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Performing Multiplications Beyond the Text of Some Sanskrit Mathematical Commentaries

Abstract: Sanskrit mathematical commentaries contain many different kinds of performance, some of which are embedded in the lists of the solved examples they almost always contain. The resolutions of problems contained in these lists of solved examples include representations of a working surface on which computations would have been carried out or diagrams drawn. This chapter discusses how the execution of multiplications can be reconstructed with the help of these representations. I examine how these representations appear in manuscripts and editions of these texts. I also reflect on the material objects on which and with which these multiplications may have been performed. The discussion is based on Pṛthūdaka's (fl. 850) commentary on the *Theoretical Treatise of the Corrected Brāhma School* (*Brāhmasphuṭasiddhānta* 628), the anonymous undated commentary on Śrīdhara's (c. 800) *Board Mathematics* (*Pāṭīgaṇita*)¹ and *An Introductory Commentary on Arithmetics to Awaken the Young* (*Bālabodhāṅkavṛtti*), Śambhūdāsa's (1428/1429) Old-Gujarātī commentary on the anonymous and undated Sanskrit *Twenty-five Rules* (*Pañcaviṃśatikā*).

1 Introduction: texts that stage performances

This chapter explores the ways in which, as historians, we use Sanskrit mathematical treatises and their commentaries to reproduce and reconstruct the execution of certain algorithms. To do so requires reflection on what the aims of Sanskrit mathematical texts were, with regard to the execution of algorithms. Was their purpose to teach how to carry out an execution and, if so, how did texts go about doing so? Furthermore, what do we know of the tools used to execute an arithmetical algorithm?

Answering these questions directly may not be possible. We can however start by noting that Sanskrit mathematical treatises and commentaries did indeed stage performances, some of which related to the execution of algorithms.

¹ This is a contested translation of the title of this text, discussed in Section 3 below.

1.1 An orally chanted text for teaching?

The first and most famous of the staged performances of treatises and commentaries was indeed the vocalisation of versified texts.² These days, a good Sanskritist, an authentic one, seems to be one who, in international conferences, can chant the Sanskrit verses of a treatise (*śāstra*). Such a Sanskritist not only displays a knowledge of versification, but also demonstrates that the study of ancient South Asian texts is more than an academic discipline: it is also a live practice.

The earliest medieval theoretical astronomical treatise handed down to us, the *Āryabhaṭīya* (499), was authored by Āryabhaṭa (b. 476), who spoke of himself in the third person as ‘proclaiming’ (*nigad-*) the topics of the treatise.³ His famous (because it is one of the earliest) ‘sine table’ is a verse which lists numerical values, using a special way of naming and noting them:⁴

Ābh.1.12.

makhi bhakhi phakhi dhakhi ṅakhi ṅakhi ṅinakhi hasjha skaki kiṣga śghaki kighva| ghlaki, kigra, hakyā, ghaki, kica, sga, śbha, ṅva, kla, ghta, cha kalār dhajyāḥ||

makhi (225), *bhakhi* (224), *phakhi* (222), *dhaki* (219), *ṅakhi* (215), *ṅakhi* (210), *ṅinakhi* (205), *hasjha* (199), *skaki* (191), *kiṣga* (183), *śghaki* (174), *kighva* (154), | *ghlaki* (143), *kigra* (131), *hakyā* (119), *ghaki* (106), *kica* (93), *sga* (79), *śbha* (65), *ṅva* (51), *kla* (37), *ghta* (22), *cha* (7) are the half-chords in minutes||.

Here Āryabhaṭa seems to display both a text and musical onomatopoeia. Perhaps this verse was composed to be performed orally, or perhaps for the reader to imagine that it was so performed. How purely oral ancient Sanskrit texts were is a question that has generated much debate.⁵ Historians, and in particular historians of Indian mathematics, often consider that treatises were first stored in a person’s memory rather than in a manuscript, and then made to be per-

² Often studies of orality focus on the sacred texts of the Vedas, which Frits Staal argues were made to be performed, chanted and recited, without concentrating on their contents (Staal et al. 1983, 256, for instance).

³ Ābh.1.cd. *āryabhaṭas trīṇi gadati gaṇitaṃ kālakriyāṃ golam*, ‘Āryabhaṭa proclaims three: Mathematics, Time-reckoning, the Sphere’. Shukla and Sarma 1976, 1.

⁴ Concerning the ‘sine table’ and its importance, see Van Brummelen 2009, 99–100. Concerning Āryabhaṭa’s notation of numerical values with syllables, see Shukla and Sarma 1976, 3–5. This notation was an extensively debated topic in the nineteenth and early twentieth century, as discussed notably in Keller 2011.

⁵ Most famously and consistently, Jack Goody believed in the Vedas’s pure orality, notably in Goody 2010, 166–169, a position he maintained despite much criticism, as found for instance in Falk 1990.

formed orally.⁶ Nonetheless, such texts describe a world where things are written and traced: noted numbers and drawings of diagrams are part of the prescriptions they contain. Furthermore, the commentaries on these oral treatises are often explicitly written (*likh-*) texts. They too can contain quoted verses and staged dialogs. Finally, whether we are looking at South Asian language manuscripts in general or manuscripts dealing with mathematical and astral texts in these languages, quite famously, we are overwhelmed by the sheer number of documents.⁷ These manuscripts are quite recent and made of fragile bark, palm leaf and paper – an abundance which suggests, in part, that transmission also involved hand copying texts. The oral performance of an astronomical or mathematical treatise thus seems to have been a standard rhetoric for the presentation of scholarly knowledge; a *mise-en-scène* of how ideally scholarly knowledge should be performed and transmitted without necessarily being the reality of how the text was actually passed down (in writing rather than through oral transmission?) or stored (in writing rather than in a person’s mind?).⁸

This paradox of our sources and our historiographic difficulties in treating them also indicates how little we know of the contexts in which astronomical and mathematical texts in Sanskrit were composed, studied, learned, performed and copied. In the case of Sanskrit mathematical texts, the context is often thought to be that of a school, or at least of a teacher and student relationship.⁹ Another possible imagined setting, from what we know of early modern literature and miniatures, is a royal court with its courtly performances and rivalries, which might echo into more rural settings in temples or on village stages.¹⁰ Finally, the vigorous debate forums of what would have been a South Asian public sphere from the eighteenth century onwards, probably also fuelled imagination on the public performance of Sanskrit mathematical texts and procedures.¹¹

6 From Filliozat 2004 to Kusuba 2018, for instance.

7 Srinivas 2019.

8 Keller 2016, 577. Of course, we can imagine that transmission involved all these acts simultaneously.

9 This historiography applied to the *Āryabhaṭīya* has been studied in Keller 2016. In Kusuba 2018, this is an underlying thread of the article, clear already from its abstract which states: ‘When students read the rules, they learned the procedure of calculation. When they read examples, they learned how to set down given numbers’. In other words, Takanori Kusuba assumes that the texts were read by students wanting to learn how to execute elementary operations.

10 For a summary of social and political issues of text, manuscript and performance in early modern India, see O’Hanlon 2013, 89–93. Concerning representations of courtly astrologers, see Sarma 2000.

11 On these forums, see Bayly 1996, Chap. 5, 181–211.

These pre-colonial and colonial debates did include questions of astronomy.¹² Mathematics were also at the time an arena of more or less public discourses, notably concerning the writing of local mathematical textbooks in regional languages.¹³ We might therefore wonder if these public performances can be projected back much further into the past.

For most texts before the second millennium, it is however often hard to know precisely in which families, places, and institutions (temples, royal courts, village schools or city forums) the texts were composed, copied, learned, commented, and studied. More often than not, if we want to know what audience the text was aimed at, and who must have performed the text, executed its algorithms and used the commentaries, we have no choice but to use the clues we get from the both texts and the manuscripts through which they have been handed down to us.

We will set aside this question of context in what follows. Instead, the performance of an algorithm, essentially the execution of elementary operations, will be our focus here. These executions are evoked in treatises and are part of the performances staged in commentaries.

1.2 Staging the resolution of problems

I have suggested elsewhere that Sanskrit mathematical commentaries might be characterised by the lists of solved problems they contain.¹⁴ There seems to be a fixed structure for how the resolution of problems are to be staged in commentaries, something that goes beyond time and regional variations, for each solved problem has a standard organization, as represented in Fig. 1:

1. The performative announcement of an example/problem (*uddeśaka, udāharaṇa*).
2. The statement of the problem, somewhat in the form of a riddle, often versified.
3. A setting (*nyāsa, sthāpana*) of the givens of the problem. This involves a performative declaration ‘setting’ (*nyāsaḥ*), which opens into the text of the commentary a representation of the surface on which mathematical non discursive actions are going to be carried out to solve the problem: diagrams can be drawn, numbers can be displayed in tabular formats, and so

¹² Minkowski 2001; Dodson 2007, 162–167.

¹³ Raina and Habib 1990.

¹⁴ Keller 2022, 100.

- on. These settings and the objects they contain translate the discursive givens of the problem into a configuration which enables the execution of actions and operations involved in the resolution.
4. The resolution (*karāṇa*) of the problem follows, in which sometimes several states of the working surface might be displayed.
 5. Sometimes a part is devoted to the explanation/proof (*vāsanā, upapatti*) of the general rule related to the problem.¹⁵
 6. The statement of the solution.

Such a structure is often made visible in modern editions, but might not be typographically set out when dealing with manuscripts. This textual organisation may vary in its subparts from text to text and manuscript to manuscript.

<p>विन्यस्याधो गुण्यं कवाटसन्धिक्रमेण गुणराशोः । गुण्येद्विलोमगत्याऽनुलोमगणैर्वा क्रमशः ॥ १८ ॥ उत्सार्योत्सार्यं ततः कवाटसन्धिर्भवेद्विदं करणम् । तस्मिंस्तत्त्वति यस्मात्प्रत्युत्पन्नस्ततस्तत्त्वः ॥ १९ ॥ रूपस्थानविभागाद् द्विधा भवेत्खण्डसंज्ञकं करणम् । प्रत्युत्पन्न विधाने करणान्येतानि चत्वारि ॥ २० ॥</p> <p>*(उदाहरणम्—</p> <p>षण्णवतिद्विक्रमेकं चैकद्विगुणानि षण्णवाष्टौ च । सप्तत्रिगुणान् पंचकषट्खाष्टौ च कुरु षष्टिगुणान् ॥ ३ ॥)</p> <p>प्रतिरूपगुणयोः राशिरद्विक्रमस्य रूपं क्वाटसन्धिक्रमेण गुण्यं गुण्यं (योरेकविंशति- षण्णवत्यधिकशतद्वादशकयोः कवाटसन्धिक्रमेण न्यासः— १३३३३३) एकस्थानस्य षट्कं रूपेण गुणितं षट् इति एकाधःस्थाने षट्, ततः द्विक्रमेण गुणितं षट्कं द्वादश इति द्विकाधःस्थाने द्वौ रूपमपि नवानामधः जातम् । न्यासः १३३३३३ । ततो दशास्थानस्य नवकं गुणयितुं संप्रति गुणराशिः । न्यासः १३३३३३ । इदानीं नवानामेकविंशतेषु गुण्यगुणकभावो जातः, रूपेण गुणितं नवकं नव, स्वाधःस्थितद्विक्रमयोगात् तत्स्थाने रूपं जायते, रूपमपि द्विकाधःस्थितरूपेण युज्यते द्वे भवतः ; द्वाभ्यां गुणितं नवके षष्टादश) 'पूर्वदेव तदधो न्यासः, षष्ट्यु ष स्वाधःस्थितद्विक्रमो तत्स्थानं शून्यं, रूपमपि द्विकाधःस्थितं रूपेण युज्यते द्वे भवतः । ततश्च दशस्थाने' द्विकं गुणयितुं संप्रति गुणराशिः । स्वाधःस्थानम् १३३३३३ । इदानीं द्वयोरेकविंशतेषु गुण्यगुणकभावो जातः, एकगुणितौ द्वौ द्वाविंश, एकाधःस्थे (शून्ये) द्विकं लिप्या जालौ द्वौ, द्वाभ्यां च द्वौ गुणितौ चत्वारः, स्वाधःस्थितद्विक्रमयोगात् षट् (१३३३३३) । ततः सहस्रस्थानस्य रूपं गुणयितुं संप्रति गुणराशिः । न्यासः (१३३३३३) इदानीमेकस्यैकविंशतेषु गुण्यगुणकभावो जातः, तदा रूपेण गुणितं रूपं रूपमेव, प्रत्यु क्षिप्तं सप्त, द्वाभ्यामेकं गुणितं द्वाविंशति । निःसोपिते गुण्यराशौ, गुणके निवृत्ते कलं तदेव २७२१६ । एवं रूपविभागे यत्त्रिंशत्विभागस्थानानि^१ तानि पृथक् पृथक् गुणकेन</p>	<p>Verses of the Treatise</p> <p>Announcement of an example</p> <p>Versified example</p> <p>Announcement of setting and setting</p> <p>Resolution</p> <p>Stating the solution</p>
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Fig. 1: The different parts of a solved problem. Execution of a multiplication in the commentary on Śrīdhara’s *Board Mathematics* as set up in Kripa Shankar Shukla’s 1959 edition.¹⁶

¹⁵ Keller 2022, 118 gives examples of proofs in commentaries made with specific givens and carried out within the resolution of examples. This is most common but often erased from the standard historiography of proofs in Sanskrit mathematical sources, as discussed in Keller 2022, 110–111.

Within this structure, therefore, different performances are staged inside the text: the performative declarations of problems and settings, the statement of the problem, the translation of the problem's givens on a working surface, the resolution of a problem, and the explanations related to the resolution. They can each be treated in a separate rubric, suggesting that the resolution of a problem involves all these sub-actions, which together form one great performance: the execution of the algorithm which solves a given problem.

The text of commentaries sometimes alludes to what is being done, orally, off-stage so to say, with what is placed on the working surface. Typically, in Bhāskara's commentary on the *Āryabhaṭīya*, oral explanations are given on diagrams. These explanations are evoked in the texts but not explicated.¹⁷

Therefore, the different actions involved in the resolution of a problem are sometimes staged partly in the text, but other actions might simply be alluded to, and others yet remain tacit.

We may wonder how these performances were to be staged. Were they to be performed theatrically with an oral vocalisation, or silently for one's self? Even as we leave orality aside, we might ask whether we can speak of the execution of an algorithm as a performance. Was it thought of in this way by the authors of Sanskrit mathematical commentaries?

1.3 Texts, algorithms, and performances

The execution of an algorithm involves much of what defines a performance:¹⁸ it is a skill that can be repeated and rehearsed to be carried out well. It can be different every time it is acted out, and it may or may not have publics, but since we learn of them through texts that evoke them, we may imagine at least two concerned with the performance: the author of the text and his or her reader. An algorithm's steps need not all be fixed precisely or in the same order, as we will see. The space in which a procedure is to be performed, and the material objects which it involves are open questions to us as historians.

Would the authors, performers, and readers of Sanskrit mathematical texts in the past endorse such points of view?

¹⁶ The publication is free of copyright, and available here <<https://archive.org/details/Patiganita>>.

¹⁷ Keller 2005, 298–299.

¹⁸ As considered in Schechner 2017, 26, 35, 49, and 52, and Fischer-Lichte 2014, 18.

The vocabulary associated with the execution of procedures in Sanskrit etymologically conveys the idea of making and of action, which is the same word (*karāṇa*). The same term can be used to evoke both the procedure itself and the resolution of a problem. Consequently, operations appear as the elementary building blocks from which algorithms are made. They are very literally perifabrications (*parikarman*). Other words can be used of course, some evoking methods which lead (*nyāya*) to a fruit/result (*phala*). Therefore, the execution of the algorithm and the algorithm are one and a same thing in the vocabulary. In theatre and dance, according to Sanskrit texts, if a performance is related to action (*kṛ-*) the performance itself has a separate name and verb (*naṭ-*). We do not find such distinctions in the resolution of problems or the execution of algorithms. However, texts that were not theatre plays but had a scholarly dimension could be associated with some kind of performance, chanting intertwined with near theatrical staging, or recitation with improvisation, possibly in Sanskrit and vernacular. This is notably the case of the historical law texts called ‘the ancient’ (*purāṇa*), which were performed by specialists, *pauraṇikas*.¹⁹ We can thus imagine that similarly astral texts or mathematical texts could be performed by the specialists of these texts. Astrologers in South Asia today are known to intertwine computations in what is sometimes a ritual performance. Such ritual performances might include the recitation of texts.²⁰ To evaluate whether they existed in the past, and in what form, we would need to document such stagings in non-mathematical and non-astral literature as well.

Historians of mathematics have taken the resolution of problems as described in mathematical commentaries as a stage, set in the text, on which the resolution is performed to teach and explain how an algorithm should be executed. In other words, more often than not, historians of mathematics have approached mathematical texts as very literally describing the execution of operations and algorithms. As such then, the *sūtras* of mathematical treatises have appeared as faulty: aphoristic and failing to describe the intricate details necessary to reproduce an execution, while commentaries would seem to rely on much tacit knowledge.

Let us first set aside, though, the idea that transmitted texts are always about enabling such performances. We have noted that some of the actions entailed by an execution might not be staged directly in the text. Some might be

¹⁹ Some of such performances are described in O’Hanlon 2013, 117–118.

²⁰ Tarabout 2006; Tarabout 2007; and Tarabout 2015 are analytical variations on an astrological ritual carried out in Kerala temples. See also Guenzi 2013, for example 116–118 translated into English in Guenzi 2021, 90–91.

tacit, and others simply alluded to. Perhaps some of what is staged in the text – including the display of working surfaces – aims at inviting the reader (directly or indirectly) to try to enact a set of actions. Text by text we should try to assess which elements may invite us to think that this is the case. In all instances, choices are made in the treatises as in the commentaries on which part of the performance is shown. Each mathematical commentary has its own way of relating to the different performances that the resolution of a problem entails. A text can never faithfully describe and/or represent the whole set of performances attached to the resolution of a problem, since these include non-discursive acts. They however necessarily give a point of view on it. More precisely, a treatise’s rules give a point of view on the steps of the procedure, while the commentary expounds on what is deemed necessary about it.²¹

In what follows, the intention of the texts in relation to the performance of the algorithm by a reader or hearer will not be discussed directly. However, we will have to touch on such an intention as we try to reconstruct the performance of a multiplication. We will see how discussing the performance in relation to the text helps us perceive what could have been some aspect of the author’s aim. We will consider the texts as more or less willing testimonies to how a given algorithm or operation was carried out, and we will use texts as the tools we have as historians to reconstruct these executions. But we will not assume that all the texts we deal with are, as Matthieu Husson and Samuel Gessner in this volume put it, ‘toolbox’ manuscripts.

This chapter tackles some of the difficult components of the reconstruction of executions, those which may specifically be concerned with performance: the material objects used, the surface on which the execution took place, and the different orders and details of those little steps in the execution of an algorithm that exhibit a know-how open to improvisation. What clues do we have about them?

In what follows, this chapter looks at the dialectics of what the text states about the resolution of a problem, the representations of different stages of the working surface within the text, and how they are articulated to one another. We will do so in the case of the execution of a multiplication. It will be a way of asking on what surface, with what tools, multiplications could have been carried out. The discussion will use Pṛthūdaka’s (fl. 850) commentary (abbrev. PBSS) on the *Theoretical Treatise of the Corrected Brāhma School* (*Brāhmasphuṭasiddhānta* 628 – abbrev. BSS), the anonymous undated commentary on Śrīdhara’s (c. 800) *Board Mathematics* (*Pāṭīgaṇita* – abbrev. PG) and the *An Introductory Commentary on*

²¹ Keller 2015a, 189–190, 210–211.

Arithmetics to Awaken the Young (*Bālabodhāṅkavṛtti* – abbrev. BBA), Śambhūdāsa’s (1428/1429) Old-Gujarātī commentary on the anonymous Sanskrit *Twenty-five Rules* (*Pañcaviṁśatikā* – abbrev. PV).²²

We first look at what the text and manuscript stage of the working surface in relation to the text and the execution, and we then raise questions about the material objects that were used to perform these executions.

2 Texts displaying the execution of multiplications

Many different techniques for multiplications are evoked in Sanskrit mathematical texts. Those pertaining to the decimal place value notation have attracted more attention in the historiography than others. Of course, this is in part due to the historiographical trope of an interest in ‘our’ method of noting numbers. However, operations using place-value resources are also those that involve representation of tabular dispositions on a working surface.²³ In what spaces were multiplications carried out? Were such spaces the same as those on which texts were written? How did multiplication executions deal with place-value and its tabular resources? Notably, how were carry-overs and intermediate steps noted, or dealt with? With these questions in mind, we will look at two different kinds of multiplication executions.

The first example comes from Pṛthūdaka’s commentary on the mathematical chapter of the *Theoretical Treatise of the Corrected Brāhma School* (*Brāhmasphuṭasiddhānta*). We will look at a multiplication whose multiplicand is made into a ‘cow’s string’ (*go-sūtrikā*). The following examples concern a well-known multiplication method called ‘door-hinges’ (*kavāṭa-sandhi*). We will look at how different texts represent different moments of multiplication execution on a working surface, focusing first on how they display the working surface and the intermediary steps.²⁴

²² All primary sources, their editions and associated abbreviations are summed up in Appendix A.

²³ Discussed in Keller and Morice-Singh 2022.

²⁴ Some elements of this and the following subsection have already been discussed in Keller and Morice-Singh 2022.

2.1 Manuscripts displaying different spaces?

Brahmagupta, the seventh-century astronomer and mathematician, defined a mathematician as someone who knew twenty different types of operation (*pari-karman*). His ninth-century commentator Pṛthūdaka noted that multiplication (*pratyutpanna*) counted among them. At the end of the mathematical chapter, multiplications with integers are evoked. One of the verses runs as follows:²⁵

BSS.12.55.

The product (*pratyutpanna*) is the multiplicand (*guṇya*), made into ‘a cow’s string’ (*go-sūtrikā*)²⁶, equal in portions (*khaṇḍa*) to the multiplier (*guṇakāra*), multiplied <and the partial products> added, or, <the multiplicand> is equal in parts (*bheda*) to the multiplier ||55||

This rule provides the gist of what we can read as two different kinds of execution of a multiplication organised around two subdivisions of the multiplier (*guṇakāra*): in portions (*khaṇḍa*) or in parts (*bheda*). The rule then relies on two specified operands: a multiplicand (*guṇya*) and a multiplier (*guṇakāra*) which are not treated as interchangeable. They do not enter the same steps in the execution procedure. It is understood that the multiplicand (*guṇya*) is repeated as many times as there are parts or portions in the multiplier.

²⁵ BSS.12-55 *guṇakāra-khaṇḍa-tulyo guṇyo gosūtrikā-kṛto guṇitaḥ| sahitaḥ pratyutpanno guṇakāraka-bheda-tulyo vā||*. Translations of this verse can also be found in Colebrooke 1817, 319; Datta and Singh 1935, 135; and Keller and Morice-Singh 2022.

²⁶ There is a certain amount of discussion on the reading and understanding of this name. Colebrooke 1817, 319, reads ‘cow’s string’ (*go-sūtrikā*) and Hayashi 2017 has recently agreed to follow him on this as this reading corresponds to what is noted in available manuscripts of the treatise and commentary. Sudhākara Dvivedin’s text, as well as Bibhutibhusan Datta and Narayan Avadesh Singh’s interpretation, suggest a different reading, *go-mūtrikā*. Datta and Singh 1935, 147, n. 4, evoke live oral traditions through ‘paṇḍits’, to justify their reading. The transition from the *devanagari* म (*ma*) to स (*sa*) is indeed very slight and could explain the corruption of the text here. The expression *go-mūtrikā* means lit. ‘cow’s urine’ and is a common word used to signify ‘zig-zag’. Takao Hayashi argues that there is another kind of multiplication that is standardly called *go-mūtrikā* and which indeed corresponds to a zig-zag; while the ‘cow’s string’, which could attach several cows in a row together, may correspond better to what is described of the display. This argument however is also debatable. First, because a same name could be used to denote different multiplication executions as shown in Hayashi 2017, 58–59 and in Keller and Morice-Singh 2022. Second, although attested in dictionary entries such as the one authored by Monier Monier-Williams, the existence and use of such strings (or of such an expression) needs to be documented. Nevertheless, since the image of three cows in a diagonal row is a plausible one for this multiplication method, and since it further retains the manuscript readings, we will adopt this name for the time being.

Pr̥thūdaka’s interpretation of Brahmagupta’s rule details in part how these multiplications should be executed. Here we concentrate on a multiplier subdivided into ‘portions’ (*khaṇḍa*), with a multiplicand shaped into a ‘cow’s string’. Multiplying with a multiplier subdivided into ‘portions’ (*khaṇḍa*) means, according to Pr̥thūdaka, splitting the multiplier according to its different powers of ten. This method then depends on the fact that counting uses base ten and that multiplication is distributive over addition. Pr̥thūdaka considers the example 235×288 . In the commentary, 288 is the multiplier and 235 the multiplicand. He will compute $235 \times 288 = 235 (2 \cdot 10^2 + 8 \cdot 10^1 + 8 \cdot 10^0) = 470 \cdot 10^2 + 1880 \cdot 10^1 + 1880 \cdot 10^0 = 67680$. A reconstruction of the process is provided in Appendix B.

Three manuscripts record Pr̥thūdaka’s commentary on the mathematical chapter of the *Theoretical Treatise of the Corrected Br̥hma School*: Henry Thomas Colebrooke’s manuscript (I_1) which served as a basis for his 1817 translation, Dvivedi’s manuscript (V_1), which served for his 1902 edition of it, and a copy of Colebrooke’s manuscript (I_2).²⁷

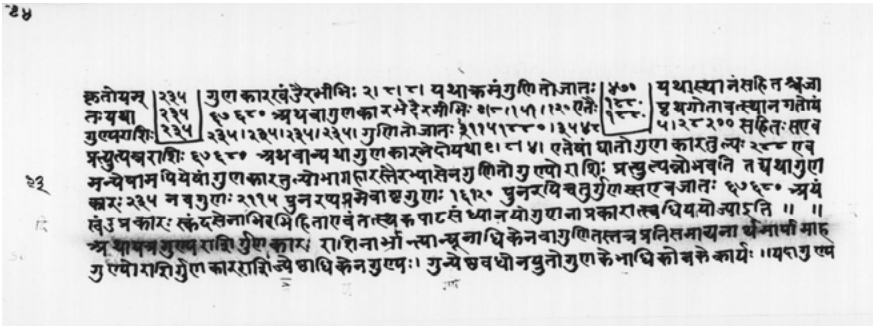


Fig. 2: I_1 , Colebrooke’s manuscript; London, British Library, IOSAN2769 = IOSAN1304, upper hand of fol. 178’. © British Library.

27 Setsuro Ikeyama very generously provided the copies of V_1 used here, while copies of I_1 and I_2 were made available by the funds of the Algo ANR.

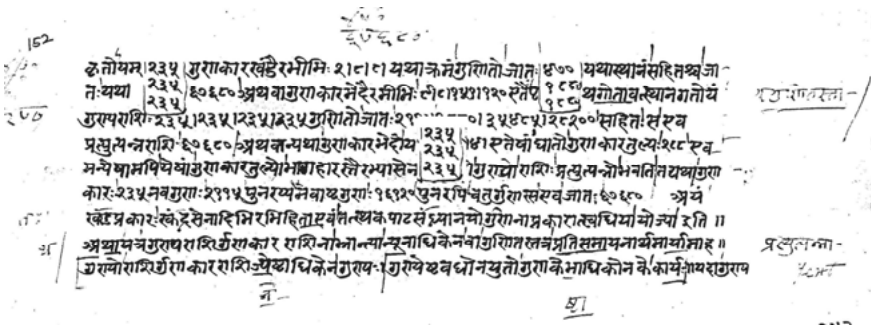


Fig. 3: V₁, Dvivedi's manuscript; Varansi, Sanskrit University Library (Sarasvatī Bhavana), 98256, upper part of fol. 52'.

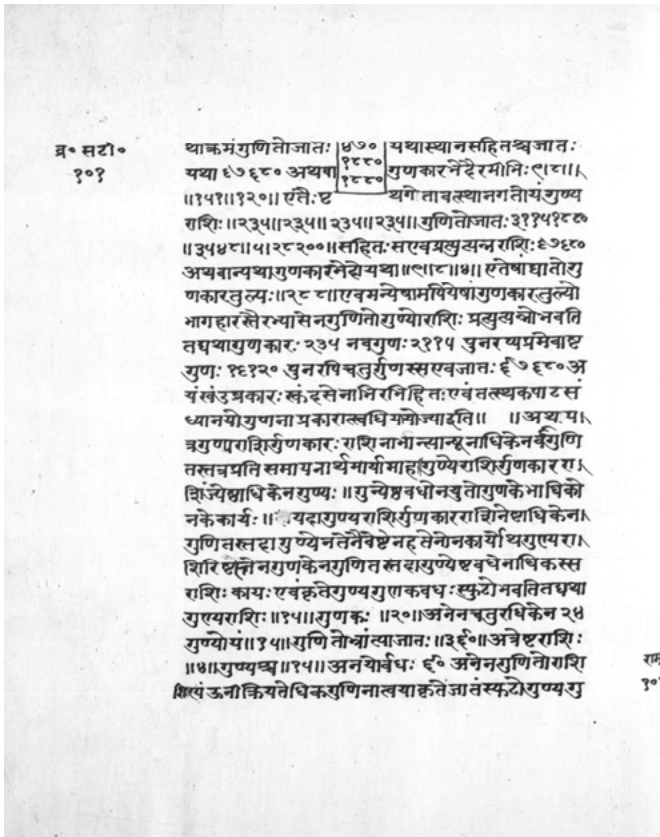


Fig. 4: I₂ a copy of I₁; London, British Library, IOSAN2770 = IOSAN2266. © British Library.

Such manuscripts are quite recent: they date from the end of the eighteenth century (I_1) and possibly the nineteenth century (I_2 , V_1), while Pṛthūdaka's commentary is probably from the ninth century. They are separated by roughly a thousand years.

Manuscripts display the multiplicand and the multiplier in separate places. The three manuscripts display the multiplicand of the example in a column, repeating it identically several times (as many times as there are digits in the multiplier), as shown in Fig. 5a–b:

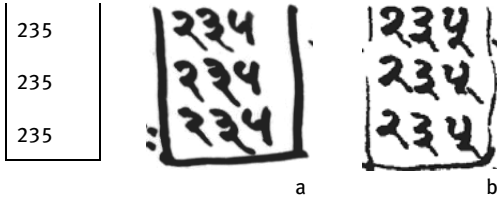


Fig. 5a–b: Multiplicand in a cow-string in I_1 when multiplying 235 by 288; London, British Library, IOSAN2769 = IOSAN1304, upper hand of fol. 178^r. © British Library (a); multiplicand in a cow-string in V_1 when multiplying 235 by 288; Varansi, Sanskrit University Library (Sarasvatī Bhavana), 98256, upper part of fol. 52^r (b).

A capsule is used to separate this display from discursive text. The verticality of the layout suggests that the working surface could be in a different space from the one in which the text itself was inscribed. However, it is also possible to imagine that the working surface was simply separated from the rest of the text because it was not meant to be read in the same order as the rest of the linear textual discourse. In such a case, the mathematical work to be carried out would have been made on the same medium of inscription as the text.

On the other hand, there is no ambiguity that the multiplier seems to be noted within the text. Like the multiplicand, the multiplier is noted using decimal place-value notation. 288 is noted as:

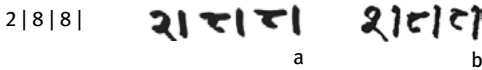


Fig. 6a–b: Multiplier into portions in I_1 when multiplying 235 by 288; London, British Library, IOSAN2769 = IOSAN1304, upper hand of fol. 178^r. © British Library (a); multiplier into portions in V_1 when multiplying 235 by 288; Varansi, Sanskrit University Library (Sarasvatī Bhavana), 98256, upper part of fol. 52^r (b).

Each digit is separated by a *daṇḍa* (the Sanskrit punctuation mark: |), in the order in which it would be noted without this separation (2 | 8 | 8 | is what is noted and not 200 | 80 | 8 |). The different powers of ten of the multiplier are laid out in a horizontal row, the same on which sentences of discursive language are made. The inscription appears as integrated in the sentences, not separated from them.

Therefore, the multiplicand seems either to be shaped on a working surface separate from the text – or at least distinguished from usual text – while the multiplier seems to be integrated within discursive text. The space and medium of the working surface in relation to the text that refers to it remains uncertain, paradoxical. There are reasons to believe that in the original text by Pṛthūdaka, representations of the working surface concerned a medium different from the one on which permanent text was written. Pṛthūdaka and the author he comments upon, Brahmagupta, both refer to ‘dust computations’ (*dhūli-karman*), suggesting that computations were performed either on a dust board, or simply on the bare ground; not on a palm leaf or bark on which manuscripts would have been inscribed in his lifetime.²⁸ But why then would the multiplier, on the other hand, be included in the text? Could this indicate that for those who copied the manuscripts in the late eighteenth century and onwards, the idea of a separate working surface represented an antiquated form and that in more modern times computations were usually integrated within the text itself? This is a hypothetical historical interpretation, but we might also be misled by thinking about material tools of computation. Perhaps that which is in a capsule represents computations to be committed to memory? We cannot answer these questions, of course, but they do show however that what is represented by encapsulated numerical tables remains undetermined, open to interpretation. Possibly, over time, for different authors and scribes it represented different things.

We might then want to retrieve what Pṛthūdaka tells us in the text about the space in which executions are carried out. But here the space of execution and place-value have the same name: *sthāna* (place/position). This indeterminacy of what place/position refers to, raises questions on the display of intermediary products during the execution of the multiplication.

Pṛthūdaka quite clearly states that places/positions (*sthāna*) are central to some steps of the execution:

²⁸ Wujastyk 2014, 166.

The product is [the multiplicand] multiplied respectively and separately by precisely those portions of the multiplier <and> added according to place (*yathā sthānam*).²⁹

Ambiguously, ‘places’ could designate the different rows of the column where the multiplicand is repeated or the positions in which the digits are written when noting down a number in decimal place-value notation. In the first case, Pṛthūdaka’s remarks concern the order in which the partial products are added; in the second, he refers to their relative values. In the reconstruction of the execution we have not been able to decide how, according to Pṛthūdaka, the multiplicand and then the partial products are laid out and then summed in the computation. If we consider that ‘place’ refers to place-value notation, it is possible to tentatively adopt an interpretation of the columnar display which is not found in the manuscripts. Each row could have been written one place to the right with respect to the previous one (as cows tied on a same string), placing the multipliers according to the value of the respective digit of the multiplier that it will be multiplied with:

2	3	5		
	2	3	5	
		2	3	5

However, in all manuscripts, the digits in the last line do not seem to be properly placed to carry out a column-by-column sum as we are used to.³⁰

²⁹ *tair eva gunakāra-khaṇḍaiḥ pṛthak pṛthag gunito yathā sthānām sahitaḥ pratyutpanno bhavaty.* The expression ‘according to place’ can be understood as referring to ‘multiplied’, ‘added’ or to both.

³⁰ The interpretation of the layouts in the manuscript is actually a bit tricky here. Although the numbers are not strictly aligned digit by digit, one can maybe read a diagonal of ‘zeros’ followed less clearly by a diagonal of ‘seven, eight, eight’. Such diagonals seem to appear in *I*₁. Strikingly enough, paleographically, the zero is usually a drawn circle as in the first row with 470 is followed for 1880 by what appears as not ‘zeros’ but simple points, which can be used in manuscripts to represent an empty space in tabular layouts. Here however it would be difficult to understand why empty spaces would be drawn for one space and not for the following ones, or not for the 470 above. They make sense if this is a way of anticipating the lack of space to draw them out in a proper diagonal, as in the other manuscripts. In the copy of *I*₁, *I*₂, such diagonals are less obvious and could have been ignored by the person who copied it. The problem might also be, as in *V*₁, a lack of space.

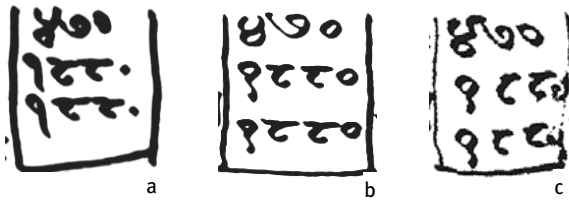


Fig. 7a–c: Partial products in I_1 when multiplying 235 by 288; London, British Library, IO-SAN2769 = IOSAN1304, upper hand of fol. 178^r. © British Library (a); partial products in I_2 when multiplying 235 by 288; London, British Library, IOSAN2770 = IOSAN2266. © British Library (b); partial Products in V_1 when multiplying 235 by 288; Varansi, Sanskrit University Library (Sarasvatī Bhavana), 98256, upper part of fol. 52^r (c).

On the other hand, if Pṛthudaka is not referring to place-value with the term ‘place’, then he is just noting that the different partial products of the multiplications are given in different rows which are ‘places’ from which a sum is then executed. Imagining that this encapsulated text refers to a gridded surface outside of the text might be a way of explaining the ambiguity: the ‘place’ would be both exterior to the text, but gridded according to place-value. Our uncertainty over the place and materiality of the working surface as represented by a capsule is linked to our uncertainty concerning whether resources of place-value notations are used in the sums of the partial products when considering this execution of a multiplication.³¹

We also note that, at the time of the execution, the order in which the partial products are computed and then summed is open to variation. This part of the execution is not exactly detailed in the commentary; the partial products are displayed altogether, and then their sum which gives the final product. Whether the partial products were made from top to bottom, bottom to top, or starting in the middle is not specified.

We also note that the execution as described by Pṛthūdaka is carried out in four distinct steps: (a) the shaping and/or displaying of the multiplicand, (b) the identification of the digits forming the multiplier, (c) the computation of the partial products, and then (d) their sum. This is not the order of execution. To properly set up the multiplicand (a) we need to already know in how many digits the multiplier is subdivided (b). In Pṛthūdaka’s commentary, (b) then appears as a justification of (a), and is not a step concerned with the execution of

³¹ All of these hesitations have been discussed in more detail in Keller and Morice-Singh 2022, 510–520, and also Chemla 2022, 107–109.

the multiplication. In other words, his description of the execution is intended not at having us reproduce it, but at explaining some of its elements.

Although many questions are raised by this analysis, let us retain two main ideas: first, that we need to better understand what encapsulated configurations of numbers represent for our authors and manuscript copiers; and second, in this case, that the order of the steps is not an issue for the commentator who describes the execution, for a degree of latitude is left to the executioner. The intention of the commentator, it seems, might be to explain to the reader some of the steps, not to guide the reader step-by-step in the execution.

In what follows, I would like to emphasise how commentaries can display different stages of the working surface while illustrating a same procedure: they may be illustrating possible differences of steps in the execution, but may also testify to different modes of execution. To do so, we will look at two different texts describing the same method of multiplication: precisely the ‘door-hinge’ method (*kavāṭa-* or *kapāṭa-* *sandhi*).³² Such a multiplication technique, known also to Pṛthūdaka, uses place-value and is described in numerous Sanskrit mathematical texts.

2.2 One procedure, two texts and many possible executions

The ‘door-hinge’ procedure for a multiplication relies on the decimal place-value notation and uses a dynamic layout. In such an algorithm, the multiplier is set above the multiplicand and made to move either from left to right, or from right to left, as the multiplicand’s digits disappear to be replaced by intermediate products until the initial multiplicand has vanished, yielding in its place the result. The process is quite consistently evoked in Sanskrit treatises devoted to mathematics.³³

We will look here at two different commentaries which detail the execution of a ‘door-hinge’ multiplication: the anonymous and undated commentary to Śrīdhara’s *Board Mathematics* (*Pāṭīgaṇita* – abbrev. PG) (c. 800), and an Old Gujarati commentary on an anonymous Sanskrit treatise, Śambhudāsa’s

³² We will see both names used in the different texts examined below.

³³ A reconstruction of two multiplications using this method is provided in Appendix C and Appendix D. Hayashi 2017, 58–59, notes the references of nine texts ranging from the ninth to the sixteenth century giving rules for such a procedure, under different names. He also notes that the name itself, ‘door-hinge’, becomes progressively attributed rather to the ‘lattice’ (also known as *gelosia*) method.

(1428/1429) *An Introductory Commentary on Arithmetics to Awaken the Young* (*Bālabodhāṅkavṛtti* – abbrev. BBA), on the *Twenty-five Rules* (*Pañcaviṁśatikā*).

2.2.1 A ‘door-hinge’ multiplication in a commentary on *Board Mathematics*

Śrīdhara, probably a near contemporary of Pṛthūdaka, details in *Board Mathematics* (*Pāṭiḡaṇita*) some of the ways to execute a multiplication. The text is known in a single incomplete manuscript, with an anonymous undated commentary. It was discovered by Avadhesh Narayan Singh (1901–1954) in the Raghunātha Temple Library in Jammu.³⁴ He obtained a copy of it from which an edition was prepared in 1959 by Kripa Shankar Shukla (1918–2007).³⁵

The rule which considers a ‘door-hinge method’ (*kavāṭa-sandhi-krama*), runs as follows:³⁶

PG.18

Having placed the multiplicand (*guṇya*) below the multiplier quantity (*guṇarāśi*), according to the ‘door-hinge’ method (*kavāṭa-sandhi-krama*), one should multiply by going indirectly (*viloma-gati*) or in a direct (*anuloma-mārga*) way, step by step.

PG.19

Having shifted again and again thus should be the door-hinge. This procedure (*karaṇa*) when it (the multiplier) is stationary is therefore a multiplication ‘as it stands’ (*tat-stha*).

Śrīdhara provides under the same name what appears during execution as two different multiplication procedures. With the multiplier above the multiplicand, the ‘direct way’ (*anuloma-mārga*) involves a multiplier moving from right to left, from the multiplicand’s digit for the smallest power of ten to the multiplicand’s digit for the highest power of ten. A reconstitution of this procedure for the computation of 1296×21 is provided in Appendix C. ‘Going indirectly’ (*viloma-gati*) involves a multiplier moving from left to right, from the multiplicand’s digit for the highest power of ten to the multiplicand’s digit for the lowest power

³⁴ Specifically Raghunātha Temple Library, Manuscript 3074 (Gha Alm 14 Shlf 1). At the time of writing (April 2023), a digital copy can be found at archive.org: <<https://archive.org/details/PatiiGanitaWithTikaSridharacharya3074GhaAlm14Shlf1DevanagariJyotish>>.

³⁵ Shukla 1959, i–ii.

³⁶ ‘PG.18 *vinyasyādho guṇyaṃ kavāṭa-sandhi-krameṇa guṇa-rāśeḥ| guṇayed viloma-gatyā ’nuloma-mārgeṇa vā kramaśaḥ ||18||* PG.19. *utsāryotsārya tataḥ kavāṭa-sandhir bhaved idaṃ karaṇam| tasmimś tiṣṭhati yasmāt pratyutpannas tatas tatsthaḥ ||19||*’. For the Sanskrit edition, see Shukla 1959, 12–13; for the English translation, see Shukla 1959, 9.

of ten. However, the multiplier need not be moved; when it is motionless, the name of the execution changes to ‘as it stands’ (*tat-stha*).

The position and regular shifting of the multiplier obviously have an important role in these methods of execution. When the multiplier moves, the position of the highest digit of the multiplier indicates the digit of the multiplicand below it on which an intermediary multiplication has to be performed.

The procedure to multiply 1296×21 in the ‘direct way’ is precisely the one whose execution is detailed in the edited commentary exhibited in Fig. 1. As edited, the displays represent different steps of the execution of the multiplication up to the numerical result. They all seem to belong to the same space as the written text of the commentary. Furthermore, the commentary seems to guide us through the different intermediary steps of the process, as if to help the reader – and anyone trying to reproduce the process. It provides little images of what the working surface should look like each time the multiplicand slides one step to the right. It sometimes even provides positions just before the sliding. But a close look at the edition and at its conventions can help us to see that this is actually an illusion. As shown in Fig. 9, even if you do not read Sanskrit, you can see that all the text between parentheses and highlighted in grey represents the editor’s additions to the commentary.³⁷

Representations of the working surface for this example in the actual manuscript of the commentary are reduced to two encapsulated dispositions, which gives an idea of the process in its middle, as seen in Fig. 10 (and explained in Appendix C).³⁸

³⁷ The editor presents the last display as a reconstruction in between parenthesis but the last display is found in the manuscript as shown below.

³⁸ The publication is free of copyright, and available here <<https://archive.org/details/PatiiGanitaWithTikaSridharacharya3074GhaAlm14Shlf1DevanagariJyotish>>.

विन्यस्याधो गुण्यं कवाटसन्धिक्रमेण गुणराशेः ।
 गुण्येद्विलोमगत्याऽनुलोममार्गेण वा क्रमशः ॥ १८ ॥
 उत्साय्योत्सार्य्य ततः कवाटसन्धिर्भवेदिवं करणम् ।
 तस्मिंस्तिष्ठति यस्मात्प्रत्युत्पन्नस्ततस्तत्स्थः ॥ १९ ॥
 रूपस्थानविभागाद् द्विधा भवेत्खण्डसंज्ञकं करणम् ।
 प्रत्युत्पन्नविधाने करणान्येतानि चत्वारि ॥ २० ॥

(उदाहरणानि—
 पण्यवतिद्विक्रमेकं चैकद्विगुणानि पण्यवाष्टौ च ।
 सप्तत्रिगुणान् पंचकषट्खाष्टौ च कुरु षष्टिगुणान् ॥ ३ ॥)

प्रतिरूपमुत्पन्नो राशिरुद्धिरूपवृन्दस्य कियान् स्यादिति गुरगुण्य (योरेकविंशति-
 पण्यवत्यधिकशतद्वादशकयोः कवाटसन्धिक्रमेण न्यासः— १२९३१ । एकस्थानस्य षट्कं रूपेण
 गुणितं षट् इति एकाधःस्थाने षट्, ततः द्विकेन गुणिते षट्के द्वादश इति द्विकाधःस्थाने द्वौ
 रूपमपि नवानामधः जातम् । न्यासः १२९३१ । ततो दशस्थानस्य नवकं गुणयितुं सर्पति
 गुणराशिः । न्यासः १२९३१ । इदानीं नवानामेकविंशतेदच गुण्यगुणकभावो जातः, रूपेण
 गुणितं नवकं नव, स्वाधःस्थितद्विकयोगात् तत्स्थाने रूपं जायते, रूपमपि द्विकाधःस्थितरूपेण
 युज्यते द्वे भवतः ; द्वाभ्यां गुणिते नवके अष्टादश) 'पूर्ववदेव तदधो न्यासः, अष्टमु च
 स्वाधःस्थितद्विकयोगे तत्स्थानं शून्यं, रूपमपि द्विकाधःस्थितं रूपेण युज्यते द्वे भवतः ।
 ततश्च शतस्थानं' द्विकं गुणयितुं सर्पति गुणराशिः । स्थापनम् १३०१६ । इदानीं
 द्वयोरेकविंशतेदच गुण्यगुणकभावो जातः, एकगुणितौ द्वौ द्वावेव, एकाधःस्ये (शून्ये)
 द्विकं क्षिप्त्वा जातौ द्वौ, द्वाभ्यां च द्वौ गुणितौ चत्वारः, स्वाधःस्थितद्विकयोगात् षट्
 (१३३१६) । ततः सहस्रस्थानस्य रूपं गुणयितुं सर्पति गुणराशिः । न्यासः (१३३१६)
 इदानीमेकस्यैकविंशतेदच गुण्यगुणकभावो जातः, तदा रूपेण गुणितं रूपं रूपमेव, षट्मु
 क्षिप्तं सप्त, द्वाभ्यामेकं गुणितं द्वाविति । निःशेषिते गुण्यराशौ, गुणके निवृत्ते, फलं
 तदेव २७२१६ । एवं रूपविभागे यत्परिमाणविभागस्थानानि^१ तानि पृथक् पृथक् गुणकेन

Fig. 8: The editor Shukla’s additions to complete the commentary on *Board Mathematics*: example of the multiplication of 1296×21 with a ‘door-hinge’ process in the direct way.

प्राप्ते पूर्ववदेव तदधो न्यासः अष्टस्रवत्त्वाधःस्थितद्विकयोगे तत्स्थानं शून्यं रूपमपि द्विकाधःस्थितं रूपेण युज्यते द्वे भव
 तः ततश्च नवस्थानस्य द्विकं गुणयितुं सर्पति गुणराशिः स्थापनम् १३०१६ इदानीं द्वादशस्थानमेकविंशतेदच गुण्यगु
 णकभावो जातः एकगुणितौ द्वौ द्वावेव एकाधःस्य द्विकं क्षिप्त्वा जातौ द्वौ, द्वाभ्यां च द्वौ गुणितौ चत्वारः स्वा
 धःस्थितद्विकयोगात् षट् ततः सहस्रस्थानस्य रूपं गुणयितुं सर्पति गुणराशिः न्यासः १३३१६ इदानीमेकस्य
 एकविंशतेदच गुण्यगुणकभावो जातः तदा रूपेण गुणितं रूपं रूपमेव षट्मु क्षिप्तं सप्त द्वाभ्यामेकं द्विगुण
 तं द्वाविति निःशेषिते गुण्यराशौ गुणके निवृत्ते फलं तदेव २७२१६ एवं रूपविभागे यत्परिमाणविभागस्थानानि

Fig. 9: The two intermediary configurations in the ‘door-hinge’ process when computing 1296×21 in the direct way in the commentary on *Board Mathematics*; Raghunātha Temple Library, Manuscript 3074 (Gha Alm 14 Shlf 1), fol. 12^v.

This representation displays how the shifting of the multiplier as well as partial sums and partial products can be dealt with. Here, the first representation is enough to tell us that the digits of the multiplicand are progressively replaced by those of the result during the multiplication. It shows us that carry-overs are placed below the multiplicand, next to the position they will be added to.³⁹ Carry-overs are stored on the working surface temporarily; before the multiplier moves from one position to another, previous carry-overs are integrated into the partial result. No previous carry-overs are left visible on the working surface. The second representation shows how the multiplier moves from right to left over the modified multiplicand displaying a partial result. The result itself is finally given within the text: it is not encapsulated and we do not know if in the end the multiplier is erased. These two only preserved windows onto the working surface are enough to tell us that the aim of the commentator is not to provide a step-by-step reconstruction of the performance, but just to indicate how carry-overs and intermediate products could be treated. Such displays suppose in a way that one has already tried to carry-out the execution and has some questions about the performance. Other executions of the same procedure are possible, as we will now see.

2.2.2 A ‘door-hinge’ multiplication detailed by Śambhudāsa

Śambhudāsa’s (1428/1429) Old Gujarati commentary *An Introductory Commentary on Arithmetics to Awaken the Young (Bālabodhāñkavṛtti* – abbrev. BBA) on an anonymous Sanskrit treatise, the *Twenty-five Rules (Pañcaviṃśatikā)*, has been edited, translated, and commented upon by Hayashi.⁴⁰ The edition has used three manuscripts, one of which (manuscript B) is dated to 1428/1429.⁴¹ Verses 4 and 5 of the treatise that it comments on evokes the ‘door-hinge’ multiplication.⁴² The method itself is specified in verse 5. The commentary on the process uses as an

³⁹ The editor Shukla places the carry-overs immediately below this position. The configuration presented in the manuscript might be approximative, but the carry-over seems to be below and slightly to the right of this position. This point is also discussed in Appendix C.

⁴⁰ Hayashi 2017.

⁴¹ Hayashi 2017, 5–6.

⁴² PV.4. *dvidhā kapāṭasaṃdhiś ca tathā gomūtrikā dvidhā| tatstho dvidhā punaḥ proktas tathā ṣaḍstridhā smṛtaḥ||*. For the English translation, see Hayashi 2017, 57: ‘There are two kinds of “door-hinges” (*kapāṭasandhi*). Likewise, there are two kinds of “zig-zags” (*gomūtrikā*). “As it stands” (*tat-stha*) has also been declared to be of two kinds, and “portions” (*khaṇḍa*) has been laid down as being of three kinds’.

example related to verse 4, the product 18×1196 . This example is detailed in Appendix D. The Sanskrit edition of the text is presented in Fig. 10.

There are several small differences with the method as we know it through Śrīdhāra's text. The inverse order here corresponds to Śrīdhāra's direct order, and what stands as a 'multiplier' (*guṇakāra*) in *Board Mathematics* is called here a 'price' (*mūlya*), while the 'multiplicand' (*guṇya*) is called 'the question' (*praśna*), which Hayashi translates as '<term in> question'. The multiplication is thus thought of here as the operation of that which determines the price of things.

प्रथमोदा उदाहरणं ।¹ रूपा गदीयाणा एक सहस्र एक सु छन्नु । प्रति द्रम्मा १८
अदार । किं फलं भवति । न्यासः । कपाटसंधि अनुलोमगति

१	८
१	१
९	६
७	४

गुणने रूपं

१	८	८	२	८
	१	९	६	
		७	४	

लब्धं द्रम्माः २१५२८ तथा कपाटसंधि विलोमगतिः

१	१	९	६	८
	८	७	४	
		८	२	

गुणने रूपं

१	१	९	६	८
	८	७	४	
		८	२	

लब्धं द्रम्माः २१५२८ ॥

¹उदाहरणं] उदारणं B(with *ha* in margin).

²

१	१	८
१	१	९
	९	६
	७	४

B.

³

१	८	८	२	८
	१	९	६	
		७	४	

B.

⁴

१	८
१	१
९	६
७	४

B.

⁵

१	१	९	६	८
	८	७	४	
		८	२	

B, which places this box between *tathā gomūtri* and *kā* two lines below.

⁶B places this box between *gomūtrikā* and *anulomagati* in the next sentence.

Fig. 10: Multiplication of 18×1196 with a 'door-hinge' in 'inverse order' and in 'direct order' in Śāmbudhāsa's commentary; Hayashi 2017, 16.

As seen in Fig. 10, both the initial disposition and the penultimate steps are presented in the manuscript for the multiplication of 18×1196 , for both the direct (line 4) and indirect way (lines 2–3). They both lead to the result, 21528. In Hayashi’s edition, the two tabular displays of the multiplication’s first and penultimate steps, *and the result*, are given in a capsule. In contrast with what we have seen with the manuscripts of Pṛthūdaka’s commentary dealing with a multiplicand shaped as a ‘cow’s string’, here the capsule seems to separate the numbers from the text, showing that they are in different separate spaces. This suggests that for the scribe at least numbers were made to be inscribed in a space different to that of the running text.

The procedure also presents some difference compared to the one in the anonymous commentary on *Board Mathematics*. As we are given to see the configuration when the multiplier is in its penultimate place, all the intermediary products are preserved diagonally over three lines. In the anonymous commentary on *Board Mathematics*, intermediate products are summed into the partial result at each step, and the result appears progressively. It is on the working surface when the last partial product is incorporated into it. In the display given in Śambhudāsa’s commentary, the partial products are each set down, and the last step involves their summing which will yield the result, the product.⁴³

We see that while documenting what the working surface might look like, commentators can choose the moment of the execution they want to display. In all cases, however – and contrary to the assumption for instance of Shukla in his edition of *Board Mathematics* – commentators do not aim to take us through each modification of the working surface, step by step. Furthermore, we have seen that to perform a ‘door-hinge’ multiplication, there was some possible latitude in intermediary steps.⁴⁴ This shows once again the treatise’s choice of spelled out steps. Thus, in an execution there seems to be at least two kinds of steps: the structurally important ones, which are spelled out, and those which are less important for the authors, which are not made explicit, although they might be crucial to perform the algorithm. Authors and their commentators *both* operate choices. The choices they make highlight the fact that they aim at mak-

⁴³ We can see that manuscript B, carefully reproduced by Hayashi, somewhat like the manuscripts of Pṛthūdaka’s commentary, does not seem to follow the spacing carefully. For this manuscript the emphasis might be on the intermediate products and sums rather than on their dispositions.

⁴⁴ Such latitude is well known and was discussed in detail with some references to specific manuscripts by Datta and Singh 1935, 135–143.

ing general statements. They spell out steps that are to be executed in all cases, rather than considering specific ones, necessary for a particular resolution.

But one may wonder whether between the anonymous commentator of *Board Mathematics* and Śambhudāsa's commentary there is not also a difference in the material used to carry out the process. *Board Mathematics* documents a process in which intermediary steps are progressively erased, as on a 'dust board'. This is not the case of Śambhudāsa's process, which is hybrid. Whereas the digits of the multiplicand are progressively erased, as when we use a dust board, the intermediary steps are preserved, just like when we write on paper.

Moreover, for whoever has tried to reproduce a 'door-hinge' multiplication, the constant movement of the multiplier over the multiplicand raises the question of the material with which the execution is to be carried out. Whether on paper or on a dust-board, the constant re-writing (and erasing) of the multiplier is indeed tedious. It would be much easier to either leave the multiplier alone (but then we are not technically in a 'door-hinge' multiplication anymore) or to execute the multiplication on a gridded space using valued tokens that could be moved around, rather than dust or chalk on a board, or pencil and ink on paper.

What then are the clues we have on the kind of material with which computations could have been performed? This is what we now explore.

3 Performing with what?

Most historians of Indian mathematics today consider that computations were carried out in arithmetics on a medium on which the intermediary steps of a procedure or a computation can be erased. This would either be the bare ground or a board using either dust or anything that might provide colour on the board and be easily erased: soapstone, coloured powder and, in more recent instances, chalk.

3.1 Historiography of the computational board

Most probably this consensus comes from the seminal study by Datta in 1928, synthesised in the reference manual he published jointly with Singh in 1935.⁴⁵ The term *pāṭiganīta*, translated by him as 'science of the calculation with a board', designates a sub-discipline of mathematics dealing with arithmetics but

⁴⁵ Datta 1928 and Datta and Singh 1935, vol. 1, Chap. II, 123–127. Datta 1928, 521, evokes notably yellow or white sandstone, ink and chalk.

also including elements of plane geometry and other specific topics. The board in question would be the one on which computations were carried out. Datta's text is partly a discussion of the philology and possible etymology of the *pāṭi*, translated with some difficulty as 'board'. Datta admits that the term is quite late, mentions that it is used for board in vernacular languages of India, and therefore thinks that it is a degraded form of the usual word for 'plank' or 'board' (*paṭṭa* and *phalaka*). He also shows that by the sixteenth century the word was taken to mean rather 'in succession'. In that case it refers to positional computations rather than to the media on which computations using positional notations could be performed.⁴⁶ It remains uncertain whether the term itself refers to a kind of (positional?) computation or the media on which computations were carried out. Most probably, the meaning of the term changed over time. Nonetheless, textual evidence of writing boards from the early centuries before the common era, as well as the use of a board by Bengali astrologers in his own day, convinced Datta and many of his readers that it was a very ancient practice that was continuous throughout the subcontinent. Datta noted (followed in this respect by the testimonies gathered by Sreeramula Rajeswara Sarma) that the board was used either with dust and a style, or with a kind of tool which could impart colour to what was often a blackened surface, be it a tablet or even in later times a sheet of paper. This would have been the medium on which 'dust work' (*dhūlikarma*) was carried out. As we have seen above, Brahmagupta in the seventh century and his commentator Pṛthūdhaka in the ninth or tenth century used these expressions in their writings about computations.⁴⁷ More extensive and historical research on the evocation of dust work in computations is no doubt needed; for now, this is the only early testimony of such explicit dust computations I know of.⁴⁸ Datta explicitly stated that the medium for writing ephemera and the medium for setting down complicated computations had to be the same.

Such a point of view involved setting aside other, early testimonies that were more fragmentary and difficult to interpret, suggesting that other objects could have been used for computing. The classic of Buddhist philosophy, Vasubandhu's *Treasury of Buddhist Philosophy (Abhidharmakośabhāṣya c. fourth or fifth century)*,

⁴⁶ Datta 1928, 521–524, 526. For information on writing material, and the use of a board or a plank for ephemeral writing, see also Sarma 1985. Hayashi 2014 chose to translate the term *pāṭi* as 'algorithm'.

⁴⁷ See notably, as suggested by Datta 1928, 522, n. 1, BSS.10.62, 66, 67, Dvivedin 1902, 143–145.

⁴⁸ Datta also mentions *Siddhāntaśiromāṇi, yantrādhyāya* 24, but in the absence of a critical edition, reference seems to be to a *paṭṭikā*, i.e. a board, rather than to dust work (Śāstri and Wilkinson 1861, 215).

refers to coloured counters (*varṭikā*), whose value change according to place. Other philosophical Buddhist compendiums, which might have been composed earlier than this text, refer to the same argument attributed to Vasumitra.⁴⁹ In mathematics, Āryabhaṭa (b. 476) calls an algebraic unknown a ‘bead’ (*gulikā*), while in the twelfth century, in Bhāskara II’s *Algebra (Bijagaṇita)*, when several unknowns are to be considered, they are named after different colours, bringing to mind the dual possibility of drawing/painting colours on a working surface or using coloured beads, seeds or shells to make computations. All of these are but vague allusions. We note however that certain objects were likely candidates to be used as common computational tools. We might imagine for instance that seeds and grains – which were sometimes also used as beads – could have such a function, since grains as units of capacity and weight are evoked throughout Sanskrit mathematical texts. Another candidate, referred to in mathematical texts although here again not as a tool for computation but as the coin, is the *cauri* (or *cowrie*) shell.⁵⁰ The use of *cauri* shells has been attested from some of the earliest archaeological excavations in South Asia. Imported mostly from the Maldives islands, they are known to have been used over a very long period of time as a multifarious object: game token, dice, money, symbol of fertility, or expensive decorative jewel, all together.⁵¹ It is noteworthy that *cauris* are currently found across all the sea routes of Asia, the Middle-East and even Europe, in areas in which we know that place value computations also travelled.⁵² Testimonies going back to the eighteenth century attest to the use of *cauris* by as-

49 Ruegg 1978, 172–175; Bronkhorst 1994, 1041; Hayashi 2001. The same image is drawn up in the seventh century, in Bhāskara I’s commentary on Āryabhaṭa’s definition of the decimal place value notation. The commentator argues in favour of the notation, evoking the fact that, where many units/shapes (*rūpa*) might be used to state a quantity, the decimal place-value notation enables such quantities to be stated with less unit/shapes. The term *rūpa* however is ambiguous, and could refer equally to ‘shapes’ as to ‘units’ noted with symbols. Shukla 1976, 46, discussed and translated in Keller 2006, vol. 1, 11–12.

50 In South Asia, *cauris* have been documented as a kind of money – small change really in contrast with metallic money which had more value – essentially in eastern India specifically in the Bay of Bengal. In eighth- to twelfth-century Bengal, metallic money seems to have been a kind of account money with which computations were made while payments were made in cowrie shells. In this context, cowries were sometimes considered as ‘broken’ money (*cūṃṃi*), together with other ‘dust money’, e.g. dusts or small amounts of gold and silver used as lesser change than metallic coins. Majumdar and Chatterjee 2014, 49, quoting Mukherjee 1993, 5, 9, 14, and 54; Yang 2018, 46–47, further develops the argument.

51 Majumdar and Chatterjee 2014, 39; Yang 2018, 41–42.

52 Heimann 1980, quoted by Majumdar and Chatterjee 2014, 47; Yang 2018, 1–2.

trologers as tools for computations.⁵³ Even today, ‘traditional’ astrologers are known to compute with cauris. There is a growing ethnography of these practices, particularly in the form of filmed documents of astrologers performing divination and computations. To my knowledge, none of these have been published yet.⁵⁴

It is thus possible to imagine the use of valued tokens of sorts for computing, and notably for executing multiplications, although the texts do not detail such media. Note that computing with such tokens might not require a board.

3.2 Grids and cloths

Computations with place value notations require a grid, some kind of ephemeral table, whether it is explicitly drawn or not.⁵⁵ Sanskrit mathematical commentaries are sometimes intent on spelling out the importance of such grids, especially when they help explain the rationale of an operation. Ephemeral tables for many other types of procedures are known to have been used in arithmetical computations.⁵⁶ The material culture of archaeology shows that grids could be found in multiple spaces and on multiple materials. Many traditional Indian cloths are known to contain grids.⁵⁷ We might immediately think of the grids of the cotton plaids known as *madrās* derived from south Indian *lungis* and *dhotis*⁵⁸. But, more broadly, weaving – which considers the making of cloth as a grid – is attested in South Asia since prehistoric times.⁵⁹ Another pervasive grid of the

⁵³ See for instance Playfair 1790, 139.

⁵⁴ Thanks to T. P. Radhakrishnan, Professor of Sanskrit at Pondicherry University and practising astrologer, Sho Hirose (Tokyo), who filmed him, and Senthil Babu (Institut français de Pondichéry, Puducherry), who was the intermediary between them, I have had access to a film made in 2019 in Puducherry in which Radhakrishnan demonstrates to Hirose how he computes with cauri shells. He notably demonstrates a ‘door-hinge’ computation for a sexagesimal computation using cauri shells: as many cauri shells as units. We can see him using the grids used to represent the different cells of a horoscope as a grid for computation. He also uses a big grey shell to represent the number 5 and sometimes 0. Radhakrishnan’s practices gives us an example of how a ‘door-hinge’ multiplication is applied today by somebody familiar with and from within the live culture of Sanskrit mathematical and astral scholarly lore.

⁵⁵ Keller, Montelle and Koolakudlu forthcoming; Keller 2015b.

⁵⁶ Keller 2015b.

⁵⁷ The terms *pāṭikā* and *pāṭalikā* are attested to mean ‘cloth’ in Pali, according to Davids and Stede 1921–1925, https://dsal.uchicago.edu/cgi-bin/app/pali_query.py?qs=pa%E1%B9%A%ADalika&searchhws=yes&matchtype=exact (accessed in July 2023).

⁵⁸ Muthian Vasantha 2016, 336–338.

⁵⁹ Fuller 2008.

material culture of South Asia are the board games that can be found through South Asian archaeological sites, from Kashmir to South India, engraved in stone.⁶⁰ Such board games contain grids that could be used with valued tokens such as seeds, stones or cauris.⁶¹ Furthermore, the boards on which games were played were known to have rarely been made of solid material; cloth, leather and other material easy to transport were known to have been used as well.⁶²

The objective of this digression is not to decide, author by author, commentator by commentator, what media would or could have been used, when and how. The aim is simply to point out that we need not assume that over time and in all of South Asia a same tool – a board also used for ephemeral writing – was used to carry out computations. We certainly need to investigate local material cultures both synchronically and diachronically, to identify objects that might testify differently, from south to west, east to north, to different mediums for writing, computing, and playing.⁶³ The same material might have been used at times to carry out these three activities, and at others not. Even though the possible and attested material objects that could have been used for computation have to be investigated more thoroughly, let us nevertheless come back to the performance of multiplication to see how imagining other media helps us look with new eyes at the performances staged in texts in manuscripts.

3.3 Historicising working surfaces represented in manuscripts

Observation of contemporary astrologers, live experiments with children, and so on are no substitutes for documents of the past. We cannot reconstruct the past from the present, nor can we do so by putting side by side reconstructed computations and fragmentary elements of past material artefacts. These experiments can nevertheless serve as a backdrop to widen our perspectives. It is thus striking that the multiplication procedures discussed in this paper can also be carried out, quite easily, with some kind of valued tokens, which could slide on

⁶⁰ Bhatta 1995; Topsfield 2006, 19, Fig 8; Fritz and Gibson 2007; Soar 2007 (notable Figs 22.2 and 22.3).

⁶¹ Topsfield 2006, 18; Fritz and Gibson 2007, 112; Jaffer 2006, 129, evokes Satyajit Ray's *Chess Players* when Mirza, after his wife had hidden his chessmen, replaced them by different vegetables – illustrating the versatility of the kinds of objects that could be used as tokens.

⁶² Topsfield 2006, 19, 22, Fig. 12; Jaffer 2006, 129, 141.

⁶³ For an example of (a nineteenth-century) board game inscribed with magic squares, see Rangachar Vasantha 2006, 149–151, Fig. 9.

a gridded surface and be replaced by others.⁶⁴ The possibilities just tell us that we need not always be attached specifically to the board, nor to the dust.

There is however something more. We may be struck by a feature of the displays of working surfaces in manuscripts. I have suggested above that Sanskrit and even Old Gujarati commentaries of Sanskrit mathematical texts make general statements about the execution of the multiplications they evoke. The representations of working surfaces found in manuscripts seem, in the same way, to depict some or all the possible media used. Capsules might be a generic way of displaying many possible working surfaces and the different kinds of numerical inscriptions they could involve. As we rethink the execution of multiplications seen above – whether we imagine the execution on paper, on a cloth or a board, with dust, coloured brushes or cauris – the representation of the working surface found in manuscripts represents each one equally and adequately. The multiplicand in a ‘cow’s string’ in Pṛthūdaka’s commentary could be either on a dust-board or on a gridded game-board using seeds. The seemingly hybrid medium of the displayed steps of the ‘door-hinge’ multiplication, as found in the manuscripts of Śambhudāsa’s commentary, might be an intentional inclusive mode of representation of different media, different possible ways of executing the multiplication.

It is striking that capsules are found in the earliest mathematical manuscripts that have come down to us and are still in use in the most recent Sanskrit mathematical manuscripts we have studied – some of which were probably written in the first half of the twentieth century. That capsules may be a generic way of representing all working surfaces is but a first approximative hypothesis. Looking back at manuscripts, and at how and whether they really encapsulate ephemeral data such as numerical tables and diagrams, reveals a more complex set of inscriptions. Thus, for instance, we have seen that the manuscript of Pṛthūdaka’s commentary contains encapsulated and unencapsulated numerical configurations. This is the case of manuscripts we have looked at, which contain a truly wide diversity of ways of displaying configurations of numbers

⁶⁴ Experimenting with French high-school children has shown that using tokens like cauris seems easier and appears more ‘natural’ in practice than paper or a kind of erasable slate. Between 2016 and 2019, in a team with Charlotte de Varent, Matthieu Husson and Barbara Jamin, we had high-school students learn different techniques to execute multiplications as found in Sanskrit texts. Among the questions debated with them, was that of the materials with which the multiplications could have been carried out: paper, dust, mentally, with tokens? The ‘door-hinge’ execution was taught directly with tokens, but no one suggested that it could have been used on paper or dust. On the other hand, several students did suggest that other techniques could be used with tokens too, notably the multiplicand in a ‘cow’s string’.

in or out of capsules, in gridded tables or as part of discursive texts. In other words, we need to go back more carefully to our manuscripts to look at the different ways in which they represent what would have been a working surface. How generic are these representations? How do they relate to discursive text? What do these diverse ways of representing configurations testify to? More largely, do manuscripts aim at staging many different types of execution?

4 Conclusion: back to ideas of performance in relation to the reconstruction of algorithms

Trying to perform multiplications as closely as possible to those described in manuscripts thus raises questions concerning the media through which operations were carried out. It also underlines how the texts and the manuscripts speak differently about performance, each choosing the image of the continuous performance they display and comment on. As we wonder what the aim of Sanskrit mathematical texts was regarding the execution of algorithms, we note that texts seem to point to an idea about the process, its correctness, and its most salient steps, rather than being a normative statement on each of its steps. Moreover, by not focusing on the executions, authors might have been intent on making statements that apply to different cases: different ways of noting numbers at times, and possibly different material tools to execute a multiplication.

Perhaps there is a voluntary silence about the tools. Using material culture to imagine them makes us realise that this material culture could have changed but the algorithms still make sense. The notation used to represent the working surface can therefore be thought of as being general and generic – in the same way that Sanskrit mathematical *sūtras* sought to make general procedures that could apply to many different cases, and have many different interpretations.

One might argue then that only general or generic elements of the executions are highlighted in treatises and their commentaries, and that the rest may have been left to everything that makes up a performance.

At the end of the verse commentary in which Pṛthūdaka comments on the multiplication shaped as a ‘cow-string’, he notes:⁶⁵ ‘In the same way, methods of multiplication such as “as it stands” (*tat-stha*) and “door-hinges” (*kapāṭa-sandhi*), should be used ingeniously’.⁶⁶

65 *evaṃ tatstha-kapāṭasandhy-ādayo guṇaṇā-prakārās svādhiyā yojya iti*].

66 Keller and Morice-Singh 2022, 542–544, with emendations suggested by Hayashi.

Ingenuity (here *svādhī*, lit. ‘well thought’) was indeed probably the most widespread quality that mathematical authors of Sanskrit commentaries associated with the practice of mathematics and the execution of algorithms. Executions could thus be spelled out to explain something about them, but not necessarily to provide the ingenious know-how which might also be required to perform the execution.

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Appendix A: Texts, editions and abbreviations

Table 1: Texts, editions and abbreviations.

Author	Date	Title	Genre	Abbrevs	Edition
Āryabhaṭa	b. 476	<i>Āryabhaṭīya</i>	Treatise	Ābh	Shukla and Sarma 1976
Brahmagupta	b. 598	<i>Brāhmasphuṭasiddhānta</i> (<i>Theoretical Treatise of the Corrected Brāhma School</i>)	Treatise	BSS	Dvivedin 1902
Bhāskara	written 629	<i>Āryabhaṭīyabhāṣya</i> (<i>Commentary on the Āryabhaṭīya</i>)	Commentary	BAB	Shukla 1976
Śrīdhara	c. 800	<i>Pāṭīgaṇita</i> (<i>Board Mathematics</i>)	Treatise	PG	Shukla 1959
Pṛthūdaka	fl. c. 850	<i>Vāsanābhāṣya</i> (<i>Commentary with Explanations</i>)	Commentary	PBSS	Ikeyama 2003
Unknown	written before 1429	<i>Pañcaviṃśatikā</i> (<i>Twenty-five Rules</i>)	Treatise	PV	Hayashi 1991
Śambhudāsa	written 1428/1429	<i>Bālabodhāṅkavṛtti</i> (<i>An Introductory Commentary on Arithmetics to Awaken the Young</i>)	Commentary	BBA	Hayashi 2017
Unknown			Commentary	APG	Shukla 1959

Appendix B: Reconstructing a multiplication with a multiplicand shaped in a ‘cow’s string’

The different steps of the execution as understood by Pṛthūdaka⁶⁷ can be reconstructed as follows:

Step 1: The multiplicand, made into a ‘cow’s string’, is equal in portions to the multiplier. The number of digits forming the multiplier determines the number of

⁶⁷ Justified in Keller and Morice-Singh 2022.

times the multiplicand is noted in a column. In the case where 235 is multiplied by 288, since 288 is made of three digits, 235 is noted three times in a tabular format. The three manuscripts concur that this refers to a display in a column:

235
235
235

Another possible interpretation could be to understand the layout with each row written one place to the right with respect to the previous one, placing the multipliers according to the value of the respective digit of the multiplier that it will be multiplied with:

2	3	5	
	2	3	5
		2	3 5

Step 2: It is multiplied in due order by the portions of the multiplier one after the other. One by one, each digit of the multiplier multiplies one of the noted multiplicands in the column. Thus, with Pṛthūdaka’s example, $2 \times 235 = 470$ and $8 \times 235 = 1880$. In manuscripts it is difficult to discern whether this layout appears as:

470
1880
1880

Or as:

4	7	0	
1	8	8	0
	1	8	8 0

Step 3: They are ‘summed according to place’. The partial products are summed according to their relative places or values, thus providing the result of the multiplication 67680.

Appendix C: Reconstructing the ‘door-hinge’ procedure in *Board Mathematics*

This is a reconstruction of the steps of 21×1296 executed with a ‘door-hinge’ (*kāvaṭa-sandhi*) procedure in the ‘direct way’ (*anuloma-mārga*) in Śrīdhara’s *Board Mathematics* (*Pāṭiḡaṇita*).

This reconstruction uses Shukla’s edition of *Board Mathematics*.⁶⁸ The process is studied by Datta and Singh in 1935 and by myself and Catherine Morice-Singh in 2022.⁶⁹ For each step we adduce the representations of the working surface that have been printed in the edition, as well as what is found in the manuscript.

Step 1: The multiplicand is set below the multiplier. Here the multiplier is 21 and the multiplicand 1296. Since the procedure is in the ‘direct way’, the multiplier is set at the top right side of 1296, with 2 above 6. This is a step that is displayed and reconstructed by Shukla in his edition but that is not found in the manuscript.



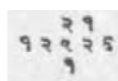
Fig. 11: ‘Door-hinge’ multiplication in the ‘direct way’ in *Board Mathematics*: initial position and 6×21 .

6 is multiplied by 21: the first product, 6, is written below the 1, the second product, 12, below the 2, after erasing the multiplicand’s 6. The carry-over 1 is placed below the next digit of the multiplicand, 9. Shukla has supplied the last configuration of this step, which is not found in the manuscript.

⁶⁸ Shukla 1959, 13–14 for the Sanskrit edition, 7 for the English translation.

⁶⁹ Datta and Singh 1935, 137–143; Keller and Morice-Singh 2022, 526–533, 548–551.

Step 2: 21 slides one step to the left and 2 is placed above 9. This new position of 21 is reconstructed in Shukla's edition and is not in the manuscript. The second digit of the multiplicand, 9, is multiplied by 21. The first product, 9, is added to the 2 which is already there, lower down. 11 is obtained, but as there is already the carried-over 1, it is replaced by 2, below 9. The same kind of process is performed for the second product, 18, 2×9 . When placing 8 it has to be added to 2, which gives a two-digit result $18 + 2 = 20$. Zero is thus noted in place of the previous 9, and a carry-over of 2 is placed to the left and on a line below the zero, under the multiplicand's 2.



21	$9 \times 1 = 9$	21	$9 \times 2 = 18$	21
12926	$9 + 2 = 11$	12916	$18 + 2 = 20$	12016
1	$1 + 1 = 2$	2		2

Fig. 12: 'Door-hinge' multiplication in the 'direct way' in *Board Mathematics*: 9×21 .

Step 3: 21 slides one step to the left, and 2 is placed above the third digit of the multiplicand 2. This configuration is found in the manuscript.⁷⁰



21	$2 \times 1 = 2$	21	$2 \times 2 = 4$	21
12016	$2 + 0 = 2$	12216	$4 + 2 = 6$	16216
2		2		

Fig. 13: 'Door-hinge' multiplication in the 'direct way' in *Board Mathematics*: 2×21 .

70 Notice that the carry-over 2, and the digit 2 of the multiplier, are aligned in a same column a bit to the right of the multiplicand's 2. Similarly, the digit 1 of the multiplier is above but in between the 0 and the 1 of the multiplicand. It is as if the multiplier was represented while in the process of sliding from right to left. The carry-over 2 is however placed rather in a space between the multiplicand's 2 and 0. In the manuscript then, the carry-over would not be placed immediately below the position of the multiplicand it should be added to, but a bit to the right of it. I have not included this slight difference to my reconstruction, which is largely faithful to the interpretation of the display by the editor, Shukla.

This configuration is also reproduced in Shukla’s edition.⁷¹ This configuration indicates that the next step is to execute the multiplication of 2 (the third digit of the multiplicand) with 21. The multiplicand’s 2 is multiplied by 1, the resulting 2 is noted in place of the zero, on the multiplicand’s line. The multiplicand’s 2 is multiplied by the multiplier’s 2, the resulting 4 is added to the carry-over 2 below, and 6 is placed instead of the 2 in the multiplicand’s line. The resulting configuration has been reconstructed by the editor Shukla. It is not found in the manuscript.

Step 4: 21 slides one step to the left and the multiplier’s 2 is above the multiplicand’s 1. This configuration is the second found in the manuscript.



Fig. 14: ‘Door-hinge’ multiplication in the ‘direct way’ in *Board Mathematics*: 1×21 .

This means that the fourth digit of the multiplicand, 1, is multiplied by 21: the process continues as before. The multiplier’s 1 is added to the multiplicand’s 6 which is then replaced by 7. The multiplier’s 2 replaces the last digit on the multiplicand’s line. The result is obtained.

Appendix D: Reconstructing the ‘door-hinge’ multiplication as detailed by Śambhudāsa

This reconstruction of the ‘door-hinge’ in ‘inverse order’ for the multiplication of 18×1196 follows Hayashi’s edition, translation and interpretation.⁷² The digits in bold are those that are being worked upon at each step.

⁷¹ In contrast to the manuscript, Shukla’s edition shows us a multiplier aligned with the multiplicand and the carry-over.

⁷² As found in Hayashi 2017, 16 Sanskrit, 63 English, 64–66 reconstruction.

1	8				
	1	1	9	6	

$1 \times 1 = 1$:

1	8				
1	1	1	9	6	

$8 \times 1 = 8$, the initial digit 1 is replaced by 8:

1	8				
1	8	1	9	6	

18 is moved to the right:

	1	8			
1	8	1	9	6	

$1 \times 1 = 1$. This time 1 is placed below the new digit 8:

	1	8			
1	8	1	9	6	
	1				

$8 \times 1 = 8$. As before, the initial digit 1 is replaced by 8:

	1	8			
1	8	8	9	6	
	1				

18 is moved to the right:

1	8				
1	8	8	9	6	
1					

$1 \times 9 = 9$ placed on a line below the multiplicand:

	1	8			
1	8	8	9	6	
1	9				

$8 \times 9 = 72$. Since there is already a carry-over where the 7 should be placed, it is placed on a line below. 2 replaces the initial digit of the multiplicand:

	1	8			
1	8	8	2	6	
1	9				
	7				

18 is moved to the right:

		1	8		
1	8	8	2	6	
1	9				
	7				

$1 \times 6 = 6$ is placed on the first line under the multiplicand:

		1	8		
1	8	8	2	6	
1	9	6			
	7				

$8 \times 6 = 48$. 4 is placed on the second line under the multiplicand and 8 in place of the initial digit:

				1	8
1	8	8	2	8	
	1	9	6		
		7	4		

Note that in the manuscript, the penultimate position given is without the multiplier:

1	8	8	2	8
	1	9	6	
		7	4	

When summing, the result is obtained:

1	8	8	2	8
	1	9	6	
		7	4	
2	1	5	2	8

