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ALMOST SHARP RATES OF CONVERGENCE FOR THE AVERAGE COST AND DISPLACEMENT IN THE OPTIMAL MATCHING PROBLEM

MICHAEL GOLDMAN MARTIN HUESMANN FELIX OTTO

ABSTRACT. In this note we prove estimates for the average cost in the quadratic optimal transport problem on the two-dimensional flat torus which are optimal up to a double logarithm. We also prove sharp estimates on the displacement. This is based on the combination of a post-processing of our quantitative linearization result together with a quasi-orthogonality property.

1. INTRODUCTION

The aim of this note is to improve the currently best known rates of convergence of the average cost in the quadratic optimal transport problem on the two-dimensional flat torus from [2]. Compared with the conjecture from Caracciolo and al. in [9], our rate is optimal up to a double logarithm. We first use our quantitative linearization result from [13] (in its post-processed version of [12]) to improve the estimates from [3] on the optimal transport map. We then combine this with a quasi-orthogonality property first observed in [14, (3.28)] and relatively standard heat kernel estimates to conclude.

To state our main result let us set some notation. We work on the 2 dimensional flat torus $\mathbb{T}_1 = (\mathbb{R}/\mathbb{Z})^2$ and consider $(X_i)_{i \geq 1}$ a family of i.i.d. uniformly distributed random variables on \mathbb{T}_1 . For $n \geq 1$ we define the empirical measure as

$$\mu_n = \frac{1}{n} \sum_{i=1}^n \delta_{X_i}.$$

The (quadratic) optimal matching problem on the torus is then

$$\mathbb{E} \left[\inf_{\pi \in \text{Cpl}(\mu_n, 1)} \int_{\mathbb{T}_1 \times \mathbb{T}_1} |x - y|^2 d\pi(x, y) \right] =: \mathbb{E} [W_2^2(\mu_n, 1)].$$

Here, we identify the Lebesgue measure on \mathbb{T}_1 with the constant density 1, $\text{Cpl}(\mu_n, 1)$ denotes the couplings between μ_n and the Lebesgue measure on \mathbb{T}_1 , and W_2 is the L^2 Kantorovich Wasserstein distance on \mathbb{T}_1 .

While it is known since the seminal article [1] that $n\mathbb{E}[W_2^2(\mu_n, 1)] \sim \log n$, it was recently conjectured in [9] that

$$(1.1) \quad \lim_{n \rightarrow \infty} \left(n\mathbb{E}[W_2^2(\mu_n, 1)] - \frac{\log n}{4\pi} \right) \in \mathbb{R}.$$

Our main result in this direction is the following theorem.

Theorem 1.1. *We haveⁱ*

$$(1.2) \quad \left| n\mathbb{E}[W_2^2(\mu_n, 1)] - \frac{\log n}{4\pi} \right| \lesssim \log \log n.$$

In light of conjecture (1.1), the error in (1.2) is optimal up to a double logarithm. This essentially improves by $\sqrt{\log n}$ the rate obtained in [2] which built on the approach from [5] where the leading order term in (1.1) was identified. Let us however point out that our result currently holds only in the case of the flat torus while [2, 5] covers any Riemannian manifold without boundary as well as the case of the unit cube. Moreover, since we mostly rely on [13], we are currently not able to treat the bi-partite problem. We refer to [20, 19, 7, 6, 11, 15, 4, 16, 8] for related results.

As in [5, 2] our proof of (1.2) is based on the linearization ansatz proposed in this context by [9] and which we now recall. If $\pi_n = (T_n, \text{Id})_{\#} 1$ is the optimal (random) coupling between μ_n and 1, i.e.

$$W_2^2(\mu_n, 1) = \int_{\mathbb{T}_1 \times \mathbb{T}_1} |x - y|^2 d\pi_n,$$

this ansatz postulate that $T_n(y) - y$ is well approximated by $\nabla f_n(y)$ where f_n solves the Poisson equation

$$-\Delta f_n = \mu_n - 1.$$

As understood since [5], this ansatz can only make sense after some regularization. For $t > 0$, let p_t be the heat kernel at time t on \mathbb{T}_1 and set $f_{n,t} = p_t * f_n$ be a solution of

$$-\Delta f_{n,t} = p_t * (\mu_n - 1).$$

As a consequence of the trace formulaⁱⁱ, see e.g. [2, Lemma 3.14],

$$(1.3) \quad n\mathbb{E} \left[\int_{\mathbb{T}_1} |\nabla f_{n,t}|^2 \right] = \frac{|\log t|}{4\pi} + O(\sqrt{t}).$$

In order to prove (1.2), it is thus 'enough' to prove that for some t_n with $|\log t_n| = \log n + O(\log \log n)$,

$$n \left| \mathbb{E}[W_2^2(\mu_n, 1)] - \mathbb{E} \left[\int_{\mathbb{T}_1} |\nabla f_{n,t}|^2 \right] \right| \lesssim \log \log n.$$

The main step to prove this is our second main result.

ⁱThe notation $A \lesssim B$, which we use in output statements, means that there exists a universal constant $C > 0$ such that $A \leq CB$.

ⁱⁱwhen integrating with respect to the Lebesgue measure we drop the factor dy

Theorem 1.2. *For any $t \geq \frac{1}{n} = r_n^2$ we have*

$$(1.4) \quad n\mathbb{E} \left[\int_{\mathbb{T}_1} |x - y - \nabla f_{n,t}(x)|^2 d\pi_n \right] \lesssim 1 + \log \left(\frac{t}{r_n^2} \right).$$

Moreover, for $t \geq t_n = n^{-1} \log^3 n$ we have

$$(1.5) \quad n\mathbb{E} \left[\int_{\mathbb{T}_1} |T_n(y) - y - \nabla f_{n,t}(y)|^2 \right] \lesssim \log \left(\frac{t}{r_n^2} \right).$$

Notice that the difference between (1.4) and (1.5) is that we replaced $\nabla f_{n,t}(x)$ in the former by $\nabla f_{n,t}(y)$ in the latter. While both have sharp dependence in t , we can go down to the microscopic scale $t = r_n^2 = n^{-1}$ in (1.4) but are restricted to mesoscopic scales $t \geq t_n \gg n^{-1}$ in (1.5). This is most likely an artefact from our proof. Indeed, we derive (1.5) from (1.4) combined with an L^∞ bound on $\nabla^2 f_{n,t_n}$. This imposes the choice $t_n \gg n^{-1}$ (see (2.2) of Lemma 2.1). Still, (1.5) improves by $\sqrt{\log n}$ a similar bound from [3] (see also the recent generalization [10]).

The proof of (1.4) is mostly based on the quantitative linearization result from [13] in its post-processed version from [12]. Let us list the differences between [12, Proposition 4.7] and (1.4). A first point is to pass from a compactly supported convolution kernel as in [12] to the heat kernel as in (1.4). This is done using Lemma 2.4 which relies on relatively standard heat/Green kernel estimates. A second difficulty is to pass from a quenched and localized estimate in [12] to an annealed and global one, see (3.1). This is obtained appealing to stationarity. The argument here is a bit more delicate than its counterpart in [12].

From this sketch of proof it is clear that the 'only' obstacle to obtain (1.5) down to $t = n^{-1}$ is the fact that [13] is currently only known for constant target measures. An extension of this result to arbitrary measures should also allow to extend our results to the bi-partite case.

Let us notice that combining (1.5), (1.3) and $n\mathbb{E} [W_2^2(\mu_n, 1)] \sim \log n$ together with Cauchy-Schwarz inequality, it is not hard to obtain

$$\left| n\mathbb{E} [W_2^2(\mu_n, 1)] - \frac{\log n}{4\pi} \right| \lesssim (\log n \log \log n)^{\frac{1}{2}}$$

which gives an alternative proof of the estimate in [2]. Similar sub-optimal error terms coming from the application of Cauchy-Schwarz inequality can be seen in [2, Theorem 1.2] for example. In order to obtain the sharper estimate (1.2) we rely instead on the quasi-orthogonality property

$$\left| n\mathbb{E} \left[\int_{\mathbb{T}_1} (T_n(y) - y - \nabla f_{n,t_n}(y)) \cdot \nabla f_{n,t_n}(y) \right] \right| \lesssim 1.$$

This type of estimates, first noticed in [14, (3.28)], see also [13, Lemma 1.7], are a central ingredient in the variational approach to the regularity theory for optimal transport maps (see also [18, 17, 21]).

2. PRELIMINARIES

In this section we gather a few technical results which follow from relatively standard heat-kernel estimates. To simplify notation and presentation we provide estimates only for moments of order four but similar bounds can be obtained for moments of arbitrary order. As in [2], for $t \geq 0$ we let

$$q_t(x) = \int_t^\infty (p_s(x) - 1) ds$$

so that

$$(2.1) \quad \nabla f_{n,t}(y) = \frac{1}{n} \sum_{i=1}^n \nabla q_t(X_i - y).$$

Lemma 2.1. *Let $t_n = n^{-1} \log^3 n$. Then,*

$$(2.2) \quad \mathbb{E}[\|\nabla^2 f_{n,t_n}\|_\infty^4]^{\frac{1}{4}} \lesssim \frac{1}{\log n}.$$

For every $n^{-1} \leq s < t < 1$ we have

$$(2.3) \quad n\mathbb{E} \left[\int_{\mathbb{T}_1} |\nabla f_{n,s} - \nabla f_{n,t}|^4 \right]^{\frac{1}{2}} \lesssim 1 + \log \left(\frac{t}{s} \right).$$

Proof. We first prove (2.2). For $\xi > 0$ define the event $A_\xi^{n,t} = \{\|\nabla^2 f_{n,t}\|_\infty \leq \xi\}$. By [2, Theorem 3.3], there exists a constant $C > 0$ such that for any $n \in \mathbb{N}$ and $0 < t < 1$ we have

$$(2.4) \quad \mathbb{P}[(A_\xi^{n,t})^c] \lesssim \begin{cases} \frac{1}{t^2 \xi^3} e^{-Cn t \xi^2} & \text{if } 0 < \xi \leq 1 \\ \frac{1}{t^2 \xi^3} e^{-Cn t \xi} & \text{if } \xi \geq 1. \end{cases}$$

The estimate for $\xi \geq 1$ is not explicitly contained in the statement of [2, Theorem 3.3] but follows by the exact same argument.

Since for a non-negative random variable X and $a > 0$, $\mathbb{E}[X^p] \lesssim a^p + \int_a^\infty \xi^{p-1} \mathbb{P}[X \geq \xi] d\xi$, (2.4) implies that for every $1 > a > 0$,

$$(2.5) \quad \mathbb{E} \left[\|\nabla^2 f_{n,t_n}\|_\infty^4 \right] \lesssim a^4 + \int_a^1 \frac{1}{t_n^2} e^{-Cn t_n \xi^2} d\xi + \int_1^\infty \frac{1}{t_n^2} e^{-Cn t_n \xi} d\xi.$$

Recalling that $t_n = n^{-1} \log^3 n$, the last integral can be estimated as

$$\frac{1}{t_n^2} \int_1^\infty e^{-Cn t_n \xi} d\xi \lesssim \frac{n^2}{\log^9 n} \exp(-C \log^3 n) \ll \frac{1}{\log^4 n}.$$

The other integral can be estimated by

$$\begin{aligned} \int_a^1 \frac{1}{t_n^2} e^{-Cn t_n \xi^2} d\xi &\lesssim \frac{n^2}{\log^6 n} \int_a^\infty \exp(-C \xi^2 \log^3 n) d\xi \\ &\lesssim_{\xi=s \log^{-3/2} n} \frac{n^2}{\log^{\frac{15}{2}} n} \int_{a \log^{3/2} n}^\infty e^{-Cs^2} ds \lesssim \frac{n^2}{\log^9 n} e^{-Ca^2 \log^3 n}. \end{aligned}$$

Choosing $a = \gamma \log^{-1} n$ with $C\gamma^2 > 2$ we get

$$e^{-Ca^2 \log^3 n} \leq n^{-2}$$

and thus

$$\int_a^1 \frac{1}{t_n^2} e^{-Cnt_n \xi^2} d\xi \lesssim \frac{1}{\log^4 n}.$$

Plugging this in (2.5) concludes the proof of (2.2).

We now turn to the proof of (2.3). By stationarity we have

$$\mathbb{E} \left[\int_{\mathbb{T}_1} |\nabla f_{n,s} - \nabla f_{n,t}|^4 \right] = \mathbb{E}[|\nabla f_{n,s}(0) - \nabla f_{n,t}(0)|^4].$$

Since $\nabla f_{n,s}(0) = n^{-1} \sum_i \nabla q_s(X_i)$ we have by Rosenthal inequality,

$$\begin{aligned} \mathbb{E}[|\nabla f_{n,s}(0) - \nabla f_{n,t}(0)|^4] &\lesssim n^{-4} \left(n \int_{\mathbb{T}_1} |\nabla q_s - \nabla q_t|^4 + \left(n \int_{\mathbb{T}_1} |\nabla q_s - \nabla q_t|^2 \right)^2 \right) \\ &= n^{-3} \int_{\mathbb{T}_1} |\nabla q_s - \nabla q_t|^4 + n^{-2} \left(\int_{\mathbb{T}_1} |\nabla q_s - \nabla q_t|^2 \right)^2. \end{aligned}$$

To estimate the first right-hand side term we recall from [2, Corollary 3.13] that for $0 < t < 1$,

$$(2.6) \quad \int_{\mathbb{T}_1} |\nabla q_t|^4 \lesssim t^{-1}.$$

Using triangle inequality we thus conclude that

$$(2.7) \quad \int_{\mathbb{T}_1} |\nabla q_s - \nabla q_t|^4 \lesssim \int_{\mathbb{T}_1} |\nabla q_s|^4 + \int_{\mathbb{T}_1} |\nabla q_t|^4 \lesssim t^{-1} + s^{-1} \lesssim n.$$

For the second right-hand side term we can argue as in [2, Proposition 3.11] using that $q_s = \int_s^\infty (p_r - 1) dr$ and $-\Delta q_s = p_s - 1$ to obtain after integration by parts,

$$\begin{aligned} \int_{\mathbb{T}_1} |\nabla q_s - \nabla q_t|^2 &= \int_{\mathbb{T}_1} (q_t - q_s)(p_t - p_s) = \int_{\mathbb{T}_1} \int_s^t p_r (p_t - p_s) dr dx \\ &= \int_s^t \int_{\mathbb{T}_1} (p_r p_t - 1) dx dr - \int_s^t \int_{\mathbb{T}_1} (p_r p_s - 1) dx dr. \end{aligned}$$

Using the semi-group property of the heat kernel together with the trace formula (see e.g. [2, Theorem 3.7]) we have

$$\int_s^t \int_{\mathbb{T}_1} (p_r p_t - 1) dx dr = \int_s^t (p_{r+t}(0) - 1) dr = \int_{s+t}^{2t} (p_r(0) - 1) dr = \frac{1}{4\pi} \log \left(\frac{2t}{s+t} \right) + O(1).$$

Arguing similarly we have

$$\int_s^t \int_{\mathbb{T}_1} (p_r p_s - 1) dx dr = \frac{1}{4\pi} \log \left(\frac{s+t}{2s} \right) + O(1)$$

so that

$$\mathbb{E} \left[\int_{\mathbb{T}_1} |\nabla q_s - \nabla q_t|^2 \right] = \frac{1}{4\pi} \log \left(\frac{t}{s} \right) + O(1).$$

Combining this with (2.7) concludes the proof of (2.3). \square

Lemma 2.2. *For $t \geq r_n^2 = \frac{1}{n}$ we have*

$$(2.8) \quad n\mathbb{E} \left[\frac{1}{|B_{r_n}|} \int_{B_{r_n}} |\nabla f_{n,t}(0) - \nabla f_{n,t}(x)|^2 d\mu_n \right] \lesssim \frac{1}{(nt)^{1/2}} \lesssim 1.$$

Proof. Put $E = \mathbb{E} \left[\int_{B_{r_n}} |\nabla f_{n,t}(0) - \nabla f_{n,t}(x)|^2 d\mu_n \right]$ and write $\mu_n = \frac{1}{n} \sum_{i=1}^n \delta_{X_i}$ to get

$$\begin{aligned} E &= \mathbb{E} \left[\chi_{B_{r_n}}(X_1) |\nabla f_{n,t}(0) - \nabla f_{n,t}(X_1)|^2 \right] = \mathbb{E} \left[\chi_{B_{r_n}}(X_1) |\nabla f_{n,t}(0)|^2 \right] \\ &\quad - 2\mathbb{E} \left[\chi_{B_{r_n}}(X_1) \nabla f_{n,t}(0) \cdot \nabla f_{n,t}(X_1) \right] + \mathbb{E} \left[\chi_{B_{r_n}}(X_1) |\nabla f_{n,t}(X_1)|^2 \right]. \end{aligned}$$

We now estimate each term separately using (2.1). For the first one we have

$$\mathbb{E} \left[\chi_{B_{r_n}}(X_1) |\nabla f_{n,t}(0)|^2 \right] = \frac{1}{n^2} \sum_{i,j} \mathbb{E} \left[\chi_{B_{r_n}}(X_1) \nabla q_t(X_i) \cdot \nabla q_t(X_j) \right].$$

By independence of the X_i and the fact that $\int_{\mathbb{T}} \nabla q_t(x) = 0$ we obtain that for $i \neq j$ the expectation is 0. Hence,

$$\begin{aligned} \mathbb{E} \left[\chi_{B_{r_n}}(X_1) |\nabla f_{n,t}(0)|^2 \right] &= \frac{n-1}{n^2} \mathbb{E} \left[\chi_{B_{r_n}}(X_1) |\nabla q_t(X_2)|^2 \right] + \frac{1}{n^2} \mathbb{E} \left[\chi_{B_{r_n}}(X_1) |\nabla q_t(X_1)|^2 \right] \\ &= \frac{n-1}{n^2} |B_{r_n}| \int_{\mathbb{T}} |\nabla q_t|^2 + \frac{1}{n^2} \int_{B_{r_n}} |\nabla q_t|^2. \end{aligned}$$

For the last term we have similarly,

$$\begin{aligned} \mathbb{E} \left[\chi_{B_{r_n}}(X_1) |\nabla f_{n,t}(X_1)|^2 \right] &= \frac{1}{n^2} \sum_{i,j} \mathbb{E} \left[\chi_{B_{r_n}}(X_1) \nabla q_t(X_i - X_1) \cdot \nabla q_t(X_j - X_1) \right] \\ &= \frac{n-1}{n^2} \mathbb{E} \left[\chi_{B_{r_n}}(X_1) |\nabla q_t(X_2 - X_1)|^2 \right] + \frac{1}{n^2} \mathbb{E} \left[\chi_{B_{r_n}}(X_1) |\nabla q_t(0)|^2 \right] \\ &= \frac{n-1}{n^2} |B_{r_n}| \int_{\mathbb{T}} |\nabla q_t|^2 + \frac{1}{n^2} |B_{r_n}| |\nabla q_t(0)|^2. \end{aligned}$$

Regarding the middle term we have

$$\begin{aligned} \mathbb{E} [\chi_{B_{r_n}}(X_1) \nabla f_{n,t}(0) \cdot \nabla f_{n,t}(X_1)] &= \sum_{i,j} \mathbb{E} [\chi_{B_{r_n}}(X_1) \nabla q_t(X_i) \cdot \nabla q_t(X_j - X_1)] \\ &= \frac{n-1}{n^2} \mathbb{E} [\chi_{B_{r_n}}(X_1) \nabla q_t(X_2) \cdot \nabla q_t(X_2 - X_1)] + \frac{1}{n^2} \mathbb{E} [\chi_{B_{r_n}}(X_1) \nabla q_t(X_1) \cdot \nabla q_t(0)] \\ &= \frac{n-1}{n^2} \int_{B_{r_n}} \int_{\mathbb{T}} \nabla q_t(z) \cdot \nabla q_t(z-x) + \frac{1}{n^2} \int_{B_{r_n}} \nabla q_t(x) \cdot \nabla q_t(0). \end{aligned}$$

Since by [2, Proposition 3.12]

$$(2.9) \quad \sup_{B_{r_n}} |\nabla q_t| \lesssim (r_n + t^{1/2})^{-1} \lesssim t^{-1/2}$$

we have

$$\frac{1}{n^2} \int_{B_{r_n}} \nabla q_t(x) \cdot \nabla q_t(0) \lesssim \frac{|B_{r_n}|}{n^2} t^{-1}$$

and similarly for the two other terms with prefactor n^{-2} . Therefore, for some $C \gg 1$,

$$\begin{aligned} nE - C \frac{|B_{r_n}|}{nt} &\lesssim \left(|B_{r_n}| \int_{\mathbb{T}} |\nabla q_t|^2 - \int_{B_{r_n}} \int_{\mathbb{T}} \nabla q_t(z) \cdot \nabla q_t(z-x) \right) \\ &= \int_{B_{r_n}} \int_{\mathbb{T}} \nabla q_t(z) \cdot (\nabla q_t(z) - \nabla q_t(z-x)) \\ &= \int_{B_{r_n}} \int_{\mathbb{T}} -\Delta q_t(z) \cdot (q_t(z) - q_t(z-x)) \\ &= \int_{B_{r_n}} \int_{\mathbb{T}} (p_t(z) - 1) \cdot (q_t(z) - q_t(z-x)) \\ &= \int_{B_{r_n}} (q_{2t}(0) - q_{2t}(x)) \leq |B_{r_n}| r_n \sup_{B_{r_n}} |\nabla q_{2t}|. \end{aligned}$$

Here we used that $-\Delta q_t = p_t - 1$ together with the semi-group property of p_t . Using (2.9) again we conclude the proof of (2.8). \square

Remark 2.3. For $t \geq t_n = n^{-1} \log^3 n$, the proof of (2.8) can be significantly simplified with the help of the Hessian bounds (2.2).

Finally, in order to translate the results from [13, 12] to the setting of [5, 2, 3] we will need to be able to switch from convolutions against compactly supported kernels to convolutions against the heat kernel.

Lemma 2.4. Let $\eta \in C_c^\infty(B_1)$ be a smooth convolution kernel. For $r > 0$, set $\eta_r = r^{-2} \eta(\cdot/r)$ and let then φ_n^r the mean-zero solution of

$$-\Delta \varphi_n^r = \eta_r * (\mu_n - 1).$$

For every $t \geq n^{-1}$ we have

$$(2.10) \quad n \mathbb{E} [|\nabla f_{n,t}(0) - \nabla \varphi_n^{\sqrt{t}}(0)|^4]^{\frac{1}{2}} \lesssim 1.$$

Proof. We start by noting that $\nabla\varphi_n^{\sqrt{t}}(0) = n^{-1}\sum_i(\eta_{\sqrt{t}} * \nabla q_0)(X_i)$. Since $\nabla f_{n,t}(0) = n^{-1}\sum_i \nabla q_t(X_i)$ we may apply as above Rosenthal inequality to obtain

$$\begin{aligned} & \mathbb{E}[|\nabla f_{n,t}(0) - \nabla\varphi_n^{\sqrt{t}}(0)|^4] \\ & \lesssim n^{-4} \left(\sum_i \mathbb{E}[|\nabla q_t(X_i) - \nabla(\eta_{\sqrt{t}} * q_0)(X_i)|^4] + \left(\sum_i \mathbb{E}[|\nabla q_t(X_i) - \nabla(\eta_{\sqrt{t}} * q_0)(X_i)|^2]^2 \right) \right) \\ & = n^{-4} \left(n \int_{\mathbb{T}_1} |\nabla q_t - \nabla(\eta_{\sqrt{t}} * q_0)|^4 + (n \int_{\mathbb{T}_1} |\nabla q_t - \nabla(\eta_{\sqrt{t}} * q_0)|^2)^2 \right) \\ & \lesssim n^{-3} \left(\int_{\mathbb{T}_1} |\nabla q_t|^4 + \int_{\mathbb{T}_1} |\nabla(\eta_{\sqrt{t}} * q_0)|^4 \right) + n^{-2} \left(\int_{\mathbb{T}_1} |\nabla q_t - \nabla(\eta_{\sqrt{t}} * q_0)|^2 \right)^2. \end{aligned}$$

The first right-hand side term is estimated by (2.6). Arguing exactly as in the proof of [12, (3.5)], we may similarly estimate the second right-hand side term by

$$\int_{\mathbb{T}_1} |\nabla(\eta_{\sqrt{t}} * q_0)|^4 \lesssim t^{-1}.$$

In order to conclude the proof of (2.10) we are left with proving

$$(2.11) \quad \int_{\mathbb{T}_1} |\nabla q_t - \nabla(\eta_{\sqrt{t}} * q_0)|^2 \lesssim 1.$$

To this aim we write

$$(2.12) \quad \int_{\mathbb{T}_1} |\nabla q_t - \nabla(\eta_{\sqrt{t}} * q_0)|^2 \lesssim \int_{B_{3\sqrt{t}}} |\nabla q_t|^2 + \int_{B_{3\sqrt{t}}} |\nabla(\eta_{\sqrt{t}} * q_0)|^2 + \int_{\mathbb{T}_1 \setminus B_{3\sqrt{t}}} |\nabla q_t - \nabla(\eta_{\sqrt{t}} * q_0)|^2$$

For the first right-hand side term we notice that by [2, Proposition 3.12] in $B_{3\sqrt{t}}$ we have

$$|\nabla q_t| \lesssim \frac{1}{\sqrt{t}}$$

so that

$$(2.13) \quad \int_{B_{3\sqrt{t}}} |\nabla q_t|^2 \lesssim \int_{B_{3\sqrt{t}}} \frac{1}{t} \lesssim 1.$$

The second right-hand side term in (2.12) is treated similarly. Indeed, arguing as in [12, (3.7)] we get that in $B_{3\sqrt{t}}$,

$$|\nabla(\eta_{\sqrt{t}} * q_0)| \lesssim \frac{1}{\sqrt{t}}$$

so that

$$(2.14) \quad \int_{B_{3\sqrt{t}}} |\nabla(\eta_{\sqrt{t}} * q_0)|^2 \lesssim 1.$$

We finally turn to the last right-hand side term in (2.12). We further decompose it as

$$(2.15) \quad \int_{\mathbb{T}_1 \setminus B_{3\sqrt{t}}} |\nabla q_t - \nabla(\eta_{\sqrt{t}} * q_0)|^2 \lesssim \int_{\mathbb{T}_1 \setminus B_{3\sqrt{t}}} |\nabla q_t - \nabla q_0|^2 + \int_{\mathbb{T}_1 \setminus B_{3\sqrt{t}}} |\nabla(\eta_{\sqrt{t}} * q_0) - \nabla q_0|^2.$$

For the first right-hand side term we have by definition of q_t ,

$$\nabla q_t - \nabla q_0 = - \int_0^t \nabla p_s ds.$$

Moreover, by [2, Theorem 3.8&3.9], we have in $\mathbb{T}_1 \setminus B_{3\sqrt{t}}$

$$|\nabla p_s|^2 \lesssim \frac{|x|^2}{s^4} \exp(-c \frac{|x|^2}{s}).$$

Using Cauchy-Schwarz, we find

$$\begin{aligned} \int_{\mathbb{T}_1 \setminus B_{3\sqrt{t}}} |\nabla q_t - \nabla q_0|^2 &\lesssim t \int_0^t \int_{\mathbb{T}_1 \setminus B_{3\sqrt{t}}} |\nabla p_s|^2 \lesssim t \int_0^t \int_{\mathbb{T}_1 \setminus B_{3\sqrt{t}}} \frac{|x|^2}{s^4} \exp(-c \frac{|x|^2}{s}) \\ &\lesssim t \int_0^t \int_{3\sqrt{t/s}}^\infty \frac{r^3}{s^2} \exp(-cr^2) dr ds. \end{aligned}$$

Using that for $A \geq 3$,

$$\int_A^\infty r^3 \exp(-cr^2) \lesssim A^2 \exp(-cA^2)$$

we find

$$\int_{\mathbb{T}_1 \setminus B_{3\sqrt{t}}} |\nabla q_t - \nabla q_0|^2 \lesssim t^2 \int_0^t s^{-3} \exp(-c \frac{t}{s}) ds \stackrel{s=tu}{=} \int_0^1 u^{-3} \exp(-\frac{1}{u}) du \lesssim 1.$$

We now estimate the second right-hand side term in (2.15). For $x \in \mathbb{T}_1 \setminus B_{3\sqrt{t}}$ and $y \in B_{\sqrt{t}}$ we have arguing exactly as in [12, Lemma 3.1]

$$|\nabla q_0(x-y) - \nabla q_0(x)| \lesssim \frac{|y|}{|x|^2}.$$

Therefore,

$$\begin{aligned} \int_{\mathbb{T}_1 \setminus B_{3\sqrt{t}}} |\nabla(\eta_{\sqrt{t}} * q_0) - \nabla q_0|^2 &= \int_{\mathbb{T}_1 \setminus B_{3\sqrt{t}}} \left| \int_{B_{\sqrt{t}}} \eta_{\sqrt{t}}(y) (\nabla q_0(x-y) - \nabla q_0(x)) dy \right|^2 dx \\ &\leq \int_{\mathbb{T}_1 \setminus B_{3\sqrt{t}}} \left(\int_{B_{\sqrt{t}}} \eta_{\sqrt{t}}(y) |y| \right)^2 |x|^{-4} dx \leq t \int_{\mathbb{T}_1 \setminus B_{3\sqrt{t}}} |x|^{-4} \lesssim 1. \end{aligned}$$

We conclude that

$$(2.16) \quad \int_{\mathbb{T}_1 \setminus B_{3\sqrt{t}}} |\nabla q_t - \nabla(\eta_{\sqrt{t}} * q_0)|^2 \lesssim 1.$$

Injecting (2.13), (2.14) and (2.16) in (2.12) we obtain the desired (2.11). \square

We recall that from the concentration properties of $W_2^2(\mu_n, 1)$ (see [5] or [12, Theorem 4.5 & Remark 4.6]) we have the following moment bound:

Lemma 2.5. *We have*

$$(2.17) \quad n\mathbb{E} \left[(W_2^2(\mu_n, 1))^2 \right]^{\frac{1}{2}} \lesssim \log n.$$

3. PROOF OF THE MAIN RESULTS

In this section we prove Theorem 1.2 and then Theorem 1.1.

Proof of Theorem 1.2. We start with (1.4). Let $t \geq r_n^2$. As in [12], we first rely on stationarity to infer that

$$\mathbb{E} \left[\int_{\mathbb{T}_1 \times \mathbb{T}_1} |x - y - \nabla f_{n,t}(x)|^2 d\pi_n \right] = \mathbb{E} \left[\frac{1}{|B_{r_n}|} \int_{B_{r_n} \times \mathbb{T}_1} |x - y - \nabla f_{n,t_n}(x)|^2 d\pi_n \right].$$

Let φ_n^r be defined as in Lemma 2.4. We now claim that in order to prove (1.4), it is enough to prove

$$(3.1) \quad n\mathbb{E} \left[\frac{1}{|B_{r_n}|} \int_{B_{r_n} \times \mathbb{T}_1} |x - y - \nabla \varphi_n^{r_n}(0)|^2 d\pi_n \right] \lesssim 1.$$

To this aim we use for $(x, y) \in B_{r_n} \times \mathbb{T}_1$ the triangle inequality in the form

$$\begin{aligned} |x - y - \nabla f_{n,t}(x)|^2 &\lesssim |x - y - \nabla \varphi_n^{r_n}(0)|^2 \\ &\quad + |\nabla \varphi_n^{r_n}(0) - \nabla f_{n,r_n^2}(0)|^2 + |\nabla f_{n,r_n^2}(0) - \nabla f_{n,t}(0)|^2 + |\nabla f_{n,t}(0) - \nabla f_{n,t}(x)|^2. \end{aligned}$$

In order to prove the claim we need to show that the contributions coming from the last three terms on the right-hand side are controlled. Observe, that $n\mu_n(B_{r_n})$ is a Binomial random variable with all moments of order 1. Hence, we can estimate

$$\begin{aligned} n\mathbb{E} \left[\frac{1}{|B_{r_n}|} \int_{B_{r_n} \times \mathbb{T}_1} |\nabla \varphi_n^{r_n}(0) - \nabla f_{n,r_n^2}(0)|^2 d\pi_n \right] \\ = \frac{n}{|B_{r_n}|} \mathbb{E} \left[\mu_n(B_{r_n}) |\nabla \varphi_n^{r_n}(0) - \nabla f_{n,r_n^2}(0)|^2 \right] \\ \leq \frac{n}{|B_{r_n}|} \mathbb{E} \left[(\mu_n(B_{r_n}))^2 \right]^{\frac{1}{2}} \mathbb{E} \left[|\nabla \varphi_n^{r_n}(0) - \nabla f_{n,r_n^2}(0)|^4 \right]^{\frac{1}{2}} \stackrel{(2.10)}{\lesssim} 1. \end{aligned}$$

Second,

$$\begin{aligned} n\mathbb{E} \left[\frac{1}{|B_{r_n}|} \int_{B_{r_n} \times \mathbb{T}_1} |\nabla f_{n,r_n^2}(0) - \nabla f_{n,t}(0)|^2 d\pi_n \right] \\ \leq \frac{n}{|B_{r_n}|} \mathbb{E} \left[(\mu_n(B_{r_n}))^2 \right]^{\frac{1}{2}} \mathbb{E} \left[|\nabla f_{n,r_n^2}(0) - \nabla f_{n,t}(0)|^4 \right]^{\frac{1}{2}} \stackrel{(2.3)}{\lesssim} \left(1 + \log \left(\frac{t}{r_n^2} \right) \right). \end{aligned}$$

Finally, by (2.8) of Lemma 2.2 we have

$$\begin{aligned} n\mathbb{E} \left[\frac{1}{|B_{r_n}|} \int_{B_{r_n} \times \mathbb{T}_1} |\nabla f_{n,t_n}(0) - \nabla f_{n,t_n}(x)|^2 d\pi_n \right] \\ = n\mathbb{E} \left[\frac{1}{|B_{r_n}|} \int_{B_{r_n}} |\nabla f_{n,t_n}(0) - \nabla f_{n,t_n}(x)|^2 d\mu_n \right] \lesssim 1. \end{aligned}$$

We now establish (3.1). After rescaling and changing $\varphi_n^{r_n}$ into $-\varphi_n^{r_n}$, [12, Proposition 4.6] yields the existence of a random radius $r_{*,n} \geq r_n$ with moments of every order i.e. for every $p \geq 1$,

$$(3.2) \quad \mathbb{E} \left[\left(\frac{r_{*,n}}{r_n} \right)^p \right] \lesssim_p 1$$

such that

$$\sup\{|x - y - \nabla \varphi_n^{r_{*,n}}(0)| : (x, y) \in \text{Spt } \pi_n \cap (B_{r_n} \times \mathbb{T}_1)\} \lesssim r_{*,n}.$$

Moreover, combining [12, Lemma 4.3 & Theorem 4.5] together with [13, Lemma 2.10] we may further assume that $r_{*,n}$ is such that for some $\alpha \in (0, 1)$,

$$|\nabla \varphi_n^{r_{*,n}}(0) - \nabla \varphi_n^{r_n}(0)| \lesssim r_{*,n} \left(\frac{r_{*,n}}{r_n} \right)^{2+\alpha}.$$

We can thus estimate using triangle inequality,

$$\begin{aligned} n\mathbb{E} \left[\frac{1}{|B_{r_n}|} \int_{B_{r_n} \times \mathbb{T}_1} |x - y - \nabla \varphi_n^{r_n}(0)|^2 d\pi_n \right] \\ \lesssim n\mathbb{E} \left[\frac{1}{|B_{r_n}|} \int_{B_{r_n} \times \mathbb{T}_1} |x - y - \nabla \varphi_n^{r_{*,n}}(0)|^2 d\pi_n \right] + n\mathbb{E} \left[\frac{\mu(B_{r_n})}{|B_{r_n}|} |\nabla \varphi_n^{r_{*,n}}(0) - \nabla \varphi_n^{r_n}(0)|^2 \right] \\ \lesssim n\mathbb{E} [r_{*,n}^4]^{\frac{1}{2}} + n\mathbb{E} \left[r_{*,n}^4 \left(\frac{r_{*,n}}{r_n} \right)^{4(2+\alpha)} \right]^{\frac{1}{2}} \stackrel{(3.2)}{\lesssim} 1. \end{aligned}$$

This concludes the proof of (3.1) and, hence, of (1.4).

To show (1.5) we use the triangle inequality to estimate for $t \geq t_n$,

$$\begin{aligned} n\mathbb{E} \left[\int_{\mathbb{T}_1} |T_n(y) - y - \nabla f_{n,t}(y)|^2 \right] \\ \lesssim n\mathbb{E} \left[\int_{\mathbb{T}_1} |\nabla f_{n,t} - \nabla f_{n,t_n}|^2 \right] + n\mathbb{E} \left[\int_{\mathbb{T}_1} |T_n(y) - y - \nabla f_{n,t_n}(y)|^2 \right]. \end{aligned}$$

Using Hölder inequality and (2.3) we see that it is enough to prove (1.5) for $t = t_n$. We then use triangle inequality again to write

$$\begin{aligned} n\mathbb{E} \left[\int_{\mathbb{T}_1} |T_n(y) - y - \nabla f_{n,t_n}(y)|^2 \right] &= n\mathbb{E} \left[\int_{\mathbb{T}_1 \times \mathbb{T}_1} |x - y - \nabla f_{n,t_n}(y)|^2 d\pi_n \right] \\ &\lesssim n\mathbb{E} \left[\int_{\mathbb{T}_1 \times \mathbb{T}_1} |x - y - \nabla f_{n,t_n}(x)|^2 d\pi_n \right] + n\mathbb{E} \left[\int_{\mathbb{T}_1 \times \mathbb{T}_1} |\nabla f_{n,t_n}(x) - \nabla f_{n,t_n}(y)|^2 d\pi_n \right]. \end{aligned}$$

Since the second right-hand side term is estimated as

$$\begin{aligned} n\mathbb{E} \left[\int_{\mathbb{T}_1 \times \mathbb{T}_1} |\nabla f_{n,t_n}(x) - \nabla f_{n,t_n}(y)|^2 d\pi_n \right] &\lesssim n\mathbb{E} \left[\|\nabla^2 f_{n,t_n}\|_\infty^4 \right]^{\frac{1}{2}} \mathbb{E} \left[(W_2^2(\mu_n, 1))^2 \right]^{\frac{1}{2}} \\ &\stackrel{(2.2) \& (2.17)}{\lesssim} \frac{\log n}{\log^2 n} = \frac{1}{\log n}, \end{aligned}$$

we conclude by (1.4). \square

We finally prove Theorem 1.1.

Proof of Theorem 1.1. We start by writing

$$|T_n(y) - y|^2 = |T_n(y) - y - \nabla f_{n,t_n}(y)|^2 + |\nabla f_{n,t_n}(y)|^2 + 2(T_n(y) - y - \nabla f_{n,t_n}(y)) \cdot \nabla f_{n,t_n}(y).$$

After integration and using (1.5) with $t = t_n$, we thus get

$$\begin{aligned} \left| n\mathbb{E} \left[\int_{\mathbb{T}_1} |T_n(y) - y|^2 \right] - \frac{\log n}{4\pi} \right| &\lesssim \log \log n + \left| n\mathbb{E} \left[\int_{\mathbb{T}_1} |\nabla f_{n,t_n}(y)|^2 \right] - \frac{\log n}{4\pi} \right| \\ &\quad + \left| n\mathbb{E} \left[\int_{\mathbb{T}_1} (T_n(y) - y - \nabla f_{n,t_n}(y)) \cdot \nabla f_{n,t_n}(y) \right] \right|. \end{aligned}$$

Since by the trace formula, see (1.3)

$$\left| n\mathbb{E} \left[\int_{\mathbb{T}_1} |\nabla f_{n,t_n}(y)|^2 \right] - \frac{\log n}{4\pi} \right| \lesssim \log \log n,$$

we are left with the proof of the quasi-orthogonality property

$$(3.3) \quad \left| n\mathbb{E} \left[\int_{\mathbb{T}_1} (T_n(y) - y - \nabla f_{n,t_n}(y)) \cdot \nabla f_{n,t_n}(y) \right] \right| \lesssim 1.$$

For this we first split the left-hand side as

$$(3.4) \quad n\mathbb{E} \left[\int_{\mathbb{T}_1} (T_n(y) - y - \nabla f_{n,t_n}(y)) \cdot \nabla f_{n,t_n}(y) \right] = n\mathbb{E} \left[\int_{\mathbb{T}_1} (T_n(y) - y) \cdot \nabla f_{n,t_n}(y) \right] - n\mathbb{E} \left[\int_{\mathbb{T}_1} |\nabla f_{n,t_n}(y)|^2 \right].$$

To estimate the first term we argue in the spirit of [18] and introduce the following notation: for $y \in \mathbb{T}_1$ let $X_s = X_s(y)$ be the constant speed geodesic with $X_0 = y$ and $X_1 = T_n(y)$. We thus have

$$\begin{aligned} \int_{\mathbb{T}_1} (T_n(y) - y) \cdot \nabla f_{n,t_n}(y) &= \int_{\mathbb{T}_1} \int_0^1 \dot{X}_s \cdot \nabla f_{n,t_n}(X_0) \\ &= \int_{\mathbb{T}_1} \int_0^1 \dot{X}_s \cdot \nabla f_{n,t_n}(X_s) + \int_{\mathbb{T}_1} \int_0^1 \dot{X}_s \cdot (\nabla f_{n,t_n}(X_0) - \nabla f_{n,t_n}(X_s)). \end{aligned}$$

For the second term we can estimate

$$(3.5) \quad n\mathbb{E} \left[\left| \int_{\mathbb{T}_1} \int_0^1 \dot{X}_s \cdot (\nabla f_{n,t_n}(X_0) - \nabla f_{n,t_n}(X_s)) \right| \right] \lesssim n\mathbb{E} \left[\|\nabla^2 f_{n,t_n}\|_\infty W_2^2(\mu_n, 1) \right] \stackrel{(2.2)\&(2.17)}{\lesssim} 1.$$

For the first term we use that $\dot{X}_s \cdot \nabla f_{n,t_n}(X_s) = \frac{d}{ds}[f_{n,t_n}(X_s)]$ to write (recall that $X_1 = T_n(y)$ and $X_0 = y$)

$$\begin{aligned} \int_{\mathbb{T}_1} \int_0^1 \dot{X}_s \cdot \nabla f_{n,t_n}(X_s) &= \int_{\mathbb{T}_1} (f_{n,t_n}(X_1) - f_{n,t_n}(X_0)) \\ &= \int_{\mathbb{T}_1} f_{n,t_n} d(\mu_n - 1) = \int_{\mathbb{T}_1} f_{n,t_n} (-\Delta f_{n,0}) = \int_{\mathbb{T}_1} \nabla f_{n,t_n} \cdot \nabla f_{n,0}. \end{aligned}$$

Using the semi-group property of the heat kernel we get

$$\int_{\mathbb{T}_1} \nabla f_{n,t_n} \cdot \nabla f_{n,0} = \int_{\mathbb{T}_1} |\nabla f_{n, \frac{t_n}{2}}|^2.$$

We thus conclude that

$$\left| n\mathbb{E} \left[\int_{\mathbb{T}_1} (T_n(y) - y) \cdot \nabla f_{n,t_n}(y) \right] - n\mathbb{E} \left[\int_{\mathbb{T}_1} |\nabla f_{n, \frac{t_n}{2}}|^2 \right] \right| \lesssim 1.$$

Plugging this back into (3.4) yields

$$\begin{aligned} \left| n\mathbb{E} \left[\int_{\mathbb{T}_1} (T_n(y) - y - \nabla f_{n,t_n}(y)) \cdot \nabla f_{n,t_n}(y) \right] \right| \\ \lesssim 1 + n \left| \mathbb{E} \left[\int_{\mathbb{T}_1} |\nabla f_{n, \frac{t_n}{2}}|^2 \right] - \mathbb{E} \left[\int_{\mathbb{T}_1} |\nabla f_{n,t_n}|^2 \right] \right| \lesssim 1 \end{aligned}$$

where in the last line we used once more (1.3). This concludes the proof of (3.3). \square

Remark 3.1. *Let us point out that the only source of suboptimality in (1.2) lies in (3.5) where we use in a crucial way the bound on $\nabla^2 f_{n,t_n}$ from Lemma 2.1. Indeed, if we knew (using the notation from the proof of Theorem 1.1) that analogously to (1.4)*

$$n\mathbb{E} \left[\int_{\mathbb{T}_1 \times \mathbb{T}_1} \int_0^1 \left| \dot{X}_s - \nabla f_{n,r_n^2}(X_s) \right|^2 ds d\pi_n \right] \lesssim 1$$

and analogously to (1.3),

$$(3.6) \quad n\mathbb{E} \left[\int_{\mathbb{T}_1 \times \mathbb{T}_1} \int_0^1 |\nabla f_{n,t}(X_s)|^2 ds d\pi_n \right] = \frac{|\log t|}{4\pi} + O(\sqrt{t})$$

then the exact same proof would yield an error term which is of order one.

Remark 3.2. In a similar direction, let us also observe that arguing as in the proof of Lemma 2.2 we can obtain the analog of (1.3) (this explicit computation cannot be done for (3.6))

$$n\mathbb{E} \left[\int_{\mathbb{T}_1} |\nabla f_{n,t}|^2 d\mu_n \right] = \frac{|\log t|}{4\pi} + O(\sqrt{t}).$$

Instead of relying on (1.5) in the proof of (1.2), we could have thus used (1.4) (with $t = t_n$). This would however make the proof slightly heavier without affecting the final result.

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(MICHAEL GOLDMAN) CMAP, CNRS, ÉCOLE POLYTECHNIQUE, INSTITUT POLYTECHNIQUE DE PARIS, 91120 PALAISEAU, FRANCE

Email address: MICHAEL.GOLDMAN@CNRS.FR

(MARTIN HUESMANN) UNIVERSITÄT MÜNSTER, GERMANY

Email address: MARTIN.HUESMANN@UNI-MUENSTER.DE

(FELIX OTTO) MAX PLANCK INSTITUTE FOR MATHEMATICS IN THE SCIENCES, 04103 LEIPZIG, GERMANY

Email address: FELIX.OTTO@MIS.MPG.DE