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# Boundary correlations for the $Z$ -invariant Ising Model

Tristan Pham-Mariotti

December 13, 2023

## Abstract

Consider the natural graph associated to a rhombus tiling of a polygonal region in the plane. The spin correlations between boundary vertices of this graph in the  $Z$ -invariant Ising model do not depend on the choice of the rhombus tiling but only on the region. We provide a matrix formula depending on the region which allows practical computations of boundary correlations in this setting, extending the results of Galashin in the critical case.

## 1 Introduction

Let  $R$  be a polygonal region in the plane and let  $\mathbb{T}$  be a rhombus tiling of  $R$  such as in Figure 1. There is a natural graph  $G_{\mathbb{T}} = (V, E)$  assigned to it, whose edges are diagonals of rhombi, see Figure 2. The *Ising model* on this graph is defined as follows. A *spin configuration*  $\sigma$  of  $G_{\mathbb{T}}$  is a function of the vertices of  $G_{\mathbb{T}}$  with values in  $\{-1, 1\}$ . The probability of a spin configuration  $\sigma$  is given by the *Ising Boltzmann measure*

$$\mathbb{P}(\sigma) := \frac{1}{Z} \prod_{e=xy \in E} \exp(j_{xy} \sigma_x \sigma_y)$$

where

$$Z := \sum_{\sigma \in \{-1, 1\}^V} \prod_{e=xy \in E} \exp(j_{xy} \sigma_x \sigma_y),$$

is the *Ising partition function* and  $(j_e)_{e \in E}$  are positive real numbers, called *coupling constants*. Let  $n \geq 1$  and let  $b_1, \dots, b_n$  be the vertices of  $G_{\mathbb{T}}$  on the boundary of  $R$  listed in counterclockwise order. In this article, we are interested in computing the *boundary spin correlations* between  $b_i$  and  $b_j$ , denoted by  $\langle \sigma_{b_i} \sigma_{b_j} \rangle$ , for all  $i, j$  in  $[n]$  where  $[n]$  is the notation for  $\{1, \dots, n\}$ . It is defined by the formula:

$$\langle \sigma_{b_i} \sigma_{b_j} \rangle := \sum_{\sigma \in \{-1, 1\}^V} \mathbb{P}(\sigma) \sigma_{b_i} \sigma_{b_j}.$$

The  *$Z$ -invariant Ising model* was introduced by Baxter in [Bax82] and consists in choosing coupling constants such that the boundary spin correlations depend only on the region  $R$  and not on the tiling  $\mathbb{T}$ . This constraint yields a set of equations called the

*Yang-Baxter equations*, see Section 2.2 for more details. A solution to these equations can be parametrized by rhombus angles  $(\theta_e)_{e \in E}$  naturally assigned to edges of  $G_{\mathbb{T}}$  and by an *elliptic modulus*  $k \in \mathbb{C}$  such that  $k^2 \in (-\infty, 1)$  as follows:

$$j_e := \frac{1}{2} \ln \left( \frac{1 + \operatorname{sn}(\theta_e|k)}{\operatorname{cn}(\theta_e|k)} \right),$$

where  $\operatorname{sn}$  and  $\operatorname{cn}$  are rescaled versions of the classical *Jacobi elliptic functions*  $\operatorname{sn}$  and  $\operatorname{dn}$  (see [AS65, Chapter 16], [Law89] and the appendix for more details on these functions) defined by, for all  $t$  in  $\mathbb{C}$ ,

$$\operatorname{sn}(t) = \operatorname{sn} \left( \frac{2K(k)}{\pi} t \middle| k \right) \quad \text{and} \quad \operatorname{cn}(t) = \operatorname{cn} \left( \frac{2K(k)}{\pi} t \middle| k \right) \quad (1)$$

where  $K(k) = \int_0^{\frac{\pi}{2}} \frac{dx}{\sqrt{1-k^2 \sin^2(x)}}$  is the *complete elliptic integral of the first kind*.

When  $k = 0$ , the functions  $\operatorname{sn}$  and  $\operatorname{cn}$  become the usual trigonometric functions  $\sin$  and  $\cos$ . The model then corresponds to the *critical*  $Z$ -invariant Ising model, whose criticality is proved in [Li12], [CDC13] and [Lis14]. In the critical case, Galashin [Gal22] proves a formula, depending only on  $R$ , for the boundary spin correlations. Our main contribution is a generalization of this result to the whole  $Z$ -invariant regime.

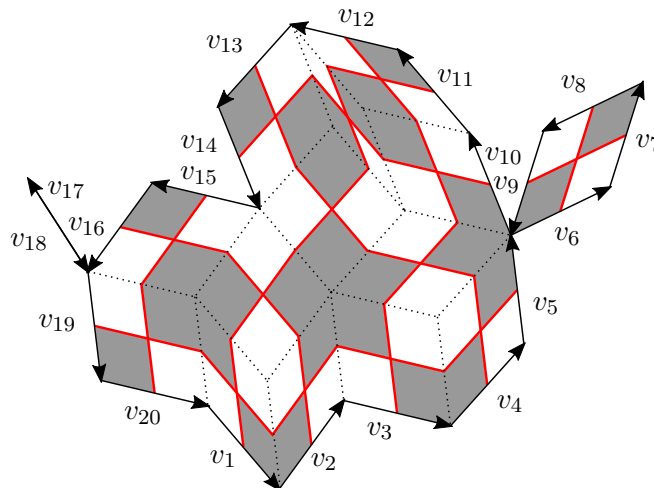


Figure 1: A region with a rhombus tiling (in dotted lines) and the traintracks (in red) associated to it

Let us fix the setting to formulate the statement of the main theorem. Let  $R$  be a polygonal region whose boundary is a closed polygonal chain composed of  $2n$  unit vectors  $v_1, \dots, v_{2n}$  listed in counterclockwise order. Let  $\mathbb{T}$  be a rhombus tiling of  $R$  and let  $\mathcal{T}$  be the set of lines (called *traintracks*, see [KS05]) which are obtained by connecting the midpoints of the opposite edges of each rhombus. For all  $j \in [n]$ , the midpoint of the unit vector  $v_j$  is linked to the midpoint of a unit vector  $v_k$  for some  $k \in [n]$  by one of these lines. These links can be recorded with a fixed-point-free involution  $\tau := \tau_R$  of

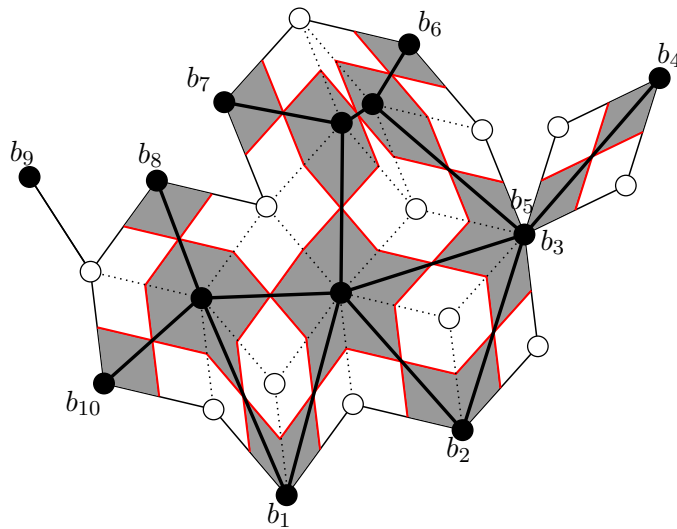


Figure 2: Graph associated to a region and a traintrack arrangement.

$[2n]$  defined as follows: for all  $j \in [2n]$ , let  $k \in [2n]$  is the unique index such that the midpoint of  $v_j$  is linked to the midpoint of  $v_k$ . We then set  $\tau(j) = k$  and  $\tau(k) = j$ . Note that we have  $v_j = -v_k$  when  $\tau(j) = k$ . Furthermore  $\tau$  depends only on the region  $R$  and not on the rhombus tiling.

For technical reasons, we first consider regions for which there are no integers  $1 \leq i < j < k < l \leq 2n$  such that  $v_i = -v_j = v_k = -v_l$ . Such regions are said to be *non-alternating*. The case of alternating regions is considered in Section 3.2. We choose angles  $\alpha_1, \dots, \alpha_{2n}$  in a particular way, and set, for all  $j$  in  $[2n]$ ,  $v_j := e^{i2\alpha_j}$ , see Definition 1 for a precise definition of these angles.

Now we define the function from which spin boundary correlations can be computed. Let  $k \in \mathbb{C}$  such that  $k^2 \in (-\infty, 1)$  and let  $k' := \sqrt{1 - k^2}$ . To a region  $R$ , associate a function  $\gamma_R : \mathbb{R} \rightarrow \mathbb{R}^{2n}$  with coordinates  $\gamma_R(t) = (\gamma_1(t), \dots, \gamma_{2n}(t))$  where

$$\forall t \in \mathbb{R}, \quad \forall p \in [2n], \quad \gamma_p(t) = (-1)^{|J_p \cap [p]|} \Pi_p(t) \prod_{j \in J_p} \text{sn}(t - \alpha_j) \quad (2)$$

with

$$\forall p \in [2n], \quad J_p := \{j \in [2n] : (p, j, \tau(j)) \text{ are in cyclic order}\}, \quad (3)$$

$$\Pi_p(t) := \prod_{j=1}^{2 \lfloor \frac{p}{2} \rfloor} \frac{\text{dn}(t - \alpha_j)}{\sqrt{k'}}$$

and  $\text{dn}$  is a rescaled version of the Jacobi elliptic function  $\text{dn}$  as in Equation (1).

As explained earlier, the spin boundary correlations of the  $Z$ -invariant Ising model depend only on the region  $R$ . Thus denote by  $M_R$  the *boundary correlations matrix* defined by, for all  $i, j$  in  $[n]$ ,  $(M_R)_{ij} = \langle \sigma_{b_i} \sigma_{b_j} \rangle$ . Following Galashin and Pylyavskyy [GP20],

we introduce a modified version of  $M_R$ . Let  $\tilde{M}_R = (\tilde{m}_{i,j})$  be the  $n \times 2n$  matrix defined in the following way: for  $i = j$ , set  $\tilde{m}_{i,2i-1} = \tilde{m}_{i,2i} = m_{i,i} = 1$ . For  $i \neq j$ , set

$$\tilde{m}_{i,2j-1} = -\tilde{m}_{i,2j} = (-1)^{i+j+1\{i < j\}} m_{i,j}.$$

**Example 1.** For instance, for  $n = 3$  the matrix  $\tilde{M}$  is equal to

$$\begin{pmatrix} 1 & 1 & m_{12} & -m_{12} & -m_{13} & m_{13} \\ -m_{12} & m_{12} & 1 & 1 & m_{23} & -m_{23} \\ m_{13} & -m_{13} & -m_{23} & m_{23} & 1 & 1 \end{pmatrix}.$$

The row span of  $\tilde{M}_R$  is called the *doubling map* and is denoted by  $\phi(M_R)$ . We can now state the main result of this paper that expresses the doubling map in terms of the function  $\gamma_R$  introduced above. We write  $\text{span}(\gamma_R)$  for the linear subspace of  $\mathbb{R}^{2n}$  containing finite linear combinations of elements  $\gamma_R(t)$  where  $t \in \mathbb{R}$ .

**Theorem 1.** *Let  $R$  be a non-alternating region composed of  $2n$  unit vectors. Then  $\text{span}(\gamma_R)$  is of dimension  $n$  and  $\text{span}(\gamma_R) = \phi(M_R)$ .*

There is an analogous result for alternating regions, see Section 3.2. For regions with distinct angles  $(\alpha_i)_{i \in [2n]}$ , a basis for  $\text{span}(\gamma_R)$  is obtained in a simple way described by the next proposition.

**Proposition 1.** *Let  $R$  be a region composed of  $2n$  unit vectors with angles  $\alpha_1, \dots, \alpha_{2n}$  such that  $\alpha_i \neq \alpha_j$  for all  $i \neq j$ . Write  $J_1 \cup \{1\} := \{j_1, \dots, j_n\}$ . Then  $\{\gamma_R(\alpha_{j_1}), \dots, \gamma_R(\alpha_{j_n})\}$  form a basis of  $\text{span}(\gamma_R)$ .*

The next proposition complements Theorem 1 by giving a practical way of computing correlations knowing  $\phi(M_R)$ . Define the  $2n \times n$  matrix  $K_n$  by

$$K_n = \frac{1}{2} \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \\ 0 & 0 & \cdots & 1 \end{pmatrix}.$$

**Proposition 2** (Galashin [Gal22]). *Let  $A$  be a  $n \times 2n$  matrix whose row span is equal to  $\phi(M_R)$ . Then the matrix  $AK_n$  is invertible and it satisfies*

$$(AK_n)^{-1}A = \tilde{M}_R.$$

So now give an example to illustrate computations.

**Example 2.** Let  $n = 2$  and  $R := R_2$  be the square region, see Figure 3. We have

$$\alpha_1 = 0, \alpha_2 = \frac{\pi}{4}, \alpha_3 = \frac{\pi}{2} \text{ and } \alpha_4 = \frac{3\pi}{4}.$$

We also have

$$\tau(1) = 3, \tau(2) = 4, \tau(3) = 1 \text{ and } \tau(4) = 2,$$

so we get

$$J_1 = \{2\}, J_2 = \{3\}, J_3 = \{4\} \text{ and } J_4 = \{1\}.$$

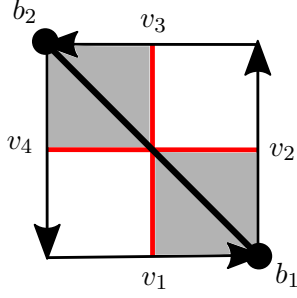


Figure 3: The region  $R_2$

Let  $A$  be the matrix with rows  $-\gamma_R(0)$  and  $-\gamma_R(\frac{\pi}{4})$  that is

$$\begin{aligned} A &= \begin{pmatrix} \operatorname{sn} \frac{\pi}{4} & \operatorname{sn}(\frac{\pi}{2}) \operatorname{dn}(\frac{\pi}{4}) \frac{1}{k'} & \operatorname{sn}(\frac{3\pi}{4}) \operatorname{dn}(\frac{\pi}{4}) \frac{1}{k'} & 0 \\ 0 & \operatorname{sn}(\frac{\pi}{4}) \operatorname{dn}(\frac{\pi}{4}) \frac{1}{k'} & \operatorname{sn}(\frac{\pi}{2}) \operatorname{dn}(\frac{\pi}{4}) \frac{1}{k'} & \operatorname{sn}(\frac{\pi}{4}) \end{pmatrix} \\ &= \begin{pmatrix} \frac{1}{\sqrt{1+k'}} & \frac{1}{\sqrt{k'}} & \frac{1}{\sqrt{k'(1+k')}} & 0 \\ 0 & \frac{1}{\sqrt{k'(1+k')}} & \frac{1}{\sqrt{k'}} & \frac{1}{\sqrt{1+k'}} \end{pmatrix} \end{aligned}$$

using elliptic functions identities (13) and (15) for the last equality. Thus

$$AK_n = \frac{1}{2} \begin{pmatrix} \frac{1}{\sqrt{1+k'}} + \frac{1}{\sqrt{k'}} & \frac{1}{\sqrt{k'(1+k')}} \\ \frac{1}{\sqrt{k'(1+k')}} & \frac{1}{\sqrt{k'}} + \frac{1}{\sqrt{1+k'}} \end{pmatrix}$$

Then we compute

$$(AK_n)^{-1} = \frac{1}{\frac{1}{1+k'} + \frac{1}{\sqrt{k'(1+k')}}} \begin{pmatrix} \frac{1}{\sqrt{1+k'}} + \frac{1}{\sqrt{k'}} & -\frac{1}{\sqrt{k'(1+k')}} \\ -\frac{1}{\sqrt{k'(1+k')}} & \frac{1}{\sqrt{k'}} + \frac{1}{\sqrt{1+k'}} \end{pmatrix}$$

and finally we obtain

$$(AK_n)^{-1}A = \begin{pmatrix} 1 & 1 & \frac{1}{\sqrt{k'+\sqrt{1+k'}}} & -\frac{1}{\sqrt{k'+\sqrt{1+k'}}} \\ -\frac{1}{\sqrt{k'+\sqrt{1+k'}}} & \frac{1}{\sqrt{k'+\sqrt{1+k'}}} & 1 & 1 \end{pmatrix}.$$

Therefore we deduce that

$$\langle \sigma_{b_1} \sigma_{b_2} \rangle = \frac{1}{\sqrt{k'} + \sqrt{1+k'}}.$$

In this simple case, the same value can be found by the following direct computation. For the  $R_2$  region, the constant  $j_e$  associated to the single edge  $e$  of the graph is equal to  $\frac{1}{2} \ln \left( \frac{1+\operatorname{sn}(\frac{\pi}{4})}{\operatorname{cn}(\frac{\pi}{4})} \right)$  by Equation (8). Then

$$\begin{aligned} \langle \sigma_{b_1} \sigma_{b_2} \rangle &= \frac{2e^J - 2e^{-J}}{2e^J + 2e^{-J}} \\ &= \left( \sqrt{\frac{1+\operatorname{sn}(\frac{\pi}{4})}{\operatorname{cn}(\frac{\pi}{4})}} - \sqrt{\frac{\operatorname{cn}(\frac{\pi}{4})}{1+\operatorname{sn}(\frac{\pi}{4})}} \right) \left( \sqrt{\frac{1+\operatorname{sn}(\frac{\pi}{4})}{\operatorname{cn}(\frac{\pi}{4})}} + \sqrt{\frac{\operatorname{cn}(\frac{\pi}{4})}{1+\operatorname{sn}(\frac{\pi}{4})}} \right)^{-1} \\ &= \frac{1+\operatorname{sn}(\frac{\pi}{4}) - \operatorname{cn}(\frac{\pi}{4})}{1+\operatorname{sn}(\frac{\pi}{4}) + \operatorname{cn}(\frac{\pi}{4})} \\ &= \frac{1+\sqrt{1+k'} - \sqrt{k'}}{1+\sqrt{1+k'} + \sqrt{k'}} \quad \text{using the identities (13) and (14)} \\ &= \frac{1+\sqrt{1+k'} - \sqrt{k'}}{(1+\sqrt{1+k'} - \sqrt{k'})(\sqrt{1+k'} + \sqrt{k'})} \\ &= \frac{1}{\sqrt{k'} + \sqrt{1+k'}}. \end{aligned}$$

*Remark 1.* In general, the method used to compute correlations in the Ising model goes through the *Kasteleyn matrix* [Fis61, Kas63, Fis66] or the *Kac-Ward matrix* [DZM<sup>+</sup>99, Lis16] associated to it. But these matrices depend on the rhombus tiling with a number of rows and columns which is of order the number of tiles that is  $O(n^2)$ . Here the matrix  $AK_n$  is only of size  $n \times n$  but the coefficients are more complex.

**Outline of the paper.** In Section 2, we introduce the model and the notations. Then we prove the main theorem of the paper in Section 3. We discuss duality and near-critical case in the rest.

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## 2 Preliminaries

We first define general polygonal regions and graphs arising from them since the proof of the main result goes through degenerate regions. Then we introduce the  $Z$ -invariant Ising model and we explain how the doubling map simplifies the computations of boundary correlations.

*Remark 2.* We follow closely the notations found in [Gal22] with a few exceptions: the angles of a region are written  $\alpha$  instead of  $\theta$ , we use the word *traintracks* instead of *pseudolines* and the semi-angle  $\theta_e$  associated to a rhombus is equal to  $\frac{\pi}{2}$  minus the one of Galashin.

## 2.1 Regions and graphs

Let  $n \geq 1$  be an integer. Let  $\tau : [2n] \rightarrow [2n]$  be a fixed-point-free involution and let  $\alpha = (\alpha_1, \dots, \alpha_{2n}) \in \mathbb{R}^{2n}$  be a sequence of  $2n$  angles satisfying the two following conditions:

$$\forall j < \tau(j), \quad \alpha_{\tau(j)} = \alpha_j + \frac{\pi}{2}, \quad (4)$$

and

$$\forall j < k < \tau(j) < \tau(k), \quad \alpha_j < \alpha_k < \alpha_{\tau(j)} < \alpha_{\tau(k)}. \quad (5)$$

**Definition 1.** A sequence of angles  $\alpha$  satisfying the conditions (4) and (5) is called a  $\tau$ -*shape*. A *region*  $R$  is a pair  $(\tau, \alpha)$  with  $\tau$  a fixed-point-free involution and  $\alpha$  a  $\tau$ -shape.

Define  $2n$  unit vectors  $v_1, \dots, v_{2n}$  by

$$\forall j \in [2n], \quad v_j := e^{i2\alpha_j}.$$

Note that the number 2 in the definition is there to simplify computations in the proof of the main result. A region  $R$  can be drawn in the plane as a closed polygonal chain with sides  $v_1, \dots, v_{2n}$  (see Figure 1). Note that in general this polygonal chain may intersect itself. If it is non-self-intersecting, we say that the region  $R$  is *simple*.

Let  $d_1, \dots, d_{2n}$  be  $2n$  points drawn counterclockwise on a disk. A *traintrack arrangement*  $\mathcal{T}$  of  $(d_1, \dots, d_{2n})$  is a set of  $n$  lines, called *traintracks*, which pair these  $2n$  points such that there is no self-intersection, any two lines intersect at most one, intersection points are in the interior of the disk and no three lines intersect at one point. The pairing induced by a traintrack arrangement  $\mathcal{T}$  defines a fixed-point-free involution  $\tau := \tau_{\mathcal{T}}$  where for all  $j, k \in [2n]$  we have  $\tau(k) = j$  and  $\tau(j) = k$  if and only if  $d_j$  and  $d_k$  are paired. Conversely, for every fixed-point-free involution  $\tau$ , there exists a traintracks arrangement  $\mathcal{T}$  such that  $\tau = \tau_{\mathcal{T}}$ : draw straight lines pairing  $d_j$  to  $d_{\tau(j)}$  for all  $j \in [n]$  for instance (avoiding three lines intersection by curving the lines if needed).

Let  $R = (\tau, \alpha)$  be a region and let  $\mathcal{T}$  be a traintrack arrangement such that  $\tau = \tau_{\mathcal{T}}$ . Note that  $\mathcal{T}$  divides the region  $R$  into faces that can be colored in black and white in a bipartite way such that, for all  $j \in [n]$ , the face next to the arc of the disk connecting  $d_{2j-1}$  and  $d_{2j}$  is black (see Figure 4).

**Definition 2.** Let  $G_{\mathcal{T}}^{\bullet} = (V, E)$  be the graph whose vertices are the black faces and whose edges join black faces that are incident to a common intersection point of traintracks (see Figure 2). Let  $G_{\mathcal{T}}^{\circ}$  be the *dual graph* of  $G_{\mathcal{T}}^{\bullet}$ , i.e., the graph whose vertices



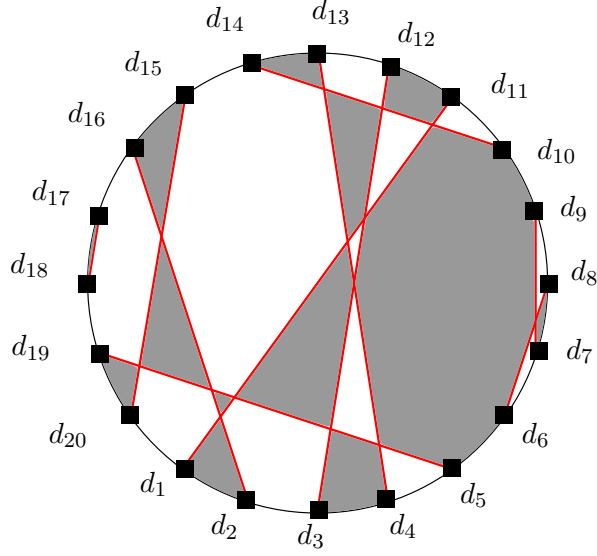


Figure 4: Another way of representing the region  $R$  of Figure 1 (with a different choice of traintrack arrangement)

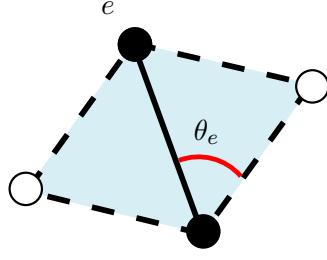
are the white faces and whose edges join white faces that are incident to a common intersection point of traintracks. Let  $G_{\mathcal{F}}^{\diamond}$  be the *diamond graph* of  $G_{\mathcal{F}}^{\bullet}$  that is the bipartite graph whose vertices are the white faces and the black faces and whose edges join white faces and black faces that are adjacent.

*Remark 3.* The choice of coloring is arbitrary and switching the role of black and white is equivalent to switching the definition of  $G_{\mathcal{F}}^{\bullet}$  and its dual. If one wants to choose the other way of coloring, it suffices to relabel  $(v_1, \dots, v_{2n})$  into  $(v_2, \dots, v_{2n}, v_1)$  to get back to our setting. This choice of labeling is related to the duality of the Ising model. For a more extensive discussion about this duality, see Section 4.

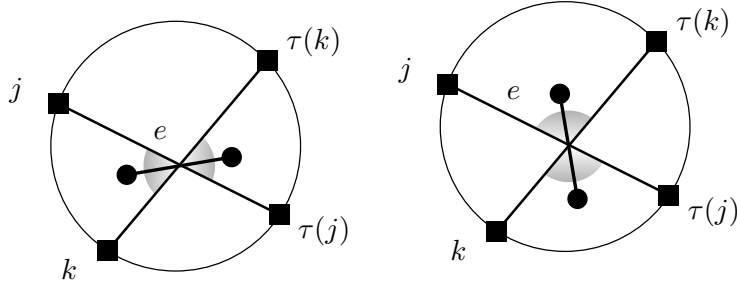
There are  $n$  boundary black faces that we denote by  $b_1, \dots, b_n$  where  $b_j$  is the black face adjacent to the vertices  $d_{2j-1}$  and  $d_{2j}$  for all  $j \in [n]$ . Note that we may have several boundary points  $b_{j_1}, \dots, b_{j_k}$  corresponding to the same black face, as  $b_3$  and  $b_5$  in Figure 4. In this case, we consider these points as if they were “contracted” into a single vertex and we set  $\langle \sigma_{b_{j_r}}, \sigma_{b_{j_s}} \rangle := 1$  for all  $r, s \in [k]$ , see [GP20, Definition 6.1] for details.

Now for each edge  $e \in E$ , we define an angle  $\theta_e$  as follows. Each edge  $e$  corresponds to the intersection point of a traintrack connecting  $d_j$  to  $d_{\tau(j)}$  and another traintrack connecting  $d_k$  to  $d_{\tau(k)}$  for some  $j < k < \tau(j) < \tau(k)$ . We set either  $\theta_e := \alpha_k - \alpha_j$  or  $\theta_e := \alpha_{\tau(j)} - \alpha_k$  depending on how the two black faces linked by  $e$  are located relative to  $j, k, \tau(j), \tau(k)$ , see Figure 5b. For simple regions, there is a *rhombus tiling* associated to a traintrack arrangement (see [KS05]) and then  $\theta_e$  corresponds to the semi-angle adjacent to  $e$  of the rhombus containing  $e$ , see Figure 5a.

It will be convenient to extend the definition of  $\tau$  and  $\alpha$ . Given a fixed-point-free involution  $\tau : [2n] \rightarrow [2n]$ , define  $\tilde{\tau} : \mathbb{Z} \rightarrow \mathbb{Z}$  to be a lift of  $\tau$  satisfying the following



(a) Angle  $\theta_e$  in the rhombus containing  $e$



(b) The two cases of the definition of  $\theta_e$ :  $\theta_e = \alpha_k - \alpha_j$  on the left and  $\theta_e = \alpha_{\tau(j)} - \alpha_k$  on the right.

Figure 5: Rhombus definition and abstract definition of the angle  $\theta_e$

conditions:

$$\forall k \in \mathbb{Z}, \quad \tilde{\tau}(k + 2n) = \tilde{\tau}(k) + 2n,$$

$$\forall k \in \mathbb{Z}, \quad k < \tilde{\tau}(k) < k + 2n,$$

and

$$\forall k \in [2n], \quad \tilde{\tau}(k) \equiv \tau(k) \pmod{2n}.$$

Similarly, given a sequence of angles  $\alpha = (\alpha_j)_{j \in [2n]}$ , define  $\tilde{\alpha} = (\tilde{\alpha}_j)_{j \in \mathbb{Z}}$  by

$$\forall k \in [2n], \quad \tilde{\alpha}_k = \alpha_k$$

and

$$\forall k \in \mathbb{Z}, \quad \tilde{\alpha}_{k+2n} = \tilde{\alpha}_k + \pi.$$

Finally, for  $k \in \mathbb{Z}$ , set  $\tilde{v}_k = \exp(2i\tilde{\alpha}_k)$ . This notation allows us to have more symmetric formulas. Define, for all  $p \in \mathbb{Z}$ ,

$$\tilde{J}_p := \{\tilde{\tau}(j) \mid j \in \mathbb{Z} \text{ such that } j < p \text{ and } \tilde{\tau}(j) > p\}. \quad (6)$$

The set  $\tilde{J}_p$  coincides modulo  $2n$  with the set  $J_p$  defined in (3) for  $p \in [2n]$ . Equality (2) can then be rewritten, for  $p \in [2n]$ ,

$$\gamma_p(t) = \Pi_p(t) \prod_{j \in \tilde{J}_p} \text{sn}(t - \tilde{\alpha}_j) \quad (7)$$

using the fact that  $\operatorname{sn}(u + \pi) = -\operatorname{sn}(u)$  for all  $u \in \mathbb{C}$ . We extend the definition of  $\gamma_p(t)$  for all  $p \in \mathbb{Z}$  using Equality (7) as a definition. Note that we have  $\gamma_{p+2n}(t) = (-1)^{n-1} \gamma_p(t)$  for all  $p \in \mathbb{Z}$ . From now on, we switch freely between the notation with or without tilde.

## 2.2 The Z-invariant Ising Model

Let  $R$  be a region and let  $\mathcal{T}$  be a traintrack arrangement. Consider the *Ising model* on  $G_{\mathcal{T}}^{\bullet}$  with *coupling constants*  $(j_e)_{e \in E}$ . Following Baxter [Bax82], the  $(j_e)_{e \in E}$  are chosen in a way that boundary correlations depend only on the region  $R$  and not on the traintrack arrangement. More precisely, we call *star* a vertex of degree 3 and we define a *star-triangle move* to be a local transformation of a triangle in the graph  $G_{\mathcal{T}}^{\bullet}$  into a star which gives a new graph  $G_{\mathcal{T}'}^{\bullet}$  (this can be understood as the movement of a traintrack over an intersection point, see Figure 6). The Ising model is said to be *Z-invariant* if,

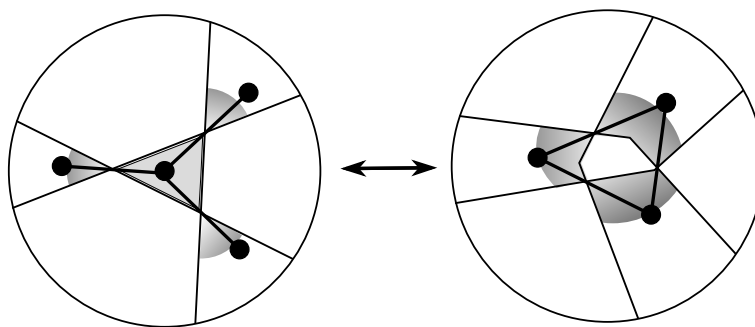


Figure 6: Star-triangle move

when decomposing the partition function according to the possible spin configurations at the three vertices of a star/triangle, it only changes by a constant when performing the star-triangle move, this constant being independent of the choice of the spins at the three vertices. These constraints can be written as a set of equations called the *Yang-Baxter equations* for the Ising model. A parametrization of the solutions of these equations was found by Baxter in [Bax82] and then rewritten in [BTR19]. Let  $k$  be a complex number such that  $k^2 \in (-\infty, 1)$ ,  $k$  is known as the *elliptic modulus*. For every edge  $e$  of  $G_{\mathcal{T}}^{\bullet}$ , set

$$j_e := \frac{1}{2} \ln \left( \frac{1 + \operatorname{sn}(\theta_e | k)}{\operatorname{cn}(\theta_e | k)} \right). \quad (8)$$

Kenyon showed in [Ken05] that any graph  $G_{\mathcal{T}}^{\bullet}$ , where  $\mathcal{T}$  is a traintrack arrangement, can be turned into a graph  $G_{\mathcal{T}'}^{\bullet}$ , with  $\mathcal{T}'$  another traintrack arrangement, applying only star-triangle moves. This implies that boundary correlations depend only on the region and not on the choice of the traintrack arrangement.

## 2.3 Removing a crossing

Following Galashin [Gal22], let us define formally the notion of crossing as well as that of descent, which corresponds to a crossing adjacent to a boundary.

**Definition 3.** Let  $R = (\tau, \alpha)$  be a region. We say that  $R$  has a *crossing* if there exists  $j, k \in [2n]$  such that  $j < k < \tau(j) < \tau(k)$ . An index  $j \in [2n]$  satisfying  $j < j + 1 < \tilde{\tau}(j) < \tilde{\tau}(j + 1)$  is called a  $\tau$ -*descent*.

*Remark 4.* Note that in [Gal22], the author claims that a region with crossings necessarily has a  $\tau$ -descent. But this is not always the case, see Figure 7 for an example. We thus consider this possibility in the proof of the main theorem.

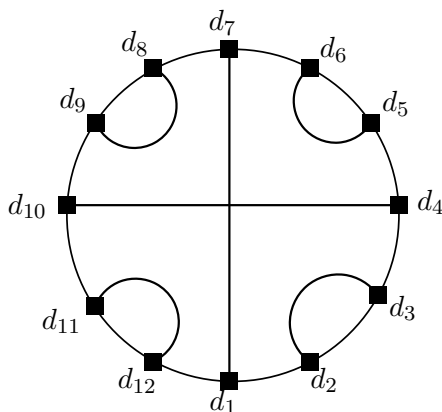


Figure 7: A region with a crossing but without a  $\tau$ -descent

**Definition 4.** Let  $R = (\tau, \alpha)$  be a region with a  $\tau$ -descent  $j \in [2n]$ . Denote by  $t_j$  the transposition of  $[2n]$  that swaps  $j$  and  $j + 1$ . Define a new region  $R' := R \cdot t_j := (\tau', \alpha')$  by  $\tau' = \tau \circ t_j$  and, for all  $l \in [2n]$ ,  $\alpha'_l = \alpha_{t_j(l)}$ . One can check that  $R'$  is actually a region. Note also that if  $R$  is non-alternating, then  $R'$  is also non-alternating.

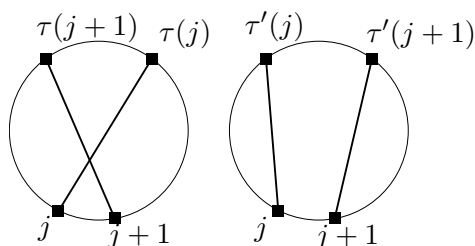


Figure 8: A region  $R$  with a  $\tau$ -descent  $j$  (left) and the region  $R' = R \cdot t_j$  (right)

We now explain how the linear space  $\phi(M_R)$  is modified when we remove a  $\tau$ -descent. Let  $R = (\tau, \alpha)$  be a region with a  $\tau$ -descent  $j \in [2n]$ . There is an edge  $e$  associated to the intersection point between the traintrack incident to  $d_j$  and the traintrack incident to  $d_{j+1}$ . Set  $s_e := (\text{ch}(2j_e))^{-1}$  and  $c_e := \tanh(2j_e)$ . We can express these quantities as functions of  $\alpha_j$  and  $\alpha_{j+1}$ . To gain in clarity, define the functions  $\text{sn}^*$ ,  $\text{cn}^*$  and  $\text{dn}^*$  as follows. Let  $k^* := i \frac{k}{\bar{k}}$  be the *dual parameter* of  $k$  and for all  $t \in \mathbb{C}$  set

$$\text{sn}^*(t) = \text{sn} \left( \frac{2K(k^*)}{\pi} t \middle| k^* \right), \quad \text{cn}^*(t) = \text{cn} \left( \frac{2K(k^*)}{\pi} t \middle| k^* \right)$$

and

$$\operatorname{dn}^*(t) = \operatorname{dn}\left(\frac{2K(k^*)}{\pi}t \mid k^*\right).$$

Note that these rescaled functions are then  $2\pi$ -periodic.

- If  $j$  is odd then  $\theta_e = \frac{\pi}{2} - (\alpha_{j+1} - \alpha_j)$ . Therefore,

$$s_e = \frac{1}{2} \left( \frac{1 + \operatorname{sn}(\theta_e)}{\operatorname{cn}(\theta_e)} + \frac{\operatorname{cn}(\theta_e)}{1 + \operatorname{sn}(\theta_e)} \right) = \operatorname{cn}(\theta_e) = \operatorname{cn}\left(\frac{\pi}{2} - (\alpha_{j+1} - \alpha_j)\right).$$

By Table 1 and the duality formula (18) we thus get

$$s_e = k' \operatorname{sd}(\alpha_{j+1} - \alpha_j) = \operatorname{sn}^*(\alpha_{j+1} - \alpha_j).$$

Similarly, using again Table 1 and the other duality formula (19), we have

$$c_e = \operatorname{sn}\left(\frac{\pi}{2} - (\alpha_{j+1} - \alpha_j)\right) = \operatorname{cd}(\alpha_{j+1} - \alpha_j) = \operatorname{cn}^*(\alpha_{j+1} - \alpha_j).$$

- If  $j$  is even then  $\theta_e = \alpha_{j+1} - \alpha_j$ . Therefore

$$s_e = \operatorname{cn}(\alpha_{j+1} - \alpha_j) \quad \text{and} \quad c_e = \operatorname{sn}(\alpha_{j+1} - \alpha_j).$$

For all  $j \in \mathbb{Z}$ , set  $c_j = \operatorname{cn}(\alpha_{j+1} - \alpha_j)$ ,  $s_j = \operatorname{sn}(\alpha_{j+1} - \alpha_j)$ ,  $c_j^* = \operatorname{cn}^*(\alpha_{j+1} - \alpha_j)$  and  $s_j^* = \operatorname{sn}^*(\alpha_{j+1} - \alpha_j)$ .

**Definition 5.** Define, for all  $j \in \mathbb{Z}$ , the  $2 \times 2$  matrix  $B_j^\alpha$  by

$$B_j^\alpha := \begin{cases} \begin{pmatrix} 1/c_j^* & s_j^*/c_j^* \\ s_j^*/c_j^* & 1/c_j^* \end{pmatrix} & \text{if } j \text{ is odd,} \\ \begin{pmatrix} 1/c_j & s_j/c_j \\ s_j/c_j & 1/c_j \end{pmatrix} & \text{if } j \text{ is even.} \end{cases}$$

For  $1 \leq j < 2n$ , define the matrix  $g_j^\alpha$  which coincides with the  $2n \times 2n$  identity matrix except for the  $2 \times 2$  block  $B_j^\alpha$  which appears in rows and columns  $j, j+1$ . For  $j = 2n$ , the matrix  $g_{2n}^\alpha$  coincides with the  $2n \times 2n$  identity matrix except for the following four entries:

$$(g_{2n}^\alpha)_{1,1} = (g_{2n}^\alpha)_{2n,2n} = \frac{1}{s_{2n}} \quad \text{and} \quad (g_{2n}^\alpha)_{1,2n} = (g_{2n}^\alpha)_{2n,1} = (-1)^{n-1} \frac{c_{2n}}{s_{2n}}.$$

Theorem 3.22 of Galashin and Pylyavskyy in [GP20] gives in a more general setting an equality between the linear spaces  $\phi(M_R)$  and  $\phi(M_{R'})$ . In our particular case, it is read as follows. For a linear subspace  $V \subset \mathbb{R}^{2n}$  of dimension  $n$  and for a matrix  $g$  of size  $2n \times 2n$ , define the linear subspace  $V \cdot g$  as  $\{v \cdot g : v \in V\}$  where  $v \in V$  is seen as a row vector.

**Theorem 2** ([GP20]). *Let  $R = (\tau, \alpha)$  be a region and  $j \in [2n]$  be a  $\tau$ -descent. Then for  $R' := R \cdot t_j$ , we have*

$$\phi(M_R) = \phi(M_{R'}) \cdot g_j^\alpha.$$

*Remark 5.*

- Note that for a  $\tau$ -descent  $j \in [2n]$ , the matrix  $g_j^\alpha$  is well-defined. Indeed, we have  $\tau(j) \neq j + 1$ . Then  $\alpha_j < \alpha_{j+1} < \alpha_j + \frac{\pi}{2}$  by Condition (5). Thus  $c_j \neq 0$  and  $c_j^* \neq 0$  which are the numbers appearing as denominators of the fractions defining the entries of  $B_j^\alpha$ . In that case,  $g_j^\alpha$  is invertible and its determinant is equal to 1.
- The critical  $Z$ -invariant Ising model corresponds to the elliptic modulus  $k = 0$ . In this case, we have  $k = k^* = 0$  so there is no need to distinguish the parity of  $j$  in the definition of  $B_j^\alpha$  and  $g_j^\alpha$  since  $c_j = c_j^*$  and  $s_j = s_j^*$ . The computations of the next section are therefore much simpler in the critical  $Z$ -invariant case.

### 3 Explicit formula for the doubling map

#### 3.1 Main result

The main result of this paper gives an explicit expression for  $\phi(M_R)$ . Recall that, to a region  $R$ , we associate a curve  $\gamma_R : \mathbb{R} \rightarrow \mathbb{R}^{2n}$  with coordinates  $\gamma_R(t) = (\gamma_1(t), \dots, \gamma_{2n}(t))$  where

$$\forall p \in [2n], \quad \gamma_p(t) = \Pi_p(t) \prod_{j \in \bar{J}_p} \text{sn}(t - \tilde{\alpha}_j) \quad (9)$$

with

$$\Pi_p(t) := \prod_{j=1}^{2\lfloor \frac{p}{2} \rfloor} \frac{\text{dn}(t - \tilde{\alpha}_j)}{\sqrt{k'}} = \prod_{j=1}^{\lfloor \frac{p}{2} \rfloor} \frac{\text{dn}(t - \tilde{\alpha}_{2j})}{\text{dn}(t - \tilde{\alpha}_{\tau(2j-1)}}. \quad (10)$$

We make several remarks concerning Formula (9).

*Remark 6.*

- The function  $\gamma_R$  depends only on the region  $R$  and not on the rhombus tiling.
- By changing the choice of the black/white coloring which leads to Definition 2, it implies a switch between “odd” and “even” in Definition 5. All results and computations remain valid by replacing in (9) and (10), but also in the coming computations, the rescaled ‘primal’ Jacobi elliptic function  $\text{sn}, \text{dn}, \text{cn}$  by their dual version  $\text{sn}^*, \text{dn}^*, \text{cn}^*$ .

The product  $\Pi_p(t)$  can be seen as a product associated to a path. Let  $R$  be a region and let  $\mathcal{S}$  be the set of  $n$  train-tracks associated to it with endpoints  $d_1, \dots, d_{2n}$  and associated angles  $\alpha_1, \dots, \alpha_{2n}$ . Let  $G_{\mathcal{S}}^\diamond$  be the diamond graph of Definition 2. Let  $f$  and

$f'$  be two vertices of  $G_{\mathcal{G}}^{\diamond}$ . To a path  $\Gamma_{f,f'}$  in  $G_{\mathcal{G}}^{\diamond}$  going from  $f$  to  $f'$  we associate the following product:

$$\prod_{e=xy \in \Gamma_{f,f'}} \omega_{(x,y)}(t)$$

where

$$\omega_{(x,y)}(t) := \frac{1}{\sqrt{k'}} \operatorname{dn}(t - \alpha_{j(x,y)}),$$

and  $j(x,y)$  is such that  $d_{j(x,y)}$  is the endpoint of the train-track crossed which is on the left of the oriented edge  $(x,y)$ . This product does not depend on the choice of path between  $f$  and  $f'$ . Indeed the product of the multipliers on a path surrounding a rhombus is equal to 1 using Table 1. Let  $f$  and  $f'$  be two vertices of  $G_{\mathcal{G}}^{\diamond}$  that are adjacent to the boundary of  $R$ . The path that goes from  $f$  to  $f'$  going only through boundary vertices exists for any rhombus tiling (see Figure 9). Thus the product associated to it is the same for any tiling. To a number  $p \in [2n]$  we associate the white face  $f_p$  touching the boundary between  $p$  and  $p+1$  if  $p$  is even and we associate the white face  $f_p$  touching the boundary between  $p-1$  and  $p$  if  $p$  is odd (modulo  $2n$ ). Then  $\Pi_p(t)$  is equal to the product associated to the path that goes from  $f_p$  to  $f_1$ .

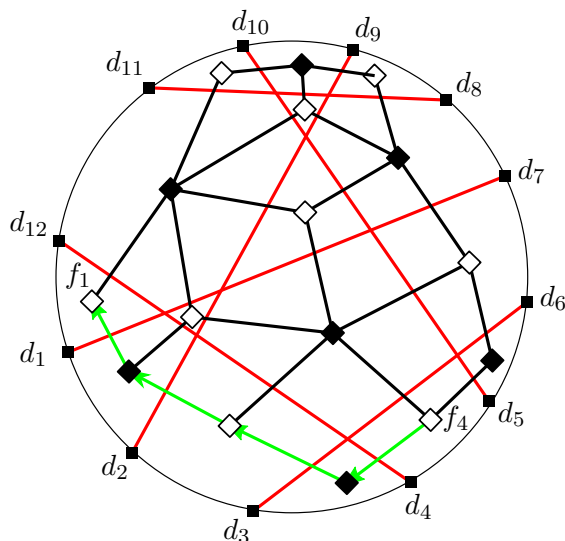


Figure 9: A path in the diamond graph going from  $f_4$  to  $f_1$

**Proposition 3.** Let  $R = (\tau, \alpha)$  be a non-alternating region and  $j \in [2n]$  be a  $\tau$ -descent. Then for  $R' = R \cdot t_j$  we have for all  $t \in \mathbb{R}$ ,

$$\gamma_R(t) = \gamma_{R'}(t) \cdot g_j^\alpha,$$

where  $g_j^\alpha$  is the matrix defined in Definition 5.

*Proof.* In this proof, we consider the tilded version of  $\tau, \alpha$  and  $J_p$  even if the tilde notation is omitted. It allows us to treat the case  $j = 2n$  with the same notation as the other cases  $j < 2n$ . For the region  $R'$ , write  $\gamma_{R'}(t) = (\gamma'_1(t), \dots, \gamma'_{2n}(t))$  and, for  $m \in [2n]$ , denote by  $J'_m$  the set obtained from  $\tau'$ . By comparing  $J_m$  and  $J'_m$  for all  $m$  different from  $j, j+1$  we see that  $\gamma_m(t) = \gamma'_m(t)$ . So it remains to consider the case  $m = j$  and the case  $m = j+1$ .

Note that

$$J_j \setminus \{j+1\} = J_{j+1} \setminus \{\tau(j)\} \text{ and } J'_j \setminus \{j+1\} = J'_{j+1} \setminus \{\tau'(j)\}.$$

Suppose first that  $j$  is even.

We have

$$(\gamma_{R'}(t) \cdot g_j^\alpha)_j = \gamma'_j(t) \frac{1}{\text{cn}(\alpha_{j+1} - \alpha_j)} + \gamma'_{j+1}(t) \frac{\text{sn}(\alpha_{j+1} - \alpha_j)}{\text{cn}(\alpha_{j+1} - \alpha_j)}$$

and

$$(\gamma_{R'}(t) \cdot g_j^\alpha)_{j+1} = \gamma'_j(t) \frac{\text{sn}(\alpha_{j+1} - \alpha_j)}{\text{cn}(\alpha_{j+1} - \alpha_j)} + \gamma'_{j+1}(t) \frac{1}{\text{cn}(\alpha_{j+1} - \alpha_j)}.$$

Since  $j$  is even, we have  $\Pi_j(t) = \Pi_{j+1}(t)$  so, by (9),  $\gamma'_j(t) = \Phi(t)\text{sn}(t - \alpha'_{j+1})$  and  $\gamma'_{j+1}(t) = \Phi(t)\text{sn}(t - \alpha'_{\tau'(j)})$  where

$$\Phi(t) := \prod_{l \in J'_j \setminus \{j+1\}} \Pi_j(t)\text{sn}(t - \alpha'_l)$$

is the common factor. Recall that, by Condition (4) on the angles and Definition 4 of  $R'$ , the identities  $\alpha'_{\tau'(j)} = \alpha_{\tau(j+1)} = \alpha_{j+1} + \frac{\pi}{2}$  and  $\alpha'_{j+1} = \alpha_j$  are satisfied. Therefore

$$\gamma'_{j+1}(t) = -\Phi(t)\text{cd}(t - \alpha_{j+1})$$

where  $\text{cd}(u) := \frac{\text{cn}(u)}{\text{sn}(u)}$  for all  $u \in \mathbb{R}$ . We also have

$$\gamma'_j(t) = \Phi(t)\text{sn}(t - \alpha_j).$$

We deduce that

$$\begin{aligned} (\gamma_{R'}(t) \cdot g_j^\alpha)_j &= \frac{\Phi(t)}{\text{cn}(\alpha_{j+1} - \alpha_j)} \\ &\times \left( \text{sn}[(t - \alpha_{j+1}) + (\alpha_{j+1} - \alpha_j)] - \text{cd}(t - \alpha_{j+1})\text{sn}(\alpha_{j+1} - \alpha_j) \right). \end{aligned}$$

Now using Relation (16)

$$\text{dn}(u)\text{sn}(u+v) = \text{cn}(u)\text{sn}(v) + \text{sn}(u)\text{cn}(v)\text{dn}(u+v),$$



we get

$$(\gamma_{R'}(t) \cdot g_j^\alpha)_j = \frac{\Phi(t)}{\text{cn}(\alpha_{j+1} - \alpha_j)} \left( \frac{\text{cn}(t - \alpha_{j+1}) \text{sn}(\alpha_{j+1} - \alpha_j)}{\text{dn}(t - \alpha_{j+1})} + \frac{\text{sn}(t - \alpha_{j+1}) \text{cn}(\alpha_{j+1} - \alpha_j) \text{dn}(t - \alpha_j)}{\text{dn}(t - \alpha_{j+1})} - \text{cd}(t - \alpha_{j+1}) \text{sn}(\alpha_{j+1} - \alpha_j) \right),$$

thus

$$(\gamma_{R'}(t) \cdot g_j^\alpha)_j = \Phi(t) \text{sn}(t - \alpha_{j+1}) \frac{\text{dn}(t - \alpha_j)}{\text{dn}(t - \alpha_{j+1})}.$$

Injecting the expression of  $\Phi(t)$  and reorganizing the terms we have

$$(\gamma_{R'}(t) \cdot g_j^\alpha)_j = \left( \prod_{l \in J'_j \setminus \{j+1\}} \text{sn}(t - \alpha'_l) \right) \text{sn}(t - \alpha_{j+1}) \times \left( \prod_{l=1}^{\lfloor \frac{j}{2} \rfloor - 1} \frac{\text{dn}(t - \alpha'_{2l})}{\text{dn}(t - \alpha'_{\tau'(2l-1)})} \right) \frac{\text{dn}(t - \alpha'_j)}{\text{dn}(t - \alpha'_{\tau'(j-1)})} \frac{\text{dn}(t - \alpha_j)}{\text{dn}(t - \alpha_{j+1})}.$$

Since for all  $l \in J'_j \setminus \{j+1\}$  we have  $\alpha'_l = \alpha_l$  and since  $J'_j = J_j$ , it gives

$$\left( \prod_{l \in J'_j \setminus \{j+1\}} \text{sn}(t - \alpha'_l) \right) \text{sn}(t - \alpha_{j+1}) = \prod_{l \in J_j} \text{sn}(t - \alpha_l).$$

Similarly, for all  $1 \leq l \leq \lfloor \frac{j}{2} \rfloor - 1$ , we have  $\alpha'_{2l} = \alpha_{2l}$  and  $\alpha'_{\tau'(2l-1)} = \alpha_{\tau(2l-1)}$ . We also have  $\alpha'_j = \alpha_{j+1}$  and  $\alpha'_{\tau'(j-1)} = \alpha_{\tau(j-1)}$  so

$$\left( \prod_{l=1}^{\lfloor \frac{j}{2} \rfloor - 1} \frac{\text{dn}(t - \alpha'_{2l})}{\text{dn}(t - \alpha'_{\tau'(2l-1)})} \right) \frac{\text{dn}(t - \alpha'_j)}{\text{dn}(t - \alpha'_{\tau'(j-1)})} \frac{\text{dn}(t - \alpha_j)}{\text{dn}(t - \alpha_{j+1})} = \prod_{l=1}^{\lfloor \frac{j}{2} \rfloor} \frac{\text{dn}(t - \alpha_{2l})}{\text{dn}(t - \alpha_{\tau(2l-1)})}. \quad (11)$$

Finally we get

$$(\gamma_{R'}(t) \cdot g_j^\alpha)_j = \prod_{l \in J_j} \text{sn}(t - \alpha_l) \prod_{l=1}^{\lfloor \frac{j}{2} \rfloor} \frac{\text{dn}(t - \alpha_{2l})}{\text{dn}(t - \alpha_{\tau(2l-1)})} = \gamma_j(t).$$

Now we turn to the coefficient  $j+1$ . Factorizing by  $\Phi(t)$  we have

$$(\gamma_{R'}(t) \cdot g_j^\alpha)_{j+1} = \frac{\Phi(t)}{\text{cn}(\alpha_{j+1} - \alpha_j)} \left( \text{sn}(t - \alpha_j) \text{sn}(\alpha_{j+1} - \alpha_j) - \text{cd}[(t - \alpha_j) - (\alpha_{j+1} - \alpha_j)] \right).$$

Then we use the following relation (17)

$$\text{cn}(u + v) = \text{cn}(u) \text{cn}(v) - \text{sn}(u) \text{sn}(v) \text{dn}(u + v)$$

to get

$$(\gamma_{R'}(t) \cdot g_j^\alpha)_{j+1} = \frac{\Phi(t)}{\operatorname{cn}(\alpha_{j+1} - \alpha_j)} \left( \operatorname{sn}(t - \alpha_j) \operatorname{sn}(\alpha_{j+1} - \alpha_j) - \frac{\operatorname{cn}(t - \alpha_j) \operatorname{cn}(\alpha_{j+1} - \alpha_j)}{\operatorname{dn}(t - \alpha_{j+1})} \right. \\ \left. - \frac{\operatorname{sn}(t - \alpha_j) \operatorname{sn}(\alpha_{j+1} - \alpha_j) \operatorname{dn}(t - \alpha_{j+1})}{\operatorname{dn}(t - \alpha_{j+1})} \right)$$

Thus

$$(\gamma_{R'}(t) \cdot g_j^\alpha)_{j+1} = \Phi(t) (-\operatorname{cd}(t - \alpha_j)) \frac{\operatorname{dn}(t - \alpha_j)}{\operatorname{dn}(t - \alpha_{j+1})} = \Phi(t) \operatorname{sn}(t - \alpha_{\tau(j)}) \frac{\operatorname{dn}(t - \alpha_j)}{\operatorname{dn}(t - \alpha_{j+1})}.$$

Reinjecting the expression of  $\Phi(t)$  and reorganizing the terms it gives

$$(\gamma_{R'}(t) \cdot g_j^\alpha)_{j+1} = \left( \prod_{l \in J'_j \setminus \{j+1\}} \operatorname{sn}(t - \alpha'_l) \right) \operatorname{sn}(t - \alpha_{\tau(j)}) \\ \times \left( \prod_{l=1}^{\lfloor \frac{j}{2} \rfloor - 1} \frac{\operatorname{dn}(t - \alpha'_{2l})}{\operatorname{dn}(t - \alpha'_{\tau'(2l-1)})} \right) \frac{\operatorname{dn}(t - \alpha'_j)}{\operatorname{dn}(t - \alpha'_{\tau'(j-1)})} \frac{\operatorname{dn}(t - \alpha_j)}{\operatorname{dn}(t - \alpha_{j+1})}.$$

Since  $J_{p+1} = J'_p \setminus \{p+1\} \cup \{\tau(p)\}$  we have

$$\left( \prod_{j \in J'_p \setminus \{p+1\}} \operatorname{sn}(t - \alpha'_j) \right) \operatorname{sn}(t - \alpha_{\tau(p)}) = \prod_{j \in J_{p+1}} \operatorname{sn}(t - \alpha_j).$$

Recalling (11) and since  $\lfloor \frac{j}{2} \rfloor = \lfloor \frac{j+1}{2} \rfloor$  for  $j$  even we finally deduce that

$$(\gamma_{R'}(t) \cdot g_j^\alpha)_{j+1} = \prod_{l \in J_{j+1}} \operatorname{sn}(t - \alpha_l) \prod_{l=1}^{\lfloor \frac{j+1}{2} \rfloor} \frac{\operatorname{dn}(t - \alpha_{2l})}{\operatorname{dn}(t - \alpha_{\tau(2l-1)})} = \gamma_{j+1}(t).$$

Suppose now that  $j$  is odd. The idea is the same but the common factor between  $\gamma'_j(t)$  and  $\gamma'_{j+1}(t)$  is different. Indeed we have,

$$\gamma'_j(t) = \Phi(t) \operatorname{sn}(t - \alpha'_{j+1}) = \Phi(t) \operatorname{sn}^*(t - \alpha'_{j+1}) \operatorname{dn}^*(t - \alpha'_{\tau'(j+1)}).$$

And using consecutively Table 1 and the dual relation (20) we have

$$\gamma'_{j+1}(t) = \Phi(t) \operatorname{sn}(t - \alpha'_{\tau'(j)}) \frac{\operatorname{dn}(t - \alpha'_{j+1})}{\operatorname{dn}(t - \alpha'_{\tau'(j)})} \\ = \Phi(t) \operatorname{sn}^*(t - \alpha'_{\tau'(j)}) \frac{1}{k'} \operatorname{dn}(t - \alpha'_{j+1}) \\ = \Phi(t) \operatorname{sn}^*(t - \alpha'_{\tau'(j)}) \operatorname{dn}^*(t - \alpha'_{\tau'(j+1)}).$$

Thus  $\gamma'_j(t) = \tilde{\Phi}(t)\text{sn}^*(t - \alpha'_{j+1})$  and  $\gamma'_{j+1}(t) = \tilde{\Phi}(t)\text{sn}^*(t - \alpha'_{\tau(j)})$  with  $\tilde{\Phi}(t) = \Phi(t)\text{dn}^*(t - \alpha'_{\tau(j+1)})$  is the common factor between  $\gamma'_j(t)$  and  $\gamma'_{j+1}(t)$ . A similar computation as above with  $\text{sn}^*$  instead of  $\text{sn}$  therefore gives

$$(\gamma_{R'}(t) \cdot g_j^\alpha)_j = \tilde{\Phi}(t)\text{sn}^*(t - \alpha_{j+1}) \frac{\text{dn}^*(t - \alpha_j)}{\text{dn}^*(t - \alpha_{j+1})}.$$

Rewriting these without  $*$ , using the duality relations we have

$$(\gamma_{R'}(t) \cdot g_j^\alpha)_j = \Phi(t)\text{sn}(t - \alpha_{j+1}) \frac{\text{dn}(t - \alpha_{\tau(j+1)})}{\text{dn}(t - \alpha'_{\tau(j+1)})} \frac{\text{dn}(t - \alpha_{j+1})}{\text{dn}(t - \alpha_j)}.$$

Since  $\alpha'_{\tau(j+1)} = \alpha_{\tau(j)}$  we deduce that

$$\frac{\text{dn}(t - \alpha_{\tau(j+1)})}{\text{dn}(t - \alpha'_{\tau(j+1)})} \frac{\text{dn}(t - \alpha_{j+1})}{\text{dn}(t - \alpha_j)} = 1.$$

Finally we get

$$(\gamma_{R'}(t) \cdot g_j^\alpha)_j = \prod_{l \in J_j} \text{sn}(t - \alpha_l) \prod_{l=1}^{\lfloor \frac{j}{2} \rfloor} \frac{\text{dn}(t - \alpha_{2l})}{\text{dn}(t - \alpha_{\tau(2l-1)})} = \gamma_j(t).$$

Now we turn to the coefficient  $j + 1$ . By the above computations we have

$$(\gamma_{R'}(t) \cdot g_j^\alpha)_{j+1} = \tilde{\Phi}(t)\text{sn}^*(t - \alpha_{\tau(j)}) \frac{\text{dn}^*(t - \alpha_j)}{\text{dn}^*(t - \alpha_{j+1})}.$$

Rewriting these using the duality relations we have

$$(\gamma_{R'}(t) \cdot g_j^\alpha)_{j+1} = \Phi(t)\text{sn}(t - \alpha_{\tau(j)}) \frac{\text{dn}(t - \alpha_j)}{\text{dn}(t - \alpha'_{\tau(j+1)})} \frac{\text{dn}(t - \alpha_{j+1})}{\text{dn}(t - \alpha_j)}.$$

Since  $\alpha'_{\tau(j+1)} = \alpha_{\tau(j)}$  we deduce that

$$(\gamma_{R'}(t) \cdot g_j^\alpha)_{j+1} = \prod_{l \in J_j} \text{sn}(t - \alpha_l) \prod_{l=1}^{\lfloor \frac{j+1}{2} \rfloor} \frac{\text{dn}(t - \alpha_{2l})}{\text{dn}(t - \alpha_{\tau(2l-1)})} = \gamma_{j+1}(t) \quad \square.$$

Now we can turn to the proof of the main theorem:

**Theorem.** *Let  $R = (\tau, \alpha)$  be a region and  $j \in [2n]$  be a  $\tau$ -descent. Then for  $R' := R \cdot t_j$ , we have*

$$\phi(M_R) = \phi(M_{R'}) \cdot g_j^\alpha$$

*Proof of Theorem 1.* As in the paper [Gal23], we proceed by induction. Our proof only differs in the induction step since we are careful with the fact that a crossing does not imply the existence of a  $\tau$ -descent, see Remark 4. We proceed by induction on the number  $\ell$  of crossings.

For  $\ell = 0$ , let  $R$  be a region without crossing. Since there is no crossing, there is always an index  $p$  such that  $\tau(p) = p + 1$ . Let us show that  $\text{span}(\gamma_R)$  contains the vector  $e_p + \varepsilon_{p,\tau(p)}e_{\tau(p)}$  where  $\varepsilon_{p,\tau(p)} = (-1)^{\frac{\tau(p)-p-1}{2}}$  and  $e_1, \dots, e_{2n}$  is the canonical basis of  $\mathbb{R}^{2n}$ .

By the non-alternating condition we have that  $\gamma_p(\alpha_p), \gamma_{\tau(p)}(\alpha_p) \neq 0$  while  $\gamma_m(\alpha_p) = 0$  for all  $m \neq p, \tau(p)$ . Note that  $J_p = J_{\tau(p)}$  and  $\Pi_p = \Pi_{\tau(p)}$  so  $\gamma_p(t) = \varepsilon_{p,\tau(p)}\gamma_{\tau(p)}(t)$ . Thus  $\gamma(\alpha_p)$  is proportional to  $e_p + \varepsilon_{p,\tau(p)}e_{\tau(p)}$  which shows the result.

Now we can modify the curve  $\gamma_R$  as follows. Define  $\gamma_R^{[p,\tau(p)]}$  from  $\gamma_R$  by setting the  $p$ -th and the  $\tau(p)$ -th coordinates to 0 and dividing the other coordinates by  $\text{sn}(t - \alpha_p)$  (which is a common factor of all  $\gamma_m(t)$  where  $m \neq p, \tau(p)$ ). Let us show that

$$\text{span}(\gamma_R) = \text{span}(\gamma_R^{[p,\tau(p)]}) \oplus \text{span}(e_p + \varepsilon_{p,\tau(p)}e_{\tau(p)}).$$

The first inclusion is clear since we have

$$\gamma_R(t) = \text{sn}(t - \alpha_p)\gamma_R^{[p,\tau(p)]}(t) + \prod_{j \in J_p} \text{sn}(t - \alpha_j)\Pi_p(e_p + \varepsilon_{p,\tau(p)}).$$

For the other inclusion it suffices to note that  $\gamma_R(\alpha_p) = c(e_p + \varepsilon_{p,\tau(p)})$  where  $c$  is a constant and that

$$\gamma_R^{[p,\tau(p)]}(t) = \frac{1}{\text{sn}(t - \alpha_p)}\gamma_R(t) - \prod_{j \in J_p} \text{sn}(t - \alpha_j)\Pi_p \frac{1}{c}\gamma_R(\alpha_p).$$

Now notice that  $\text{span}(\gamma_R^{[p,\tau(p)]})$  with two of its coordinates set to 0 is the same as  $\text{span}(\gamma_{\tilde{R}})$  where  $\tilde{R}$  is the region obtained from  $R$  by removing the arc  $p, \tau(p)$ . Indeed the product  $\Pi_m$  associated to  $m$  for all  $m$  not equal to  $p, \tau(p)$  is the same in  $R$  and in  $\tilde{R}$ . This can be seen for instance using the path notation of the product  $\Pi_m$  explained in Remark 6 and using the fact that we can find a path that does not cross the traintrack linking  $d_p$  to  $d_{\tau(p)}$ . Therefore we can repeat the process for the region  $\tilde{R}$  and eventually decompose  $\text{span}(\gamma_R)$  as a direct sum of  $\text{span}(e_p + \varepsilon_{p,\tau(p)}e_{\tau(p)})$  over all pairs  $p < \tau(p)$ . By definition of  $\phi(M_R)$  as the span of the matrix  $\tilde{M}$ , this direct sum is equal to  $\phi(M_R)$ . It also proves that for  $\ell = 0$ ,  $\text{span}(\gamma_R)$  is of dimension  $n$ . Since for all  $j \in [2n]$  and for all  $\alpha \in \mathbb{R}^{2n}$ , the matrix  $g_j^\alpha$  is invertible, we can conclude that  $\text{span}(\gamma_R)$  is of dimension  $n$  for all  $\ell \geq 0$ .

Now for the induction step, suppose that the result is true for  $\ell$  crossings. Let  $R$  be a region with  $\ell + 1$  crossings. Let  $j < k$  be two indices of  $[2n]$  such that the traintrack linking  $d_j$  and  $d_{\tau(j)}$  crosses the traintrack linking  $d_k$  and  $d_{\tau(k)}$ . Assume the difference  $k - j$  is minimal in the sense that no other  $k, j$  with a smaller difference satisfy the above condition.

- If  $j$  is a  $\tau$ -descent (i.e.  $k = j + 1$ ). Then let  $R' = R \cdot t_j$ .  $R'$  has  $\ell$  crossings so by the induction hypothesis we have  $\text{span}(\gamma_{R'}) = \phi(M_{R'})$ . Since  $\phi(M_R) = \phi(M_{R'}) \cdot g_j^\alpha$  by Theorem 2 and  $\text{span}(\gamma_R) = \text{span}(\gamma_{R'}) \cdot g_j^\alpha$  by Proposition 3 we deduce that  $\phi(M_R) = \text{span}(\gamma_R)$ .
- If  $j$  is not a  $\tau$ -descent, we transform the region  $R$  into a region  $\hat{R}$  such that  $j$  is a  $\tau$ -descent in  $\hat{R}$  as follows. First note that, by minimality of the difference  $k - j$ , it means that there is no crossing in the area between  $j + 1$  and  $k - 1$ , see Figure 10.

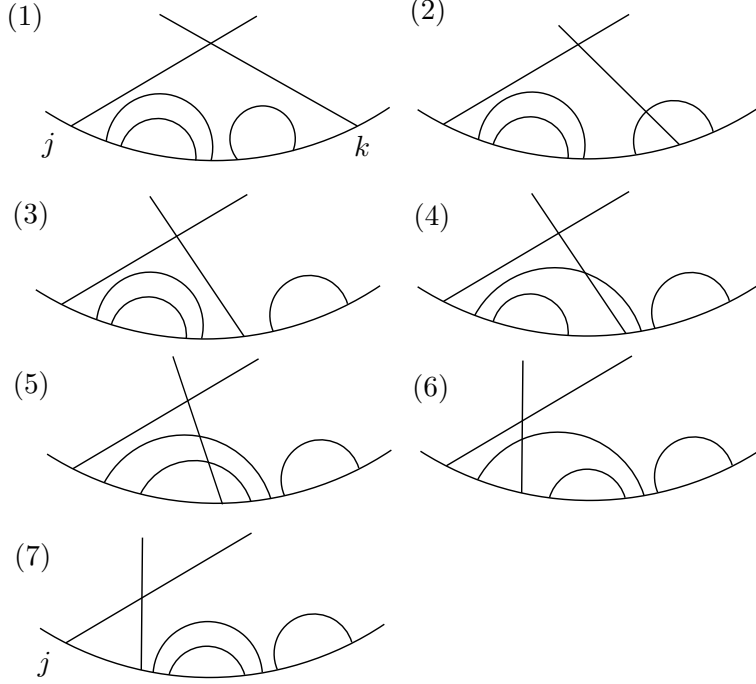


Figure 10: Procedure to make  $j$  into a  $\tau$ -descent (from left to right and top to bottom)

Define, by induction, a sequence of regions  $(R_i)_{0 \leq i \leq k-(j+1)} = (\tau_i, \alpha_i)$  by  $R_0 := R$  and for  $i \geq 1$ :

- If  $k - i$  is a  $\tau_{i-1}$ -descent, then  $R_i = R_{i-1} \cdot t_{k-i}$ .
- Otherwise, set  $R_i$  to be the region obtained by crossing  $k - i$  and  $k - (i - 1)$ .

The region  $\hat{R} := R_{k-(j+1)}$  is such that  $j$  is a  $\hat{\tau}$ -descent. And we have  $\phi(M_R) = \phi(M_{\hat{R}}) \cdot g$  and  $\text{span}(\gamma_R) = \text{span}(\gamma_{\hat{R}}) \cdot g$  where  $g$  is an invertible matrix. By the first step applied to  $\hat{R}$  we know that  $\text{span}(\gamma_{\hat{R}}) = \phi(M_{\hat{R}})$  so we deduce the result for  $R$ .  $\square$

We end with a proof of Proposition 1.

*Proof of Proposition 1.* Let  $A$  be the matrix with rows  $\gamma_R(\alpha_{j_1}), \dots, \gamma_R(\alpha_{j_n})$ . Note that, since all angles are distinct, we have  $\gamma_{j_l}(\alpha_{j_l}) \neq 0$  for all  $l \in [n]$ . Therefore, the principal

minor of  $A$  composed by the  $n$  first columns of  $A$  is an upper-triangular matrix with non-zero diagonal coefficients. It is thus invertible which implies that  $A$  is of rank  $n$ .  $\square$

### 3.2 Alternating regions

In this section we extend our result to alternating regions.

**Definition 6.** Let  $R = (\tau, \alpha)$  and  $v_j = e^{i2\alpha_j}$  for all  $j \in [2n]$ . An *alternating region* is a region such that there exist integers  $1 \leq i < j < k < l \leq 2n$  such that  $v_i = -v_j = v_k = -v_l$ .

We adapt the ideas of [Gal22] to our case. Let  $R = (\tau, \alpha)$  be a region. For  $p \in [2n]$ , define  $m_p := \#\{j \in J_p : v_j = v_p\}$ . For  $p \in [2n]$ , let  $\text{supp}_\tau(p)$  be the set of indices that we encounter by going from  $p$  to  $\tau(p)$  in the trigonometric direction, that is

$$\text{supp}_\tau(p) := \begin{cases} \{p, p+1, \dots, \tau(p)\} & \text{if } p < \tau(p) \\ \{p, p+1, \dots, n, 1, \dots, \tau(p)\} & \text{if } p > \tau(p). \end{cases}$$

For a vector  $x \in \mathbb{C}^{2n}$  and a set  $S \subset [2n]$ , define the vector  $x_{|S}$  to be the vector obtained from  $x$  by

$$(x_{|S})_p := \begin{cases} x_p & \text{if } p \in S \\ 0 & \text{if } p \notin S. \end{cases}$$

Finally, for  $p \in [2n]$ , define

$$u_R^{(p)} := \gamma_R^{(m_p)}(\alpha_p)_{| \text{supp}_\tau(p)}$$

where  $\gamma_R^{(m_p)}$  is the  $m_p$ -th derivative of  $\gamma_R$ .

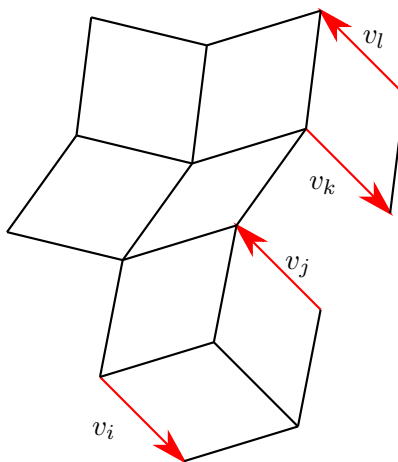


Figure 11: Example of an alternating region

**Proposition 4.** Let  $R = (\tau, \alpha)$  be a region. The  $n$  vectors  $\{u_R^{(j)} : j \in J_1 \cup \{1\}\}$  form a basis of  $\phi(M_R)$ .

*Proof.* Let  $(R_N = (\tau, \boldsymbol{\alpha}^N)_{N \geq 1})$  be a sequence of regions such that for all  $N \geq 1$ , the angles  $(\alpha_j^N)_{j \in [2n]}$  are all distinct (so that in particular  $R_N$  is non-alternating for all  $N$ ) and  $\alpha_j^N$  tends to  $\alpha_j$  when  $N$  goes to infinity. The function  $R \rightarrow M_R$  is continuous in  $\boldsymbol{\alpha}$  so  $M_{R_N} \rightarrow M_R$  as  $N \rightarrow \infty$ . Furthermore the doubling map  $\phi$  is continuous so  $\phi(M_{R_N})$  consists of vectors of  $\mathbb{R}^{2n}$  that are limits of sequences of vectors  $x^N$  of  $\phi(M_{R_N}) = \text{span}(\gamma_{R_N})$ . Therefore we should prove that the vectors  $\{u_R^{(j)} : j \in J_1 \cup \{1\}\}$  are obtained as such limits and that they are linearly independent.

We first show that these vectors are linearly independent. Denote  $J_1 \cup \{1\} := \{j_1 < j_2 < \dots < j_n\}$ . Let  $U$  be the matrix with lines  $u_R^{j_1}, u_R^{j_2}, \dots, u_R^{j_n}$ . Consider the  $n \times n$  principal minor indexed by the columns  $J_1 \cup \{1\}$ . We claim that this minor is an upper-triangular matrix with non zero diagonal coefficients. Indeed, since we differentiate exactly the right number of times, we have  $(u_R^{j_l})_{j_l} = \gamma_{j_l}^{(m_{j_l})}(\alpha_{j_l}) \neq 0$ . For  $1 \leq l < l' \leq n$ , we have  $l \notin \text{supp}_\tau(l')$  thus  $(u_R^{j_{l'}})_{j_l} = 0$ .

Now we show that for each  $j \in [2n]$ ,  $u_R^j$  is a limit of a sequence  $u^{(j,N)} \in \text{span}(\gamma_{R_N})$ . In other words, we want to approximate the derivatives of  $\gamma_R$  by linear combinations of  $\gamma_{R_N}$ . To do that, we introduce the *discrete derivative operator*. For a smooth function  $f$  and for  $a \in \mathbb{R}$  define

$$\bar{\partial}_a f(x) := \frac{f(x) - f(a)}{x - a}$$

For  $(a_1, a_2, \dots, a_m)$  distinct real numbers we set

$$\bar{\partial}_{(a_1, \dots, a_m)} f(x) := \bar{\partial}_{a_1} \circ \bar{\partial}_{a_2} \circ \dots \circ \bar{\partial}_{a_m}.$$

We have the following property that can be proved by induction, using Taylor expansion:

$$\lim_{a_1, \dots, a_m, a \rightarrow a_0} [(\bar{\partial}_{(a_1, \dots, a_m)} f(x))|_{x=a}] = \frac{1}{m!} f^{(m)}(a_0). \quad (12)$$

Let  $j \in [2n]$ . Define

$$J'_j = \{l \in J_j \cap \text{supp}_\tau(j) : \alpha_l = \alpha_j\} \text{ and } J''_j = \{l \in J_j \setminus \text{supp}_\tau(j) : \alpha_l = \alpha_j\}.$$

Write  $J'_j := \{l'_1, \dots, l'_{m'}\}$  and  $J''_j := \{l''_1, \dots, l''_{m''}\}$ . We have  $m_j = m' + m''$ . Now define

$$\gamma_{R_N}^b(t) = \frac{1}{\prod_{s=1}^{m''} \text{sn}(t - \alpha_{l''_s}^N)} \gamma_{R_N}(t).$$

Finally, let

$$w^{(j,N)} := \bar{\partial}_{(\alpha_{l'_1}^N, \dots, \alpha_{l'_{m'}}^N)} \gamma_{R_N}^b(\alpha_j^N).$$

These vectors are linear combinations of vectors of  $\text{span}(\gamma_{R_N})$  and we now show that they tend to  $u_R^{(j)}$ , up to a multiplicative constant, when  $N$  goes to infinity. First let us check that the coordinates that are not in  $\text{supp}_\tau(j)$  are equal to 0. Let  $p \notin \text{supp}_\tau(j)$ . Then  $j \in J_p$  thus  $\gamma_p(\alpha_j) = 0$ . Furthermore, for all  $s \in [m']$ , we have  $l'_s \in J'_j$  thus

$\text{supp}_\tau(l'_s) \subset \text{supp}_\tau(j)$ . Therefore  $p \notin \text{supp}_\tau(l'_s)$  and  $\gamma_p(\alpha_{l'_s}) = 0$ . So for  $p \notin \text{supp}_\tau(j)$ ,  $w_p^{(j,N)} \rightarrow 0$  when  $N$  goes to infinity. It remains to show that the other coordinates converge to the expected result. We know, by the property (12), that  $m'!w^{(j,N)}$  converges to  $(\gamma_R^b)^{(m')}(\alpha_j)$  when  $N \rightarrow \infty$ . Denote by  $g_N(t)$  the function  $\prod_{s=1}^{m''} \text{sn}(t - \alpha_{l'_s}^N)$  and by  $g(t)$  the limit of  $g_N(t)$  as  $N \rightarrow \infty$ . By Leibniz formula, we have

$$(\gamma_{R_N})^{(m_j)}(\alpha_j) = \sum_{s=0}^{m_j} \binom{m_j}{s} (\gamma_{R_N}^b)^{(s)}(\alpha_j) g_N^{(m_j-s)}(\alpha_j).$$

Note that for  $m_j - s < m''$  i.e.  $s > m'$ , we have  $g_N^{(m_j-s)}(\alpha_j) \rightarrow 0$  as  $N \rightarrow \infty$ . And for  $s < m'$  we have  $(\gamma_{R_N}^b)^{(s)}(\alpha_j) \rightarrow 0$  as  $N \rightarrow \infty$ . Therefore

$$\gamma_R^{(m_j)}(\alpha_j) = \binom{m_j}{m'} g^{(m'')}(\alpha_j) (\gamma_R^b)^{m'}(\alpha_j).$$

We have therefore proved that  $u_R^{(j)}$  is a limit of vectors of  $\text{span}(\gamma_{R_N})$  which finishes the proof.  $\square$

## 4 Kramers-Wannier Duality

Let  $\mathcal{T}$  be a traintrack arrangement. We have considered an Ising model on  $G_{\mathcal{T}}^\bullet$  with weights  $(j_e)$  defined by (8). There is a dual Ising model defined on the dual graph  $G_{\mathcal{T}}^\circ$  with weights  $(j_e^*)$  that satisfy the Kramers-Wannier duality relation:

$$\forall e \in E \quad \sinh(2j_e) \sinh(2j_e^*) = 1.$$

It was proved in [BTR19, Section 4] that the dual weights  $j_e^*$  have the same expression than  $j_e$  with the parameter  $k$  replaced by the dual parameter  $k^*$ . In our setting, an Ising model on the dual graph  $G_{\mathbb{A}}^\circ$  can be obtained by switching the black and white colors of the vertices of the diamond graph and relabeling the vectors  $(v_1, \dots, v_{2n})$  into  $(v_2, \dots, v_{2n}, v_1)$ . Both ways give in fact the same model on the dual graph. More precisely, let  $S : \mathbb{R}^{2n} \rightarrow \mathbb{R}^{2n}$  be the *cyclic shift operator* defined by  $S(x_1, x_2, \dots, x_{2n}) = ((-1)^{n-1}x_{2n}, x_1, x_2, \dots, x_{2n-1})$ . The following theorem is proved in [GP20, Theorem 3.4].

**Theorem 3** ([GP20]). *Let  $R = (\tau, \alpha)$  be a region and let  $\mathcal{T}$  be a traintrack arrangement. Let  $(G_{\mathcal{T}}^\bullet, j)$  be an Ising model and  $(G_{\mathcal{T}}^\circ, j^*)$  be the Kramers-Wannier dual Ising model. Let  $M$  and  $M^*$  be the respective correlation matrices. Then*

$$\phi(M) \cdot S = \phi(M^*).$$

For all  $p \in [2n]$ , define

$$\gamma_p^*(t) = \prod_{j=1}^{2\lfloor \frac{p}{2} \rfloor} \frac{\text{dn}^*(t - \tilde{\alpha}_j)}{\sqrt{(k^*)'}} \prod_{j \in \tilde{J}_p} \text{sn}^*(t - \tilde{\alpha}_j).$$



As noticed in Remark 6, computations work in the same way with the dual parameter  $k^*$ , in particular the proof of Theorem 1 is well-adapted and we have  $\phi(M^*) = \text{span}(\gamma_R^*)$ . Note that, thanks to the tilded notation, we get the equality

$$(\gamma_1(t) \ \gamma_2(t) \ \dots \ \gamma_{2n}(t)) \cdot S = (\gamma_0(t) \ \gamma_1(t) \ \dots \ \gamma_{2n-1}(t)).$$

Let us check that our formula for boundary correlations is coherent with the above theorem, and in fact provide with a constructive alternative proof for these particular coupling constants. In fact, we prove something stronger:

**Lemma 1.** *For all  $t \in \mathbb{R}$ , there exists a non-zero real number  $c(t)$  such that*

$$\forall p \in \mathbb{Z}, \quad \gamma_p(t) = c(t)\gamma_{p+1}^*(t).$$

*In particular, we recover that  $\text{span}(\gamma_R) \cdot S = \text{span}(\gamma_R^*)$ .*

*Proof.* Let  $p \in \mathbb{Z}$ . Suppose  $\tau(p) \neq p + 1$ . Then  $J_{p+1} = (J_p \setminus \{p + 1\}) \cup \{\tau(p)\}$ . If  $p$  is even then

$$\frac{\gamma_p(t)}{\gamma_{p+1}^*(t)} = (k')^{p+n-1} \prod_{j=1}^p \frac{1}{\text{dn}(t - \alpha_j)} \prod_{j=0}^{p-1} \frac{1}{\text{dn}(t - \alpha_j)} \prod_{j \in J_p} \frac{1}{\text{dn}(t - \alpha_j)},$$

and

$$\begin{aligned} \frac{\gamma_{p+1}(t)}{\gamma_{p+2}^*(t)} &= (k')^{p+1+n-1} \prod_{j=1}^p \frac{1}{\text{dn}(t - \alpha_j)} \prod_{j=0}^{p+1} \frac{1}{\text{dn}(t - \alpha_j)} \prod_{j \in J_{p+1}} \frac{1}{\text{dn}(t - \alpha_j)} \\ &= (k')^{p+1+n-1} \prod_{j=1}^p \frac{1}{\text{dn}(t - \alpha_j)} \prod_{j=0}^{p+1} \frac{1}{\text{dn}(t - \alpha_j)} \prod_{j \in J_p} \frac{1}{\text{dn}(t - \alpha_j)} \frac{\text{dn}(t - \alpha_{p+1})}{\text{dn}(t - \alpha_{\tau(p)})} \\ &= \frac{\gamma_p(t)}{\gamma_{p+1}^*(t)} \quad \text{since } \text{dn}(t - \alpha_p)\text{dn}(t - \alpha_{\tau(p)}) = k'. \end{aligned}$$

Similarly when  $p$  is odd, we have  $\frac{\gamma_p(t)}{\gamma_{p+1}^*(t)} = \frac{\gamma_{p+1}(t)}{\gamma_{p+2}^*(t)}$ . We also get the result when  $\tau(p) = p + 1$  using the fact that  $\text{dn}(t - \alpha_p)\text{dn}(t - \alpha_{p+1}) = k'$  instead of the previous relation between  $J_p$  and  $J_{p+1}$ . It implies that there exists a constant  $c \in \mathbb{R}^*$  such that for all  $p \in \mathbb{Z}$  we have  $\frac{\gamma_p(t)}{\gamma_{p+1}^*(t)} = c$ .  $\square$

*Remark 7.* When the region is a regular  $2n$ -gon, the constant  $c$  is equal to the product  $\prod_{j=n+1}^{2n-1} \text{dn}(t - \alpha_j)$  and it is immediately clear that it is independent of  $p$ .

## 5 Near critical case

We now explain the computations that one should make in order to study the near critical case, that is to say the case when  $k$  tends to 0. Let  $p \in [2n]$  and let  $t \in \mathbb{R}$ . We do

a Taylor expansion of  $\gamma_p(t, k)$ , which is defined as  $\gamma_p(t)$  but we underline the dependence in  $k$ . First we get, using Equations (24) and (25),

$$\text{sn}(t - \alpha_j) = \sin(t - \alpha_j) \left( 1 + \frac{k^2}{4} \cos^2(t - \alpha_j) \right) + O(k^4)$$

and

$$\text{dn}(t - \alpha_j) = 1 - \frac{1}{2} k^2 \sin^2(t - \alpha_j) + O(k^4).$$

Finally, using the fact that  $k' = \sqrt{1 - k^2}$  we get

$$\frac{1}{\sqrt{k'}} = \frac{1}{(1 - k^2)^{\frac{1}{4}}} = 1 + \frac{k^2}{4} + O(k^4).$$

Combining all the above expansions and expanding the product, we have the following expansion for  $\gamma_p(t, k)$ :

$$\begin{aligned} \gamma_p(t, k) &= \gamma_p(t, 0) \\ &+ \frac{k^2}{4} \prod_{j \in J_p} \sin(t - \alpha_j) \left[ \sum_{j \in J_p} \cos^2(t - \alpha_j) + 2 \sum_{j=1}^{2 \lfloor \frac{p}{2} \rfloor} \cos(2(t - \alpha_j)) \right] + O(k^4). \end{aligned}$$

Let  $R$  be a region  $R = (\tau, \boldsymbol{\alpha})$  where the angles  $(\alpha_i)_{i \in [2n]}$  are all distinct. Recall that, by Proposition 2, we get the spin boundary correlations by computing  $(\Gamma(k)K_n)^{-1}\Gamma(k)$ , where  $\Gamma(k)$  is the matrix with rows  $\gamma_R(\alpha_{j_1}), \dots, \gamma_R(\alpha_{j_n})$  with  $J_1 \cup \{1\} := \{j_1, \dots, j_n\}$  and  $k$  is the elliptic parameter. In the critical case, an explicit expression for the coefficients of  $(\Gamma(0)K_n)^{-1}\Gamma(0)$  is known, see [Gal22, Theorem 1.1]. In the general  $Z$ -invariant case, we have

$$\begin{aligned} (\Gamma(k)K_n)^{-1}\Gamma(k) &= [(\Gamma(0) + (\Gamma(k) - \Gamma(0)))K_n]^{-1}\Gamma(k) \\ &= [(\Gamma(0)K_n)^{-1} - (\Gamma(k) - \Gamma(0))K_n \\ &\quad + O(\|\Gamma(k) - \Gamma(0)\|^2)](\Gamma(0) + (\Gamma(k) - \Gamma(0))) \\ &= (\Gamma(0)K_n)^{-1}\Gamma(0) + (\Gamma(0)K_n)^{-1}(\Gamma(k) - \Gamma(0)) \\ &\quad - (\Gamma(k) - \Gamma(0))K_n\Gamma(0) + O(\|\Gamma(k) - \Gamma(0)\|^2). \end{aligned}$$

Notice that we have an explicit expression for every term in this expansion except  $(\Gamma(0)K_n)^{-1}$ . But note that for  $k = 0$  we have

$$\begin{aligned} \gamma_p(t, 0) &= \prod_{j \in J_p} \sin(t - \alpha_j) \\ &= \prod_{j \in J_p} (\sin(t) \cos(\alpha_j) - \cos(t) \sin(\alpha_j)) \\ &= \sum_{l=1}^n (-\cos(t))^{l-1} \sin(t)^{n-l} \sum_{\substack{H \subset J_p \\ |H|=l+1}} \prod_{j \in H} \sin(\alpha_j) \prod_{j \in J_p \setminus H} \cos(\alpha_j). \end{aligned}$$

Therefore  $\Gamma(0) = V \cdot F$  where  $V$  is an invertible Vandermonde-like matrix with entries  $((-\cos(t_{j_i}))^l \sin(t_{j_i})^{n-l})_{1 \leq i, l \leq n}$  and  $F$  is the  $n \times 2n$  matrix defined by, for all  $1 \leq l \leq n$  and for all  $1 \leq p \leq 2n$ ,

$$F_{l,p} := \sum_{\substack{H \subset J_p \\ |H|=l+1}} \prod_{j \in H} \sin(\alpha_j) \prod_{j \in J_p \setminus H} \cos(\alpha_j).$$

In [Gal22], it is proven that  $F = D \cdot B$  where  $D$  is a diagonal matrix and  $B$  is a  $n \times 2n$  matrix with entries  $B_{lj} = z_l^{j-1}$  with  $z_l = e^{\frac{\pi i}{2n}(2l-n-1)}$ . The inverse of  $BK_n$  can then be computed explicitly, see again [Gal22] for details. Therefore  $(\Gamma(0)K_n)^{-1}$  also has an explicit expression but considering the complexity of this expression, we did not manage to get information from it.

## A Useful identities involving elliptic functions

We state here facts about Jacobi elliptic functions that we use in this article (without proofs). More details can be found in [AS65] and [Law89].

Let  $k \in \mathbb{C}$  such that  $k^2 \in (-\infty, 1)$ . Define  $K = K(k) := \int_0^{\frac{\pi}{2}} \frac{1}{\sqrt{1-k^2 \sin^2(x)}} dx$ , it is called the *complete elliptic integral of the first kind*. Let  $k' := \sqrt{1-k^2}$ , and define the dual elliptic parameter  $k^*$  by  $k^* := i \frac{k}{k'}$ .

We recall our rescaling convention for elliptic functions: if  $p$  and  $q$  are distinct letters among  $c, s, d, n$ , we set for all  $u \in \mathbb{C}$ :

$$\text{pq}(u) = \text{pq}\left(\frac{2K(k)u}{\pi} | k\right), \quad \text{pq}^*(u) = \text{pq}\left(\frac{2K(k^*)u}{\pi} | k^*\right).$$

**Change of arguments** Jacobi's elliptic functions satisfy addition formula by quarter-periods and half-periods (see [AS65, Table 16.8] for the unrescaled version), among which:

$u$	$-u$	$u \pm \frac{\pi}{2}$	$u + \pi$
sn	-sn	$\pm \text{cd}$	-sn
cn	cn	$\mp k' \text{sd}$	-cn
dn	dn	$k' \frac{1}{\text{dn}}$	dn
cd	cd	$\mp \text{sn}$	-cd

Table 1: Formulas for quarter-periods and half-periods

Some values of these functions are known explicitly:

$$\text{sn}\left(\frac{\pi}{4}\right) = \frac{1}{\sqrt{1+k'}} \quad \text{and} \quad \text{sn}\left(\frac{\pi}{2}\right) = 1, \quad (13)$$

$$\operatorname{cn}\left(\frac{\pi}{4}\right) = \frac{\sqrt{k'}}{\sqrt{1+k'}}, \quad (14)$$

$$\operatorname{dn}\left(\frac{\pi}{4}\right) = \frac{1}{\sqrt{k'}}. \quad (15)$$

**Addition formulas** Some versions of addition formulas used in the proof of the main lemma are

$$\forall u, v \in \mathbb{C} \quad \operatorname{dn}(u)\operatorname{sn}(u+v) = \operatorname{cn}(u)\operatorname{sn}(v) + \operatorname{sn}(u)\operatorname{cn}(v)\operatorname{dn}(u+v), \quad (16)$$

$$\forall u, v \in \mathbb{C} \quad \operatorname{cn}(u+v) = \operatorname{cn}(u)\operatorname{cn}(v) - \operatorname{sn}(u)\operatorname{sn}(v)\operatorname{dn}(u+v). \quad (17)$$

They are consequences of more classical addition formulas (see [AS65, 16.17.1–16.17.3]), and are listed in their unrescaled version as exercises in [Law89, Chapter 2, Exercise 32].

**Dual formulas** Some formulas that link elliptic functions and dual elliptic functions are:

$$\forall u \in \mathbb{C} \quad k'\operatorname{sd}(u) = \operatorname{sn}^*(u), \quad (18)$$

$$\forall u \in \mathbb{C} \quad \operatorname{cd}(u) = \operatorname{cn}^*(u), \quad (19)$$

$$\forall u \in \mathbb{C} \quad \frac{1}{\operatorname{dn}(u)} = \operatorname{dn}^*(u). \quad (20)$$

See [AS65, Section 16.10].

**Taylor expansion when  $k \rightarrow 0$**

$$K(k) = \frac{\pi}{2} + \frac{\pi}{8}k^2 + O(k^4), \quad (21)$$

$$\forall u \in \mathbb{C} \quad \operatorname{sn}(u) = \sin(u) - \frac{1}{4}k^2(u - \sin(u)\cos(u))\cos(u) + O(k^4), \quad (22)$$

$$\forall u \in \mathbb{C} \quad \operatorname{dn}(u) = 1 - \frac{1}{2}k^2\sin^2(u) + O(k^4). \quad (23)$$

See [AS65, Section 16.13]. From by (21), we have

$$\sin\left(\frac{2K(k)u}{\pi}\right) = \sin(u) + \frac{k^2}{4}u\cos(u) + O(k^4), \quad \cos\left(\frac{2K(k)u}{\pi}\right) = \cos(u) - \frac{k^2}{4}u\sin(u) + O(k^4).$$

Injecting these expansions in the rescaled functions  $\operatorname{sn}$  and  $\operatorname{dn}$ , one gets:

$$\begin{aligned}
\operatorname{sn}(u) &= \operatorname{sn}\left(\frac{2K(k)}{\pi}u|k\right) \\
&= \sin\left(\frac{2K(k)u}{\pi}\right) - \frac{k^2}{4} \left( \frac{2K(k)u}{\pi} - \sin\left(\frac{2K(k)u}{\pi}\right) \cos\left(\frac{2K(k)u}{\pi}\right) \right) \cos\left(\frac{2K(k)u}{\pi}\right) + O(k^4) \\
&= \sin(u)\left(1 + \frac{k^2}{4} \cos^2(u)\right) + O(k^4)
\end{aligned} \tag{24}$$

and

$$\begin{aligned}
\operatorname{dn}(t - \alpha_j) &= 1 - \frac{1}{2}k^2 \sin^2\left(\frac{2K(k)}{\pi}(t - \alpha_j)\right) + O(k^4) \\
&= 1 - \frac{1}{2}k^2 \sin^2(t - \alpha_j) + O(k^4).
\end{aligned} \tag{25}$$

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