Mass transfer development over large rectangular wall region governed by secondary flows

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Mass transfer probes: measurement of wall shear stress

- Advantages:
  - can be used in a wide variety of flow;
  - no interference with the flow;
  - possibility of measuring time-varying flows;
  - in the case of electro-chemical probes: calibration is not necessary.
- Two basic spatial orientations between probe main edge and fluid flow:
  - perpendicular transfer surface, see Fig. 1b);
  - slanted transfer surface, see Fig. 1c).
- Governing equation of mass transport:

$$\frac{\partial c}{\partial t}+u_x\frac{\partial c}{\partial x}+u_y\frac{\partial c}{\partial y}+u_z\frac{\partial c}{\partial z}=D\frac{\partial^2 c}{\partial x^2}+D\frac{\partial^2 c}{\partial y^2}+D\frac{\partial^2 c}{\partial z^2} \quad (1)$$

- boundary conditions:

$$\begin{array}{lll} x\to-\infty & z>0 & c=c_\infty \\ x\in\langle 0,L_x\rangle & z=0 & c=c_w \\ x<0\ \cup\ x>L_x & z=0 & \frac{\partial c}{\partial z}=0 \\ \forall x & z\to\infty & c=c_\infty \end{array} \quad (2)$$

- Assumptions:
  1. steady or pseudo-steady state → $\frac{\partial c}{\partial t}=0$;
  2. the flow is homogeneous over the surface of the probe;
  3. convection is large enough that diffusion in the x direction can be neglected → $D\frac{\partial^2 c}{\partial x^2}\approx 0$;
  4. concentration is changed negligibly in direction y → $D\frac{\partial^2 c}{\partial y^2}\approx 0$;
  5. diffusion boundary layer thickness, δ_D, is smaller than the thickness of hydrodynamical boundary layer, δ_H → $u_x=S_x\,z$; $u_y=S_y\,z$, see Fig. 1a);
  6. a) for perpendicular transfer surface: $u_x\gg u_y, u_z \to u_y, u_z\approx 0$;
     b) for slanted transfer surface: $u_x, u_y\gg u_z \to u_z\approx 0$.

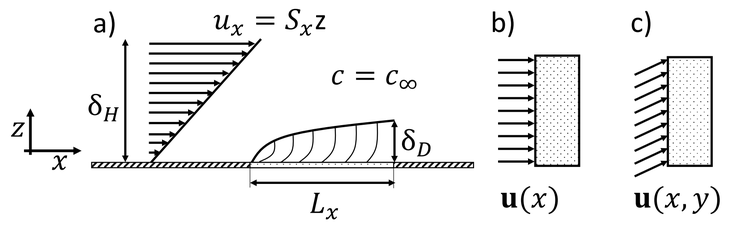

Figure 1: a) Description of mass transfer in the close of the probe; b) perpendicular transfer surface; c) slanted transfer surface.

Perpendicular transfer surface

- Simplification of eq. (1) according to the assumptions (1 - 6):

$$S_x\,z\frac{\partial c}{\partial x}=D\frac{\partial^2 c}{\partial z^2}\ ; \quad (3)$$

- Analytical solution of eq. (3):

$$\frac{c-c_w}{c_\infty-c_w}=\frac{1}{\Gamma(4/3)}\int_0^{\eta}e^{-\eta^2}d\eta;\qquad \eta=\frac{z}{{L_x}^{1/3}}\left[\frac{S_x}{9D}\right]^{\frac{1}{3}} \quad (4)$$

- Evaluation of mass transfer coefficient (Lévêque solution) on the base of eqs. (3) and (4):

$$\bar{k}_z=\frac{3}{2\Gamma(4/3)9^{1/3}}\left(\frac{D^2S_x}{L_x}\right)^{\frac{1}{3}}=0.8075\left(\frac{D^2S_x}{L_x}\right)^{\frac{1}{3}} \quad (5)$$

- Normalized Sherwood number:

$$\mathrm{Sh}^*=\frac{\mathrm{Sh}}{\mathrm{Pe_s}^{1/3}}=\frac{\frac{\bar{k}_z L_x}{D}}{\left(\frac{L_x^2 S_x}{D}\right)^{1/3}}=\bar{k}_z\left(\frac{L_x}{D^2S_x}\right)^{1/3}=0.8075 \quad (6)$$

Slanted transfer surface

- Simplification of eq.(1) according to the assumptions (1 - 6):

$$S_x z\frac{\partial c}{\partial x}+S_y z\frac{\partial c}{\partial y}=D\frac{\partial^2 c}{\partial z^2} \quad (7)$$

- No analytical solution of eq. (7), but it is possible to find out:
  - analytical solution of l_{eq} and Sh^*;
  - numerical solution of concentration profiles and Sh^*.

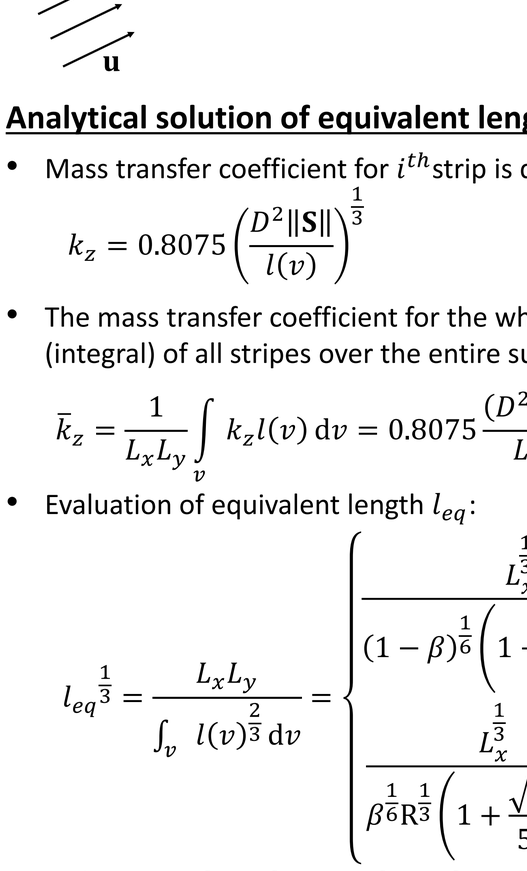

Figure 2: Scheme for derivation of analytical l_{eq} and Sh^*

Analytical solution of equivalent length and normalized Sherwood number

- Mass transfer coefficient for i^{th} strip is defined as:

$$k_z=0.8075\left(\frac{D^2\|\mathbf{S}\|}{l(v)}\right)^{\frac{1}{3}} \quad (8)$$

- The mass transfer coefficient for the whole probe is evaluation on the base of summation (integral) of all stripes over the entire surface of the probe. :

$$\bar{k}_z=\frac{1}{L_xL_y}\int_v k_z l(v)\,dv=0.8075\frac{(D^2\|\mathbf{S}\|)^{\frac{1}{3}}}{L_xL_y}\int_v l(v)^{\frac{2}{3}}dv=0.8075\left(\frac{D^2\|\mathbf{S}\|}{l_{eq}}\right)^{\frac{1}{3}} \quad (9)$$

- Evaluation of equivalent length l_{eq}:

$${l_{eq}}^{\frac{1}{3}}=\frac{L_xL_y}{\int_v l(v)^{\frac{2}{3}}dv}=\begin{cases}\dfrac{L_x^{\frac{1}{3}}}{(1-\beta)^{\frac{1}{6}}\left(1+\dfrac{\mathrm{R}\sqrt{\beta}}{5\sqrt{1-\beta}}\right)} & \text{for } \beta\le\beta_c \\ \dfrac{L_x^{\frac{1}{3}}}{\beta^{\frac{1}{6}}\mathrm{R}^{\frac{1}{3}}\left(1+\dfrac{\sqrt{1-\beta}}{5R\sqrt{\beta}}\right)} & \text{for } \beta>\beta_c\end{cases} \quad \text{where} \quad \beta_c=\frac{1}{1+\mathrm{R}^2} \quad (10)$$

- The normalize Sherwood number Sh^* is defined as:

$$\mathrm{Sh}^*=\frac{\mathrm{Sh}}{\mathrm{Pe_s}^{1/3}}=\bar{k}_z\left(\frac{l}{D^2\|\mathbf{S}\|}\right)^{\frac{1}{3}}=0.8075\left(\frac{D^2\|\mathbf{S}\|}{l_{eq}}\right)^{\frac{1}{3}}\left(\frac{l}{D^2\|\mathbf{S}\|}\right)^{\frac{1}{3}}=0.8075\left(\frac{l}{l_{eq}}\right)^{\frac{1}{3}} \quad (11)$$

- The resulting value of the Sh^* is dependent on the choice of length l in the eq. (11), see Fig. 3:

$$\mathrm{Sh}^*=\begin{cases}0.8075\,(1-\beta)^{\frac{1}{6}}\left(1+\dfrac{\mathrm{R}\sqrt{\beta}}{5\sqrt{1-\beta}}\right) & \text{for } l=L_x;\beta\le\beta_c \\ 0.8075\,\beta^{\frac{1}{6}}\mathrm{R}^{\frac{1}{3}}\left(1+\dfrac{\sqrt{1-\beta}}{5R\sqrt{\beta}}\right) & \text{for } l=L_x;\ \beta>\beta_c \\ 0.8075 & \text{for } l=l_{eq};\forall\beta\end{cases} \quad (12)$$

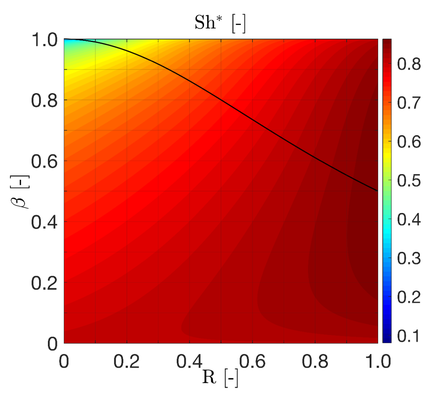

Figure 3: Normalized Sherwood number evaluated for length $l=L_x$ in dependence on the parameters β and ratio R. Black curve describes the value of critical β_c.

Numerical solution of concentrations profile and normalized Sherwood number

- Definition of hydrodynamical parameter β and non-dimensional characteristics of the system:

$$\beta=\frac{S_y^2}{S_x^2+S_y^2}=\frac{S_y^2}{\|\mathbf{S}\|^2}\Longrightarrow S_x=\sqrt{1-\beta}\|\mathbf{S}\|;\ S_y=\sqrt{\beta}\|\mathbf{S}\| \quad (13)$$

$$c^+=\frac{c}{c_\infty};\ x^+=\frac{x}{L_x};y^+=\frac{y}{L_y};z^+=\frac{z}{\delta_D};\delta_D=\left(\frac{Dl}{\|\mathbf{S}\|}\right)^{\frac{1}{3}};\mathrm{R}=\frac{L_x}{L_y} \quad (14)$$

- Modification of eq. (7):

$$\frac{\partial \mathrm{c}^+}{\partial \mathrm{x}^+}+\frac{\sqrt{\beta}\,R}{\sqrt{1-\beta}}\frac{\partial \mathrm{c}^+}{\partial \mathrm{y}^+}=\frac{1}{\mathrm{z}^+\sqrt{1-\beta}}\frac{L_x}{l}\frac{\partial^2 \mathrm{c}^+}{(\partial \mathrm{z}^+)^2} \quad (15)$$

- Numerical solution of eq. (7), see Fig. 4. The shape of the concentration profiles is dependent on the parameters β and ratio R.
- Sh^* is evaluated from derivative of concentration in the z direction on the probe surface using a relation, see Fig. 5:

$$\mathrm{Sh}^*=\int_0^1\int_0^1-\frac{\partial \mathrm{c}^+}{\partial \mathrm{z}^+}\,dx^+dy^+ \quad (16)$$

- Final values of Sh^* is dependent on the choice of length l . For example, the length l can be replaced by the width of the electrode L_x or by the equivalent length:
  - $l=L_x$ and for $\forall\beta$:

$$\frac{\partial \mathrm{c}^+}{\partial \mathrm{x}^+}+\frac{\sqrt{\beta}\,R}{\sqrt{1-\beta}}\frac{\partial \mathrm{c}^+}{\partial \mathrm{y}^+}=\frac{1}{\mathrm{z}^+\sqrt{1-\beta}}\frac{\partial^2 \mathrm{c}^+}{(\partial \mathrm{z}^+)^2}\Longrightarrow \mathrm{Sh}^*(\mathrm{R},\beta),\text{ see Fig. 3} \quad (17)$$

  - $l=l_{eq}$ and for $\beta\le\beta_c$:

$$\frac{\partial \mathrm{c}^+}{\partial \mathrm{x}^+}+\frac{\sqrt{\beta}\,R}{\sqrt{1-\beta}}\frac{\partial \mathrm{c}^+}{\partial \mathrm{y}^+}=\frac{1}{\mathrm{z}^+\sqrt{1-\beta}}\sqrt{1-\beta}\left(1+\frac{\mathrm{R}\sqrt{\beta}}{5\sqrt{1-\beta}}\right)^3\frac{\partial^2 \mathrm{c}^+}{(\partial \mathrm{z}^+)^2}\Longrightarrow \mathrm{Sh}^*=0.8075 \quad (18)$$

  - $l=l_{eq}$ and for $\beta>\beta_c$:

$$\frac{\partial \mathrm{c}^+}{\partial \mathrm{x}^+}+\frac{\sqrt{\beta}\,R}{\sqrt{1-\beta}}\frac{\partial \mathrm{c}^+}{\partial \mathrm{y}^+}=\frac{1}{\mathrm{z}^+\sqrt{1-\beta}}\sqrt{\beta}\mathrm{R}\left(1+\frac{\sqrt{1-\beta}}{5R\sqrt{\beta}}\right)^3\frac{\partial^2 \mathrm{c}^+}{(\partial \mathrm{z}^+)^2}\Longrightarrow \mathrm{Sh}^*=0.8075 \quad (19)$$

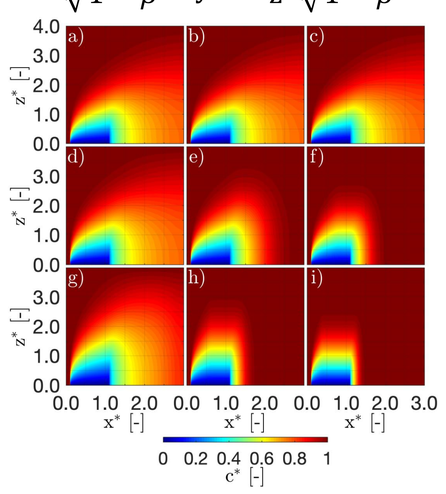

Figure 4: Concentration profile in the plane $x^+z^+$for $y^+=L_y/2$: 1st row - $\beta=0$; 2nd row - $\beta=0.4$; 3th row - $\beta=0.8$; 1st column - $\mathrm{R}=0.1$; 2nd column - $\mathrm{R}=0.4$; 3th column $\mathrm{R}=0.8$.

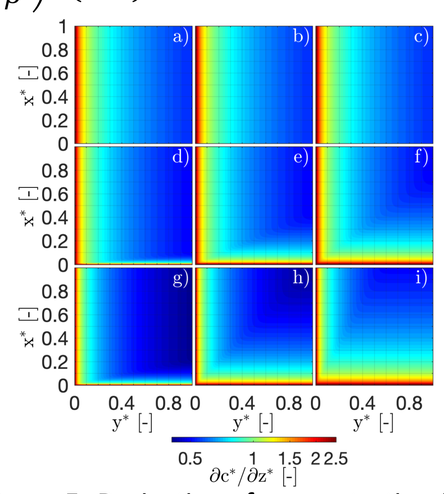

Figure 5: Derivative of concentration in the z direction on the probe surface (logarithmic colorbar): 1st row - $\beta=0$; 2nd row - $\beta=0.4$; 3th row - $\beta=0.8$; 1st column - $\mathrm{R}=0.1$; 2nd column - $\mathrm{R}=0.4$; 3th column $\mathrm{R}=0.8$.

Conclusion:

We have derived an equation describing the equivalent length and normalized Sherwood number as a function of the fluid flow angle (characterized by parameter β) and the aspect ratio of the rectangular mass transfer probe ($R = L_x/L_y$). The given equations could serve for better interpretation of the measurement of wall shear stress (ED or hot-film method) especially for fluid flow associated with secondary flows.

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