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# Singularity Robust Inverse Kinematics of Serial Manipulators by means of a Joint Arc Length Parameterization

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**Abstract.** The inverse kinematics problem is ill-posed near kinematic singularities leading to extremely large joint velocities. This classical problem is addressed from a different perspective; instead of prescribing the end-effector (EE) motion in terms of a path parameter, the motion is parameterized in terms of the arc length in joint space. This allows tackling the inverse kinematics problem when approaching kinematic singularities. Thereupon a singularity consistent sampling scheme can easily be devised. A sampling of the arc length gives direct control over the joint motion for prescribed EE motion. The arc length parameterization can be used for solving the (time) optimal path following problem, avoiding the ill-conditioning upfront.

**Keywords:** Serial robots · inverse kinematics · singularities · path planning · arc length parameterization · non-equidistant discretization

## 1 Introduction

A solution of the inverse kinematics problem is a relation of the EE-motion and the joint motion. Near a kinematic singularity, small EE-velocities lead to extremely large joint velocities, which is a well-known problem for the robot control. To mitigate this problem, path planning strategies for avoiding singularities exist [9]. The problem of large velocities (causing wear of mechanical components and possibly reaching the rated actuator limits) remains, however, even when the robot is only near a singularity. Moreover, often, a prescribed EE-path is deduced from process requirement (e.g. welding, polishing) without taking singularities into account. The velocity inverse kinematics map (defined by the generalized inverse of the forward kinematics Jacobian) transforms EE-velocities to joint velocities, and its metric properties change drastically near singularities. The EE-motion along a prescribed path is described by a path parameter  $\sigma$ . An equidistant sampling of the EE-path is translated to extreme changes in the sampling intervals in joint space. The length of the corresponding path in joint space is described by the arc length  $s$ . Now an obvious approach to

avoid high joint velocities, i.e. large changes of  $s$  within a time step, is to adaptively reduce the EE-velocity, i.e. the change of  $\sigma$ , when traversing the path. Moreover, the EE-motion could be described in terms of  $s$ , and thus be sampled according to prescribed (e.g. equidistant) increments of  $s$ , allowing to prescribe the joint velocity while following a given EE-path.

In this paper, the inverse kinematics problem is addressed for the 1) parameterization of an EE-path  $\mathbf{z}(\sigma)$  in terms of the path parameter  $\sigma$  [3,7], and 2) the parameterization of the joint motion  $\mathbf{q}(s)$  in terms of the (kinetic) arc length  $s$ . Each of these parameterizations gives rise to a particular inverse kinematics problem. Both are derived in this paper, and computational methods for their solution are proposed. The two parameterizations refer exclusively to the geometry of motion, and are independent of the temporal execution of the motion. Clearly, a smooth motion in terms of joint arc length  $s$  is desirable (rather than in the path parameter  $\sigma$ , as it is the standard approach). It thus provides an avenue for smooth singularity-robust optimal motion planning.

## 2 Kinematics in Terms of Path Parameter

### 2.1 Forward kinematics

Denote with  $\mathbf{q} \in \mathbb{R}^n$  the vector of joint variables, and with  $\mathbf{z} \in SE(3)$  the EE-pose. Throughout the paper, the EE-path  $\mathbf{z}(\sigma)$  is assumed to be given, e.g. from a path planning, and is parameterized in terms of a path parameter  $\sigma(t) \in [0, 1]$ . Since the joint motion is determined from the EE-motion, this induces a parameterization of joint coordinates in terms of  $\sigma$ . The latter is usually used for motion planning and control. Notice, that the relation of path parameter in work space and joint space is generally not unique. With slight abuse of notation, the same symbol is used throughout the paper. The geometric forward kinematics map

$$\mathbf{z} = f(\mathbf{q}) \quad (1)$$

determines the EE-pose for prescribed joint coordinates  $\mathbf{q}$  [14]. The EE-twist, containing the translation and angular velocity, is determined by the velocity forward kinematics

$$\mathbf{V}(\mathbf{q}, \dot{\mathbf{q}}) = \begin{pmatrix} \mathbf{v} \\ \boldsymbol{\omega} \end{pmatrix} = \mathbf{J}(\mathbf{q}) \dot{\mathbf{q}} \quad (2)$$

where  $\mathbf{J}(\mathbf{q}) = \partial \mathbf{V} / \partial \dot{\mathbf{q}}$  is the geometric Jacobian. As the EE-twist depends linearly on the joint rates, and since  $\dot{\mathbf{q}} = \mathbf{q}' \dot{\sigma}$ , the velocity forward kinematics (2) can be expressed in terms of  $\dot{\sigma}$  as

$$\mathbf{V} = \frac{\partial \mathbf{V}}{\partial \dot{\sigma}} \dot{\sigma} = \mathbf{J} \mathbf{q}' \dot{\sigma}. \quad (3)$$

This yields  $\mathbf{V}(\sigma, \dot{\sigma}) = \mathbf{h}(\sigma) \dot{\sigma}$  and the relation

$$\mathbf{h}(\sigma) = \mathbf{J}(\mathbf{q}(\sigma)) \mathbf{q}'(\sigma) \quad (4)$$

where  $\mathbf{h}(\sigma) := \partial \mathbf{V} / \partial \dot{\sigma}$  is the *partial path-velocity*. The relation (4) is referred to as the *velocity forward path-kinematics problem*, which will be central for the singularity-robust adaptive sampling. Here  $\mathbf{z}(\sigma(t))$ ,  $\mathbf{h}(\sigma)$ ,  $\dot{\sigma}$ , and thus  $\mathbf{V}(\sigma, \dot{\sigma})$  are given from the trajectory planning (which is not the topic of this paper).

## 2.2 Inverse kinematics

The standard inverse kinematics problem consists in finding the joint coordinates for given an EE-pose  $\mathbf{z}$ . This can be solved analytically for particular robots, such as wrist-partitioned serial manipulators [1]. In general, however, it can only be solved numerically. Various different computational methods were discussed in [1,6,11,15], including nonlinear root-finding, fixed point iteration, integration of the joint velocity. In this paper, the numerical integration of the joint velocity is pursued. This amounts to solving the relation (2) for  $\mathbf{q}(t)$  with given EE-twist  $\mathbf{V}(t)$ . When the path is parameterized with  $\sigma(t)$  the EE-twist is determined by the partial path-velocity via (3). Then the curve, and thus  $\mathbf{q}$ , is uniquely determined by (4), where the partial path velocity  $\mathbf{h}(\sigma)$  describes the motion along the path.

The *velocity inverse path-kinematics problem* is to determine  $\mathbf{q}'$  for given partial path velocity  $\mathbf{h}$ . A solution to the velocity inverse path-kinematics is

$$\mathbf{q}' = \mathbf{J}(\mathbf{q})^+ \mathbf{h} \quad (5)$$

where  $\mathbf{J}(\mathbf{q})^+$  is a generalized pseudo inverse [2,10]. The *geometric inverse path-kinematics problem* is to determine the joint motion  $\mathbf{q}(\sigma)$  in terms of the path parameter  $\sigma$ . The solution of the inverse path-kinematics problem is the solution of the initial value problem (IVP) (5) with  $\mathbf{q}(0) = \mathbf{q}_0$  as initial value. When numerically solving the IVP (5), numerical drifts are unavoidable. In order to reduce the drift, the partial path-velocity error is introduced as

$$\mathbf{e}' = \mathbf{h}_d - \mathbf{J}(\mathbf{q}) \mathbf{q}' \quad (6)$$

with  $\mathbf{h}_d$  denoting the prescribed partial path-velocity. Therewith, the system (5) is amended as

$$\mathbf{q}' = \mathbf{J}(\mathbf{q})^+ (\mathbf{h}_d + \mathbf{K}\mathbf{e}) \quad (7a)$$

$$\mathbf{0} = \mathbf{e}' + \mathbf{K}\mathbf{e} \quad (7b)$$

where (7b) governs the error dynamics. The latter asymptotically converges to zero for  $\mathbf{K} > 0$  [4]. The geometric error  $\mathbf{e}$  in (7b) comprises the position error and orientation error, and can be expressed as

$$\mathbf{e} = \begin{pmatrix} \mathbf{e}_p \\ \mathbf{e}_o \end{pmatrix} = \begin{pmatrix} \mathbf{r}_d - \mathbf{r}(\mathbf{q}) \\ \frac{1}{2}(\tilde{\mathbf{n}}(\mathbf{q})\mathbf{n}_d + \tilde{\mathbf{s}}(\mathbf{q})\mathbf{s}_d + \tilde{\mathbf{a}}(\mathbf{q})\mathbf{a}_d) \end{pmatrix} \quad (8)$$

where  $\mathbf{r}_d$  is the prescribed EE-position, and  $\mathbf{R}_d = [\mathbf{n}_d \ \mathbf{s}_d \ \mathbf{a}_d]$  the prescribed EE-orientation, with columns  $\mathbf{n}, \mathbf{s}, \mathbf{a}$  [13], and  $\tilde{\mathbf{x}} \in \mathbb{R}^{3,3}$  is the skew symmetric (cross

product) matrix associated to  $\mathbf{x} \in \mathbb{R}^3$ . With these definitions and a proper choice of  $\mathbf{K}$ , the velocity inverse path-kinematics problem can be solved numerically stable using time integration schemes. In summary, the solution of (5) yields a parameterization of the joint coordinates  $\mathbf{q}(\sigma)$  in terms of the parameter  $\sigma$ .

### 3 Kinematics in Terms of Arc Length

#### 3.1 Arc length parameterization

The inverse path-kinematics problem is ill-posed near singularities which not only leads to numerical problems, but also causes extremely fast and undesirable joint motions. Following a path with constant speed  $\dot{\sigma}_0$ , for instance, leads to an EE-motion with a well-defined bounded EE-velocity. The corresponding inverse kinematics solution  $\mathbf{q}(t)$  will not have well-defined bounds, especially at forward kinematics singularities, and the joint velocity  $\dot{\mathbf{q}}(t)$  tends to infinity. As a consequence, when prescribing the EE-motion by an equidistant sampling of the path parameter  $\sigma$ , the 'distance' of joint coordinates  $\mathbf{q}$  corresponding to two consecutive values of  $\sigma$  tend to vary drastically. A measure of 'distance' of two joint coordinate vectors is the arc length  $s$  defined by the line element  $ds = \sqrt{d\mathbf{q}^T \mathbf{M} d\mathbf{q}}$  with metric  $\mathbf{M}$  (using the generalized mass matrix  $\mathbf{M}$  of the robot would yield the kinetic line element and ensure homogeneity). The arc length gives rise to a well-defined discretization of the curve in joint space in terms of the step size. For the rest of this paper  $\mathbf{M}$  is chosen as identity matrix.

The arc length path-kinematics problem is to determine the EE-motion  $\mathbf{z}(s)$  as a function of  $s$ , and thus to derive parameterizations  $\sigma = \sigma(s)$  and  $\mathbf{q} = \mathbf{q}(s)$ . It is known from the theory of metric curves that there is no such explicit relation in general. Therefore, the problem is formulated as an ordinary differential equation (ODE). With the path-parameterization  $\mathbf{q}(\sigma)$ , the arc length in joint space is [5]

$$s = \int_0^\sigma \|\mathbf{q}'\|_{\mathbf{M}} d\xi \quad (9)$$

with  $\|\mathbf{q}'\|_{\mathbf{M}} := \sqrt{\mathbf{q}'^T \mathbf{M} \mathbf{q}'}$  denoting the *joint path speed*. This leads to  $\partial s / \partial \sigma = \|\mathbf{q}'\|_{\mathbf{M}}$ , and with (4) follows

$$\frac{\partial \sigma}{\partial s} = \frac{1}{\|\mathbf{J}^+ \mathbf{h}\|_{\mathbf{M}}} = \frac{1}{\|\mathbf{q}'\|_{\mathbf{M}}}, \quad \frac{\partial \mathbf{q}}{\partial s} = \frac{\mathbf{J}^+ \mathbf{h}}{\|\mathbf{J}^+ \mathbf{h}\|_{\mathbf{M}}} = \frac{\mathbf{q}'}{\|\mathbf{q}'\|_{\mathbf{M}}} \quad (10)$$

with  $\mathbf{q}'$  as (5). This ODE system can be solved with given initial conditions  $\sigma_0$  and  $\mathbf{q}_0$  for  $s = 0$ .

#### 3.2 Numerical solution of the initial value problem

The system (10) can be solved for a given parameter interval  $s \in [0, s_T]$ . This necessitates knowledge of the terminal arc length  $s_T$ , which is unknown, however. The condition on the terminal arc length is that

$$\sigma(s_T) = 1. \quad (11)$$

This serves as indicator function to terminate the integration. That is, the system (10) is solved numerically, until  $\sigma(s) = 1$ . This can be carried out numerically stable using integration schemes with zero-crossing event detection using advanced root finding methods. For practical applications, the integration can be stopped simply when  $\sigma > 1$ . Notice that the solution  $\sigma(s)$  merely serves for this detection of terminal arc length.

As for the standard geometric inverse path-kinematics problem, the numerical drift of the arc length parameterized inverse path-kinematics subsystem (10) must be stabilized. This is again achieved by introducing the path-velocity error

$$\frac{\partial \mathbf{e}}{\partial s} = \frac{\partial \mathbf{h}_d}{\partial s} - \mathbf{J} \frac{\partial \mathbf{q}}{\partial s}, \quad (12)$$

with  $\partial \mathbf{h}_d / \partial s = \mathbf{h}'_d / \|\mathbf{q}'\|_{\mathbf{M}}$ . The system (10) is amended as

$$\frac{\partial \mathbf{q}}{\partial s} = \frac{\mathbf{J}^+ \mathbf{h}}{\|\mathbf{J}^+ \mathbf{h}\|_{\mathbf{M}}} + \mathbf{J}^+ \mathbf{K} \mathbf{e} \quad (13a)$$

$$\mathbf{0} = \frac{\partial \mathbf{e}}{\partial s} + \mathbf{K} \mathbf{e} \quad (13b)$$

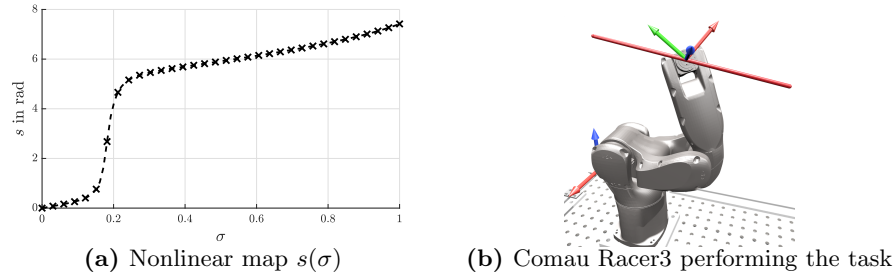
where (13b) governs the error dynamics. Asymptotically convergence to zero is ensured for  $\mathbf{K} > 0$ .

The numerical solution must take into account situations where  $\mathbf{q}' = \mathbf{0}$ , due to the division by  $\|\mathbf{q}'\|_{\mathbf{M}}$ . This can happen for two reasons. The first case, when  $\mathbf{h} = \mathbf{0}$  is avoided by assuming a regular EE-path so that always  $\mathbf{h} \neq \mathbf{0}$ . Secondly, the partial path-velocity  $\mathbf{h}$  is an element of the null space  $\mathbf{h} \in \ker \mathbf{J}^+$ . As a consequence, the corresponding EE movement has no effect on any joint coordinates. This, then is a contradiction to the forward kinematics.

## 4 A singularity-consistent sampling scheme

The adaptive sampling strategy is based on the velocity inverse path-kinematics problem. It is assumed that the EE-path using  $s$  as parameter is equivalent to the curve parameterized with the arc length  $s$ . As mentioned earlier, prescribing the EE-motion by an equidistant sampling  $\Delta\sigma$  inevitably causes drastically varying joint coordinates  $\mathbf{q}$ , of two consecutive samples, when solving the inverse path kinematics problem (5). The 'distance' between consecutive values of  $\mathbf{q}$  is measured by the arc length (9). The latter is a nonlinear relation of  $s$  and  $\sigma$ . For illustration purpose, Fig. 1a shows the mapping  $s(\sigma)$  for a 6-DOF *Comau Racer3* robot following a straight line in the work space with constant orientation and speed  $\dot{\sigma}_0$ , Fig. 1b. It can be seen that the manipulator passes near a singularity at  $\sigma \approx 0.15 \dots 0.25$ , indicated by the drastically increase of the joint arc length.

By parameterizing the prescribed path using the joint arc length  $s$ , Sec. 3, the evolution of the joint coordinates  $\mathbf{q}$  and the relation between  $s$  and  $\sigma$  is maintained. Following the trajectory then with constant *arc length speed*  $\dot{s}_0$  and equidistant sampling for  $s$ , a varying speed  $\dot{\sigma}$  due to the nonlinear mapping



**Fig. 1:** 6-DOF Comau Racer3 following a straight line in task-space with constant orientation and nonlinear relation between  $s$  and  $\sigma$ .

(9) is induced. As a consequence of the relation, smaller speeds  $\dot{\sigma}$  are achieved at higher rates  $\partial s/\partial\sigma$  leading to more sampling points in  $\sigma$  (locally). Since  $\dot{\mathbf{q}} = \mathbf{q}'\dot{\sigma}$ , lower joint velocities are expected in such regions. Moreover, this can be used as an adaptive sampling strategy for  $\sigma$  due to the nonlinear mapping. This discretization scheme, then can be applied for solving time optimal path following problems, e.g. [8,12,16].

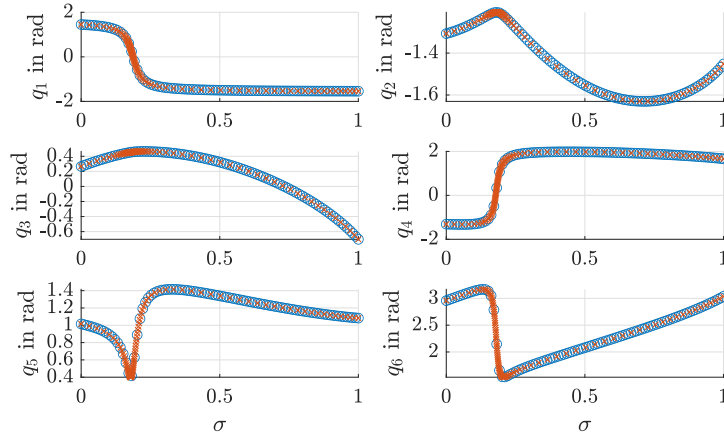
## 5 Simulation results

Again considering the 6-DOF *Comau Racer3* robotic arm manipulator following a straight line with constant orientation, Fig. 1. In order to get comparable results, the EE follows the prescribed path with 1) constant speed  $\dot{\sigma}_0$  and equidistant sampling in  $\sigma$  when using the velocity inverse path-kinematics (5), and 2) constant arc length speed  $\dot{s}_0$  with equidistant discretization in  $s$  for the arc length parameterized velocity inverse path-kinematics (10). As shown in Fig. 2, the same joint coordinate samples are obtained with both strategies. The arc length parameterized solution in Fig. 1a shows an accumulation of sampling points at  $\sigma \in [0.15, 0.25]$ , which indicates the passing near a singularity (Sec. 4).

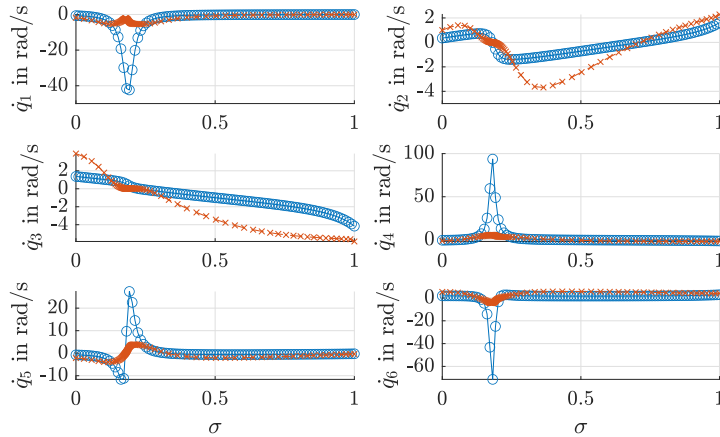
As previously discussed, for fixed step size  $\Delta s$  (which means fixed time step size due to constant  $\dot{s}_0$ ), higher sampling rates of the joint coordinates in this area are induced and hence lower joint velocities  $\dot{\mathbf{q}} = \mathbf{q}'\dot{\sigma}$  as Fig. 3 shows. This is particularly pronounced for  $\dot{q}_4$ . The arc length parameterized velocity inverse path-kinematic solution yields a joint velocity  $\dot{q}_4$  more than 10 times lower than the solution obtained with (5).

## 6 Conclusion and outlook

The inverse kinematics problem was expressed in terms of the arc length  $s$  in joint space when the EE-motion is parameterized by a path parameter  $\sigma$ . This leads to an ODE system for  $\mathbf{q}(s)$  and  $\sigma(s)$  which is solved for  $s \in [0, s_T]$ . It is shown that using the arc length as independent parameter (instead of  $\sigma$ ) allows



**Fig. 2:** Samples of the joint coordinates  $\mathbf{q}$ . Blue circles: using solution (5) for the inverse path kinematic with constant speed  $\dot{\sigma}_0$ . Orange crosses: Using solution (10) for arc length parameterized inverse path kinematics with constant arc length speed  $\dot{s}_0$



**Fig. 3:** Resulting joint velocities  $\dot{\mathbf{q}}$ . Blue circles: using (5) for the inverse path kinematics with constant speed  $\dot{\sigma}_0$ . Orange crosses: using (10) for the inverse path kinematics with constant speed  $\dot{s}_0$ .

for coping with inverse kinematic singularities. Moreover, the joint motion can be fully controlled, instead of controlling the EE-speed. Instead of large joint motions near singularities, the method leads to reduced EE-motions. As shown in the example, the arc length parameterized velocity inverse path-kinematics solution (10) gives rise to an adaptive sampling scheme that yields significantly smaller joint velocities. The introduced approach is applicable to time-optimal



path following that is robust against singularities. This will be addressed in future work. Preliminary results indicate that the optimization problem is better conditioned and excessive joint velocities are avoided.

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## References

1. Aristidou, A., Lasenby, J.: In: Inverse kinematics: a review of existing techniques and introduction of a new fast iterative solver (2009)
2. Ben-Israel, A., Greville, T.N.: Generalized inverses: theory and applications, vol. 15. Springer Science & Business Media (2003)
3. Breivik, M., Fossen, T.: Path following for marine surface vessels. In: Oceans '04 MTS/IEEE Techno-Ocean '04 (IEEE Cat. No.04CH37600). vol. 4, pp. 2282–2289 Vol.4 (2004)
4. Bruno Siciliano, Lorenzo Sciavicco, L.V.G.O. (ed.): Robotics, chap. Differential Kinematics and Statics, pp. 105–160. Springer London, London (2009)
5. Carmo, M.P.d.: Differential Geometry of Curves and Surfaces: Revised and Updated Second Edition. Courier Dover Publications (2016)
6. Goldenberg, A.A., Benhabib, B., Fenton, R.: A Complete Generalized Solution to the Inverse Kinematics of Robots. Robotics and Automation, IEEE Journal of **1**, 14–20 (Apr 1985)
7. Hladio, A., Nielsen, C., Wang, D.: Path following for a class of mechanical systems. IEEE Transactions on Control Systems Technology **21**(6), 2380–2390 (2013)
8. Kaserer, D., Gatringer, H., Müller, A.: Time optimal motion planning and admittance control for cooperative grasping. IEEE Robotics and Automation Letters **5**(2), 2216–2223 (2020)
9. Marani, G., Kim, J., Yuh, J., Chung, W.K.: A real-time approach for singularity avoidance in resolved motion rate control of robotic manipulators. Proceedings 2002 IEEE International Conference on Robotics and Automation (Cat. No.02CH37292) **2**, 1973–1978 vol.2 (2002)
10. Penrose, R.: A generalized inverse for matrices. Mathematical Proceedings of the Cambridge Philosophical Society p. 406–413 (1955)
11. Reiter, A., Mueller, A., Gatringer, H.: On higher-order inverse kinematics methods in time-optimal trajectory planning for kinematically redundant manipulators. IEEE Transactions on Industrial Informatics **14**, 1681 – 1690 (01 2018)
12. Shin, K., McKay, N.: A dynamic programming approach to trajectory planning of robotic manipulators. IEEE Transactions on Automatic Control **31**(6), 491–500 (1986)
13. Siciliano, B.: A closed-loop inverse kinematic scheme for on-line joint-based robot control. Robotica **8**(3), 231–243 (1990)
14. Siciliano, B., Khatib, O. (eds.): Springer Handbook of Robotics, chap. Kinematics, pp. 11–36. Springer (2016)
15. Thomopoulos, S.C., Tam, R.Y.: An iterative solution to the inverse kinematics of robotic manipulators. Mechanism and Machine Theory **26**(4), 359–373 (1991)
16. Verscheure, D., Demeulenaere, B., Swevers, J., De Schutter, J., Diehl, M.: Time-optimal path tracking for robots: A convex optimization approach. IEEE Transactions on Automatic Control **54**(10), 2318–2327 (2009)