

A note on the Casas-Alvero Conjecture

Daniel Schaub, Univ Angers, CNRS, LAREMA, SFR MATHSTIC
F-49000 Angers, France
email: daniel.schaub@univ-angers.fr

Mark Spivakovsky, Univ Paul Sabatier, CNRS, IMT UMR 5219
F-31062 Toulouse, France and
CNRS, LaSol UMI 2001, UNAM.
email: mark.spivakovsky@math.univ-toulouse.fr

November 7, 2024

Abstract

The Casas–Alvero conjecture predicts that every univariate polynomial f over a field K of characteristic zero having a common factor with each of its derivatives $H_i(f)$ is a power of a linear polynomial. Let $f = x^d + a_1x^{d-1} + \cdots + a_1x \in K[a_1, \dots, a_{d-1}][x]$ and let $R_i = \text{Res}(f, H_i(f)) \in K[a_1, \dots, a_{d-1}]$ be the resultant of f and $H_i(f)$, $i \in \{1, \dots, d-1\}$. The Casas–Alvero Conjecture for algebraically closed fields is equivalent to saying that R_1, \dots, R_{d-1} are “independent” in a certain sense, namely that the height $ht(R_1, \dots, R_{d-1}) = d-1$ in $K[a_1, \dots, a_{d-1}]$. In this paper we prove a partial result in this direction: if $i \in \{d-3, d-2, d-1\}$ then $R_i \notin \sqrt{(R_1, \dots, \check{R}_i, \dots, R_{d-1})}$.

1 Introduction

In the year 2001 Eduardo Casas–Alvero published a paper on higher order polar germs of plane curve singularities [1]. His work on polar germs inspired him to make the following conjecture (according to the testimony of José Manuel Aroca, E. Casas communicated the problem orally well before 2001).

A lot of work on the Casas–Alvero conjecture has been done since 2001. Some of the most important partial results can be found in [1–7].

Let K be a field, d a strictly positive integer and $f = x^d + a_1x^{d-1} + \cdots + a_{d-1}x + a_d$ a monic univariate polynomial of degree d over K . Let

$$H_i(f) = \binom{d}{i}x^{d-i} + \binom{d-1}{i}a_1x^{d-i-1} + \cdots + \binom{i}{i}a_{d-i}$$

be the i -th Hasse derivative of f .

Definition 1 *The polynomial f is said to be a **Casas–Alvero polynomial** if for each $i \in \{1, \dots, d-1\}$ it has a non-constant common factor with its i -th Hasse derivative $H_i(f)$.*

Note that, by definition, a Casas–Alvero polynomial f has a common root with $H_{d-1}(f)$. In particular, if $\text{char } K = 0$, it has at least one root $b \in K$, regardless of whether or not K is algebraically closed. Making the change of variables $x \rightsquigarrow x - b$, we may assume that 0 is a

root of f , in other words, $a_d = 0$. In the sequel, we will always make this assumption without mentioning it explicitly.

Conjecture 1 (Casas–Alvero) *Assume that $\text{char } K = 0$. If $f \in K[x]$ is a Casas–Alvero polynomial of degree d with $a_d = 0$, then $f(x) = x^d$.*

For $i \in \{1, \dots, d-1\}$, let $R_i = \text{Res}(f, H_i(f)) \in K[a_1, \dots, a_{d-1}]$ be the resultant of f and $H_i(f)$. The polynomials f and $H_i(f)$ have a common factor if and only if $R_i = 0$. Thus f is Casas–Alvero if and only if the point $(a_1, \dots, a_{d-1}) \in K^{d-1}$ belongs to the algebraic variety $V(R_1, \dots, R_{d-1}) \subset K^{d-1}$. In those terms the Conjecture can be reformulated as follows:

Conjecture 2 *Let $V = V(R_1, \dots, R_{d-1}) \subset K^{d-1}$. Then $V = \{0\}$.*

When K is algebraically closed, this is also equivalent to

Conjecture 3

$$\sqrt{(R_1, \dots, R_{d-1})} = (a_1, \dots, a_{d-1}) \quad (1)$$

or, equivalently,

$$a_i^N \in (R_1, \dots, R_{d-1}) \text{ for all } i \in \{1, \dots, d-1\} \text{ and some } N \in \mathbb{N}. \quad (2)$$

Remark 2 (i) *When K is not algebraically closed, Conjecture 3 is a priori stronger.*

(ii) *Let $K \subset K'$ be a field extension. The induced extension*

$$K[a_1, \dots, a_{d-1}] \subset K'[a_1, \dots, a_{d-1}]$$

is faithfully flat. Since the polynomials R_1, \dots, R_{d-1} have coefficients in \mathbb{Z} , (2) holds in $K[a_1, \dots, a_{d-1}]$ if and only if it holds in $K'[a_1, \dots, a_{d-1}]$. Hence the truth of Conjecture 3 depends only on the characteristic of K but not on the choice of the field K itself. Because of this, we will take $K = \mathbb{C}$ in the sequel.

Remark 3 *Formulae (1) and (2) can be interpreted in terms of Gröbner bases. Namely, (1) and (2) are equivalent to saying that for any choice of monomial ordering and Gröbner basis (f_1, \dots, f_s) of (R_1, \dots, R_{d-1}) , after renumbering the f_j , the leading monomial of f_j is a power of a_j for $j \in \{1, \dots, d-1\}$.*

Remark 4 *Conjecture 3 and Remark 3 say that, as polynomials in $K[a_1, \dots, a_{d-1}]$, the resultants R_1, \dots, R_{d-1} are “independent” in a certain sense.*

Each of the following statements is also equivalent to the Casas–Alvero Conjecture.

(a) *For each $i \in \{1, \dots, d-2\}$, the element R_{i+1} is not a zero divisor modulo (R_1, \dots, R_i) (in other words, R_1, \dots, R_{d-1} form a regular sequence in $K[a_1, \dots, a_{d-1}]$).*

(b) *For each $i \in \{1, \dots, d-2\}$,*

$$R_{i+1} \notin \bigcup_{\mathfrak{p} \in \text{Ass}((R_1, \dots, R_i))} \mathfrak{p}.$$

where $\text{Ass}((R_1, \dots, R_i))$ is the set of associated primes of the ideal (R_1, \dots, R_i) .

Moreover, the above statements (a) and (b) are independent of the numbering of the R_i (since the R_i are quasi-homogeneous); a permutation of the R_i yields equivalent statements.

Notation. We will denote by $(R_1, R_2, \dots, \check{R}_i, \dots, R_{d-1})$ the ideal of $K[a_1, \dots, a_{d-1}]$ generated by the set $\{R_1, R_2, \dots, R_{d-1}\} \setminus \{R_i\}$.

The main theorem of this paper is the following partial result in the direction of Conjecture 3 and statements (a) and (b) of Remark 4.

Theorem 5 *Take an element $i \in \{d-3, d-2, d-1\}$. We have*

$$R_i \notin \sqrt{(R_1, R_2, \dots, \check{R}_i, \dots, R_{d-1})}.$$

2 Ideals generated by all the resultants but one

In this section we prove Theorem 5 after recalling some preliminary results.

Proposition 6 *Let f be a polynomial of degree d with real roots $\beta_1 \leq \beta_2 \leq \dots \leq \beta_d$. Then $H_1(f)$ has real roots $\gamma_1 \leq \gamma_2 \leq \dots \leq \gamma_{d-1}$, counted with multiplicity, where $\gamma_i \in]\beta_i, \beta_{i+1}[$ if $\beta_i < \beta_{i+1}$ and $\gamma_i = \beta_i$ if $\beta_i = \beta_{i+1}$.*

Proof: Assume that f has s distinct roots $\delta_1 < \delta_2 < \dots < \delta_s$ of multiplicities m_1, \dots, m_s , respectively. Then δ_j is a root of $H_1(f)$ of multiplicity $m_j - 1$, where we say that δ_j is a root of multiplicity 0 if it is not a root of $H_1(f)$.

By Rolle's theorem, there is at least one root of $H_1(f)$ in each of the $s-1$ open intervals $]\delta_1, \delta_2[, \dots,]\delta_{s-1}, \delta_s[$.

Notation. Let $\text{Int}(\beta_i, \beta_{i+1}) :=]\beta_i, \beta_{i+1}[$ if $\beta_i < \beta_{i+1}$ and $\text{Int}(\beta_i, \beta_{i+1}) := \{\beta_i\}$ if $\beta_i = \beta_{i+1}$.

According to the above, there is at least one real root of $H_1(f)$ in each of $\text{Int}(\beta_i, \beta_{i+1})$, $i \in \{1, \dots, d-1\}$, where $\gamma_1 \in \text{Int}(\beta_1, \beta_2), \dots, \gamma_{m_1-1} \in \text{Int}(\beta_{m_1-1}, \beta_{m_1})$ are the first $m_1 - 1$ roots of $H_1(f)$ (in fact, the same root counted with multiplicity $m_1 - 1$) and similarly for the other multiple roots of f .

We have accounted for a total of $s-1 + (m_1-1) + \dots + (m_s-1) = m_1 + \dots + m_s - 1 = d-1$ real roots of $H_1(f)$ counted with multiplicities. Hence $H_1(f)$ has no roots, real or complex, other than the ones listed above, and the result follows. \square

Corollary 7 *Let f be a polynomial of degree d with d real roots, counted with multiplicities. Then each of the $H_i(f)$, $i \in \{1, \dots, d-1\}$, has $d-i$ real roots, counted with multiplicity. In other words, all the roots of $H_i(f)$ are real.*

Next, we recall a result from [4] on almost counterexamples to the Casas-Alvero conjecture.

Definition 8 *Fix an $i \in \{1, \dots, d-1\}$. An almost counterexample to the Casas-Alvero conjecture of level i is a polynomial f that has a common root with $H_j(f)$ for all $j \in \{1, \dots, d-1\} \setminus \{i\}$ but is not a power of a linear polynomial.*

Notation. Given a polynomial f of degree d with d real roots, for a pair (k, m) of integers with $1 \leq k \leq d-1$ and $1 \leq m \leq d-k$ we write $\alpha_{k,m}(f)$ for the m -th root of $H_k(f)$, where the roots of $H_k(f)$ are ordered (weakly) increasingly.

We state the next theorem in a somewhat stronger form than in [4]: the extra information about the recycled roots $\alpha_{k_j, m_j}(f)$ does not appear explicitly in the statement of the result in [4], but is shown in the course of its proof.

Theorem 9 (J. Draisma–J. P. de Jong [4], Theorem 5) *Fix $d - 2$ pairs of integers*

$$(k_j, m_j), \quad j \in \{1, \dots, d - 2\},$$

with

$$1 \leq k_1 < k_2 < \dots < k_{d-2} \leq d - 1$$

and $1 \leq m_j \leq d - k_j$. There exists a polynomial $f \in \mathbb{R}[x]$ with $f(0) = f(1) = 0$, all of whose roots are real and lie in $[0, 1]$, such that $\alpha_{k_j, m_j}(f)$ is a root of f for all $j \in \{1, \dots, d - 2\}$ (in particular, f is an almost counterexample to the Casas-Alvero conjecture of level i , where i is the unique element of the set $\{1, \dots, d - 1\} \setminus \{k_1, \dots, k_{d-2}\}$).

We also recall the following result, Theorem 13 of [2]:

Theorem 10 *Assume that f is a counterexample to the Casas-Alvero Conjecture. Then f has at least five distinct roots.*

Proof of Theorem 5: We argue by contradiction. Assume that

$$R_i \in \sqrt{(R_1, \dots, \check{R}_i, \dots, R_{d-1})}. \quad (3)$$

Let f be an almost counterexample of level i to the Casas-Alvero conjecture with $m_j = 1$ for all $j \in \{1, \dots, d - 2\}$, given by Theorem 9. By (3), f is a Casas-Alvero polynomial that is a counterexample to the Casas-Alvero conjecture. By definition of f , $\alpha_{k_j 1}(f)$ is the first root of $H_{k_j}(f)$ and also a root of f , for all $j \in \{1, \dots, d - 2\}$. In particular, since $i \in \{d - 3, d - 2, d - 1\}$, $\alpha_{\ell 1}(f)$ is the first root of $H_\ell(f)$ and a root of f for all $\ell \in \{1, \dots, d - 4\}$.

By Proposition 6, $\alpha_{11}(f)$ is also the first root of f (so $\alpha_{11}(f) = 0$). Let m denote the multiplicity of this root of f . By Theorem 10, f has at least 5 distinct roots, hence $m \leq d - 4$. Let β denote the first strictly positive root of f .

By Proposition 6, there is a unique root $\beta^{(1)}$ of $H_1(f)$ with $\beta^{(1)} \in]0, \beta[$. If $m > 2$, again by Proposition 6, there is a unique root $\beta^{(2)}$ of $H_2(f)$ with $\beta^{(2)} \in]0, \beta^{(1)}[\subset]0, \beta[$. We continue like this recursively until $H_{m-1}(f)$, to show that there is a unique root $\beta^{(m-1)}$ of $H_{m-1}(f)$ with $\beta^{(m-1)} \in]0, \beta^{(m-2)}[\subset]0, \beta[$. Now, $H_m(f)$ has no root at 0, hence its first root $\alpha_{m1}(f)$ belongs to the open interval $]0, \beta^{(m-1)}[\subset]0, \beta[$.

This contradicts the fact that $\alpha_{m1}(f)$ is a root of f . This completes the proof. \square

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