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# A note on the Casas-Alvero Conjecture

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## Abstract

The Casas–Alvero conjecture predicts that every univariate polynomial  $f$  over a field  $K$  of characteristic zero having a common factor with each of its derivatives  $H_i(f)$  is a power of a linear polynomial. Let  $f = x^d + a_1x^{d-1} + \cdots + a_1x \in K[a_1, \dots, a_{d-1}][x]$  and let  $R_i = \text{Res}(f, H_i(f)) \in K[a_1, \dots, a_{d-1}]$  be the resultant of  $f$  and  $H_i(f)$ ,  $i \in \{1, \dots, d-1\}$ . The Casas-Alvero Conjecture for algebraically closed fields is equivalent to saying that  $R_1, \dots, R_{d-1}$  are “independent” in a certain sense, namely that the height  $ht(R_1, \dots, R_{d-1}) = d-1$  in  $K[a_1, \dots, a_{d-1}]$ . In this paper we prove a partial result in this direction: if  $i \in \{d-3, d-2, d-1\}$  then  $R_i \notin \sqrt{(R_1, \dots, \check{R}_i, \dots, R_{d-1})}$ .

## 1 Introduction

In the year 2001 Eduardo Casas–Alvero published a paper on higher order polar germs of plane curve singularities [1]. His work on polar germs inspired him to make the following conjecture (according to the testimony of José Manuel Aroca, E. Casas communicated the problem orally well before 2001).

A lot of work on the Casas-Alvero conjecture has been done since 2001. Some of the most important partial results can be found in [1–7].

Let  $K$  be a field,  $d$  a strictly positive integer and  $f = x^d + a_1x^{d-1} + \cdots + a_{d-1}x + a_d$  a monic univariate polynomial of degree  $d$  over  $K$ . Let

$$H_i(f) = \binom{d}{i}x^{d-i} + \binom{d-1}{i}a_1x^{d-i-1} + \cdots + \binom{i}{i}a_{d-i}$$

be the  $i$ -th Hasse derivative of  $f$ .

**Definition 1** *The polynomial  $f$  is said to be a Casas–Alvero polynomial if for each  $i \in \{1, \dots, d-1\}$  it has a non-constant common factor with its  $i$ -th Hasse derivative  $H_i(f)$ .*

Note that, by definition, a Casas-Alvero polynomial  $f$  has a common root with  $H_{d-1}(f)$ . In particular, if  $\text{char } K = 0$ , it has at least one root  $b \in K$ , regardless of whether or not  $K$  is algebraically closed. Making the change of variables  $x \rightsquigarrow x - b$ , we may assume that 0 is a

root of  $f$ , in other words,  $a_d = 0$ . In the sequel, we will always make this assumption without mentioning it explicitly.

**Conjecture 1 (Casas–Alvero)** *Assume that  $\text{char } K = 0$ . If  $f \in K[x]$  is a Casas–Alvero polynomial of degree  $d$  with  $a_d = 0$ , then  $f(x) = x^d$ .*

For  $i \in \{1, \dots, d-1\}$ , let  $R_i = \text{Res}(f, H_i(f)) \in K[a_1, \dots, a_{d-1}]$  be the resultant of  $f$  and  $H_i(f)$ . The polynomials  $f$  and  $H_i(f)$  have a common factor if and only if  $R_i = 0$ . Thus  $f$  is Casas–Alvero if and only if the point  $(a_1, \dots, a_{d-1}) \in K^{d-1}$  belongs to the algebraic variety  $V(R_1, \dots, R_{d-1}) \subset K^{d-1}$ . In those terms the Conjecture can be reformulated as follows:

**Conjecture 2** *Let  $V = V(R_1, \dots, R_{d-1}) \subset K^{d-1}$ . Then  $V = \{0\}$ .*

When  $K$  is algebraically closed, this is also equivalent to

**Conjecture 3**

$$\sqrt{(R_1, \dots, R_{d-1})} = (a_1, \dots, a_{d-1}) \quad (1)$$

or, equivalently,

$$a_i^N \in (R_1, \dots, R_{d-1}) \text{ for all } i \in \{1, \dots, d-1\} \text{ and some } N \in \mathbb{N}. \quad (2)$$

**Remark 2** (i) *When  $K$  is not algebraically closed, Conjecture 3 is a priori stronger.*

(ii) *Let  $K \subset K'$  be a field extension. The induced extension*

$$K[a_1, \dots, a_{d-1}] \subset K'[a_1, \dots, a_{d-1}]$$

*is faithfully flat. Since the polynomials  $R_1, \dots, R_{d-1}$  have coefficients in  $\mathbb{Z}$ , (2) holds in  $K[a_1, \dots, a_{d-1}]$  if and only if it holds in  $K'[a_1, \dots, a_{d-1}]$ . Hence the truth of Conjecture 3 depends only on the characteristic of  $K$  but not on the choice of the field  $K$  itself. Because of this, we will take  $K = \mathbb{C}$  in the sequel.*

**Remark 3** *Formulae (1) and (2) can be interpreted in terms of Gröbner bases. Namely, (1) and (2) are equivalent to saying that for any choice of monomial ordering and Gröbner basis  $(f_1, \dots, f_s)$  of  $(R_1, \dots, R_{d-1})$ , after renumbering the  $f_j$ , the leading monomial of  $f_j$  is a power of  $a_j$  for  $j \in \{1, \dots, d-1\}$ .*

**Remark 4** *Conjecture 3 and Remark 3 say that, as polynomials in  $K[a_1, \dots, a_{d-1}]$ , the resultants  $R_1, \dots, R_{d-1}$  are “independent” in a certain sense.*

*Each of the following statements is also equivalent to the Casas–Alvero Conjecture.*

(a) *For each  $i \in \{1, \dots, d-2\}$ , the element  $R_{i+1}$  is not a zero divisor modulo  $(R_1, \dots, R_i)$  (in other words,  $R_1, \dots, R_{d-1}$  form a regular sequence in  $K[a_1, \dots, a_{d-1}]$ ).*

(b) *For each  $i \in \{1, \dots, d-2\}$ ,*

$$R_{i+1} \notin \bigcup_{\mathfrak{p} \in \text{Ass}((R_1, \dots, R_i))} \mathfrak{p}.$$

*where  $\text{Ass}((R_1, \dots, R_i))$  is the set of associated primes of the ideal  $(R_1, \dots, R_i)$ .*

*Moreover, the above statements (a) and (b) are independent of the numbering of the  $R_i$  (since the  $R_i$  are quasi-homogeneous); a permutation of the  $R_i$  yields equivalent statements.*

**Notation.** We will denote by  $(R_1, R_2, \dots, \check{R}_i, \dots, R_{d-1})$  the ideal of  $K[a_1, \dots, a_{d-1}]$  generated by the set  $\{R_1, R_2, \dots, R_{d-1}\} \setminus \{R_i\}$ .

The main theorem of this paper is the following partial result in the direction of Conjecture 3 and statements (a) and (b) of Remark 4.

**Theorem 5** *Take an element  $i \in \{d-3, d-2, d-1\}$ . We have*

$$R_i \notin \sqrt{(R_1, R_2, \dots, \check{R}_i, \dots, R_{d-1})}.$$

## 2 Ideals generated by all the resultants but one

In this section we prove Theorem 5 after recalling some preliminary results.

**Proposition 6** *Let  $f$  be a polynomial of degree  $d$  with real roots  $\beta_1 \leq \beta_2 \leq \dots \leq \beta_d$ . Then  $H_1(f)$  has real roots  $\gamma_1 \leq \gamma_2 \leq \dots \leq \gamma_{d-1}$ , counted with multiplicity, where  $\gamma_i \in ]\beta_i, \beta_{i+1}[$  if  $\beta_i < \beta_{i+1}$  and  $\gamma_i = \beta_i$  if  $\beta_i = \beta_{i+1}$ .*

Proof: Assume that  $f$  has  $s$  distinct roots  $\delta_1 < \delta_2 < \dots < \delta_s$  of multiplicities  $m_1, \dots, m_s$ , respectively. Then  $\delta_j$  is a root of  $H_1(f)$  of multiplicity  $m_j - 1$ , where we say that  $\delta_j$  is a root of multiplicity 0 if it is not a root of  $H_1(f)$ .

By Rolle's theorem, there is at least one root of  $H_1(f)$  in each of the  $s-1$  open intervals  $]\delta_1, \delta_2[, \dots, ]\delta_{s-1}, \delta_s[$ .

**Notation.** Let  $\text{Int}(\beta_i, \beta_{i+1}) := ]\beta_i, \beta_{i+1}[$  if  $\beta_i < \beta_{i+1}$  and  $\text{Int}(\beta_i, \beta_{i+1}) := \{\beta_i\}$  if  $\beta_i = \beta_{i+1}$ .

According to the above, there is at least one real root of  $H_1(f)$  in each of  $\text{Int}(\beta_i, \beta_{i+1})$ ,  $i \in \{1, \dots, d-1\}$ , where  $\gamma_1 \in \text{Int}(\beta_1, \beta_2), \dots, \gamma_{m_1-1} \in \text{Int}(\beta_{m_1-1}, \beta_{m_1})$  are the first  $m_1 - 1$  roots of  $H_1(f)$  (in fact, the same root counted with multiplicity  $m_1 - 1$ ) and similarly for the other multiple roots of  $f$ .

We have accounted for a total of  $s-1 + (m_1-1) + \dots + (m_s-1) = m_1 + \dots + m_s - 1 = d-1$  real roots of  $H_1(f)$  counted with multiplicities. Hence  $H_1(f)$  has no roots, real or complex, other than the ones listed above, and the result follows.  $\square$

**Corollary 7** *Let  $f$  be a polynomial of degree  $d$  with  $d$  real roots, counted with multiplicities. Then each of the  $H_i(f)$ ,  $i \in \{1, \dots, d-1\}$ , has  $d-i$  real roots, counted with multiplicity. In other words, all the roots of  $H_i(f)$  are real.*

Next, we recall a result from [4] on almost counterexamples to the Casas-Alvero conjecture.

**Definition 8** *Fix an  $i \in \{1, \dots, d-1\}$ . An **almost counterexample to the Casas-Alvero conjecture of level  $i$**  is a polynomial  $f$  that has a common root with  $H_j(f)$  for all  $j \in \{1, \dots, d-1\} \setminus \{i\}$  but is not a power of a linear polynomial.*

**Notation.** Given a polynomial  $f$  of degree  $d$  with  $d$  real roots, for a pair  $(k, m)$  of integers with  $1 \leq k \leq d-1$  and  $1 \leq m \leq d-k$  we write  $\alpha_{k,m}(f)$  for the  $m$ -th root of  $H_k(f)$ , where the roots of  $H_k(f)$  are ordered (weakly) increasingly.

We state the next theorem in a somewhat stronger form than in [4]: the extra information about the recycled roots  $\alpha_{k_j, m_j}(f)$  does not appear explicitly in the statement of the result in [4], but is shown in the course of its proof.

**Theorem 9 (J. Draisma–J. P. de Jong [4], Theorem 5)** Fix  $d - 2$  pairs of integers

$$(k_j, m_j), \quad j \in \{1, \dots, d - 2\},$$

with

$$1 \leq k_1 < k_2 < \dots < k_{d-2} \leq d - 1$$

and  $1 \leq m_j \leq d - k_j$ . There exists a polynomial  $f \in \mathbb{R}[x]$  with  $f(0) = f(1) = 0$ , all of whose roots are real and lie in  $[0, 1]$ , such that  $\alpha_{k_j, m_j}(f)$  is a root of  $f$  for all  $j \in \{1, \dots, d - 2\}$  (in particular,  $f$  is an almost counterexample to the Casas-Alvero conjecture of level  $i$ , where  $i$  is the unique element of the set  $\{1, \dots, d - 1\} \setminus \{k_1, \dots, k_{d-2}\}$ ).

We also recall the following result, Theorem 13 of [2]:

**Theorem 10** Assume that  $f$  is a counterexample to the Casas-Alvero Conjecture. Then  $f$  has at least five distinct roots.

Proof of Theorem 5: We argue by contradiction. Assume that

$$R_i \in \sqrt{(R_1, \dots, \check{R}_i, \dots, R_{d-1})}. \quad (3)$$

Let  $f$  be an almost counterexample of level  $i$  to the Casas-Alvero conjecture with  $m_j = 1$  for all  $j \in \{1, \dots, d - 2\}$ , given by Theorem 9. By (3),  $f$  is a Casas-Alvero polynomial that is a counterexample to the Casas-Alvero conjecture. By definition of  $f$ ,  $\alpha_{k_j, 1}(f)$  is the first root of  $H_{k_j}(f)$  and also a root of  $f$ , for all  $j \in \{1, \dots, d - 2\}$ . In particular, since  $i \in \{d - 3, d - 2, d - 1\}$ ,  $\alpha_{\ell, 1}(f)$  is the first root of  $H_\ell(f)$  and a root of  $f$  for all  $\ell \in \{1, \dots, d - 4\}$ .

By Proposition 6,  $\alpha_{11}(f)$  is also the first root of  $f$  (so  $\alpha_{11}(f) = 0$ ). Let  $m$  denote the multiplicity of this root of  $f$ . By Theorem 10,  $f$  has at least 5 distinct roots, hence  $m \leq d - 4$ . Let  $\beta$  denote the first strictly positive root of  $f$ .

By Proposition 6, there is a unique root  $\beta^{(1)}$  of  $H_1(f)$  with  $\beta^{(1)} \in ]0, \beta[$ . If  $m > 2$ , again by Proposition 6, there is a unique root  $\beta^{(2)}$  of  $H_2(f)$  with  $\beta^{(2)} \in ]0, \beta^{(1)}[ \subset ]0, \beta[$ . We continue like this recursively until  $H_{m-1}(f)$ , to show that there is a unique root  $\beta^{(m-1)}$  of  $H_{m-1}(f)$  with  $\beta^{(m-1)} \in ]0, \beta^{(m-2)}[ \subset ]0, \beta[$ . Now,  $H_m(f)$  has no root at 0, hence its first root  $\alpha_{m1}(f)$  belongs to the open interval  $]0, \beta^{(m-1)}[ \subset ]0, \beta[$ .

This contradicts the fact that  $\alpha_{m1}(f)$  is a root of  $f$ . This completes the proof.  $\square$

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