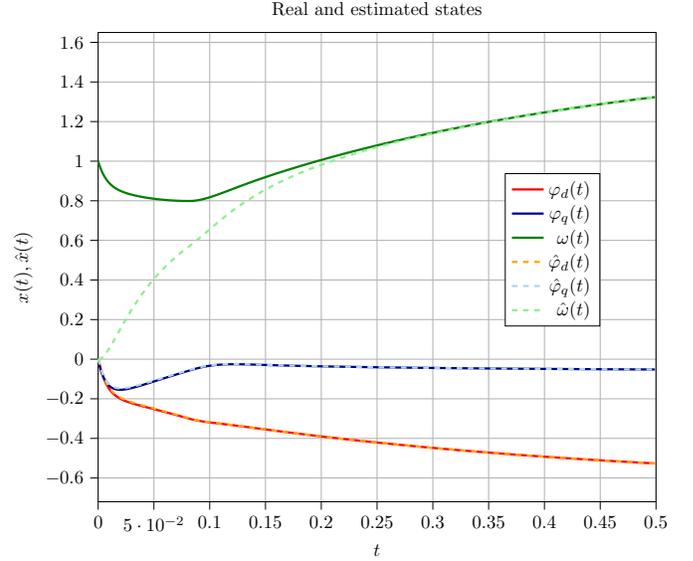


Fig. 2. System and Observer states evolution with  $u = (v_d, v_q)$  constant  $(-4V, 3V)$ .

the corresponding states evolution is depicted in Fig.2, while the observer error evolution in Fig.1. Then we consider the case in which the input voltage has a cosinusoidal evolution, i.e.,  $(v_d(t), v_q(t)) = (4 \cos(10t)V, 3 \cos(3t)V)$  as depicted in Fig.3, and the corresponding states evolution is depicted in Fig.4, while the observer error evolution in Fig.5. In it worth to notice, that the observer convergence is independent from the input signal as a consequence of the constant input matrix  $g$ .

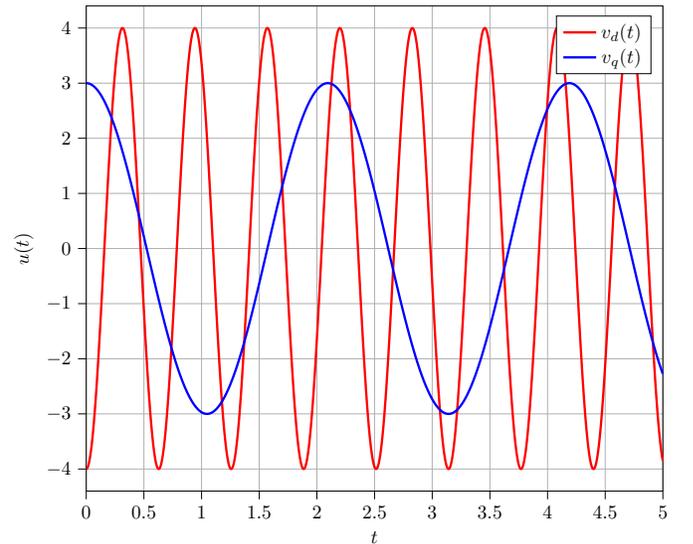


Fig. 3. Cosinusoidal inputs  $(v_d, v_q) = (-4 \cos(10t)V, 3 \cos(3t)V)$ .

## VIII. CONCLUSIONS

In this work we presented an general overview of the structure-preserving observer for port-Hamiltonian systems. In particular, we analyzed the observer structure that preserves the passivity for this class of systems and we then analyzed

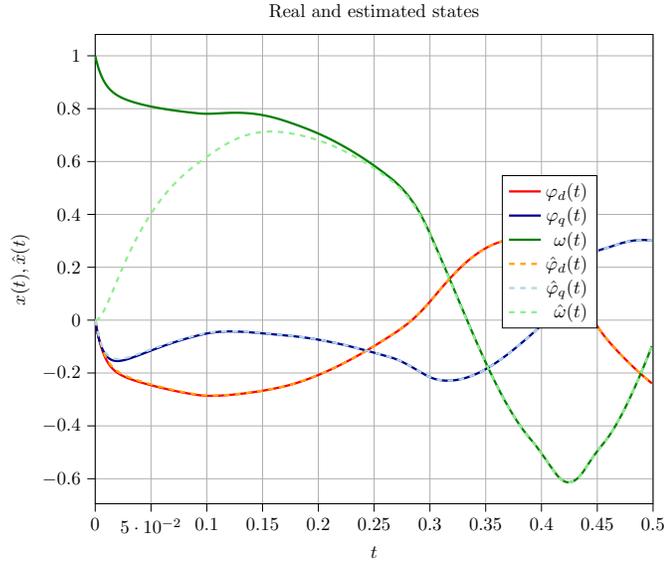


Fig. 4. System and Observer states evolution with cosinusoidal inputs  $(v_d, v_q) = (-4 \cos(10t)V, 3 \cos(3t)V)$ , depicted in Fig.3.

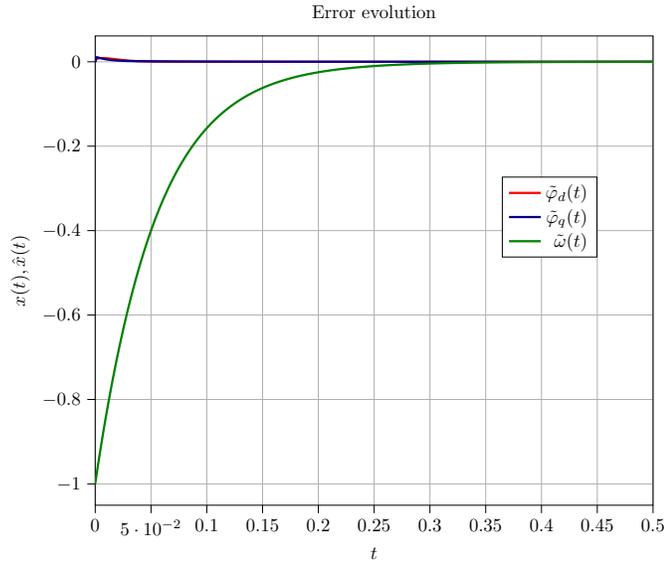


Fig. 5. Error evolution with cosinusoidal inputs  $(v_d, v_q) = (-4 \cos(10t)V, 3 \cos(3t)V)$ , depicted in Fig.3.

the conditions for passivity of the system - error dynamics cascade. We then exploit the convergence/contractive systems approach to determine the convergence of the observer dynamics onto the system dynamics for a class a port-Hamiltonian systems whose input matrix is constant in time. We conclude the work with an example of industrial interest in which we apply the design paradigm to a PMSM system in order to reconstruct the system state under two different classes on input signals, thus showing the effectiveness of the proposed approach.

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