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DISCONTINUOUS SOLUTIONS FOR THE NAVIER-STOKES EQUATIONS WITH DENSITY-DEPENDENT VISCOSITY

SAGBO MARCEL ZODJI

Abstract. We prove existence and uniqueness of global weak solutions of the Navier-Stokes equations that govern the motion of a compressible viscous fluid with density-dependent viscosity in two space dimensions. The initial velocity belongs to the Sobolev space $H^1(\mathbb{R}^2)$ and the initial fluid density is Hölder continuous on both sides of a $\mathscr{C}^{1+\alpha}$ interface with some geometrical assumption. We show that this configuration persists over time: the initial interface is transported by the flow to an interface that is as regular as the initial one.

Our result generalizes previous known of Hoff [21], Hoff and Santos [22] concerning the propagation of regularity for discontinuity surfaces by allowing more general nonlinear pressure law and density-dependent viscosity. Moreover, it supplements the work by Danchin, Fanelli and Paicu [7] with global well-posedness, even for density-dependent viscosity and we obtain uniqueness in a large space.

1. Introduction and main result

In this paper, we study the problem of existence and uniqueness of global weak solutions, with intermediate regularity, for the Navier-Stokes equations describing the motion of compressible fluid with density-dependent viscosity in $\mathbb{R}^2$. Our main interest is to generalize the result by Hoff [21], Hoff and Santos [22] about propagation of discontinuous surfaces by allowing nonlinear pressure law and density-dependent viscosity. On the other hand, this result supplements the work by Danchin, Fanelli, Paicu [7] with global well-posedness, even for density-dependent viscosity. Indeed, we consider the following system:

$$\begin{cases} \partial_t \rho + \operatorname{div}(\rho u) = 0, \\ \partial_t(\rho u) + \operatorname{div}(\rho u \otimes u) + \nabla P(\rho) = \operatorname{div}(2\mu(\rho)\mathbb{D}u) + \nabla(\lambda(\rho)\operatorname{div} u) \end{cases} \tag{1.1}$$

describing the motion of a compressible viscous fluid at constant temperature. Above, $\rho = \rho(t,x) \geqslant 0$ and $u = u(t,x) \in \mathbb{R}^2$ are respectively the density and the velocity of the fluid and they are the unknowns of the problem. Also, $P = P(\rho)$, $\mu = \mu(\rho)$, $\lambda = \lambda(\rho)$ are respectively the pressure, dynamic and kinematic viscosity law of the fluid and they are given $\mathscr{C}^2$ functions of the density. The equations (1.1) is supplemented with initial data

$$\rho_{|t=0} = \rho_0 \quad \text{and} \quad u_{|t=0} = u_0 \tag{1.2}$$

with the following behavior at infinity:

$$\lim_{|x|\to\infty} \rho_0(x) = \widetilde{\rho} \quad \text{and} \quad \lim_{|x|\to\infty} u_0(x) = 0$$

for some $\widetilde{\rho} > 0$ and $\widetilde{P} = P(\widetilde{\rho})$. The initial velocity u_0 belongs to the Sobolev space $H^1(\mathbb{R}^2)$ and the initial density is α-Hölder continuous on both sides of an curve $\mathcal{C}(0) \in \mathscr{C}^{1+\alpha}$ defined by:

$$\mathcal{C}(0) := \left\{\gamma_0(s) \in \mathbb{R}^2 \,:\, s \in [0, 2\pi]\right\}. \tag{1.3}$$

In particular, ρ_0 can experiences discontinuity across the curve $\mathcal{C}(0)$. The curve $\mathcal{C}(0)$ is assumed to be the boundary of a bounded simply connected set $D(0) \subset \mathbb{R}^2$ and verifies the following non-self interception condition:

$$\exists\, c_\gamma(0) > 0, \quad \forall\, (s,s') \in [0,2\pi] \times [0,2\pi], \quad |\gamma_0(s) - \gamma_0(s')| \geqslant c_\gamma(0)^{-1}|s - s'|. \tag{1.4}$$

The purpose of this paper is to establish existence and uniqueness of global weak solution of the system (1.1) *with initial data* (ρ_0, u_0) *in the spirit of the works by Hoff* [21]*, Hoff and Santos* [22]*. The regularity of the solution helps to propagate the* $\mathscr{C}^{1+\alpha}$ *and the non-self interception* (1.4) *properties of the initial curve at all over time. We obtain that discontinuities in the initial density persist all over the time. The extension to density-dependent viscosity is not trivial and the analysis of the model is more subtle. We obtain uniqueness in a large space. On*

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the other hand this result supplements the work by Danchin, Fanelli and Paicu [7] *with global well-posedness even with density-dependent viscosity.*

We will now proceed with the review of known results on the propagation of discontinuity surfaces in the mathematical analysis of the Navier-Stokes equations for compressible fluids.

1.1. **Review of known results.** Since its definitive formulation in the mid-19 th century, the Navier-Stokes equations have consistently captivated the attention of numerous mathematicians. The inaugural achievement in this realm is attributed to [32] who proved a local well-posedness of strong solution in the whole $\mathbb{R}^3$. The density belongs to $\mathscr{C}^{1+\alpha}$ while the velocity belongs to $\mathscr{C}^{2+\alpha}$ for some $\alpha \in (0,1)$. We also refer to the work by Solonnikov [36] which provided local well-posedness of system (1.1) in $\mathscr{C}^2$ bounded subset $\Omega \subset \mathbb{R}^d$, $d \in \{2,3\}$, with Sobolev regularity. In fact the initial density is bounded away from vacuum and belongs to $W^{1,p}(\Omega)$ for some $p > d$ whereas the initial velocity belongs to the Sobolev-Slobodetskii space $W^{2,1}_q(\Omega)$. Nash's work takes into account heat conducting fluids with viscosity laws which may depend on density or temperature. On the other hand, Solonnikov did not take into account temperature and the viscosities are constant. The first global result for system (1.1) is obtained by Matsumura and Nishida [30] for small initial data in the whole space $\mathbb{R}^3$. In the case of constant viscosity, the initial data is required to exhibit smallness in $H^3(\mathbb{R}^3)$, *whereas for non-constant viscosity, smallness in the space $H^4(\mathbb{R}^3)$ is necessary for the initial data.* Later, regularity of the initial data was weaken in [3, 6, 18] with small initial data in the critical Besov space, *for constant viscosity setting.* But still, density remains a continuous function in space.

In the constant viscosity setting, the classical solutions constructed in the referenced papers, come with the following energy balance:

$$E(t) + \mu \int_0^t \|\nabla u\|^2_{L^2(\mathbb{R}^d)} + (\mu+\lambda) \int_0^t \|\operatorname{div} u\|_{L^2(\mathbb{R}^d)} = E(0) =: E_0. \tag{1.5}$$

Above $E(t)$ is the total energy functional defined by

$$E(t) := \int_{\mathbb{R}^d} \rho(t,x) \left[\frac{|u(t,x)|^2}{2} + \int_{\widetilde{\rho}}^{\rho(t,x)} s^{-2} \left(P(s) - \widetilde{P} \right) ds \right] dx. \tag{1.6}$$

In the particular case when the pressure law is of the form $P(\rho) = a\rho^\gamma$, global weak solution of the system (1.1) is obtained for the first time by P-L Lions [29] and Feireisl, Novotný, Petzeltová [13] with some restriction on the adiabatic constant γ. The initial data is assumed to have finite initial energy, that is $E(0) < \infty$ and the solution verify (1.5) with inequality. In [2], Bresch and Desjardins established existence of a global weak solution for the Navier-Stokes equation with density-dependent viscosity. Their result requires some Sobolev regularity assumption on the initial density, which goes beyond our framework since the density is assumed to be discontinuous across a hyper-surface such that its weak gradient contains Dirac masses.

In the last three decades, people have been interested in studying the propagation of discontinuity surfaces for models coming from fluid mechanics as the Euler or Navier-Stokes equations. These discontinuity surfaces are the set of singularity points of certain quantities such as vorticity for Euler equations or density for Navier-Stokes equations. For instance, we refer to the so-called density-patch problem proposed by P-L Lions [28] for the incompressible Navier-Stokes equations: *assuming $\rho_0 = \mathbb{1}_{D_0}$ for some domain $D_0 \subset \mathbb{R}^2$, the question is whether or not for any time $t > 0$, the density is $\rho(t) = \mathbb{1}_{D(t)}$, with $D(t)$ a domain with the same regularity as the initial one.* This problem has been considered in [8, 9, 10, 12, 14, 16, 25, 26, 27, 34] where a satisfactory answer was given for various regularities of D_0 including density-dependent viscosity.

As far as we know, there are not enough results in the literature regarding the similar density-patch problem for the Navier-Stokes equations for compressible fluids. In fact, on the one hand, classical solutions are too regular and they do not take into account discontinuous initial density. On the other hand, the weak solutions constructed by P-L Lions [29] or Feireisl, Novotný, Petzeltvá [13] allow discontinuous initial density, but the velocity is too weak in order to track down discontinuity in the density. As explained, for instance in [16], a good approach to tracking discontinuous surfaces is to construct weak solutions for the full model in a class that allows the study of its dynamics. The first result regarding the propagation of discontinuous surfaces for the compressible model is credited to Hoff in his work of 2002 [21], which is in continuation of his works [19, 20]. Indeed, in his pioneer work [19], Hoff provided bounds for the following functionals

$$\mathcal{A}_1(t) = \sup_{[0,t]} \sigma \|\nabla u\|^2_{L^2(\mathbb{R}^d)} + \int_0^t \sigma \|\sqrt{\rho}\dot{u}\|^2_{L^2(\mathbb{R}^d)} \quad \text{and} \quad \mathcal{A}_2(t) = \sup_{[0,t]} \sigma^d \|\sqrt{\rho}\dot{u}\|^2_{L^2(\mathbb{R}^d)} + \int_0^t \sigma^d \|\nabla \dot{u}\|^2_{L^2(\mathbb{R}^d)} \tag{1.7}$$

owing to some smallness condition on the initial data. Above, the material derivative of the velocity $\dot{u}$, and the time weight σ are defined as follows:

$$\dot{u} := \partial_t u + (u \cdot \nabla) u \quad \text{and} \quad \sigma(t) := \min\{1, t\}.$$

He observed that the so-called effective flux defined by

$$F := (2\mu + \lambda) \operatorname{div} u - P(\rho) + \widetilde{P} \tag{1.8}$$

and the vorticity $\operatorname{curl} u$ solve the following elliptic equations:

$$\Delta F = \operatorname{div}(\rho \dot{u}) \quad \text{and} \quad \mu \Delta \operatorname{curl} u = \operatorname{curl}(\rho \dot{u}). \tag{1.9}$$

Let us notice that the effective flux was discovered by Hoff and Smoller in [23] and it plays a crucial role in the mathematical analysis of the compressible model, for instance, in the propagation of oscillations [35] or the construction of weak solutions [20, 29, 13]. By using the regularity of $\dot{u}$ provided by the functionals $\mathcal{A}_1$ and $\mathcal{A}_2$ in (1.7), he obtained the fact that the effective flux and vorticity belong, at least, to $L^{8/3}((1,\infty), L^\infty(\mathbb{R}^d))$, and this helps him to propagate the $L^\infty(\mathbb{R}^d)$ norm of the density. Thereby, he obtained existence of global weak solution for the Navier-Stokes equations with linear pressure law in a first paper [19] before considering nonlinear pressure law (gamma-law) in a second paper [20]. The initial density is assumed to be small around a positive state in $L^2(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$, far from zero and the initial velocity is small in $L^2(\mathbb{R}^d)$ but can be large in $L^{2^d}(\mathbb{R}^d)$. These weak solutions are in between the unique, global, classical one constructed by Matsumura and Nishida in [31] and the one with finite initial energy constructed by P-L Lions [29] or Feireisl, Novotný and Petzeltová [13]. In particular, discontinuous initial density are allowed and the regularity of the velocity at positive time helps to track down discontinuity in the density. As an example, in 2008, Hoff and Santos [22] studied the Lagrangian structure for such weak solutions of the Navier-Stokes equations. Basically, they write the velocity as sum of two terms: the first one is associated with the effective flux and the vorticity and is, at least, Lipschitz at positive time while the second one is associated with the pressure and is less regular in space than the first one. In fact its gradient belongs to $L^\infty((0,\infty), BMO(\mathbb{R}^d))$ space. They need the initial velocity to be a little more regular ($u_0 \in H^s(\mathbb{R}^d)$ for $s > 0$ in $d = 2$ and $s > 1/2$ in $d = 3$) in order to lower the singularity at initial time of the first term of the velocity. Hence, the velocity gradient belongs to $L^1_{\text{loc}}([0,\infty), BMO(\mathbb{R}^d))$ and this is sufficient to define a continuous flow map for the velocity field u and it turns out that continuous manifold maintains its regularity over time. However, a bit more regularity is lost, for instance, an initial $\mathscr{C}^\alpha$-manifold is transported to some $\mathscr{C}^{\alpha(t)}$-manifold with $\alpha(t)$ decaying exponentially to zero. Let us notice that they constructed solution of the heat equation with specific initial velocity $u_0 \in H^s(\mathbb{R}^3)$ for $s < 1/2$, that has infinitely many integral curves approaching $x = 0$ as t goes to 0.

This decomposition of the velocity was used some years before by Hoff [21] to propagate regularity of discontinuity surfaces in $\mathbb{R}^2$. In fact, for initial velocity in $H^\beta(\mathbb{R}^2)$, both the effective flux and vorticity belong to $L^1_{\text{loc}}([0,\infty), \mathscr{C}^\alpha(\mathbb{R}^2))$, and as a result, the gradient of the regular part of the velocity does as well, for all $0 < \alpha < \beta < 1$. Assuming that initially the density is Hölder continuous on both sides of a $\mathscr{C}^{1+\alpha}$ interface (with geometrical assumption (1.4)) across which it is discontinuous, then he showed that the second part of the velocity is at least Lipschitz. In fact, its regularity allows to show that the initial interface is transported to an interface that is as regular as the initial one. In particular, the gradient of the second term of the velocity (associated with the pressure) is Hölder continuous in the tangential direction of the transported interface.

It is important to notice that the approximate sequence of the density is constructed in a large space that precludes any non-linear pressure law. More recently, we have established in [38], the existence of local weak solutions for the two-fluid model with density-dependent viscosity and with discontinuous initial data. In particular, the regularity of the local solution is close to that needed in order to propagate the regularity of the interface. These local solutions allow general nonlinear pressure law and will be used as building blocks for global solutions in this paper. On the other hand, in [21], the analysis requires the kinematic viscosity λ to be small (see [21, Section 2.3]). These constraints will be removed in this paper.

In 2020, Danchin, Fanelli and Paicu [7] have shown that, under assumptions of tangential regularity of the initial density, the less regular part of the velocity is Lipschitz. They achieved this by investigating tangential regularity estimates in the spirit of the works by Chemin [4, 5] for ideal flows. In particular they propagate the $W^{1,p}$ regularity for hyper-surfaces up to a finite time: global propagation of the interface's regularity does not hold even for small initial data.

Finally, we may notice that in [24], the same question was investigated in a two-dimensional ball. The authors considered the case of an initial density and initial velocity that are small in H^4 on both sides of an interface, which is nearly circular and $H^{9/2}$ regular.

All of the above results concern the case of constant viscosity. However, when the viscosity becomes density-dependent, the effective flux and the vorticity lose regularity, and the analysis becomes more subtle.

In this part, we will make different observations to show that for the Navier-Stokes equations with density-dependent viscosity, the effective flux and the vorticity are not smooth as in the constant viscosity setting. In particular, we will show that at the interface, these quantities are continuous exactly where the dynamic viscosity is continuous.

The first observation consists in applying the divergence and the rotational operators to the momentum equation in order to express the effective flux and the vorticity as follows:

$$F := (2\mu(\rho) + \lambda(\rho)) \operatorname{div} u - P(\rho) + \widetilde{P} = -(-\Delta)^{-1} \operatorname{div}(\rho \dot{u}) + K\{(\mu(\rho) - \widetilde{\mu})\mathbb{D}u\}, \tag{1.10}$$

$$\mu(\rho) \operatorname{curl} u = -(-\Delta)^{-1} \operatorname{curl}(\rho \dot{u}) + K'\{(\mu(\rho) - \widetilde{\mu})\mathbb{D}u\}. \tag{1.11}$$

We refer to Section 3.1 for technical details leading to these expressions. Above, K and K' are second-order Riesz operators well-known to map linearly $L^p(\mathbb{R}^d)$ into itself for $1 < p < \infty$. At the end point they map $L^\infty(\mathbb{R}^d)$ into the $BMO(\mathbb{R}^d)$ space. The first terms of the effective flux and vorticity are regular as in constant viscosity setting and the second terms are the same order as the gradient of u. Even if the gradient of u belongs to $L^\infty(\mathbb{R}^d)$, the effective flux will just belong to $BMO(\mathbb{R}^d)$ and this is not sufficient to obtain a $L^\infty(\mathbb{R}^d)$ bound for the density. This observation is also done by Bresch and Burtea in [1] in the study of the an-isotropic Navier-Stokes equations.

The second observation will involve computing the exact jump of the effective flux, vorticity, and velocity at the interface. The discussion follows the same lines as in [33] where the case of constant viscosity is analyzed. First, we observe that there is a balance of forces applied to the interface, which suggests the continuity of the stress tensor in the normal direction of the interface, that is:

$$[\![\Pi^j]\!] \cdot n_x = 0, \quad \text{where} \quad \Pi^{jk} = 2\mu(\rho)\mathbb{D}^{jk}u + \Big(\lambda(\rho) \operatorname{div} u - P(\rho) + \widetilde{P}\Big)\delta^{jk}, \tag{1.12}$$

is the stress tensor, n_x is the outward normal vector field of the interface and $[\![g]\!]$ denotes the jump of g at the interface:

$$[\![g]\!](\sigma) = \underbrace{\lim_{r\to 0} g(\sigma + rn_x(\sigma))}_{g_+(\sigma)} - \underbrace{\lim_{r\to 0} g(\sigma - rn_x(\sigma))}_{g_-(\sigma)}, \quad \forall \sigma \in \mathcal{C}.$$

Next, since the velocity u is continuous in the whole space, and its gradient is assumed to be continuous on both sides of the interface, then the velocity gradient is also continuous in the tangential direction of the interface. Basically, discontinuities in the gradient of u will only occur in the normal direction n_x of the interface. In other words, there exists a vector field $\mathrm{a} = \mathrm{a}(t, \sigma) \in \mathbb{R}^2$ such that

$$[\![\nabla u]\!] = \mathrm{a} \otimes n_x^t. \tag{1.13}$$

From this, one easily deduces that such vector field is given by:

$$\mathrm{a} = (\mathrm{a} \cdot \tau_x)\tau_x + (\mathrm{a} \cdot n_x)n_x = [\![\operatorname{curl} u]\!]\tau_x + [\![\operatorname{div} u]\!]n_x, \tag{1.14}$$

where τ_x is the tangential vector field of the interface. By considering (1.13), one can rewrite (1.12) as follows[1]:

$$< \mu(\rho) > \left(\mathrm{a}^j + \mathrm{a} \cdot n_x n_x^j\right) + 2[\![\mu(\rho)]\!] < \mathbb{D}^{jk}u > n_x^k + [\![\lambda(\rho) \operatorname{div} u - P(\rho)]\!] n_x^j = 0. \tag{1.15}$$

Next, one multiplies the above by n_x^j before summing over j in order to obtain the following:

$$2 < \mu(\rho) > \mathrm{a} \cdot n_x + 2[\![\mu(\rho)]\!] < \mathbb{D}^{jk}u > n_x^j n_x^k + [\![\lambda(\rho) \operatorname{div} u - P(\rho)]\!] = 0$$

and, since $\mathrm{a} \cdot n_x = [\![\operatorname{div} u]\!]$, then the jump of the effective flux at the interface is given by the following:

$$[\![(2\mu(\rho) + \lambda(\rho)) \operatorname{div} u - P(\rho)]\!] = 2[\![\mu(\rho)]\!] \left(< \operatorname{div} u > - < \mathbb{D}^{jk}u > n_x^j n_x^k\right). \tag{1.16}$$

As above, we take the scalar product of (1.15) with the tangential vector τ_x and use the fact that $[\![\operatorname{curl} u]\!] = \mathrm{a} \cdot \tau_x$ in order to obtain the following:

$$[\![\mu(\rho) \operatorname{curl} u]\!] = [\![\mu(\rho)]\!] \left(< \operatorname{curl} u > -2 < \mathbb{D}^{jk}u > n_x^k \tau_x^j\right). \tag{1.17}$$

It then turns out that when the dynamic viscosity is continuous at the interface, for instance, when it is constant, the effective flux and vorticity are also continuous at the interface. Another condition for these quantities to be continuous at the interface is to show that the terms in brackets vanish, but the latter seems not true in general.

[1] $< a > := \dfrac{a_+ + a_-}{2}$ is the average of a at the interface.

In view of all the above observations, it is less clear whether the effective flux and the vorticity are continuous at the interface. However, their jumps are expressed in terms of the jump of the viscosity at the interface. As we will see in Section 3.2, the viscous damping of the density will cause the jump of the viscosity to decrease exponentially in time. Consequently, the jumps of the effective flux, the vorticity, and the gradient of the velocity will also decay exponentially in time, as observed in [21, 22, 33] *in case of constant viscosity.* We are in position to state our main result.

1.2. **Statement of the main result.** We consider the classical Navier-Stokes equations

$$\begin{cases} \partial_t \rho + \operatorname{div}(\rho u) = 0, \\ \partial_t(\rho u) + \operatorname{div}(\rho u \otimes u) + \nabla P(\rho) = \operatorname{div}(2\mu(\rho)\mathbb{D}u) + \nabla(\lambda(\rho)\operatorname{div} u), \end{cases} \tag{1.18}$$

in $d = 2$, with initial data (ρ_0, u_0) that fulfills,

$$u_0 \in H^1(\mathbb{R}^2) \quad \text{and} \quad \rho_0 - \widetilde{\rho} \in L^2(\mathbb{R}^2) \cap \mathscr{C}^{\alpha}_{pw,\gamma_0}(\mathbb{R}^2). \tag{1.19}$$

Above, $\widetilde{\rho} > 0$ is the density reference and we set $P(\widetilde{\rho}) = \widetilde{P}$ and $\mu(\widetilde{\rho}) = \widetilde{\mu}$. We assume that the initial density is far away from vacuum that is, there exist constants $0 < \rho_* < \rho^*$ such that:

$$0 < \rho_* < \rho_0 < \rho^*. \tag{1.20}$$

The pressure law $P = P(\rho)$ and viscosity laws $\mu = \mu(\rho)$, $\lambda = \lambda(\rho)$ are assumed to be regular functions of the density, that is:

$$P,\ \mu,\ \lambda \in \mathscr{C}^2([\rho_*/4, 4\rho^*]). \tag{1.21}$$

Additionally, the pressure is assumed to be an increasing function of the density and the viscosity are far away from vacuum, that is, there exist constants $0 < \mu_* < \mu^*$ such that

$$P'_{[\rho_*/4,4\rho^*]} > 0 \quad \text{and} \quad \mu_* \leqslant \mu_{[\rho_*/4,4\rho^*]},\ \lambda_{[\rho_*/4,4\rho^*]} \leqslant \mu^*. \tag{1.22}$$

On the other hand the initial interface $\mathcal{C}(0) \in \mathscr{C}^{1+\alpha}$ is:

$$\mathcal{C}(0) = \{\gamma_0(s) \in \mathbb{R}^2 \colon s \in [0, 2\pi]\} \tag{1.23}$$

where $\mathcal{C}(0)$ is the boundary of an open, bounded, simply connected set $D(0)$ and fulfills the condition (1.4). The initial data will be measured by the following constant:

$$c_0 := \|u_0\|^2_{H^1(\mathbb{R}^2)} + \|\rho_0 - \widetilde{\rho}\|^2_{L^2(\mathbb{R}^2)\cap\mathscr{C}^{\alpha}_{pw,\gamma_0}(\mathbb{R}^2)} + \|[\![\rho_0]\!]\|^2_{L^2(\mathcal{C}(0))\cap L^\infty(\mathcal{C}(0))}. \tag{1.24}$$

Theorem 1.1. *Let (ρ_0, u_0) be an initial data associated with the Navier-Stokes equations (1.18) and fulfills (1.19) and (1.20). Assume that the pressure law $P = P(\rho)$ and the viscosity laws $\mu = \mu(\rho)$ and $\lambda = \lambda(\rho)$ satisfy the assumptions as described above.*

There are constants $c > 0$ and $[\mu]_0 > 0$ such that if:

$$c_0 \leqslant c \quad \text{and} \quad \|\mu(\rho_0) - \widetilde{\mu}\|_{\mathscr{C}^{\alpha}_{pw,\gamma_0}(\mathbb{R}^2)} \leqslant [\mu]_0$$

then, there exists an unique solution (ρ, u) to the Cauchy problem associated to (1.18) with initial data (ρ_0, u_0) verifying:

$$\forall\, t > 0, \quad \mathcal{A}_1(t) + \mathcal{A}_2(t) + \mathcal{A}_3(t) + \sqrt{\beta(t)} + \sqrt{\vartheta(t)} + \sup_{[0,t]}\|u,\ \rho - \widetilde{\rho}\|^2_{L^2(\mathbb{R}^2)} + \int_0^t \|\nabla u\|^2_{L^2(\mathbb{R}^2)} \leqslant C c_0, \tag{1.25}$$

where

$$\begin{cases} \mathcal{A}_1(t) = \sup_{[0,t]}\|\nabla u\|^2_{L^2(\mathbb{R}^2)} + \int_0^t \|\sqrt{\rho}\dot{u}\|^2_{L^2(\mathbb{R}^2)}, & \mathcal{A}_2(t) = \sup_{[0,t]} \sigma\|\sqrt{\rho}\dot{u}\|^2_{L^2(\mathbb{R}^2)} + \int_0^t \sigma\|\nabla\dot{u}\|^2_{L^2(\mathbb{R}^2)}, \\ \mathcal{A}_3(t) = \sup_{[0,t]} \sigma^2\|\nabla\dot{u}\|^2_{L^2(\mathbb{R}^2)} + \int_0^t \sigma^2\|\sqrt{\rho}\ddot{u}\|^2_{L^2(\mathbb{R}^2)}, & \vartheta(t) = \int_0^t \sigma^{r_\alpha}\|\nabla u(\tau)\|^4_{\mathscr{C}^{\alpha}_{pw,\gamma(\tau)}(\mathbb{R}^2)} d\tau, \\ \beta(t) = \sup_{[0,t]}\|f(\rho)\|^4_{\mathscr{C}^{\alpha}_{pw,\gamma}(\mathbb{R}^2)} + \int_0^t \|f(\rho(\tau))\|^4_{\mathscr{C}^{\alpha}_{pw,\gamma(\tau)}(\mathbb{R}^2)} d\tau, & f(\rho) := \int_{\widetilde{\rho}}^{\rho} \frac{2\mu(s) + \lambda(s)}{s} ds \end{cases} \tag{1.26}$$

and $r_\alpha := \max\{7/5;\ 1 + 2\alpha\} < 3$. The constant C in (1.25) depends on $\alpha, \rho_*, \rho^*, \mu_*, \mu^*$ and $c_\gamma(0)$ and the regularity of the initial interface.

Remark 1.2. On the one hand, Theorem 1.1 generalises the works by Hoff [21], Hoff and Santos [22] by allowing nonlinear pressure law and density-dependent viscosity. By propagating interface regularity over time, we complete Danchin, Fanelli, and Paicu's work [7].

On the other hand, our result takes into account the viscosity of the form $1+\mu(\rho)$, which fall out of the framework of the entropy of Bresch-Desjardins and which is relevant for the models of suspensions, see for example [17].

Remark 1.3. The fact that $r_\alpha < 3$ makes the velocity gradient belongs to $L^1_{\mathrm{loc}}([0,\infty),\mathscr{C}^\alpha_{pw,\gamma}(\mathbb{R}^2))$ and this is sufficient in order to propagate the regularity of the initial curve. We have not been able to establish that ∇u belongs to $L^1([0,\infty),\mathscr{C}^\alpha_{pw,\gamma}(\mathbb{R}^2))$[2]. Consequently, the characteristic $c_\gamma(t)$ and the $\mathscr{C}^{1+\alpha}$ norm of the interface $\gamma(t)$ grow exponentially in time as:

$$c_\gamma(t) + \|\nabla\gamma(t)\|_{\mathscr{C}^\alpha} \leqslant Ce^{Ct^{3/4}}$$

but less than the growth obtained in [21].

Remark 1.4. The function $f(\rho)$ defined in (1.26) verifies the following:

$$[\![f(\rho(t,\gamma(t,s)))]\!] = [\![f(\rho_0,\gamma_0(s))]\!] \exp\left[\int_0^t -\left(g(\tau,s) + 2h(\tau,s) < \operatorname{div} u(\tau,\gamma(\tau,s)) > \right.\right.$$
$$\left.\left. - < \mathbb{D}^{jk}u(\tau,\gamma(\tau,s)) > n^j_x n^k_x(\tau,s)\right) d\tau\right] \tag{1.27}$$

where the functions g and h are defined as:

$$g(t,s) = \frac{[\![P(\rho(t,\gamma(t,s)))]\!]}{[\![f(\rho(t,\gamma(t,s)))]\!]} \quad \text{and} \quad h(t,s) = \frac{[\![\mu(\rho(t,\gamma(t,s)))]\!]}{[\![f(\rho(t,\gamma(t,s)))]\!]}.$$

For constant viscosity, f is the logarithm function, $h=0$, and the exponential decay rate is immediate as soon as the pressure is an increasing function of the density. This observation is made by Hoff in [21] and Hoff and Santos [22].

The assumption of increasing pressure ensures that the function g is lower bounded away from zero by a constant that depends on the lower and upper bounds of the density and viscosity. On the one hand, the function h is upper bounded by a constant depending of the lower and upper bounds on the density and viscosity. Thus, from the estimate of the $L^4((1,\infty),L^\infty(\mathbb{R}^2))$ norm of the velocity gradient, given by the functional ϑ, we deduce the exponential decay rate of the jump of $[\![f(\rho)]\!]$. On the other hand, the upper bounds for g and h allow us to deduce the exponential decay rate of the jump of the pressure and the viscosity. Furthermore, the expressions for the jumps of the effective flux in (1.16) and the vorticity in (1.17) at the interface lead to the exponential decay of the jump of these quantities. In consequence, the vector field a given by (1.14) decays exponentially in time so as for the jump of the gradient of velocity since (1.13) holds.

It is important to notice that if initially the density is continuous at a point $\gamma_0(s)$ of the interface, then the density and the velocity gradient at time t are continuous at $\gamma(t,s)$.

Remark 1.5. As we will see in (3.45), we write from the momentum equation (1.18) the following:

$$\widetilde{\mu}\Delta u^j = \rho\dot{u} - \partial_j\left(\frac{\widetilde{\mu}+\lambda(\rho)}{2\mu(\rho)+\lambda(\rho)}F\right) + \partial_j\left(\frac{2\mu(\rho)-\widetilde{\mu}}{2\mu(\rho)+\lambda(\rho)}(P(\rho)-\widetilde{P})\right) - \partial_k\{2(\mu(\rho)-\widetilde{\mu})\mathbb{D}^{jk}u\}$$

from which we deduce that the velocity gradient can be expressed as the sum of four terms:

$$\nabla u = \nabla u_* + \nabla u_F + \nabla u_P + \nabla u_\delta. \tag{1.28}$$

The first term of the above expression is regular in the whole space, for instance, $\nabla u_* \in L^1_{\mathrm{loc}}([0,\infty),\mathscr{C}^\alpha(\mathbb{R}^2))$. This is due to the regularity of $\dot{u}$ provided by functionals $\mathcal{A}_1$ and $\mathcal{A}_2$. The other terms are less regular: there are discontinuous across the interface $\mathcal{C}$ and we need the quantitative Hölder estimate from Theorem A.1 and the extension result of piecewise Hölder functions Lemma A.3 in order to estimate their norms. But it is less clear, whether $\nabla u_P \in L^2((1,\infty),\mathscr{C}^\alpha_{pw,\gamma}(\mathbb{R}^2))$, even though this holds locally in time. Of course, a uniform estimate is required, as our goal is to close all estimates using a bootstrap argument. So the idea is to achieve more integrability at positive times for ∇u_*. This is the key role of the functional $\mathcal{A}_3$. A similar functional was derived in [11] in the context of the incompressible Navier-Stokes model.

[2] global in time

Outline of the paper. The rest of this paper is structured as follows. In the following section, Section 2, we present the main ideas of the proof of Theorem 1.1, while technical details are postponed in Section 3.

2. Sketch of the proof of the main result

In this section, we present the main ideas of the proof of Theorem 1.1. In fact, we derive some a priori estimates for weak solutions for the following system.

$$\begin{cases} \partial_t \rho + \operatorname{div}(\rho u) = 0, \\ \partial_t(\rho u) + \operatorname{div}(\rho u \otimes u) + \nabla P(\rho) = \operatorname{div}(2\mu(\rho)\mathbb{D}u) + \nabla(\lambda(\rho)\operatorname{div} u), \end{cases} \tag{2.1}$$

The existence and uniqueness of such a local solution is the purpose of our recent contribution [38], and our result can be summarized in the following theorem.

Theorem 2.1. *Let (ρ_0, u_0) be initial data associated with the system (2.1) that satisfies the conditions (1.19),*

$$0 < \rho_* \leqslant \rho_0 \leqslant \rho^*, \quad 0 < \mu_* \leqslant \mu(\rho_0),\, \lambda(\rho_0) \leqslant \mu^*$$

and the compatibility condition

$$(\rho \dot{u})_{|t=0} = \operatorname{div}(\Pi)_{|t=0} \in L^2(\mathbb{R}^2). \tag{2.2}$$

There exists $[\mu] > 0$ such that if

$$\frac{4}{\widetilde{\mu}} \|\mu(\rho_0) - \widetilde{\mu}\|_{\mathscr{C}^\alpha_{pw,\gamma_0}(\mathbb{R}^2)} \leqslant [\mu], \tag{2.3}$$

then, there exists a time $T > 0$ and an unique solution (ρ, u) of the system (2.1) verifying the following:

1. $\forall\, 0 \leqslant t \leqslant T$, *with* $\partial D(t) =: \mathcal{C}(t) = \{\gamma(t,s) \in \mathbb{R}^2,\ \ s \in [0, 2\pi]\}$ *and* $\gamma(t) \in \mathscr{C}^{1+\alpha}$;
2. $P(\rho) - \widetilde{P} \in \mathscr{C}([0,T], L^2(\mathbb{R}^2) \cap \mathscr{C}^\alpha_{pw,\gamma}(\mathbb{R}^2))$;
3. $u \in \mathscr{C}([0,T], H^1(\mathbb{R}^2)) \cap L^\infty((0,T), \dot{W}^{1,6}(\mathbb{R}^2)) \cap L^{16}((0,T), \dot{W}^{1,8}(\mathbb{R}^2))$, $\sigma^{r/4}\nabla u \in L^4((0,T), \mathscr{C}^\alpha_{pw,\gamma}(\mathbb{R}^2))$ *for* $0 < r < 2$;
4. $\dot{u} \in \mathscr{C}([0,T], L^2(\mathbb{R}^2)) \cap L^2((0,T), \dot{H}^1(\mathbb{R}^2))$, $\sqrt{\sigma}\nabla \dot{u} \in L^\infty((0,T), L^2(\mathbb{R}^2)) \cap L^4((0,T), L^4(\mathbb{R}^2))$;
5. $\sqrt{\sigma}\ddot{u} \in L^2((0,T) \times \mathbb{R}^2)$, $\sigma \ddot{u} \in L^\infty((0,T), L^2(\mathbb{R}^2)) \cap L^2((0,T), \dot{H}^1(\mathbb{R}^2))$,

where $\sigma(t) = \min\{1, t\}$.

Remark 2.2. The velocity u and its the material derivatives $\dot{u}$ are continuous in the whole space and their gradients, ∇u and $\nabla \dot{u}$ are Hölder continuous on both sides of the interface γ. Moreover, the second material derivative of the velocity, $\ddot{u}$ is continuous at least across the interface γ.

Let us notice that Theorem 2.1 comes with the following blow-up criterion:

Theorem 2.3 (Blow-up criterion)**.** *Let (ρ, u) be the solution constructed in Theorem 2.1 defined up to a maximal time T^*. Assume that, for all $0 < t < T^*$ we have:*

$$\begin{aligned} &\limsup_{t \to T^*} \left\{ c_\gamma(t) + \|\nabla \gamma(t)\|_{\mathscr{C}^\alpha} + \left\| \frac{1}{\rho(t)},\ \frac{1}{\mu(\rho(t))},\ \frac{1}{\lambda(\rho(t))} \right\|_{L^\infty(\mathbb{R}^2)} \right\} \\ &+ \limsup_{t \to T^*} \left\{ \|u(t)\|_{H^1(\mathbb{R}^2)} + \|(\rho \dot{u})(t)\|_{L^2(\mathbb{R}^2)} + \|P(\rho(t)) - \widetilde{P}\|_{\mathscr{C}^\alpha_{pw,\gamma(t)}(\mathbb{R}^2)} \right\} < \infty \end{aligned}$$

and

$$\frac{1}{\widetilde{\mu}} \sup_{[0,t]} \|\mu(\rho, c) - \widetilde{\mu}\|_{\mathscr{C}^\alpha_{pw,\gamma}(\mathbb{R}^2)} \leqslant [\mu]$$

then $T^* = \infty$.

Let us notice that the solution constructed in Theorem 2.1 is not smooth: the density can be discontinuous across a curve and the second weak derivative of the velocity is not locally integrable in space; it could contains Dirac masses. But the regularity of u is sufficient in order to use u, $\dot{u}$ and $\sigma^2\ddot{u}$ as a test function in the following various computations. The a priori estimates we will derive will involve lower regularity on the initial data; in particular, the compatibility condition (2.2) will not be required.

The first one is the classical energy, which consists in taking the scalar product of the momentum equation $(2.1)_2$ with the velocity u, and then integrating it over time and space. Due to the continuity of the velocity u and the normal component of the stress tensor, one obtains the so-called classical energy:

$$\sup_{s\in[0,t]} E(s) + \int_0^t \int_{\mathbb{R}^2} \{2\mu(\rho)|\mathbb{D}u|^2 + \lambda(\rho)|\mathrm{div}\, u|^2\} \leqslant E(0), \tag{2.4}$$

where $E(t)$ is the total energy at time t, defined by:

$$E(t) = \int_{\mathbb{R}^2} \left\{ \rho \frac{|u|^2}{2} + H_1(\rho) \right\} (t,x)dx.$$

Above, H_1 stands for the potential energy that solves the following ODE :

$$\rho H_1'(\rho) - H_1(\rho) = P(\rho) - \widetilde{P}.$$

The potential energy H_1 is given by the formula:

$$H_1(\rho) = \rho \int_{\widetilde{\rho}}^{\rho} s^{-2}(P(s) - \widetilde{P})ds.$$

More generally, one can define, as in [1], the potential energy H_l that solves the ODE

$$\rho H_l'(\rho) - H_l(\rho) = |P(\rho) - \widetilde{P}|^{l-1}(P(\rho) - \widetilde{P})$$

and given by the formula:

$$H_l(\rho) = \rho \int_{\widetilde{\rho}}^{\rho} s^{-2}|P(s) - \widetilde{P}|^{l-1}(P(s) - \widetilde{P})ds.$$

The next step is devoted to providing bounds for the Hoff functionals $\mathcal{A}_1$ and $\mathcal{A}_2$, which, in Eulerian coordinates, are analogous to the energies obtained in the proof of the local well-posedness result in Lagrangian coordinates. These functionals yield estimates for the material acceleration (see [19, 21, 1]) that is,

$$\dot{u} = \partial_t u + (u \cdot \nabla)u,$$

and consequently for the velocity. The first Hoff functional $\mathcal{A}_1$ is obtained while taking the scalar product of the momentum equation (2.1) written in the form

$$\rho \dot{u}^j = \mathrm{div}(\Pi^j),$$

with the material acceleration $\dot{u}$. The second Hoff functional $\mathcal{A}_2$ is obtained by first, taking the material derivative

$$\partial_t \cdot + \mathrm{div}(\cdot\, u)$$

of the momentum equation, before taking the scalar product of the obtained equation with the material acceleration $\sigma\dot{u}$. Under the assumption that the density and viscosity are lower bounded away from vacuum and upper bound, and that the dynamic viscosity is close to a constant state, we have the following estimates for the functionals $\mathcal{A}_1$ and $\mathcal{A}_2$.

Proposition 2.4. Assume that the density and viscosity are uniformly lower and upper bounded:

$$0 < \underline{\rho} \leqslant \rho(t,x) \leqslant \overline{\rho}, \quad 0 < \underline{\mu} \leqslant \mu(\rho) \leqslant \overline{\mu} \quad \text{and} \quad 0 < \underline{\lambda} \leqslant \lambda(\rho) \leqslant \overline{\lambda} \tag{2.5}$$

and let

$$\delta := \frac{1}{\underline{\mu}} \sup_{[0,t]} \|\mu(\rho) - \widetilde{\mu}\|_{L^\infty(\mathbb{R}^2)}.$$

There exists a positive function $\kappa = \kappa(l)$ for $l \in (1,\infty)$, such that, if

$$\delta \left(\frac{2\underline{\mu} + \underline{\lambda}}{2\overline{\mu} + \overline{\lambda}} \right)^{-\frac{1}{l+1}} < \kappa(l),$$

for $l \in \{2,3\}$ then, the following bounds hold true for the Hoff functionals:

$$\mathcal{A}_1(t) \leqslant M \left[C_0 + \mathcal{A}_1(t)^{1/2} \left(C_0 + \mathcal{A}_1(t)\right) + \mathcal{A}_1(t) \left(C_0 + \mathcal{A}_1(t)\right) \right], \tag{2.6}$$

$$\mathcal{A}_2(t) \leqslant M \left[C_0 + \mathcal{A}_1(\sigma(t)) + \mathcal{A}_1(t) \left(C_0 + \mathcal{A}_1(t)\right) \right], \tag{2.7}$$

for some constant M which depends on the lower and upper bounds of the density and viscosity as given in (2.5), as well as on δ. Hereafter, C_0 denotes a sum of powers of c_0, which is defined in (1.24) as the norm of the initial data.

The proof of Proposition 2.4 is given in Section 3.1. In the following proposition we estimate the L^p norm of the density and velocity gradient jump at the interface.

Proposition 2.5. Assume that the density is bounded far away from vacuum and upper bounded as:

$$0 < \underline{\rho} \leqslant \rho(t,x) \leqslant \overline{\rho}. \tag{2.8}$$

Then, there are constants $0 < \underline{\nu} < \overline{\nu}$ and $M > 0$ depending on $\underline{\rho}$ and $\overline{\rho}$ such that the following inequalities hold true:

$$\|[\![f(\rho(t))]\!]\|_{L^p(\mathcal{C}(t))} \leqslant \|[\![f(\rho_0)]\!]\|_{L^p(\mathcal{C}(0))} \exp\left(-\underline{\nu}\, t + (4\overline{\nu} + 1/p)\int_0^t \|\nabla u(\tau)\|_{L^\infty(\mathbb{R}^2)} d\tau\right), \tag{2.9}$$

$$\|[\![\nabla u(t)]\!]\|_{L^p(\mathcal{C}(t))} \leqslant M\|[\![f(\rho_0)]\!]\|_{L^p(\mathcal{C}(0))}\left(1 + \|\nabla u(t)\|_{L^\infty(\mathbb{R}^2)}\right) \exp\left(-\underline{\nu}\, t + (4\overline{\nu} + 1/p)\int_0^t \|\nabla u(\tau)\|_{L^\infty(\mathbb{R}^2)} d\tau\right), \tag{2.10}$$

for all $1 \leqslant p \leqslant \infty$ where f is defined in (1.26).

In the ideal setting where we have a priori estimate for

$$\int_0^t \|\nabla u(\tau)\|_{L^\infty(\mathbb{R}^2)} d\tau$$

that is uniform with respect to time, the exponential decay of the jump of the velocity gradient and $f(\rho)$ follows immediately. Of course, in that case, there is no need to use the decay of jumps since the characteristics of the interface will be uniformly bounded in time. Unfortunately, we are not able to obtain such an estimate, and the exponential decay of the density and velocity gradient jump at the interface turns out to be crucial in order to compensate for the growth of the characteristics of the interface. Indeed, as we will see in (3.45), we have from the momentum equation (1.18) the following expression for the velocity gradient:

$$\nabla u = \nabla u_* + \nabla u_F + \nabla u_P + \nabla u_\delta \tag{2.11}$$

and it is not clear whether we can obtain an $L^2((\sigma(t),t), \mathscr{C}^\alpha_{pw,\gamma}(\mathbb{R}^2))$ estimate for ∇u_P uniformly with respect to time. However, as soon as we obtain a uniform estimate for $\vartheta(t)$ with respect to time (see Proposition 2.8), then, thanks to Hölder and Young inequalities:

$$\int_0^t \|\nabla u(\tau)\|_{L^\infty(\mathbb{R}^2)} d\tau \leqslant \frac{\underline{\nu}}{4(4\overline{\nu} + 1/p)} t + M\vartheta(t),$$

we can deduce from the proposition above the exponential decay in time of the density and velocity gradient jump. The proof of Theorem 2.5 is given in Section 3.2.

We notice that the functionals $\mathcal{A}_1$ and $\mathcal{A}_2$ obtained in Proposition 2.4 do not yield significant time integrability for the material derivative $\dot{u}$ such that we can control the $L^4((\sigma(t),t), \mathscr{C}^\alpha_{pw,\gamma}(\mathbb{R}^2))$ norm of ∇u_*. Thus, this requires us to perform another energy estimate. On the other hand, to achieve exponential decay in the case of constant viscosity [21], the velocity gradient was required to be small in $L^\infty_{t,x}$ relative to $\underline{\nu}$. To obtain the $L^\infty((\tau,\infty)\times\mathbb{R}^2)$ $(\tau > 0)$ norm estimate for the velocity gradient, the author performs an energy estimate that consists in applying the material derivative $\partial_t \cdot + \operatorname{div}(\cdot\, u)$ to the momentum equation $(2.1)_2$ before multiplying the obtained equation by $\dot{u}|\dot{u}|^{s-2}$. This energy estimate yields the following bound

$$\sup_{t \geqslant \tau} \|\dot{u}(t)\|^s_{L^s(\mathbb{R}^2)} \leqslant C(\tau)C_0, \tag{2.12}$$

for all $\tau > 0$, from which he deduces the $L^\infty((\tau,T)\times\mathbb{R}^2)$ norm of the gradient of the regular part of the velocity. However, a smallness condition on the kinematic viscosity [21, Section 2.3] has been required. Alternatively, we will derive functional $\mathcal{A}_3$, (see (1.26) for the definition), that controls the following norm of $\dot{u}$

$$\sup_{[0,T]} \sigma\|\nabla\dot{u}\|_{L^2(\mathbb{R}^2)}.$$

This implies in particular (2.12) and the bound of the energy $\mathcal{A}_3$ does not require any smallness condition on the kinematic viscosity. A similar functional was considered in [11] in the context of the Navier-Stokes equations for incompressible fluids. To obtain this functional, we first apply the material derivative $\partial_t \cdot + \operatorname{div}(\cdot\, u)$ to the momentum equation $(2.1)_2$ and obtain the fact that $\dot{u}$ verifies:

$$\partial_t(\rho\dot{u}^j) + \operatorname{div}(\rho\dot{u}^j u) = \partial_k(\dot{\Pi}^{jk}) + \partial_k(\Pi^{jk}\operatorname{div} u) - \operatorname{div}(\partial_k u \Pi^{jk}). \tag{2.13}$$

Since the left hand side above can be reduced to $\rho\ddot{u}$, we can test the above equation with $\sigma^2\ddot{u}$ and we obtain the functional $\mathcal{A}_3$. Moreover, the following bound holds for $\mathcal{A}_3$.

Proposition 2.6. Assume that the hypotheses of Proposition 2.4 hold true for $l \in \{2,3,5\}$. Then, the following bound holds true with a constant M that depends on the lower and upper bounds of the density and the viscosity, as well as δ:

$$\mathcal{A}_3(t) \leqslant M\left[C_0 + \mathcal{A}_1(t) + \mathcal{A}_2(t) + \mathcal{L}(C_0, \mathcal{A}_1(t), \mathcal{A}_2(t), \mathcal{A}_3(t))\right]. \tag{2.14}$$

Above, the last term is super-linear with respect to C_0, $\mathcal{A}_1(t)$, $\mathcal{A}_2(t)$ and $\mathcal{A}_3(t)$.

The proof is given is Section 3.3. Let us notice that, the functionals $\mathcal{A}_1$, $\mathcal{A}_2$ and $\mathcal{A}_3$, are under control as long as the density and the viscosity are upper bounded and bounded away from vacuum as well, the dynamic viscosity is close to $\widetilde{\mu}$. The final step of our analysis consists in translating (2.6), (2.7), (2.14), (2.9) and (2.10) into piecewise Hölder estimates for the density ρ and the velocity gradient ∇u.

In general, the $L^\infty((0,T)\times\mathbb{R}^2)$ estimate for the density follows as soon as the effective flux belongs to $L^1((0,T), L^\infty(\mathbb{R}^2))$. As it is well-explained in [1], the algebraic structure of the Navier-Stokes equations with density-dependent viscosity does not allow to propagate the L^∞ norm of the density as it has been done by Hoff [19, 20] for the isotropic case. The issue arises from the fact that the effective flux, which plays the role of connecting the momentum equation and the mass equation, does not belong to $L^1_{\text{loc}}((0,\infty), L^\infty(\mathbb{R}^2))$ when the density is only bounded. In particular, the effective flux and the velocity gradient are of the same order. In fact, the effective flux F is given by:

$$F = -(-\Delta)^{-1}\operatorname{div}(\rho\dot{u}) + K((\mu(\rho) - \widetilde{\mu})\mathbb{D}u)$$

where K is a singular integral operator, well-known to map $L^\infty(\mathbb{R}^2)$ into $BMO(\mathbb{R}^2)$ space. Thus, even if the velocity gradient is bounded, it is not obvious that the effective flux will also be bounded. It is straightforward to show the following quantitative Hölder estimate by combining Theorem A.2 and Lemma A.3.

Theorem 2.7. *Let $0<\alpha<1$, $1\leqslant p<\infty$ and K an even higher-order Riesz operator. We consider a curve $\mathcal{C}\in\mathscr{C}^{1+\alpha}$ that satisfies (1.4), and for $g\in L^p(\mathbb{R}^2)\cap\mathscr{C}^\alpha_{pw,\gamma}(\mathbb{R}^2)$ we define the following function:*

$$w(x) = \int_{\mathbb{R}^2} K(x-y)g(y)dy.$$

There exists a constant C depending on α and p such that:

$$\|w\|_{\mathscr{C}^\alpha_{pw,\gamma}(\mathbb{R}^2)} \leqslant C\left[\|g\|_{L^p(\mathbb{R}^2)} + \|[\![g]\!]\|_{L^p(\mathcal{C})} + \|g\|_{\mathscr{C}^\alpha_{pw,\gamma}(\mathbb{R}^2)} + \|[\![g]\!]\|_{L^\infty(\mathcal{C})}\mathcal{P}(\mathcal{C})\right] \tag{2.15}$$

where

$$\mathcal{P}(\mathcal{C}) := c_\gamma^2\|\nabla\gamma\|^{1/\alpha}_{\mathscr{C}^\alpha} + (1+|\partial D|)\,\mathrm{P}(c_\gamma + \|\nabla\gamma\|_{L^\infty})\left(1+\|\nabla\gamma\|_{\dot{\mathscr{C}}^\alpha}\right)$$

and P a polynomial that depends on K.

Theorem 2.7 appears as an interpolation of the fact that singular integral operators of Calderón-Zygmung type map linearly $L^p(\mathbb{R}^2)$ into itself for $1<p<\infty$ and $\dot{\mathscr{C}}^\alpha(\mathbb{R}^2)$ into itself. By the help of this quantitative Hölder estimate, one has the following estimates for functionals ϑ and β.

Proposition 2.8. Assume that the density is bounded away from vacuum and upper bounded (2.8). There are constants M and M' such that the following estimates hold true:

$$\beta(t) \leqslant M' e^{M'\vartheta(t)}\left(\|f(\rho_0)\|^4_{\mathscr{C}^\alpha_{pw,\gamma_0}(\mathbb{R}^2)} + \int_0^t \sigma^{r_\alpha}\|F(\tau)\|^4_{\mathscr{C}^\alpha_{pw,\gamma(\tau)}(\mathbb{R}^2)}d\tau\right), \tag{2.16}$$

$$\begin{aligned}\vartheta(t) \leqslant M\Bigg[&C_0^2 + \mathcal{A}_1^2(t) + \mathcal{A}_2^2(t) + \mathcal{A}_3(t)^2 + \beta(t)(1+\vartheta(t)) + (1+\beta(t))\int_0^t \sigma^{r_\alpha}\|F(\tau)\|^4_{\mathscr{C}^\alpha_{pw,\gamma(\tau)}(\mathbb{R}^2)}d\tau \\ &+(1+K_0)e^{M'\vartheta(t)}\|[\![f(\rho_0)]\!]\|^4_{L^\infty\cap L^2(\mathcal{C}(0))}\left(1+\vartheta(t)+\int_0^t \sigma^{r_\alpha}\|F(\tau)\|^4_{\mathscr{C}^\alpha_{pw,\gamma(\tau)}(\mathbb{R}^2)}d\tau\right)\Bigg],\end{aligned} \tag{2.17}$$

with

$$\int_0^t \sigma^{r_\alpha}\|F(\tau)\|^4_{\mathscr{C}^\alpha_{pw,\gamma(\tau)}(\mathbb{R}^2)}d\tau \leqslant M'\left[C_0^2+\mathcal{A}_1(t)^2+\mathcal{A}_2(t)^2+\mathcal{A}_3(t)^2+\beta(t)\vartheta(t)\right.$$
$$\left.+(1+K_0)\|[\![f(\rho_0)]\!]\|^4_{L^\infty\cap L^2(\mathcal{C}(0))}\left(1+\vartheta(t)\right)e^{M'\vartheta(t)}\right]. \tag{2.18}$$

The constant M depends on δ, α and the upper and lower bounds of the density and viscosity whereas the constants M' depends on the same quantities except δ.

The proof of this proposition is given in Section 3.4. It is obvious that the estimate for β also holds for the dynamic viscosity $\mu(\rho)$. We finally close the estimates in Proposition 2.4, Theorem 2.6 and Proposition 2.8 with the help of the bootstrap argument similar to the one in [1], and we do not present the proof.

Lemma 2.9. *Let (ρ,u) be a local solution of the equations (2.1) with initial data (ρ_0,u_0) that verifies (1.19) and (1.20) and the compatibility condition:*

$$\operatorname{div}\{2\mu(\rho_0)\mathbb{D}u_0+(\lambda(\rho_0)\operatorname{div}u_0-P(\rho_0)+\widetilde{P})I_d\}\in L^2(\mathbb{R}^2).$$

Assume that the solution (ρ,u) is defined up to a maximal time T^*. There exist constants $c>0$ and $[\mu]_0>0$, such that if

$$c_0:=\|u_0\|^2_{H^1(\mathbb{R}^2)}+\|\rho_0-\widetilde{\rho}\|^2_{L^2(\mathbb{R}^2)\cap\mathscr{C}^\alpha_{pw,\gamma_0}(\mathbb{R}^2)}+\|[\![\rho_0]\!]\|^2_{L^2(\mathcal{C}(0))\cap L^\infty(\mathcal{C}(0))}\leqslant c$$

and

$$\|\mu(\rho_0)-\widetilde{\mu}\|_{\mathscr{C}^\alpha_{pw,\gamma_0}(\mathbb{R}^2)}\leqslant[\mu]_0,$$

then, there exists a constant C depending on the lower and upper bounds of the density and viscosity, as well on the initial interface $\mathcal{C}(0)$ such that for all $0<t<T^*$, the followings hold:

$$0<\rho^*/4\leqslant\rho(t,x)\leqslant 4\rho^*,\qquad \mu_*\leqslant\mu(\rho),\ \lambda(\rho)\leqslant\mu^*,\qquad \sup_{[0,t]}\|\mu(\rho)-\widetilde{\mu}\|_{\mathscr{C}^\alpha_{pw,\gamma}(\mathbb{R}^2)}\leqslant C[\mu]_0,$$

and

$$E(t)+\mathcal{A}_1(t)+\mathcal{A}_2(t)+\mathcal{A}_3(t)+\sqrt{\beta(t)}+\sqrt{\vartheta(t)}+\int_0^t\|\nabla u\|^2_{L^2(\mathbb{R}^2)}\leqslant Cc_0.$$

To complete the proof of Theorem 1.1, we have two remaining steps. First, we need to construct a sequence (ρ^δ,u^δ) that is globally defined in time, and converges to (ρ,u), which is a solution of the Navier-Stokes equations (2.1). Usually, this sequence (ρ^δ,u^δ) corresponds to the solution of the Cauchy problem (2.1) with initial data $(\rho_0^\delta,u_0^\delta)$ where $(\rho_0^\delta,u_0^\delta)$ is obtained by smoothing (ρ_0,u_0). In our case, this does not seem to be possible. The problem arises because smoothing the initial data would lead to a loss of density discontinuity. The approach we employed is more subtle and it is presented in Section 3.5 along with the proof of the stability of the approximate sequence. The second step will be devoted to showing that the regularity of the limit (ρ,u) is sufficient to establish it uniqueness. As in the proof of the local well-posedness result, the proof of uniqueness follows the same lines as in [7] since the velocity limit u comes with the following regularity:

$$\nabla u\in L^1_{\mathrm{loc}}([0,\infty),L^\infty(\mathbb{R}^2))\quad\text{and}\quad\sqrt{\sigma}\nabla u\in L^2_{\mathrm{loc}}([0,\infty),L^\infty(\mathbb{R}^2)). \tag{2.19}$$

3. Proofs

3.1. Proof of Proposition 2.4.

Proof. Let us begin the proof of this proposition by recalling that the first functional is obtained while multiplying the momentum equation, written in the form

$$\rho\dot{u}^j=\operatorname{div}\Pi^j \tag{3.1}$$

by $\dot{u}^j$, then summing over j and integrating on $\mathbb{R}^2$ in order to obtain:

$$\int_{\mathbb{R}^2}\rho|\dot{u}^j|^2+\frac{d}{dt}\int_{\mathbb{R}^2}\left\{\mu(\rho)|\mathbb{D}u|^2+\frac{\lambda(\rho)}{2}|\operatorname{div}u|^2\right\}=-\int_{\mathbb{R}^2}2\mu(\rho)\mathbb{D}^{jk}u\partial_k u^l\partial_l u^j+\int_{\mathbb{R}^2}(\rho\mu'(\rho)-\mu(\rho))|\mathbb{D}u|^2\operatorname{div}u$$
$$+\frac{1}{2}\int_{\mathbb{R}^2}(\rho\lambda'(\rho)-\lambda(\rho))(\operatorname{div}u)^3-\int_{\mathbb{R}^2}\lambda(\rho)\operatorname{div}u\nabla u^l\partial_l u+\frac{d}{dt}\int_{\mathbb{R}^2}\left(\operatorname{div}u\{P(\rho)-\widetilde{P}\}\right)+\int_{\mathbb{R}^2}\nabla u^l\partial_l u\left(P(\rho)-\widetilde{P}\right)$$
$$+\int_{\mathbb{R}^2}(\operatorname{div}u)^2\left(\rho P'(\rho)-P(\rho)+\widetilde{P}\right).$$

The above equality follows from the continuity of $\dot{u}$ and of the normal component of the stress tensor at the interface. Next, we integrate the above in time and we obtain:

$$\mathcal{A}_1(t) \lesssim C_0 + \sup_{[0,t]} \|P(\rho) - \widetilde{P}\|^2_{L^2(\mathbb{R}^2)} + \int_0^t \|\nabla u\|^3_{L^3(\mathbb{R}^2)} + \int_0^t \|P(\rho) - \widetilde{P}\|_{L^p(\mathbb{R}^2)} \|\nabla u\|^2_{L^{2p'}(\mathbb{R}^2)} \tag{3.2}$$

where we have used the classical energy (2.4) and where $p \geqslant 3$. On the other hand, we take the material derivative of the momentum equation, written in the form (3.1), before multiplying by $\dot{u}$ and space integration yields:

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^2} \rho|\dot{u}^j|^2 + \int_{\mathbb{R}^2} \left\{2\mu(\rho)|\mathbb{D}^{jk}\dot{u}|^2 + \lambda(\rho)|\operatorname{div}\dot{u}|^2\right\} &= \int_{\mathbb{R}^2} \partial_k \dot{u}^j \left\{\mu(\rho)\partial_j u^l \partial_l u^k + \mu(\rho)\partial_k u^l \partial_l u^j + 2\rho\mu'(\rho)\mathbb{D}^{jk}u \operatorname{div} u\right\} \\ &+ \int_{\mathbb{R}^2} \operatorname{div}\dot{u}\left\{\lambda(\rho)\nabla u^l \partial_l u + \rho\lambda'(\rho)(\operatorname{div} u)^2 - \rho P'(\rho)\operatorname{div} u\right\} - \int_{\mathbb{R}^2} \partial_k \dot{u}^j \Pi^{jk} \operatorname{div} u + \int_{\mathbb{R}^2} \partial_l \dot{u}^j \partial_k u^l \Pi^{jk}.\end{aligned}$$

The computations leading to the above equality can be found in Appendix B.1. Next, we multiply the above by σ before integrating in space, then Hölder and Young inequalities yield:

$$\mathcal{A}_2(t) \lesssim C_0 + \mathcal{A}_1(\sigma(t)) + \int_0^t \sigma\|\nabla u\|^4_{L^4(\mathbb{R}^2)} + \int_0^t \sigma\|P(\rho) - \widetilde{P}\|^4_{L^4(\mathbb{R}^2)}. \tag{3.3}$$

We turn to the estimation of the norm of the gradient of the velocity and the pressure that appear in the estimate (3.2) and (3.3). First, we apply the divergence operator to the momentum equation (3.1) in order to obtain an elliptic equation on the so called effective flux:

$$\operatorname{div}(\rho\dot{u}) = \operatorname{div}\operatorname{div}(2\mu(\rho)\mathbb{D}u) + \Delta\{\lambda(\rho)\operatorname{div} u - P(\rho) + \widetilde{P}\}$$

from which we deduce that

$$\lambda(\rho)\operatorname{div} u - P(\rho) + \widetilde{P} = (-\Delta)^{-1}\operatorname{div}\operatorname{div}(2\mu(\rho)\mathbb{D}u) - (-\Delta)^{-1}\operatorname{div}(\rho\dot{u}). \tag{3.4}$$

By the help of the fundamental solution of the Laplacian, we express the first term of the right hand side above by the following,

$$\begin{aligned}(-\Delta)^{-1}\operatorname{div}\operatorname{div}(2\mu(\rho)\mathbb{D}u) &= -p.v.\int_{\mathbb{R}^2} \Gamma(x-y)\operatorname{div}\operatorname{div}(2\mu(\rho)\mathbb{D}u)(y)dy \\ &= -2\mu(\rho)\operatorname{div} u - p.v.\int_{\mathbb{R}^2} \partial^2_{ij}\Gamma(x-y)(2\mu(\rho)\mathbb{D}^{ij}u)(y)dy\end{aligned}$$

from which, we deduce the following representation of the effective flux:

$$F = (2\mu(\rho) + \lambda(\rho))\operatorname{div} u - P(\rho) + \widetilde{P} = K\{(\mu(\rho) - \widetilde{\mu})\mathbb{D}u\} - (-\Delta)^{-1}\operatorname{div}(\rho\dot{u}). \tag{3.5}$$

Above, we have used the fact that

$$p.v.\int_{\mathbb{R}^2} \partial^2_{ij}\Gamma(x-y)\mathbb{D}^{ij}u(y)dy = 0.$$

On the other hand, we take the rotational operator to the momentum equation (3.1) in order to obtain:

$$\operatorname{curl}_{jk}(\rho\dot{u}) = \operatorname{curl}_{jk}\operatorname{div}(2\mu(\rho)\mathbb{D}u) = \operatorname{curl}_{jk}\operatorname{div}(2(\mu(\rho) - \widetilde{\mu})\mathbb{D}u) + \widetilde{\mu}\Delta\operatorname{curl}_{jk} u.$$

From which, we deduce

$$\begin{aligned}\widetilde{\mu}\operatorname{curl}_{jk} u &= (-\Delta)^{-1}\operatorname{curl}_{jk}\operatorname{div}(2(\mu(\rho) - \widetilde{\mu})\mathbb{D}u) - (-\Delta)^{-1}\operatorname{curl}_{jk}(\rho\dot{u}) \\ &= -(\mu(\rho) - \widetilde{\mu})\operatorname{curl}_{jk} + K'_{jklm}\{(\mu(\rho) - \widetilde{\mu})\mathbb{D}^{lm}u\} - (-\Delta)^{-1}\operatorname{curl}_{jk}(\rho\dot{u}).\end{aligned}$$

It then holds that the vorticity takes the following form:

$$\mu(\rho)\operatorname{curl}_{jk} u = K'_{jklm}\{(\mu(\rho) - \widetilde{\mu})\mathbb{D}^{lm}u\} - (-\Delta)^{-1}\operatorname{curl}_{jk}(\rho\dot{u}).$$

Let us notice that the singular integral operators K and K' are homogeneous with symbols of degree zero, such that, in view of Calderón Zygmung theory, see [37], they map the Lebesgue space $L^p(\mathbb{R}^d)$ into itself for $1 < p < \infty$, with norm $\kappa(p)$. It turns out that the following L^p estimates hold true for the vorticity and the effective flux:

$$\|F\|_{L^p(\mathbb{R}^2)} \leqslant \kappa(p)\|\mu(\rho) - \widetilde{\mu}\|_{L^\infty(\mathbb{R}^2)}\|\mathbb{D}u\|_{L^p(\mathbb{R}^2)} + \|(-\Delta)^{-1}\operatorname{div}(\rho\dot{u})\|_{L^p(\mathbb{R}^2)}, \tag{3.6}$$

$$\|\mu(\rho)\operatorname{curl} u\|_{L^p(\mathbb{R}^2)} \leqslant \kappa(p)\|\mu(\rho) - \widetilde{\mu}\|_{L^\infty(\mathbb{R}^2)}\|\mathbb{D}u\|_{L^p(\mathbb{R}^2)} + \|(-\Delta)^{-1}\operatorname{curl}(\rho\dot{u})\|_{L^p(\mathbb{R}^2)} \tag{3.7}$$

for all $1 < p < \infty$. Also, theory of singular integrals operators, together with the above estimates imply:

$$\begin{aligned}\|\nabla u\|_{L^p(\mathbb{R}^2)} &\leqslant \kappa(p)\left(\|\operatorname{div} u\|_{L^p(\mathbb{R}^2)} + \|\operatorname{curl} u\|_{L^p(\mathbb{R}^2)}\right)\\ &\leqslant \kappa(p)\left(\frac{1}{2\underline{\mu}+\underline{\lambda}}\|F + P(\rho) - \widetilde{P}\|_{L^p(\mathbb{R}^2)} + \frac{1}{\underline{\mu}}\|\mu(\rho)\operatorname{curl} u\|_{L^p(\mathbb{R}^2)}\right)\\ &\leqslant \frac{\kappa(p)^2}{\underline{\mu}}\|\mu(\rho) - \widetilde{\mu}\|_{L^\infty(\mathbb{R}^2)}\|\mathbb{D}u\|_{L^p(\mathbb{R}^2)} + \frac{\kappa(p)}{2\underline{\mu}+\underline{\lambda}}\|P(\rho) - \widetilde{P}\|_{L^p(\mathbb{R}^2)}\\ &+ \frac{\kappa(p)}{\underline{\mu}}\left(\|(-\Delta)^{-1}\operatorname{div}(\rho\dot{u})\|_{L^p(\mathbb{R}^2)} + \|(-\Delta)^{-1}\operatorname{curl}(\rho\dot{u})\|_{L^p(\mathbb{R}^2)}\right).\end{aligned}$$

Assume that

$$\delta := \frac{\|\mu(\rho) - \widetilde{\mu}\|_{L^\infty(\mathbb{R}^2)}}{\underline{\mu}} < \frac{1}{\kappa(p)^2} \tag{3.8}$$

then, the first term of the right hand side of the above inequality can be absorbed in the left hand side and it yields:

$$\begin{aligned}\|\nabla u\|_{L^p(\mathbb{R}^2)} &\leqslant \frac{\kappa(p)}{\underline{\mu}(1-\delta\kappa(p)^2)}\left(\|(-\Delta)^{-1}\operatorname{div}(\rho\dot{u})\|_{L^p(\mathbb{R}^2)} + \|(-\Delta)^{-1}\operatorname{curl}(\rho\dot{u})\|_{L^p(\mathbb{R}^2)}\right)\\ &+ \frac{\kappa(p)}{(2\underline{\mu}+\underline{\lambda})(1-\delta\kappa(p)^2)}\|P(\rho) - \widetilde{P}\|_{L^p(\mathbb{R}^2)}.\end{aligned} \tag{3.9}$$

We turn to the estimate of the L^p norm of the pressure that appears in the above inequality with the help of the potential energies. First, we recall that the potential energy H_l satisfies:

$$\frac{d}{dt}\int_{\mathbb{R}^2} H_l(\rho) + \int_{\mathbb{R}^2}|P(\rho) - \widetilde{P}|^{l-1}(P(\rho) - \widetilde{P})\operatorname{div} u = 0.$$

One can substitute the divergence of the velocity in the expression of the effective flux (3.5) as:

$$\operatorname{div} u = (2\mu(\rho) + \lambda(\rho))^{-1}\left(F + P(\rho) - \widetilde{P}\right)$$

and then obtains, after Hölder inequality:

$$\begin{aligned}&\frac{d}{dt}\int_{\mathbb{R}^2} H_l(\rho) + \int_{\mathbb{R}^2}(2\mu(\rho) + \lambda(\rho))^{-1}|P(\rho) - \widetilde{P}|^{l+1}\\ &\qquad = -\int_{\mathbb{R}^2}(2\mu(\rho) + \lambda(\rho))^{-1}|P(\rho) - \widetilde{P}|^{l-1}(P(\rho) - \widetilde{P})F\\ &\qquad \leqslant \frac{l}{l+1}\int_{\mathbb{R}^2}(2\mu(\rho) + \lambda(\rho))^{-1}|P(\rho) - \widetilde{P}|^{l+1} + \frac{1}{l+1}\int_{\mathbb{R}^2}(2\mu(\rho) + \lambda(\rho))^{-1}|F|^{l+1}.\end{aligned}$$

The first term in the right hand side can be absorbed in the left hand side and it follows that:

$$\frac{d}{dt}\int_{\mathbb{R}^2} H_l(\rho) + \frac{1}{l+1}\int_{\mathbb{R}^2}(2\mu(\rho) + \lambda(\rho))^{-1}|P(\rho) - \widetilde{P}|^{l+1} \leqslant \frac{1}{l+1}\int_{\mathbb{R}^2}(2\mu(\rho) + \lambda(\rho))^{-1}|F|^{l+1}. \tag{3.10}$$

In order to express the L^{l+1} norm of the effective flux, in the above inequality, one goes back to (3.6) and then uses (3.9) and obtains the following:

$$\begin{aligned}\int_{\mathbb{R}^2}(2\mu(\rho) + \lambda(\rho))^{-1}|F|^{l+1} &\leqslant \frac{1}{2\underline{\mu}+\underline{\lambda}}\|K\{(\mu(\rho) - \widetilde{\mu})\mathbb{D}u\}\|^{l+1}_{L^{l+1}(\mathbb{R}^2)} + \frac{1}{2\underline{\mu}+\underline{\lambda}}\int_{\mathbb{R}^2}|(-\Delta)^{-1}\operatorname{div}(\rho\dot{u})|^{l+1}\\ &\leqslant \frac{\kappa(l+1)^{l+1}}{2\underline{\mu}+\underline{\lambda}}\|\mu(\rho) - \widetilde{\mu}\|^{l+1}_{L^\infty(\mathbb{R}^2)}\|\mathbb{D}u\|^{l+1}_{L^{l+1}(\mathbb{R}^2)} + \frac{1}{2\underline{\mu}+\underline{\lambda}}\int_{\mathbb{R}^2}|(-\Delta)^{-1}\operatorname{div}(\rho\dot{u})|^{l+1}\\ &\leqslant \frac{\kappa(l+1)^{l+1}}{2\underline{\mu}+\underline{\lambda}}\left(\frac{\kappa(l+1)}{2\underline{\mu}+\underline{\lambda}}\frac{\|\mu(\rho) - \widetilde{\mu}\|_{L^\infty}}{1-\delta\kappa(l+1)^2}\right)^{l+1}\|P(\rho) - \widetilde{P}\|^{l+1}_{L^{l+1}(\mathbb{R}^2)}\\ &+ \frac{\kappa(l+1)^{l+1}}{2\underline{\mu}+\underline{\lambda}}\left(\frac{\kappa(l+1)}{\underline{\mu}}\frac{\|\mu(\rho) - \widetilde{\mu}\|_{L^\infty(\mathbb{R}^2)}}{1-\delta\kappa(l+1)^2}\right)^{l+1}\Big(\|(-\Delta)^{-1}\operatorname{div}(\rho\dot{u})\|^{l+1}_{L^{l+1}(\mathbb{R}^2)}\\ &+\|(-\Delta)^{-1}\operatorname{curl}(\rho\dot{u})\|^{l+1}_{L^{l+1}(\mathbb{R}^2)}\Big) + \frac{1}{2\underline{\mu}+\underline{\lambda}}\|(-\Delta)^{-1}\operatorname{div}(\rho\dot{u})\|^{l+1}_{L^{l+1}(\mathbb{R}^2)}.\end{aligned}$$

In the following lines, we will replace the above estimate for the effective flux in (3.10) and absorb the norm of the pressure that is the first term of the right hand side of the above estimate in the left side of (3.10). And so that, we need for example, for $l > 1$, the following:

$$\delta \left(\frac{2\underline{\mu} + \underline{\lambda}}{2\overline{\mu} + \overline{\lambda}} \right)^{-\frac{1}{l+1}} < \frac{1}{3\sqrt{(l+1)}\kappa(l+1)^2}.$$

In conclusion, for all $l > 1$ there exists a constant $\kappa = \kappa(l)$ such that if

$$\delta \left(\frac{2\underline{\mu} + \underline{\lambda}}{2\overline{\mu} + \overline{\lambda}} \right)^{-\frac{1}{l+1}} < \kappa(l) \tag{3.11}$$

then, (3.9) holds for $p = l + 1$ and the following inequality also:

$$\frac{d}{dt} \int_{\mathbb{R}^2} H_l(\rho) + \|P(\rho) - \widetilde{P}\|^{l+1}_{L^{l+1}(\mathbb{R}^2)} \lesssim \|(-\Delta)^{-1} \operatorname{div}(\rho \dot{u})\|^{l+1}_{L^{l+1}(\mathbb{R}^2)} + \|(-\Delta)^{-1} \operatorname{curl}(\rho \dot{u})\|^{l+1}_{L^{l+1}(\mathbb{R}^2)}. \tag{3.12}$$

The objective of the above computations is to know the appropriate smallness condition on the viscosity (3.11). Let us notice that, here, in order to close the estimates for functional $\mathcal{A}_1$ and $\mathcal{A}_2$, we will only need the smallness condition on the viscosity (3.11) for $l \in \{2, 3\}$ however, latter, one will need the case $l = 5$. The constants involved in (3.9) and (3.11) and the latter estimates depend not only on the lower and upper bounds of the density, viscosity but also on δ. In particular, when δ approaches

$$\kappa(l) \left(\frac{2\underline{\mu} + \underline{\lambda}}{2\overline{\mu} + \overline{\lambda}} \right)^{\frac{1}{l+1}},$$

the underlying constants become large. Assuming the smallness condition (3.11) for $l = 2$, the inequality (3.12) holds true for $l = 2$ and one can integrate it in time in order to obtain:

$$\sup_{[0,t]} \int_{\mathbb{R}^2} H_2(\rho) + \int_0^t \|P(\rho) - \widetilde{P}\|^3_{L^3(\mathbb{R}^2)} \lesssim \int_{\mathbb{R}^2} H_2(\rho_0) + \int_0^t \left(\|(-\Delta)^{-1} \operatorname{div}(\rho \dot{u})\|^3_{L^3(\mathbb{R}^2)} + \|(-\Delta)^{-1} \operatorname{curl}(\rho \dot{u})\|^3_{L^3(\mathbb{R}^2)} \right). \tag{3.13}$$

Next, we assume again that the smallness condition (3.11) holds for $l = 3$, then, one can multiply (3.12) by σ before integrating in time and obtains:

$$\sup_{[0,t]} \sigma \int_{\mathbb{R}^2} H_3(\rho) + \int_0^t \sigma \|P(\rho) - \widetilde{P}\|^4_{L^4(\mathbb{R}^2)} \lesssim \int_0^{\sigma(t)} \int_{\mathbb{R}^2} H_3(\rho) + \int_0^t \sigma \left(\|(-\Delta)^{-1} \operatorname{div}(\rho \dot{u})\|^4_{L^4(\mathbb{R}^2)} + \|(-\Delta)^{-1} \operatorname{curl}(\rho \dot{u})\|^4_{L^4(\mathbb{R}^2)} \right).$$

By combining the above estimate, the inequality (3.9), Gagliardo-Nirenberg inequality and the fact that $H_3(\rho) \lesssim_{\overline{\rho}} H_1(\rho)$, one obtains the following:

$$\int_0^t \sigma \left(\|\nabla u\|^4_{L^4(\mathbb{R}^2)} + \|P(\rho) - \widetilde{P}\|^4_{L^4(\mathbb{R}^2)} \right) \lesssim C_0 + \mathcal{A}_1(t) \left(C_0 + \mathcal{A}_1(t) \right). \tag{3.14}$$

Above, we also use the fact that

$$\sup_{[0,t]} \|P(\rho) - \widetilde{P}\|^2_{L^2(\mathbb{R}^2)} \lesssim_{\overline{\rho}} \sup_{[0,t]} \int_{\mathbb{R}^2} H_1(\rho).$$

We now go back to (3.3) from which we deduce the following bound for the second Hoff functional:

$$\mathcal{A}_2(t) \lesssim C_0 + \mathcal{A}_1(\sigma(t)) + \mathcal{A}_1(t) \left(C_0 + \mathcal{A}_1(t) \right)$$

and we turn to the bound for the first Hoff functional. We may first recall that from (3.2), one has the following:

$$\mathcal{A}_1(t) \lesssim C_0 + \int_0^t \|\nabla u\|^3_{L^3(\mathbb{R}^2)} + \int_0^t \|P(\rho) - \widetilde{P}\|_{L^p(\mathbb{R}^2)} \|\nabla u\|^2_{L^{2p'}(\mathbb{R}^2)}$$

for some $p > 3$ and all integral above will be splitted into two

$$\int_0^t = \int_0^{\sigma(t)} + \int_{\sigma(t)}^t.$$

For the lower time, we take $p = 3$ and by Hölder and Young inequalities, we have from (3.9) and (3.13) the following:

$$\int_0^{\sigma(t)} \left(\|\nabla u\|^3_{L^3(\mathbb{R}^2)} + \|P(\rho) - \widetilde{P}\|^3_{L^3(\mathbb{R}^2)} \right) \lesssim C_0 + \mathcal{A}_1(t)^{1/2} \left(C_0 + \mathcal{A}_1(t) \right). \tag{3.15}$$

For large time, we will take $p = 4$, and by Hölder, Young, Gagliardo-Nirenberg and interpolation inequalities, we have:

$$\int_{\sigma(t)}^t \|\nabla u\|^3_{L^3(\mathbb{R}^2)} + \int_{\sigma(t)}^t \|P(\rho) - \widetilde{P}\|_{L^4(\mathbb{R}^2)} \|\nabla u\|^2_{L^{8/3}(\mathbb{R}^2)} \lesssim C_0 + \mathcal{A}_1(t) \left(C_0 + \mathcal{A}_1(t) \right). \tag{3.16}$$

Finally, we have the following estimate for the first Hoff functional:

$$\mathcal{A}_1(t) \lesssim C_0 + \mathcal{A}_1(t)^{1/2} \left(C_0 + \mathcal{A}_1(t) \right) + \mathcal{A}_1(t) \left(C_0 + \mathcal{A}_1(t) \right).$$

The functional $\mathcal{A}_1$ and $\mathcal{A}_2$ are under control as long as the density and the viscosity are bounded away from vacuum and upper bounded. Additionally, the dynamic viscosity should be a small perturbation of the constant state $\widetilde{\mu}$. This ends the proof of Proposition 2.4. □

3.2. **Proof of Theorem 2.5.**

Proof. Let us begin the proof of Theorem 2.5 by rewriting the mass equation $(2.1)_1$. We recall that it can be written as follows:

$$\partial_t \log \rho + u \cdot \nabla \log \rho + \operatorname{div} u = 0.$$

We multiply the above by $2\mu(\rho) + \lambda(\rho)$ and substitute the last term by the help of the effective flux in order to obtain:

$$\partial_t f(\rho) + u \cdot \nabla f(\rho) + P(\rho) - \widetilde{P} = -F$$

where $f(\rho)$ is:

$$f(\rho) = \int_{\widetilde{\rho}}^{\rho} \frac{2\mu(s) + \lambda(s)}{s} ds. \tag{3.17}$$

In particular, if X is the flow of the velocity u, then for all $x \in \mathbb{R}^2$, one has:

$$\frac{d}{dt} f(\rho(t, X(t,x))) + P(\rho(t, X(t,x))) - \widetilde{P} = -F(t, X(t,x)). \tag{3.18}$$

In particular along the interface $\mathcal{C}(t)$ parameterized by $\gamma(t,s) = X(t, \gamma_0(s))$, one has:

$$\frac{d}{dt} f(\rho(t, \gamma(t,s))) + P(\rho(t, \gamma(t,s))) - \widetilde{P} = -F(t, \gamma(t,s)) \tag{3.19}$$

and then, by taking the jump at $\gamma(t,s)$, one has:

$$\begin{aligned} \frac{d}{dt} [\![f(\rho(t, \gamma(t,s)))]\!] + g(t,s) [\![f(\rho(t, \gamma(t,s)))]\!] &= -[\![F(t, \gamma(t,s))]\!] \\ &= -2h(t,s) [\![f(\rho(t, \gamma(t,s)))]\!] \left(< \operatorname{div} u > - < \mathbb{D}^{jk} u > n_x^j n_x^k \right) (t, \gamma(t,s)) \end{aligned}$$

where we have used the expression of the jump of the effective flux given in (1.16). Above, the function g and h are defined as:

$$g(t,s) := \frac{[\![P(\rho(t, \gamma(t,s)))]\!]}{[\![f(\rho(t, \gamma(t,s)))]\!]} \quad \text{and} \quad h(t,s) := \frac{[\![\mu(\rho(t, \gamma(t,s)))]\!]}{[\![f(\rho(t, \gamma(t,s)))]\!]}.$$

To achieve an exponential time decay of the jump of $f(\rho)$ and subsequently deduce the exponential time decay of the pressure jump, we require that g is lower and upper bounded, and h is just bounded, that is, for all $0 < \underline{\rho} \leqslant \rho,\ \rho' \leqslant \overline{\rho}$:

$$0 < \underline{\mathrm{v}} \leqslant \frac{P(\rho) - P(\rho')}{f(\rho) - f(\rho')} \leqslant \overline{\mathrm{v}} \quad \text{and} \quad \left| \frac{\mu(\rho) - \mu(\rho')}{f(\rho) - f(\rho')} \right| \leqslant \overline{\mathrm{v}} \tag{3.20}$$

for some constant $0 < \underline{\mathrm{v}} < \overline{\mathrm{v}}$ that depend on the lower and upper bound of the density. We observe that when the pressure law and viscosity laws are proportional, the condition (3.20) obviously holds. In fact the above condition states somehow that the difference of the pressure, of the dynamic viscosity μ and of $f(\rho)$ vanish at the same state to which the density converges exponentially. The first condition in (3.20) naturally holds due to the fact that the

pressure and $f(\rho)$ are increasing functions of ρ whereas the second inequality (3.20) holds as soon as the viscosity is an increasing function of ρ. The same condition holds also for the kinematic viscosity λ. It then follows that

$$\frac{d}{dt}\left\{e^{\int_0^t g(\tau,s)d\tau}[\![f(\rho(t,\gamma(t,s)))]\!]\right\} = -2h(t,s)e^{\int_0^t g(\tau,s)d\tau}[\![f(\rho(t,\gamma(t,s)))]\!]\left(<\operatorname{div} u> - <\mathbb{D}^{jk}u> n_x^j n_x^k\right)(t,\gamma(t,s))$$

and consequently:

$$|[\![f(\rho(t,\gamma(t,s)))]\!]| \leqslant |[\![f(\rho_0(\gamma_0(s)))]\!]| \exp\left(-\underline{\mathrm{v}}t + 4\overline{\mathrm{v}}\int_0^t \|\nabla u(\tau)\|_{L^\infty(\mathbb{R}^2)}d\tau\right).$$

Thus, the $L^\infty(\mathcal{C}(t))$ norm of the jump of $f(\rho)$ is bounded as follows:

$$\|[\![f(\rho(t))]\!]\|_{L^\infty(\mathcal{C}(t))} \leqslant \|[\![f(\rho_0,)]\!]\|_{L^\infty(\mathcal{C}(0))} \exp\left(-\underline{\mathrm{v}}t + 4\overline{\mathrm{v}}\int_0^t \|\nabla u(\tau)\|_{L^\infty(\mathbb{R}^2)}d\tau\right).$$

Moreover, for all $1 \leqslant p \leqslant \infty$, one has the following $L^p(\mathcal{C}(t))$ norm of $[\![f(\rho(t))]\!]$:

$$\|[\![f(\rho(t))]\!]\|_{L^p(\mathcal{C}(t))} \leqslant \|[\![f(\rho_0)]\!]\|_{L^p(\mathcal{C}(0))} \exp\left(-\underline{\mathrm{v}}t + (4\overline{\mathrm{v}} + 1/p)\int_0^t \|\nabla u(\tau)\|_{L^\infty(\mathbb{R}^2)}d\tau\right). \tag{3.21}$$

From the condition (3.20), a similar estimate holds for the jump of the viscosity $\mu(\rho)$ and the pressure $P(\rho)$. Since the kinematic viscosity λ fulfills the same condition as the dynamic viscosity μ in (3.20) a similar estimate also holds for the jump of $\lambda(\rho)$. Next, we go back to (1.16) and (1.17) in order to express the jump of $\operatorname{div} u$ and $\operatorname{curl} u$ as follows:

$$<2\mu(\rho) + \lambda(\rho)> [\![\operatorname{div} u]\!] = [\![P(\rho)]\!] - [\![\lambda(\rho)]\!] <\operatorname{div} u> -2[\![\mu(\rho)]\!] <\mathbb{D}^{jk}u> n_x^j n_x^k,$$
$$<\mu(\rho)> [\![\operatorname{curl} u]\!] = -2[\![\mu(\rho)]\!] <\mathbb{D}^{jk}u> n_x^k \tau_x^j.$$

From these expressions and the $L^p(\mathcal{C}(t))$ estimate (3.21), we deduce the $L^p(\mathcal{C}(t))$ norm of the jump of $\operatorname{div} u$ and $\operatorname{curl} u$ before deriving the following estimate for the jump of the velocity gradient using (1.13) and (1.14):

$$\begin{aligned}\|[\![\nabla u(t)]\!]\|_{L^p(\mathcal{C}(t))} \leqslant M\|[\![f(\rho_0)]\!]\|_{L^p(\mathcal{C}(0))}&\left(1 + \|\nabla u(t)\|_{L^\infty(\mathbb{R}^2)}\right)\\ &\exp\left(-\underline{\mathrm{v}}t + (4\overline{\mathrm{v}} + 1/p)\int_0^t \|\nabla u(\tau)\|_{L^\infty(\mathbb{R}^2)}d\tau\right),\ p\in[1,\infty].\end{aligned} \tag{3.22}$$

Above the constant M depends only on the lower and upper bounds of the density and viscosity. This ends the proof of Theorem 2.5. □

3.3. **Proof of Theorem 2.6.**

. Let us recall that the functional $\mathcal{A}_3$ naturally comes while testing the equation (2.13) with $\sigma^2\ddot{u}$. By doing so, we obtain the following:

$$\begin{aligned}\int_0^t \sigma^2\|\sqrt{\rho}\ddot{u}\|^2_{L^2(\mathbb{R}^2)} + \sigma^2(t)\int_{\mathbb{R}^2}\left\{\mu(\rho)|\mathbb{D}\dot{u}|^2 + \frac{\lambda(\rho)}{2}|\operatorname{div}\dot{u}|^2\right\} &= 2\int_0^{\sigma(t)}\sigma\int_{\mathbb{R}^2}\left\{\mu(\rho)|\mathbb{D}\dot{u}|^2 + \frac{\lambda(\rho)}{2}|\operatorname{div}\dot{u}|^2\right\}\\ &- \sigma^2\int_{\mathbb{R}^2}\rho P'(\rho)\operatorname{div}\dot{u}\operatorname{div}u + \int_0^{\sigma(t)}\sigma\int_{\mathbb{R}^2}\rho P'(\rho)\operatorname{div}\dot{u}\operatorname{div}u + \int_0^t\sigma^2\int_{\mathbb{R}^2}\rho P'(\rho)(\operatorname{div}\dot{u})^2\\ &+ \int_0^t\sigma^2\int_{\mathbb{R}^2}\operatorname{div}\dot{u}(\operatorname{div}u)^2(\rho P'(\rho) - P(\rho) + \widetilde{P}) - \int_0^t\sigma^2\int_{\mathbb{R}^2}\partial_l\dot{u}^j\operatorname{div}u\partial_j u^l(\rho P'(\rho) - P(\rho) + \widetilde{P})\\ &+ \sigma^2(t)I_1(t) - 2\int_0^{\sigma(t)}\sigma I_1(s)ds + \int_0^t\sigma^2 I_2(s)ds + \int_0^t\sigma^2 I_3(s)ds\end{aligned} \tag{3.23}$$

where terms $I_1,\ I_2,\ I_3$ are of the type

$$I_1 = \int_{\mathbb{R}^2}\varphi(\rho)\partial_{j_1}\dot{u}^{j_2}\partial_{j_3}u^{j_4}\partial_{j_5}u^{j_6} + \int_{\mathbb{R}^2}\partial_{j_1}\dot{u}^{j_2}\partial_{j_3}u^{j_4}(P(\rho) - \widetilde{P}), \tag{3.24}$$

$$I_2 = \int_{\mathbb{R}^2}\varphi(\rho)\partial_{j_1}\dot{u}^{j_2}\partial_{j_3}\dot{u}^{j_4}\partial_{j_5}u^{j_6} + \int_{\mathbb{R}^2}\partial_{j_1}\dot{u}^{j_2}\partial_{j_3}\dot{u}^{j_4}(P(\rho) - \widetilde{P}) + \int_{\mathbb{R}^2}\psi(\rho)\partial_{j_1}\dot{u}^{j_2}\partial_{j_3}u^{j_4}\partial_{j_5}\dot{u}^{j_6}, \tag{3.25}$$

$$I_3 = \int_{\mathbb{R}^2}\varphi(\rho)\partial_{j_1}\dot{u}^{j_2}\partial_{j_3}u^{j_4}\partial_{j_5}u^{j_6}\partial_{j_7}u^{j_8} + \int_{\mathbb{R}^2}\partial_{j_1}\dot{u}^{j_2}\partial_{j_3}u^{j_4}\partial_{j_5}u^{j_6}(P(\rho) - \widetilde{P}), \tag{3.26}$$

and where φ is either the viscosity μ or λ or $\rho\mu'$, $\rho\lambda'$, $\rho^2\mu''$ or $\rho^2\lambda''$ whereas ψ is either $\rho P'$ or $\rho^2 P''$. The computations leading to (3.23) can be found in Appendix B.2 and use the fact that the local solution (ρ, u) given by Theorem 2.1 enjoys the following Rankine Hugoniot condition at the interface (see Appendix B.1):

$$[\![\dot{\Pi}^{jk}]\!]n_x^k + [\![\Pi^{jk}\operatorname{div} u]\!]n_x^k - [\![\partial_k u^l \Pi^{jk}]\!]n_x^l = 0$$

and the fact that the second derivative of the velocity, $\ddot{u}$, is continuous at the interface, see Remark 2.2. In the following, we will estimate the terms appearing in the left hand side of (3.23) in term of C_0 and functional $\mathcal{A}_1$, $\mathcal{A}_2$ and $\mathcal{A}_3$. It is clear that the first term on the right hand side of (3.23) is controlled by the second Hoff functional:

$$2\int_0^{\sigma(t)} \sigma \int_{\mathbb{R}^2} \left\{ \mu(\rho)|\mathbb{D}\dot{u}|^2 + \frac{\lambda(\rho)}{2}|\operatorname{div}\dot{u}|^2 \right\} \leqslant \mathcal{A}_2(\sigma(t)),$$

whereas the second term on the right hand side can be estimated as follows:

$$\sigma^2(t)\left| \int_{\mathbb{R}^2} \rho P'(\rho)\operatorname{div}\dot{u}\operatorname{div} u \right| \leqslant \eta\mathcal{A}_3(t) + C_\eta \mathcal{A}_1(t)$$

for some positive η small in order to absorb the first term in the right hand side of (3.23). Next, the third term of the right hand side of (3.23) can be bounded by the help of the initial energy and the second Hoff functional:

$$\left| \int_0^{\sigma(t)} \sigma \int_{\mathbb{R}^2} \rho P'(\rho)\operatorname{div}\dot{u}\operatorname{div} u \right| \lesssim E_0^{1/2}\mathcal{A}_2(t)^{1/2},$$

and it is obvious that the fifth term is controlled by the second functional:

$$\left| \int_0^t \sigma^2 \int_{\mathbb{R}^2} \rho P'(\rho)(\operatorname{div}\dot{u})^2 \right| \lesssim \mathcal{A}_2(t).$$

We observe the following bound for the sixth and seventh term of the right hand side of (3.23)

$$\begin{aligned}\left| \int_0^t \sigma^2 \int_{\mathbb{R}^2} \operatorname{div}\dot{u}(\operatorname{div} u)^2(\rho P'(\rho) - P(\rho) + \widetilde{P}) \right| + \left| \int_0^t \sigma^2 \int_{\mathbb{R}^2} \partial_l \dot{u}^j \operatorname{div} u \partial_j u^l (\rho P'(\rho) - P(\rho) + \widetilde{P}) \right| \\ \lesssim \left[\int_0^t \sigma \|\nabla\dot{u}\|^2_{L^2(\mathbb{R}^2)} \right]^{1/2} \left[\int_0^t \sigma \|\nabla u\|^4_{L^4(\mathbb{R}^2)} \right]^{1/2} \\ \lesssim \mathcal{A}_2(t)^{1/2} \left(C_0 + \mathcal{A}_1(t) \left(C_0 + \mathcal{A}_1(t) \right) \right)^{1/2},\end{aligned} \tag{3.27}$$

where we have used the estimate (3.14) to control the $L^4((0,T)\times\mathbb{R}^2)$ norm of the gradient of the velocity. We now turn to the last four terms of (3.23). The first term among the underlying terms can be bounded, thanks to Hölder and Young's inequalities, by:

$$\left| \sigma^2(t) I_1(t) \right| \leqslant \eta\sigma^2(t)\|\nabla\dot{u}\|^2_{L^2(\mathbb{R}^2)} + C_\eta \sigma^2(t) \left(\|\nabla u\|^4_{L^4(\mathbb{R}^2)} + \|P(\rho) - \widetilde{P}\|^4_{L^4(\mathbb{R}^2)} \right)$$

for some η small in order to absorb the first term on the right hand side in the left hand side of (3.23). On the other hand, owing to the fact that the $L^\infty((0,T), L^4(\mathbb{R}^2))$ norm of the pressure is controlled by the classical energy, we deduce from (3.9) the following:

$$\begin{aligned}\sup_{[0,t]} \sigma \left(\|\nabla u\|^4_{L^4(\mathbb{R}^2)} + \|P(\rho) - \widetilde{P}\|^4_{L^4(\mathbb{R}^2)} \right) &\lesssim C_0 + \sup_{[0,t]} \sigma \left(\|(-\Delta)^{-1}\operatorname{div}(\rho\dot{u})\|^4_{L^4(\mathbb{R}^2)} + \|(-\Delta)^{-1}\operatorname{curl}(\rho\dot{u})\|^4_{L^4(\mathbb{R}^2)} \right) \\ &\lesssim C_0 + \sup_{[0,t]} \sigma \|\rho\dot{u}\|^2_{L^2(\mathbb{R}^2)} \left(\|\nabla u\|^2_{L^2(\mathbb{R}^2)} + \|P(\rho) - \widetilde{P}\|^2_{L^2(\mathbb{R}^2)} \right) \\ &\lesssim C_0 + \mathcal{A}_2(t) \left(C_0 + \mathcal{A}_1(t) \right).\end{aligned} \tag{3.28}$$

Thus, we have

$$\sup_{[0,t]} |\sigma I_1| \leqslant \eta\mathcal{A}_3(t) + C_\eta \left(C_0 + \mathcal{A}_2(t) \left(C_0 + \mathcal{A}_1(t) \right) \right). \tag{3.29}$$

Following (3.27), the second term can be bounded by:

$$\left| \int_0^{\sigma(t)} s I_1(s) ds \right| \lesssim \mathcal{A}_2(t)^{1/2} \left(C_0 + \mathcal{A}_1(t) \left(C_0 + \mathcal{A}_1(t) \right) \right)^{1/2}.$$

Next, the penultimate term of (3.23) is, thanks to Hölder and interpolation inequalities:

$$\begin{aligned}\left|\int_0^t \sigma^2 I_2(s)ds\right| &\lesssim \int_0^t \sigma^2 \|\nabla \dot{u}\|^2_{L^{8/3}(\mathbb{R}^2)} \left(\|\nabla u\|_{L^4(\mathbb{R}^2)} + \|P(\rho)-\widetilde{P}\|_{L^4(\mathbb{R}^2)}\right)\\ &\lesssim \int_0^t \sigma^2 \|\nabla \dot{u}\|^{3/2}_{L^3(\mathbb{R}^2)} \|\nabla \dot{u}\|^{1/2}_{L^2(\mathbb{R}^2)} \left(\|\nabla u\|_{L^4(\mathbb{R}^2)} + \|P(\rho)-\widetilde{P}\|_{L^4(\mathbb{R}^2)}\right)\\ &\lesssim \left[\int_0^t \sigma \left(\|\nabla u\|^4_{L^4(\mathbb{R}^2)} + \|P(\rho)-\widetilde{P}\|^4_{L^4(\mathbb{R}^2)}\right)\right]^{1/4} \left[\int_0^t \sigma \|\nabla \dot{u}\|^2_{L^2(\mathbb{R}^2)}\right]^{1/4} \left[\int_0^t \sigma^3 \|\nabla \dot{u}\|^3_{L^3(\mathbb{R}^2)}\right]^{1/2}\\ &\lesssim \mathcal{A}_2(t)^{1/4} \left(C_0 + \mathcal{A}_1(t)\left(C_0+\mathcal{A}_1(t)\right)\right)^{1/4} \left[\int_0^t \sigma^3 \|\nabla \dot{u}\|^3_{L^3(\mathbb{R}^2)}\right]^{1/2}\end{aligned}$$

where we have used the inequality (3.9) to control the $L^4((0,T)\times\mathbb{R}^2)$ norm of the pressure and of the velocity gradient. Finally, with the help of the Hölder inequality, the last term of equation (3.23) can be bounded as follows:

$$\begin{aligned}\left|\int_0^t \sigma^2 I_3(s)ds\right| &\lesssim \int_0^t \sigma^2 \|\nabla \dot{u}\|_{L^2(\mathbb{R}^2)} \left(\|\nabla u\|^3_{L^6(\mathbb{R}^2)} + \|P(\rho)-\widetilde{P}\|^3_{L^6(\mathbb{R}^2)}\right)\\ &\lesssim \left[\int_0^t \sigma \|\nabla \dot{u}\|^2_{L^2(\mathbb{R}^2)}\right]^{1/2} \left[\int_0^t \sigma^3 \left(\|\nabla u\|^6_{L^6(\mathbb{R}^2)} + \|P(\rho)-\widetilde{P}\|^6_{L^6(\mathbb{R}^2)}\right)\right]^{1/2}\\ &\lesssim \mathcal{A}_2(t)^{1/2} \left[\int_0^t \sigma^3 \left(\|\nabla u\|^6_{L^6(\mathbb{R}^2)} + \|P(\rho)-\widetilde{P}\|^6_{L^6(\mathbb{R}^2)}\right)\right]^{1/2}.\end{aligned}$$

In order to control the $L^6((0,T)\times\mathbb{R}^2)$ norm of the pressure and of the velocity gradient, we will assume that the smallness condition for the viscosity (3.11) holds for $l=5$, multiply (3.12) by σ and integrate in time and finally obtain, after considering (3.9):

$$\begin{aligned}\int_0^t \sigma \left(\|\nabla u\|^6_{L^6(\mathbb{R}^2)} + \|P(\rho)-\widetilde{P}\|^6_{L^6(\mathbb{R}^2)}\right) &\lesssim C_0 + \int_0^t \sigma \left(\|(-\Delta)^{-1}\operatorname{div}(\rho\dot{u})\|^6_{L^6(\mathbb{R}^2)} + \|(-\Delta)^{-1}\operatorname{curl}(\rho\dot{u})\|^6_{L^6(\mathbb{R}^2)}\right)\\ &\lesssim C_0 + \int_0^t \sigma \|\rho\dot{u}\|^4_{L^2(\mathbb{R}^2)} \left(\|\nabla u\|^2_{L^2(\mathbb{R}^2)} + \|P(\rho)-\widetilde{P}\|^2_{L^2(\mathbb{R}^2)}\right)\\ &\lesssim C_0 + \mathcal{A}_1(t)\mathcal{A}_2(t)\left(\mathcal{A}_1(t)+C_0\right). \qquad (3.30)\end{aligned}$$

In sum, we estimate all terms in the right hand side of (3.23) in term of C_0, functional $\mathcal{A}_1$, $\mathcal{A}_2$, $\mathcal{A}_3$ and the following term

$$\int_0^t \sigma^3 \|\nabla \dot{u}\|^3_{L^3(\mathbb{R}^2)},$$

that we will be analyzed in the following lines. The idea is similar to what has be done previously to estimate the $L^3((0,T)\times\mathbb{R}^2)$ norm of ∇u and $L^4((0,T)\times\mathbb{R}^2))$ norm of $\sigma^{1/4}\nabla u$. We begin by recalling that the equation (2.13) can be written as follows:

$$\rho \ddot{u}^j = \partial_k(\dot{\Pi}^{jk}) + \partial_k(\Pi^{jk}\operatorname{div} u) - \operatorname{div}(\partial_k u \Pi^{jk}) \qquad (3.31)$$

and we apply the divergence operator in order to observe that F_* defined by

$$F_* := (2\mu(\rho)+\lambda(\rho))\operatorname{div}\dot{u} - \lambda(\rho)\nabla u^l \partial_l u - \rho\lambda'(\rho)(\operatorname{div} u)^2 + \rho P'(\rho)\operatorname{div} u$$

can be expressed as follows:

$$\begin{aligned}F_* = &-(-\Delta)^{-1}\operatorname{div}(\rho\ddot{u}) - (-\Delta)^{-1}\partial_{jk}\{\mu(\rho)\partial_j u^l \partial_l u^k + \mu(\rho)\partial_k u^l \partial_l u^j + 2\rho\mu'(\rho)\mathbb{D}^{jk}u \operatorname{div} u\}\\ &+ (-\Delta)^{-1}\partial_{jk}(\Pi^{jk}\operatorname{div} u) - (-\Delta)^{-1}\partial_j \operatorname{div}(\partial_k u \Pi^{jk}) + K\{(\mu(\rho)-\widetilde{\mu})\mathbb{D}\dot{u}\}.\end{aligned} \qquad (3.32)$$

On the other hand, we can apply the curl to the equation (3.31) and obtain that the curl of the material derivative $\dot{u}$ can be expressed as:

$$\begin{aligned}\mu(\rho)\operatorname{curl}\dot{u} = &-(-\Delta)^{-1}\operatorname{curl}(\rho\ddot{u}) + K'\{(\mu(\rho)-\widetilde{\mu})\mathbb{D}\dot{u}\} + (-\Delta)^{-1}\operatorname{curl}\partial_k(\Pi^{jk}\operatorname{div} u)\\ &- (-\Delta)^{-1}\operatorname{curl}\operatorname{div}(\partial_k u \Pi^{jk}) - (-\Delta)^{-1}\operatorname{curl}\partial_k\{\mu(\rho)\partial_j u^l \partial_l u^k + \mu(\rho)\partial_k u^l \partial_l u^j + 2\rho\mu'(\rho)\mathbb{D}^{jk}u\operatorname{div} u\}.\end{aligned} \qquad (3.33)$$

Since the smallness condition (3.9) holds true for $p=3$, one deduces from singular operators theory that:

$$\|\nabla\dot{u}\|_{L^3(\mathbb{R}^2)} \lesssim \|(-\Delta)^{-1}\operatorname{div}(\rho\ddot{u})\|_{L^3(\mathbb{R}^2)} + \|(-\Delta)^{-1}\operatorname{curl}(\rho\ddot{u})\|_{L^3(\mathbb{R}^2)} + \|\nabla u\|^2_{L^6(\mathbb{R}^2)} + \|P(\rho)-\widetilde{P}\|^2_{L^6(\mathbb{R}^2)} + \|\nabla u\|_{L^3(\mathbb{R}^2)}.$$

It then turns out that the aforementioned term is bounded as follows:

$$\int_0^t \sigma^{5/2}\|\nabla\dot{u}\|^3_{L^3(\mathbb{R}^2)} \lesssim \int_0^t \sigma^{5/2}\left(\|(-\Delta)^{-1}\operatorname{div}(\rho\ddot{u})\|^3_{L^3(\mathbb{R}^2)} + \|(-\Delta)^{-1}\operatorname{curl}(\rho\ddot{u})\|^3_{L^3(\mathbb{R}^2)}\right) + \int_0^t \sigma^{5/2}\|\nabla u\|^3_{L^3(\mathbb{R}^2)} + \int_0^t \sigma^{5/2}\left(\|\nabla u\|^6_{L^6(\mathbb{R}^2)} + \|P(\rho)-\widetilde{P}\|^6_{L^6(\mathbb{R}^2)}\right). \tag{3.34}$$

Let us notice that the last two terms of the above inequality are controlled by (3.30) for the last term, and (3.15), (3.16) for the previous one. The first two terms on the right hand side of (3.34) can be bounded using the Gagliardo-Nirenberg inequality:

$$\begin{aligned}
&\int_0^t \sigma^{5/2}\left(\ \|(-\Delta)^{-1}\operatorname{div}(\rho\ddot{u})\|^3_{L^3(\mathbb{R}^2)} + \|(-\Delta)^{-1}\operatorname{curl}(\rho\ddot{u})\|^3_{L^3(\mathbb{R}^2)}\right)\\
&\qquad\lesssim \int_0^t \sigma^{5/2}\|\rho\ddot{u}\|_{L^2(\mathbb{R}^2)}\left(\|(-\Delta)^{-1}\operatorname{div}(\rho\ddot{u})\|^2_{L^2(\mathbb{R}^2)} + \|(-\Delta)^{-1}\operatorname{div}(\rho\ddot{u})\|^2_{L^2(\mathbb{R}^2)}\right)\\
&\qquad\lesssim \int_0^t \sigma^{5/2}\|\rho\ddot{u}\|_{L^2(\mathbb{R}^2)}\left(\|\nabla\dot{u}\|^2_{L^2(\mathbb{R}^2)} + \|\nabla u\|^4_{L^4(\mathbb{R}^2)} + \|P(\rho)-\widetilde{P}\|^4_{L^4(\mathbb{R}^2)} + \|\nabla u\|^2_{L^2(\mathbb{R}^2)}\right).
\end{aligned}$$

Above we have used (3.32) and (3.33) in order to express

$$\|(-\Delta)^{-1}\operatorname{div}(\rho\ddot{u})\|_{L^2(\mathbb{R}^2)} \quad \text{and} \quad \|(-\Delta)^{-1}\operatorname{curl}(\rho\ddot{u})\|_{L^2(\mathbb{R}^2)}.$$

Finally, we have:

$$\begin{aligned}
&\int_0^t \sigma^{5/2}\left(\|(-\Delta)^{-1}\operatorname{div}(\rho\ddot{u})\|^3_{L^3(\mathbb{R}^2)} + \|(-\Delta)^{-1}\operatorname{curl}(\rho\ddot{u})\|^3_{L^3(\mathbb{R}^2)}\right)\\
&\qquad\lesssim \left[\int_0^t \sigma^2\|\sqrt{\rho}\ddot{u}\|^2_{L^2(\mathbb{R}^2)}\right]^{1/2}\left[\int_0^t \sigma^3\left(\|\nabla\dot{u}\|^4_{L^2(\mathbb{R}^2)} + \|\nabla u\|^8_{L^4(\mathbb{R}^2)} + \|P(\rho)-\widetilde{P}\|^8_{L^4(\mathbb{R}^2)} + \|\nabla u\|^4_{L^2(\mathbb{R}^2)}\right)\right]^{1/2}\\
&\qquad\lesssim \mathcal{A}_3(t)\mathcal{A}_2(t)^{1/2} + C_0^{1/2}\mathcal{A}_1(t)^{1/2}\mathcal{A}_3(t)^{1/2}\\
&\qquad+ \mathcal{A}_3(t)^{1/2}\sup_{[0,t]}\sigma^{1/2}\left(\|\nabla u\|^2_{L^4(\mathbb{R}^2)} + \|P(\rho)-\widetilde{P}\|^2_{L^4(\mathbb{R}^2)}\right)\left[\int_0^t \sigma\left(\|\nabla u\|^4_{L^4(\mathbb{R}^2)} + \|P(\rho)-\widetilde{P}\|^4_{L^4(\mathbb{R}^2)}\right)\right]^{1/2}.
\end{aligned}$$

The norm of the pressure and the velocity gradient are controlled in (3.14) and (3.28). The estimate of the functional $\mathcal{A}_3$ is then achieved. To obtain regularity about higher-order material derivatives of velocity, one can follow the procedure as before by applying the material derivative $\partial_t\ \cdot + \operatorname{div}(\ \cdot\, u)$ to (3.31) and so on. It is clear that the computations are quite lengthy, but they follow the same principle. □

3.4. **Proof of Proposition 2.8.**

. The proof of Proposition 2.8 is divided in two parts: in the first one we obtain estimates for the functional β and the second part is devoted to the estimates for ϑ and for the effective flux F. We begin the first part of the proof by noticing that from the fact that the pressure and f are increasing functions of the density on $[\underline{\rho},\overline{\rho}]$ and the fact that the viscosity $2\mu(\rho)+\lambda(\rho)$ is bounded away from vacuum, we have the following:

$$\forall\ \underline{\rho} \leqslant \rho \leqslant \overline{\rho}, \qquad 0 < \underline{\nu} \leqslant \frac{P(\rho)-\widetilde{P}}{f(\rho)} \leqslant \overline{\nu}. \tag{3.35}$$

The constants $\underline{\nu}$, $\overline{\nu}$ depend on the lower and upper bounds of the density and viscosity. In fact, (3.35) can be obtained, for example, with the help of the Cauchy mean value theorem. We deduce from (3.18) the following:

$$f(\rho(\tau,X(\tau,x)) = f(\rho_0(x))e^{-\int_0^\tau g_1(\tau',x)d\tau'} - \int_0^\tau e^{-\int_{\tau'}^\tau g_1(\tau'',x)d\tau''}F(\tau',X(\tau',x))d\tau'$$

where the function g_1 is:

$$g_1(t,x) := \frac{P(\rho(t,X(t,x)))-\widetilde{P}}{f(\rho(t,X(t,x)))} \in [\underline{\nu},\,\overline{\nu}].$$

Thus, we have:

$$\|f(\rho(\tau))\|_{L^\infty(\mathbb{R}^2)} \leqslant e^{-\underline{\nu}\,\tau}\|f(\rho_0)\|_{L^\infty(\mathbb{R}^2)} + \int_0^\tau e^{\underline{\nu}\,(\tau'-\tau)}\|F(\tau')\|_{L^\infty(\mathbb{R}^2)}d\tau', \tag{3.36}$$

from which we deduce:

$$\sup_{[0,t]}\|f(\rho)\|^4_{L^\infty(\mathbb{R}^2)} \leqslant \|f(\rho_0)\|^4_{L^\infty(\mathbb{R}^2)} + M\int_0^t \sigma^{r_\alpha}\|F\|^4_{L^\infty(\mathbb{R}^2)} \tag{3.37}$$

for all $0 < r_\alpha < 3$ with M depending on r_α and $\underline{\nu}$. To obtain the above estimate, we express the last term of (3.36) into two terms as follows:

$$\int_0^\tau e^{\underline{\nu}(\tau'-\tau)}\|F(\tau')\|_{L^\infty(\mathbb{R}^2)}d\tau' = \int_0^\tau e^{\underline{\nu}(\tau'-\tau)}\|F(\tau')\|_{L^\infty(\mathbb{R}^2)}\mathbb{1}_{\{\tau'\leqslant 1\}}d\tau' + \int_0^\tau e^{\underline{\nu}(\tau'-\tau)}\|F(\tau')\|_{L^\infty(\mathbb{R}^2)}\mathbb{1}_{\{\tau'>1\}}d\tau'.$$

Thus by the help of Hölder's inequality we have the following:

$$\int_0^\tau e^{\underline{\nu}(\tau'-\tau)}\|F(\tau')\|_{L^\infty(\mathbb{R}^2)}\mathbb{1}_{\{\tau'\leqslant 1\}}d\tau' \leqslant \left(\frac{3}{3-r_\alpha}\right)^{3/4}\|\sigma^{r_\alpha/4}F\|_{L^4((0,\sigma(t)),L^\infty(\mathbb{R}^2))},$$

and

$$\int_0^\tau e^{\underline{\nu}(\tau'-\tau)}\|F(\tau')\|_{L^\infty(\mathbb{R}^2)}\mathbb{1}_{\{\tau'>1\}}d\tau' \leqslant \left(\frac{3}{4\underline{\nu}}\right)^{3/4}\|F\|_{L^4((\sigma(t),t),L^\infty(\mathbb{R}^2))}.$$

Moreover, by Young's inequality for convolution, one has:

$$\left\|\int_0^\tau e^{\underline{\nu}(\tau'-\tau)}\|F(\tau')\|_{L^\infty(\mathbb{R}^2)}\mathbb{1}_{\{\tau'\leqslant 1\}}d\tau'\right\|_{L^4(0,t)} \leqslant \left(\frac{1}{4\underline{\nu}}\right)^{1/4}\left(\frac{3}{3-r_\alpha}\right)^{3/4}\|\sigma^{r_\alpha/4}F\|_{L^4((0,\sigma(t)),L^\infty(\mathbb{R}^2))},$$

and

$$\left\|\int_0^\tau e^{\underline{\nu}(\tau'-\tau)}\|F(\tau')\|_{L^\infty(\mathbb{R}^2)}\mathbb{1}_{\{\tau'>1\}}d\tau'\right\|_{L^4(0,t)} \leqslant \frac{1}{\underline{\nu}}\|F\|_{L^4((\sigma(t),t),L^\infty(\mathbb{R}^2))}.$$

Thus, we have the following:

$$\int_0^t \|f(\rho(\tau))\|^4_{L^\infty(\mathbb{R}^2)}d\tau \lesssim \|f(\rho_0)\|^4_{L^\infty(\mathbb{R}^2)} + \int_0^t \sigma^{r_\alpha}\|F(\tau)\|^4_{L^\infty(\mathbb{R}^2)}d\tau. \tag{3.38}$$

We turn to the estimate of the $\dot{\mathscr{C}}^\alpha_{pw,\gamma}(\mathbb{R}^2)$ norm for the $f(\rho)$. Let x_i^0, $i\in\{1,2\}$ two points lying on the same side of the interface and let $x_i(t) := X(t,x_i)$. From the mass equation (3.19), one has the following:

$$\frac{d}{dt}f(\rho(t,x_i(t)))\Big|_{i=1}^{i=2} + P(\rho(t,x_i(t)))\Big|_{i=1}^{i=2} = -F(t,x_i(t))\Big|_{i=1}^{i=2}. \tag{3.39}$$

Using the fact that

$$g_2(t,x,y) := \frac{P(\rho(t,x)) - P(\rho(t,y))}{f(\rho(t,x)) - f(\rho(t,y))} \in [\underline{\nu},\,\overline{\nu}] \tag{3.40}$$

we have from (3.39):

$$f(\rho(\tau,x_i(\tau)))\Big|_{i=1}^{i=2} = f(\rho_0(x_i^0))\Big|_{i=1}^{i=2} e^{-\int_0^\tau g_2(\tau',x_2(\tau'),x_1(\tau'))d\tau'} - \int_0^\tau e^{-\int_{\tau'}^\tau g_2(\tau'',x_2(\tau''),x_1(\tau''))d\tau''}F(\tau',x_i(\tau'))\Big|_{i=1}^{i=2}d\tau'. \tag{3.41}$$

On the other hand, we have for all $0\leqslant\tau'\leqslant\tau$, the following:

$$|x_2(\tau') - x_1(\tau')| \leqslant e^{\int_{\tau'}^\tau\|\nabla u(\tau'')\|_{L^\infty(\mathbb{R}^2)}d\tau''}|x_2(\tau) - x_1(\tau)|$$

and thus (3.41) implies:

$$\begin{aligned}\|f(\rho(\tau))\|_{\dot{\mathscr{C}}^\alpha_{pw,\gamma(\tau)}(\mathbb{R}^2)} &\leqslant e^{\int_0^\tau\left[-g_2(\tau',x_2(\tau'),x_1(\tau'))+\|\nabla u(\tau')\|_{L^\infty(\mathbb{R}^2)}\right]d\tau'}\|f(\rho_0)\|_{\dot{\mathscr{C}}^\alpha_{pw,\gamma_0}(\mathbb{R}^2)}\\ &+ \int_0^\tau e^{\int_{\tau'}^\tau\left[-g_2(\tau'',x_2(\tau''),x_1(\tau''))+\|\nabla u(\tau'')\|_{L^\infty(\mathbb{R}^2)}d\tau''\right]}\|F(\tau')\|_{\dot{\mathscr{C}}^\alpha_{pw,\gamma(\tau)}(\mathbb{R}^2)}d\tau'.\end{aligned} \tag{3.42}$$

Also, we have for all $0 < r_\alpha < 3$, the following:

$$\int_{\tau'}^\tau\|\nabla u(\tau'')\|_{\mathscr{C}^\alpha_{pw,\gamma(\tau'')}(\mathbb{R}^2)}d\tau'' \leqslant \varepsilon(\tau-\tau') + C_\varepsilon\vartheta(t) \tag{3.43}$$

for $\varepsilon > 0$ small and a constant C_ε depending on ε and r_α. Here we choose $\varepsilon = \frac{\nu}{4}$ and we deduce from the lower bound of g_2 in (3.40), (3.42) and (3.43) the following estimates for $f(\rho)$:

(3.44)
$$\sup_{[0,t]}\|f(\rho)\|^4_{\dot{\mathscr{C}}^\alpha_{pw,\gamma}(\mathbb{R}^2)} + \int_0^t \|f(\rho(\tau))\|^4_{\dot{\mathscr{C}}^\alpha_{pw,\gamma(\tau)}(\mathbb{R}^2)}d\tau \leqslant M' e^{M'\vartheta(t)}\left(\|f(\rho_0)\|^4_{\dot{\mathscr{C}}^\alpha_{pw,\gamma_0}(\mathbb{R}^2)} + \int_0^t \sigma^{r_\alpha}\|F(\tau)\|^4_{\dot{\mathscr{C}}^\alpha_{pw,\gamma(\tau)}(\mathbb{R}^2)}d\tau\right).$$

Finally, by gathering (3.37), (3.38) and (3.44) we obtain the estimates for the functional β and this closes the first part of the proof of Proposition 2.8. The second part concerns the estimates for the functional ϑ and the Hölder norm of the effective flux. For this purpose, we recall that from the momentum equation, we have the following expression for the velocity gradient:

$$\begin{aligned}\widetilde{\mu}\Delta u^j &= \rho\dot{u} - \partial_j\left((\widetilde{\mu}+\lambda(\rho))\operatorname{div} u - P(\rho) + \widetilde{P}\right) - \partial_k\{(2\mu(\rho)-\widetilde{\mu})\mathbb{D}^{jk}u\}\\ &= \rho\dot{u} - \partial_j\left(\frac{\widetilde{\mu}+\lambda(\rho)}{2\mu(\rho)+\lambda(\rho)}F\right) + \partial_j\left(\frac{2\mu(\rho)-\widetilde{\mu}}{2\mu(\rho)+\lambda(\rho)}(P(\rho)-\widetilde{P})\right) - \partial_k\{2(\mu(\rho)-\widetilde{\mu})\mathbb{D}^{jk}u\}.\end{aligned} \tag{3.45}$$

Thus, the velocity gradient can be expressed as the sum of four terms:

$$\nabla u = \nabla u_* + \nabla u_F + \nabla u_P + \nabla u_\delta$$

and consequently, we have:

$$\begin{aligned}\vartheta(t) \leqslant &\int_0^t \sigma^{r_\alpha}\|\nabla u_*(\tau)\|^4_{\mathscr{C}^\alpha(\mathbb{R}^2)}d\tau + \int_0^t \sigma^{r_\alpha}\|\nabla u_P(\tau)\|^4_{\mathscr{C}^\alpha_{pw,\gamma(\tau)}(\mathbb{R}^2)}d\tau\\ &+ \int_0^t \sigma^{r_\alpha}\|\nabla u_F(\tau)\|^4_{\mathscr{C}^\alpha_{pw,\gamma(\tau)}(\mathbb{R}^2)}d\tau + \int_0^t \sigma^{r_\alpha}\|\nabla u_\delta(\tau)\|^4_{\mathscr{C}^\alpha_{pw,\gamma(\tau)}(\mathbb{R}^2)}d\tau.\end{aligned} \tag{3.46}$$

Notice that u_* is the regular part of the velocity associated to the first term of the right hand side of (3.45). The gradient of this regular part is Hölder continuous in the whole $\mathbb{R}^2$ with a norm that can be estimated as follows:

$$\begin{aligned}\int_0^t \sigma^{r_\alpha}\|\nabla u_*\|^4_{\mathscr{C}^\alpha(\mathbb{R}^2)} &\lesssim \int_0^t \sigma^{r_\alpha}\|\rho\dot{u}\|^{12/5}_{L^6(\mathbb{R}^2)}\left(\|\nabla u\|^{8/5}_{L^2(\mathbb{R}^2)} + \|P(\rho)-\widetilde{P}\|^{8/5}_{L^2(\mathbb{R}^2)}\right) + \int_0^t \sigma^{r_\alpha}\|\nabla\dot{u}\|^{4\alpha}_{L^2(\mathbb{R}^2)}\|\dot{u}\|^{4(1-\alpha)}_{L^2(\mathbb{R}^2)}\\ &\lesssim \left(\mathcal{A}_1(t)^{4/5} + E_0^{4/5}\right)\int_0^t \sigma^{r_\alpha}\|\nabla\dot{u}\|^{8/5}_{L^2(\mathbb{R}^2)}\|\dot{u}\|^{4/5}_{L^2(\mathbb{R}^2)}\\ &\quad + \left(\int_0^t \sigma^{1+(r_\alpha-1)/\alpha}\|\nabla\dot{u}\|^4_{L^2(\mathbb{R}^2)}\right)^\alpha\left(\int_0^t \sigma\|\dot{u}\|^4_{L^2(\mathbb{R}^2)}\right)^{1-\alpha}.\end{aligned} \tag{3.47}$$

We then choose $r_\alpha \geqslant 7/5$ in order to control the integral in the first term above by

$$\int_0^t \sigma^{r_\alpha}\|\nabla\dot{u}\|^{8/5}_{L^2(\mathbb{R}^2)}\|\dot{u}\|^{4/5}_{L^2(\mathbb{R}^2)} \lesssim \mathcal{A}_2(t)\mathcal{A}_1(t)^{1/5}$$

and, in order to control the singularity at initial time for the second term of (3.47), we choose $r_\alpha \geqslant 2\alpha+1$ and then, we obtain:

$$\left(\int_0^t \sigma^{1+(r_\alpha-1)/\alpha}\|\nabla\dot{u}\|^4_{L^2(\mathbb{R}^2)}\right)^\alpha\left(\int_0^t \sigma\|\dot{u}\|^4_{L^2(\mathbb{R}^2)}\right)^{1-\alpha} \lesssim \mathcal{A}_3(t)^\alpha\mathcal{A}_2(t)^\alpha\mathcal{A}_2(t)^{1-\alpha}\mathcal{A}_1(t)^{1-\alpha}.$$

Finally, we choose

$$r_\alpha = \max\{7/5;\, 1+2\alpha\} < 3$$

and then, we obtain the following estimate for the regular part of the velocity:

$$\int_0^t \sigma^{r_\alpha}\|\nabla u_*(\tau)\|^4_{\mathscr{C}^\alpha(\mathbb{R}^2)}d\tau \leqslant M'\left[C_0^2 + \mathcal{A}_1(t)^2 + \mathcal{A}_2(t)^2 + \mathcal{A}_3(t)^2\right]. \tag{3.48}$$

The others terms of the expression of the velocity gradient (3.45) are less regular than ∇u_*, in particular, they are discontinuous across the interface γ and we need the quantitative Hölder estimate Theorem 2.7 to bound their piecewise Hölder norms. For this purpose, we will first estimate $\mathcal{P}(\mathcal{C}(\tau))$. Obviously, we have:

$$\begin{cases} c_\gamma(\tau) \leqslant c_\gamma(0)\exp\left(\int_0^\tau \|\nabla u(\tau')\|_{L^\infty(\mathbb{R}^2)}d\tau'\right), \qquad |\mathcal{C}(\tau)| \leqslant |\mathcal{C}(0)|\exp\left(\int_0^\tau \|\nabla u(\tau')\|_{L^\infty(\mathbb{R}^2)}d\tau'\right),\\ \|\nabla\gamma(\tau)\|_{\mathscr{C}^\alpha} \leqslant \left(\|\nabla\gamma_0\|_{\mathscr{C}^\alpha} + \|\nabla\gamma_0\|^{1+\alpha}_{L^\infty}\int_0^\tau \|\nabla u(\tau')\|_{\dot{\mathscr{C}}^\alpha_{pw,\gamma(\tau')}(\mathbb{R}^2)}d\tau'\right)\exp\left((2+\alpha)\int_0^\tau \|\nabla u(\tau')\|_{L^\infty(\mathbb{R}^2)}d\tau'\right).\end{cases}$$

Thus, the following estimate holds:

$$\mathcal{P}(\mathcal{C}(\tau)) \leqslant K_0 \left(1 + \int_0^\tau \|\nabla u(\tau')\|_{\dot{\mathscr{C}}^\alpha_{pw,\gamma(\tau')}(\mathbb{R}^2)} d\tau'\right) \exp\left(m \int_0^\tau \|\nabla u(\tau')\|_{L^\infty(\mathbb{R}^2)} d\tau'\right) \tag{3.49}$$

with the constant K_0 depending on the regularity of the initial interface and m a positive constant that depends on polynomial in the definition of $\mathcal{P}(\mathcal{C})$ in Theorem 2.7.

By applying the quantitative Hölder estimate, the last term of (3.46) can be bounded as follows:

$$\begin{aligned}\|\nabla u_\delta(\tau)\|_{\mathscr{C}^\alpha_{pw,\gamma(\tau)}(\mathbb{R}^2)} &\lesssim \|\nabla u(\tau)\|_{L^2(\mathbb{R}^2)} + \|[\![(\mu(\rho(\tau)) - \widetilde{\mu})\mathbb{D}u(\tau)]\!]\|_{L^2(\mathcal{C}(\tau))} + \|2(\mu(\rho(\tau)) - \widetilde{\mu})\mathbb{D}u\|_{\mathscr{C}^\alpha_{pw,\gamma(\tau)}(\mathbb{R}^2)} \\ &\quad + \|[\![(\mu(\rho(\tau)) - \widetilde{\mu})\mathbb{D}u(\tau)]\!]\|_{L^\infty(\mathcal{C}(\tau))}\mathcal{P}(\mathcal{C}(\tau)).\end{aligned} \tag{3.50}$$

Obviously, we have

$$\int_0^t \sigma^{r_\alpha} \|\nabla u\|^4_{L^2(\mathbb{R}^2)} \leqslant M' \mathcal{A}_1(t) E_0$$

and on the other hand we consider (3.43) with $\varepsilon = \dfrac{\underline{\mathrm{v}}}{4(4\overline{\mathrm{v}} + 1/2)}$ in order to obtain a decay rate for the L^2 norm of the jump of $f(\rho)$ (3.21), and for the L^2 norm of the jump of the velocity gradient (3.22) from which we deduce the following:

$$\int_0^t \sigma^{r_\alpha} \|[\![(\mu(\rho(\tau)) - \widetilde{\mu})\mathbb{D}u(\tau)]\!]\|^4_{L^2(\mathcal{C}(\tau))} d\tau \leqslant M' \|[\![f(\rho_0)]\!]\|^4_{L^2(\mathcal{C}(0))}(1 + \vartheta(t))e^{M'\vartheta(t)}. \tag{3.51}$$

In a similar way, the following estimate holds by gathering (3.21), (3.22), (3.43) and (3.49):

$$\int_0^t \sigma^{r_\alpha} \|[\![(\mu(\rho(\tau)) - \widetilde{\mu})\mathbb{D}u(\tau)]\!]\|^4_{L^\infty(\mathcal{C}(\tau))}\mathcal{P}(\mathcal{C}(\tau))^4 d\tau \leqslant M' K_0 \|[\![f(\rho_0)]\!]\|^4_{L^\infty(\mathcal{C}(0))} \left(1 + \vartheta(t)\right) e^{M'\vartheta(t)}. \tag{3.52}$$

Finally, the penultimate term of (3.50) is bounded by:

$$\int_0^t \sigma^{r_\alpha} \|2(\mu(\rho(\tau)) - \widetilde{\mu})\mathbb{D}u(\tau)\|^4_{\mathscr{C}^\alpha_{pw,\gamma(\tau)}(\mathbb{R}^2)} d\tau \leqslant M' \beta(t)\vartheta(t). \tag{3.53}$$

All the previous computations give the following estimate:

$$\begin{aligned}&\int_0^t \sigma^{r_\alpha} \|\nabla u_\delta(\tau)\|^4_{\mathscr{C}^\alpha_{pw,\gamma(\tau)}(\mathbb{R}^2)} d\tau \\ &\qquad \leqslant M' \left[C_0 + \mathcal{A}_1(t)^2 + \beta(t)\vartheta(t) + (1 + K_0)\|[\![f(\rho_0)]\!]\|^4_{L^\infty(\mathcal{C}(0)) \cap L^2(\mathcal{C}(0))} \left(1 + \vartheta(t)\right) e^{M'\vartheta(t)}\right].\end{aligned} \tag{3.54}$$

To estimate the third term of the right-hand side of (3.46), we make use of the quantitative Hölder estimate Theorem 2.7 and we obtain:

$$\|\nabla u_F(\tau)\|_{\mathscr{C}^\alpha_{pw,\gamma(\tau)}(\mathbb{R}^2)} \lesssim \|\mathscr{F}(\tau)\|_{L^4(\mathbb{R}^2)} + \|[\![\mathscr{F}(\tau)]\!]\|_{L^4(\mathcal{C}(\tau))} + \|\mathscr{F}(\tau)\|_{\mathscr{C}^\alpha_{pw,\gamma(\tau)}(\mathbb{R}^2)} + \|[\![\mathscr{F}(\tau)]\!]\|_{L^\infty(\mathcal{C}(\tau))}\mathcal{P}(\mathcal{C}(\tau)) \tag{3.55}$$

with

$$\mathscr{F} = \frac{\widetilde{\mu} + \lambda(\rho)}{2\mu(\rho) + \lambda(\rho)} F$$

and we recall that the effective flux is:

$$F = -(-\Delta)^{-1} \operatorname{div}(\rho \dot{u}) + K\{(\mu(\rho) - \widetilde{\mu})\mathbb{D}u\}. \tag{3.56}$$

Obviously from (3.14) and Gagliardo-Nirenberg inequality, we have:

$$\int_0^t \sigma^{r_\alpha} \|\mathscr{F}\|^4_{L^4(\mathbb{R}^2)} \lesssim \int_0^t \sigma^{r_\alpha} \left(\|(-\Delta)^{-1} \operatorname{div}(\rho \dot{u})\|^4_{L^4(\mathbb{R}^2)} + \|\nabla u\|^4_{L^4(\mathbb{R}^2)}\right) \leqslant M\left(C_0 + (C_0 + \mathcal{A}_1(t))\,\mathcal{A}_1(t)\right). \tag{3.57}$$

On the other hand, we notice that the jump of $\mathscr{F}$ verifies:

$$|[\![\mathscr{F}]\!]| \lesssim |[\![f(\rho)]\!]| |< F >| + |[\![F]\!]| \quad \text{and} \quad [\![F]\!] = 2[\![\mu(\rho)]\!]\left(< \operatorname{div} u > - < \mathbb{D}^{jk} u > n^j_x n^k_x\right). \tag{3.58}$$

Thus, we have:

$$\int_0^t \sigma^{r_\alpha} \|[\![\mathscr{F}(\tau)]\!]\|^4_{L^4(\mathcal{C}(\tau))} d\tau \leqslant M' \|[\![f(\rho_0)]\!]\|^4_{L^4(\mathcal{C}(0))} \left(\vartheta(t) + \int_0^t \sigma^{r_\alpha} \|F\|^4_{L^\infty(\mathbb{R}^2)}\right) e^{M'\vartheta(t)}. \tag{3.59}$$

Also, from the expression of $\mathscr{F}$ we have:

(3.60)
$$\int_0^t \sigma^{r_\alpha}\|\mathscr{F}(\tau)\|^4_{\mathscr{C}^\alpha_{pw,\gamma(\tau)}(\mathbb{R}^2)}d\tau \leqslant M'\int_0^t \sigma^{r_\alpha}\left(\|F(\tau)\|^4_{\mathscr{C}^\alpha_{pw,\gamma(\tau)}(\mathbb{R}^2)}d\tau + \|f(\rho(\tau))\|^4_{\dot{\mathscr{C}}^\alpha_{pw,\gamma(\tau)}(\mathbb{R}^2)}\|F(\tau)\|^4_{L^\infty(\mathbb{R}^2)}d\tau\right).$$

Finally, as it has been done previously, we combine from (3.58), (3.49), (3.43) and (3.21) in order to obtain the following:

$$\int_0^t \sigma^{r_\alpha}\|[\![\mathscr{F}(\tau)]\!]\|^4_{L^\infty(\mathcal{C}(\tau))}\mathcal{P}(\mathcal{C}(\tau))^4 d\tau \leqslant M'K_0\|[\![f(\rho_0)]\!]\|^4_{L^\infty(\mathcal{C}(0))}e^{M'\vartheta(t)}\left(\vartheta(t)+\int_0^t \sigma^{r_\alpha}\|F\|^4_{L^\infty(\mathbb{R}^2)}\right). \tag{3.61}$$

We sum (3.57), (3.59), (3.60) and (3.61) and we obtain the following estimate for the third term of (3.46):

$$\begin{aligned}\int_0^t \sigma^{r_\alpha}\|\nabla u_F(\tau)\|^4_{\mathscr{C}^\alpha_{pw,\gamma(\tau)}(\mathbb{R}^2)}d\tau \leqslant M\Big[&C_0+(C_0+\mathcal{A}_1(t))\,\mathcal{A}_1(t)+(1+\beta(t))\int_0^t \sigma^{r_\alpha}\|F(\tau)\|^4_{\mathscr{C}^\alpha_{pw,\gamma(\tau)}(\mathbb{R}^2)}d\tau\\ &+\|[\![f(\rho_0)]\!]\|^4_{L^2\cap L^\infty(\mathcal{C}(0))}(1+K_0)e^{M'\vartheta(t)}\left(\vartheta(t)+\int_0^t \sigma^{r_\alpha}\|F\|^4_{L^\infty(\mathbb{R}^2)}\right)\Big].\end{aligned} \tag{3.62}$$

To obtain an estimate for the remainder term of (3.46), we apply again the quantitative Hölder estimate Theorem 2.7:

$$\|\nabla u_P(\tau)\|_{\mathscr{C}^\alpha_{pw,\gamma(\tau)}(\mathbb{R}^2)} \lesssim \|\mathscr{P}(\tau)\|_{L^4(\mathbb{R}^2)} + \|[\![\mathscr{P}(\tau)]\!]\|_{L^4(\mathcal{C}(\tau))} + \|\mathscr{P}(\tau)\|_{\mathscr{C}^\alpha_{pw,\gamma(\tau)}(\mathbb{R}^2)} + \|[\![\mathscr{P}(\tau)]\!]\|_{L^\infty(\mathcal{C}(\tau))}\mathcal{P}(\mathcal{C}(\tau))$$

where

$$\mathscr{P} = \frac{2\mu(\rho)-\widetilde{\mu}}{2\mu(\rho)+\lambda(\rho)}(P(\rho)-\widetilde{P}).$$

On the one hand, from (3.14), the following estimate holds:

$$\int_0^t \sigma^{r_\alpha}\|\mathscr{P}\|^4_{L^4(\mathbb{R}^2)}d\tau \leqslant M\left(C_0+\mathcal{A}_1(t)(C_0+\mathcal{A}_1(t))\right)$$

and on the other hand from (3.43) and (3.21) we have:

$$\int_0^t \sigma^{r_\alpha}\|[\![\mathscr{P}(\tau)]\!]\|^4_{L^4(\mathcal{C}(\tau))}d\tau \leqslant M'\|[\![f(\rho_0)]\!]\|^4_{L^4(\mathcal{C}(0))}e^{M'\vartheta(t)}.$$

As it has been done previously, we have from (3.49), (3.43) and (3.21) the following estimate:

$$\int_0^t \sigma^{r_\alpha}\|[\![\mathscr{P}(\rho(\tau))]\!]\|^4_{L^\infty(\mathcal{C}(\tau))}\mathcal{P}(\mathcal{C}(\tau))^4 d\tau \leqslant M'K_0\|[\![f(\rho_0)]\!]\|^4_{L^\infty(\mathcal{C}(0))}e^{M'\vartheta(t)}.$$

Finally, we have:

$$\int_0^t \|\mathscr{P}(\tau)\|^4_{\mathscr{C}^\alpha_{pw,\gamma(\tau)}(\mathbb{R}^2)}d\tau \leqslant M'\beta(t).$$

Gathering all of these estimates, we have the following estimate for the remainder term of (3.46):

(3.63)
$$\int_0^t \sigma^{r_\alpha}\|\nabla u_P(\tau)\|^4_{\mathscr{C}^\alpha_{pw,\gamma(\tau)}(\mathbb{R}^2)}d\tau \leqslant M\left[C_0+\mathcal{A}_1(t)(C_0+\mathcal{A}_1(t))+(1+K_0)e^{M'\vartheta(t)}\|[\![f(\rho_0)]\!]\|^4_{L^2\cap L^\infty(\mathcal{C}(0))}+\beta(t)\right].$$

The Hölder norm of the effective flux that appears in (3.62) and (3.63) can be estimaed by summing (3.48) and (3.54) since the effective flux is given by (3.56):

$$\begin{aligned}\int_0^t \sigma^{r_\alpha}\|F(\tau)\|^4_{\mathscr{C}^\alpha_{pw,\gamma(\tau)}(\mathbb{R}^2)}d\tau \leqslant M'\Big[&C_0^2+\mathcal{A}_1(t)^2+\mathcal{A}_2(t)^2+\mathcal{A}_3(t)^2+\beta(t)\vartheta(t)\\ &+(1+K_0)\|[\![f(\rho_0)]\!]\|^4_{L^\infty\cap L^2(\mathcal{C}(0))}\left(1+\vartheta(t)\right)e^{M'\vartheta(t)}\Big].\end{aligned}$$

Finally the proof of Proposition 2.8 follows by summing (3.48), (3.54), (3.62) and (3.63):

$$\begin{aligned}\vartheta(t) \leqslant M\Big[&C_0^2+\mathcal{A}_1^2(t)+\mathcal{A}_2^2(t)+\mathcal{A}_3(t)^2+\beta(t)(1+\vartheta(t))+(1+\beta(t))\int_0^t \sigma^{r_\alpha}\|F(\tau)\|^4_{\mathscr{C}^\alpha_{pw,\gamma(\tau)}(\mathbb{R}^2)}d\tau\\ &+(1+K_0)e^{M'\vartheta(t)}\|[\![f(\rho_0)]\!]\|^4_{L^\infty\cap L^2(\mathcal{C}(0))}\left(1+\vartheta(t)+\int_0^t \sigma^{r_\alpha}\|F(\tau)\|^4_{\mathscr{C}^\alpha_{pw,\gamma(\tau)}(\mathbb{R}^2)}d\tau\right)\Big].\end{aligned}$$

□

3.5. **Construction and stability of approximate sequence.** We begin the last step of the proof of Theorem 1.1 by identifying which initial datum to smooth and which not to smooth. First, the initial interface γ_0 and the initial density and velocity have the required regularity as stated in the local well-posedness theorem, Theorem 2.1. Therefore, it is not necessary to smooth them. However, the stress tensor at the initial time only fulfills

$$\operatorname{div}(\Pi_0) = \operatorname{div}(2\mu(\rho_0)\mathbb{D}u_0 + (\lambda(\rho_0)\operatorname{div} u_0 - P(\rho_0) + \widetilde{P})I_2) \in H^{-1}(\mathbb{R}^2).$$

In order to preserve the discontinuity in the initial density, we will consider, for each $\delta > 0$, (ρ_0, u_0^δ) as initial data for the equations (2.1) where u_0^δ solves the following elliptic equation:

$$-\operatorname{div}(2\mu(\rho_0)\mathbb{D}u_0^\delta + (\lambda(\rho_0)\operatorname{div} u_0^\delta - P(\rho_0) + \widetilde{P})I_2) + c^\delta u_0^\delta = -\operatorname{div}(w_\delta * \Pi_0). \tag{3.64}$$

Above, $w_\delta := \delta^{-2}w(\cdot/\delta)$ with w a smooth non negative function supported in the unit ball centered at the origin and whose integral equal 1. Also the constant c^δ is defined by:

$$c^\delta := \|w_\delta * \Pi_0 - \Pi_0\|_{L^2(\mathbb{R}^2)} \quad \text{and verifies} \quad c^\delta \xrightarrow{\delta\to 0} 0.$$

Since the viscosity are bounded away from vacuum and upper bounded and the pressure $P(\rho_0) - \widetilde{P}$ belongs to $L^2(\mathbb{R}^2)$, the existence of unique solution $u_0^\delta \in H^1(\mathbb{R}^2)$ for the elliptic equation (3.64) follows from the Lax-Milgram theorem. Moreover the sequence $(u_0^\delta)_\delta$ comes with the following estimate:

$$M^{-1}\|\nabla u_0^\delta\|^2_{L^2(\mathbb{R}^2)} + c^\delta\|u_0^\delta\|^2_{L^2(\mathbb{R}^2)} \leqslant M\left(\|\Pi_0\|^2_{L^2(\mathbb{R}^2)} + \|P(\rho_0) - \widetilde{P}\|^2_{L^2(\mathbb{R}^2)}\right) \tag{3.65}$$

with a constant $M > 1$ depending on the lower bound of the initial viscosity. To show that the sequence $(u_0^\delta)_\delta$ converges strongly to u_0 in $H^1(\mathbb{R}^2)$, we add $\operatorname{div}(\Pi_0)$ to both sides of (3.64) and we obtain:

$$-\operatorname{div}\{2\mu(\rho_0)\mathbb{D}(u_0^\delta - u_0) + \lambda(\rho_0)\operatorname{div}(u_0^\delta - u_0)I_2\} + c^\delta u_0^\delta = -\operatorname{div}(w_\delta * \Pi_0 - \Pi_0).$$

Next, we test the above equation with $u_0^\delta - u_0$ and we obtain:

$$2\int_{\mathbb{R}^2}\mu(\rho_0)|\mathbb{D}(u_0^\delta - u_0)|^2 + \int_{\mathbb{R}^2}\lambda(\rho_0)|\operatorname{div}(u_0^\delta - u_0)|^2 + c^\delta\int_{\mathbb{R}^2}u_0^\delta(u_0^\delta - u_0) = \int_{\mathbb{R}^2}\nabla(u_0^\delta - u_0)\colon(w_\delta * \Pi_0 - \Pi_0).$$

The following estimate holds from Hölder and Young inequalities:

$$M^{-1}\|\nabla(u_0^\delta - u_0)\|^2_{L^2(\mathbb{R}^2)} + \frac{1}{2}c^\delta\|u_0^\delta\|^2_{L^2(\mathbb{R}^2)} \leqslant M\|w_\delta * \Pi_0 - \Pi_0\|^2_{L^2(\mathbb{R}^2)} + \frac{1}{2}c^\delta\|u_0\|^2_{L^2(\mathbb{R}^2)}, \tag{3.66}$$

from which we deduce, owing to the definition of c^δ, see (3.65), the following convergence:

$$u_0^\delta \xrightarrow{\delta\to 0} u_0 \quad \text{in} \quad \dot{H}^1(\mathbb{R}^2).$$

As well, we have from (3.66) that

$$\limsup_{\delta\to 0}\|u_0^\delta\|^2_{L^2(\mathbb{R}^2)} \leqslant \|u_0\|^2_{L^2(\mathbb{R}^2)}$$

from which we deduce:

$$u_0^\delta \xrightarrow{\delta\to 0} u_0 \quad \text{in} \quad L^2(\mathbb{R}^2).$$

In conclusion, we recover the strong convergence of the sequence of initial velocity (u_0^δ) to u_0 in $H^1(\mathbb{R}^2)$. Also from (3.64), we have, for all $\delta > 0$, the compatibility condition:

$$\operatorname{div}(2\mu(\rho_0)\mathbb{D}u_0^\delta + (\lambda(\rho_0)\operatorname{div} u_0^\delta - P(\rho_0) + \widetilde{P})I_2) = -c^\delta u_0^\delta + \operatorname{div}(w_\delta * \Pi_0) \in L^2(\mathbb{R}^2). \tag{3.67}$$

With regard to the regularity of the initial density and interface γ_0, the fact that the initial viscosity $\mu(\rho_0)$ is a small perturbation around the constant state $\widetilde{\mu}$, along with the regularity of u_0^δ and (3.67) for each $\delta > 0$, Theorem 2.1 guarantee the existence of a local solution (ρ^δ, u^δ) for the equation (2.1) with initial data:

$$\rho^\delta_{|t=0} = \rho_0 \quad \text{and} \quad u^\delta_{|t=0} = u_0^\delta.$$

The solution is defined up to a time $T_\delta > 0$ and enjoys the regularity in Theorem 2.1. This regularity is sufficient for the computations carried out in the previous sections to make sense. As a consequence, Lemma 2.9 holds true for solution (ρ^δ, u^δ). In particular, all the conditions outlined in Theorem 2.3 are fulfilled, resulting in $T_\delta = +\infty$. We turn to the final step of this section, which consists of showing the stability of the sequence (ρ^δ, u^δ).

We begin by writing the equations solved by (ρ^δ, u^δ):

$$\begin{cases}\partial_t\rho^\delta + \operatorname{div}(\rho^\delta u^\delta) = 0,\\ \partial_t(\rho^\delta u^\delta) + \operatorname{div}(\rho^\delta u^\delta \otimes u^\delta) + \nabla P(\rho^\delta) = \operatorname{div}(2\mu(\rho^\delta)\mathbb{D}u^\delta) + \nabla(\lambda(\rho^\delta)\operatorname{div} u^\delta).\end{cases} \tag{3.68}$$

The velocity sequence $(u^\delta)_\delta$ comes with the following estimate: for all $T>0$ and $\delta>0$

$$\|\rho^\delta-\widetilde{\rho}\|^2_{L^\infty((0,T),L^2(\mathbb{R}^2))}+\|\rho^\delta-\widetilde{\rho}\|^2_{L^\infty((0,T)\times\mathbb{R}^2)}+\|u^\delta\|^2_{L^\infty((0,T),H^1(\mathbb{R}^2))} + \|\nabla u^\delta\|^2_{L^3((0,T)\times\mathbb{R}^2)}+\|\dot{u}^\delta\|^2_{L^2((0,T)\times\mathbb{R}^2)}\leqslant MC_0. \tag{3.69}$$

Hereafter, M is a constant that depends on the lower and upper bounds of the density and viscosity. Thus, there exist $\rho-\widetilde{\rho}\in L^\infty((0,\infty),L^2(\mathbb{R}^2))\cap L^\infty((0,\infty)\times\mathbb{R}^2)$ and $u\in L^\infty((0,\infty),H^1(\mathbb{R}^2))$ such that:

$$\begin{cases}\rho^\delta-\widetilde{\rho} & \rightharpoonup^* \quad \rho-\widetilde{\rho} \quad \text{in} \quad L^\infty((0,T)\times\mathbb{R}^2),\\ \rho^\delta-\widetilde{\rho} & \to \quad \rho-\widetilde{\rho} \quad \text{strongly in} \quad \mathscr{C}([0,T],L^2_w(\mathbb{R}^2)),\\ u^\delta & \rightharpoonup^* \quad u \quad \text{in } L^\infty((0,T),H^1(\mathbb{R}^2)),\\ u^\delta & \to \quad u \quad \text{strongly in } \mathscr{C}([0,T],L^2_{\mathrm{loc}}(\mathbb{R}^2)).\end{cases} \tag{3.70}$$

Moreover, by interpolation, we have, for all $2\leqslant q<\infty$,

$$u^\delta\to u \quad \text{strongly in} \quad L^\infty_{\mathrm{loc}}((0,\infty),L^q_{\mathrm{loc}}(\mathbb{R}^2)).$$

The density ρ^δ is discontinuous across the interface $\mathcal{C}^\delta$ transported by the velocity u^δ and given by:

$$\mathcal{C}^\delta(t)=\left\{\gamma^\delta(t,s)\in\mathbb{R}^2\colon s\in[0,2\pi]\right\}$$

with

$$\gamma^\delta(t,s)=\gamma_0(s)+\int_0^t u^\delta(\tau,\gamma^\delta(\tau,s))d\tau. \tag{3.71}$$

As well, the velocity u^δ comes with the following estimate: for all $T>0$, $\delta>0$

$$\sup_{[0,T]}\sigma\|\nabla u^\delta\|^4_{L^4(\mathbb{R}^2)}+\int_0^T\sigma^{r_\alpha}\|\nabla u^\delta\|^4_{\mathscr{C}^\alpha_{pw,\gamma^\delta}(\mathbb{R}^2)}\leqslant MC_0 \tag{3.72}$$

from which we deduce, together with (3.69), that the sequence $(\gamma^\delta)_\delta$ verifies: for all $T>0$ and $n\in\mathbb{N}^*$,

$$\sup_\delta\|\nabla\gamma^\delta\|_{L^\infty((0,T)\times(0,2\pi))}\leqslant M(T)C_0 \quad \text{and} \quad \sup_\delta\|\partial_t\gamma^\delta\|_{L^\infty((1/n,T)\times(0,2\pi))}\leqslant M(n)C_0.$$

The constant $M(T)$ and $M(n)$ depend not only on the lower and upper bounds of the density and viscosity but also on time T and n respectively. So, up to a subsequence, $(\gamma^\delta)_\delta$ convergences uniformly in $[0,T]\times[0,2\pi]$ to some $\gamma\in W^{1,\infty}((0,T)\times(0,2\pi))$. Moreover, from (3.69) and embedding, the velocity sequence verifies:

$$\sup_\delta\|u^\delta\|_{L^3((0,\infty),\mathscr{C}^{1/3}(\mathbb{R}^2))}\leqslant C_0$$

such that we can let δ goes to zero in (3.71) and obtain the fact that the interface limit γ verifies:

$$\gamma(t,s)=\gamma_0(s)+\int_0^t u(\tau,\gamma(\tau,s))d\tau. \tag{3.73}$$

On the other hand, we have:

$$\sup_\delta\int_0^\infty\sigma^{r_\alpha}\|\nabla u^\delta\|^4_{\mathscr{C}^\alpha_{pw,\gamma^\delta}(\mathbb{R}^2)}\leqslant C_0$$

in such way that, by lower semi-continuity of the norm and the fact that $r_\alpha<3$ we have $\nabla u\in L^1_{\mathrm{loc}}([0,\infty),\mathscr{C}^\alpha_{pw,\gamma}(\mathbb{R}^2))$. Thus, the regularity of the velocity ensures the uniqueness of γ that satisfies (3.73). Moreover, the interface has the following regularity,

$$\gamma\in\mathscr{C}([0,\infty),\mathscr{C}^{1+\alpha}([0,2\pi]))$$

and verifies the non-intersection condition with a constant $c_\gamma(t)$ that satisfies:

$$c_\gamma(t)\leqslant c_{\gamma_0}\exp\left(\int_0^t\|\nabla u(\tau)\|_{L^\infty(\mathbb{R}^2)}d\tau\right).$$

We turn to the proof of the strong convergence of the density sequence $(\rho^\delta)_\delta$. For this purpose, we define, for all $\delta>0$

$$\varrho^\delta(t,x)=\rho^\delta(t,X^\delta(t,x))$$

where X^δ is the flow of the velocity u^δ. As above, we can show that the sequence $(X^\delta)_\delta$ converges uniformly on compact set in $[0,\infty)\times\mathbb{R}^2$ to $X\in\mathscr{C}([0,\infty),W^{1,\infty}(\mathbb{R}^2))$, the flow of the limit velocity u, given by:

$$X(t,x)=x+\int_0^t u(\tau,X(\tau,x))d\tau.$$

We notice that the density sequence $(\rho^\delta)_\delta$ comes with the following bound

$$\sup_\delta\|\rho^\delta-\widetilde{\rho}\|^4_{L^\infty((0,\infty),\mathscr{C}^\alpha_{pw,\gamma^\delta}(\mathbb{R}^2))}\leqslant MC_0$$

from which, we deduce, together with (3.72) and (3.73), the following: for all $T>0$,

$$\sup_\delta\|\varrho^\delta-\widetilde{\rho}\|^4_{L^\infty((0,T),\mathscr{C}^\alpha_{pw,\gamma_0}(\mathbb{R}^2))}+\sup_\delta\|\partial_t\varrho^\delta\|^4_{L^q((0,T),L^\infty(\mathbb{R}^2))}\leqslant M(T,q)C_0,$$

for all $1<q<4/(1+r_\alpha)$ and constant $M(T,q)$ depending on q and time T. Since $\mathscr{C}^\alpha_{pw,\gamma_0}(\mathbb{R}^2)$ embeds compactly in $L^\infty_{\text{loc}}(\mathbb{R}^2)$, we deduce from Aubin-Lions Lemma that $(\varrho^\delta)_\delta$ converges uniformly on compact set in $\{(t,x)\colon x\in\mathbb{R}^2\setminus\mathcal{C}(0)\}$ to some $\varrho\in\mathscr{C}([0,\infty),\mathscr{C}^\alpha_{pw,\gamma_0}(\mathbb{R}^2))$. To deduce the strong convergence of the density, from the above analysis, we will show that $(\rho^\delta)_\delta$ converges to $\overline{\varrho}$ strongly in $L^2_{\text{loc}}([0,\infty)\times\mathbb{R}^2)$ where

$$\overline{\varrho}(t,x)=\varrho(t,X^{-1}(t,x)).$$

Above, $X^{-1}\in\mathscr{C}([0,\infty),W^{1,\infty}(\mathbb{R}^2))$ satisfies, for all $t\geqslant 0$ and $x\in\mathbb{R}^2$:

$$X^{-1}(t,x)=x-\int_0^t u(\tau,X^{-1}(\tau,x))d\tau,\qquad X(t,X^{-1}(t,x))=x\quad\text{and}\quad X^{-1}(t,X(t,x))=x. \tag{3.74}$$

In fact, for every $T>0$ and bounded set B, we have:

$$\begin{aligned}\int_0^T\int_B|\rho^\delta(t,x)-\overline{\varrho}(t,x)|^2dtdx&=\int_0^T\int_{X^\delta(t)B}|\rho^\delta(t,X^\delta(t,x))-\overline{\varrho}(t,X^\delta(t,x))|^2J^\delta(t,x)dtdx\\&\leqslant M\int_0^T\int_{X^\delta(t)B}|\varrho^\delta(t,x)-\varrho(t,x)|^2dtdx+M\int_0^T\int_{X^\delta(t)B}|\overline{\varrho}(t,X^\delta(t,x))-\varrho(t,x)|^2dtdx\end{aligned} \tag{3.75}$$

where J^δ comes from the change of variable $x\mapsto X^\delta(t,x)$ and satisfies[3]:

$$\sup_{[0,T]}\|J^\delta\|_{L^\infty(\mathbb{R}^2)}\leqslant M.$$

On the other hand, since the sequence $(X^\delta)_\delta$ converges to X, uniformly on compact set in $[0,\infty)\times\mathbb{R}^2$, there exists $\delta_0>0$ such that for all $\delta<\delta_0$, we have:

$$X^\delta(t)B\subset X(t)B+B(0,1).$$

Moreover, due to the uniform continuity of ϱ on the both sides of $\mathcal{C}(0)$, and (3.74) we have:

$$\overline{\varrho}(t,X^\delta(t,x))=\varrho(t,X^{-1}(t,X^\delta(t,x)))\quad\xrightarrow{\delta\to 0}\quad\varrho(t,x)\quad\text{a.e.}$$

As a result, thanks to the dominated convergence theorem, we have:

$$\lim_{\delta\to 0}\int_0^T\int_{X^\delta(t)B}|\overline{\varrho}(t,X^\delta(t,x))-\varrho(t,x)|^2dtdx=0. \tag{3.76}$$

As well, since the sequence $(\varrho^\delta)_\delta$ converges to ϱ, uniformly in compact set, we have:

$$\lim_{\delta\to 0}\int_0^T\int_{X^\delta(t)B}|\varrho^\delta(t,x)-\varrho(t,x)|^2dtdx=0. \tag{3.77}$$

Thus, from (3.75), (3.76) and (3.77), the strong convergence of the sequence $(\rho^\delta)_\delta$ to $\overline{\varrho}$ in $L^2_{\text{loc}}((0,\infty)\times\mathbb{R}^2)$ follows. In particular we have $\rho=\overline{\varrho}\in\mathscr{C}([0,\infty),\mathscr{C}^\alpha_{pw,\gamma}(\mathbb{R}^2))$ and by interpolation, it holds that the sequence of density $(\rho^\delta)_\delta$ converges to ρ strongly in $L^q_{\text{loc}}((0,\infty)\times\mathbb{R}^2)$ for all $1\leqslant q<\infty$.

[3]we refer to [19, Lemma 3.2]

Gathering all of these strong and weak convergences of the sequence $(\rho^\delta, u^\delta)_\delta$ and classical arguments, we can let δ goes to zero in (3.68) and we obtain that (ρ, u) solves the following:

$$\begin{cases} \partial_t \rho + \operatorname{div}(\rho u) = 0, \\ \partial_t(\rho u) + \operatorname{div}(\rho u \otimes u) + \nabla P(\rho) = \operatorname{div}(2\mu(\rho)\mathbb{D}u) + \nabla(\lambda(\rho)\operatorname{div} u). \end{cases}$$

This ends the proof of Theorem 1.1.

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Appendix A. Quantitative Hölder estimates

In this section, we present three results concerning the regularity of even higher-order Riesz transform of discontinuous functions. The first result is a recent contribution by Gancedo and García-Juárez [15] regarding quantitative Hölder estimates for even singular integral operators on patches. This is the key element for their recent result [16] concerning the global well-posedness for the 2d Navier-Stokes free boundary problem for incompressible fluids. Theorem A.2 summarizes their computations in step 4 and step 5. Finally, the last result we state in this section is from [21] and addresses the extension of a piecewise Hölder continuous function to a function that is Hölder continuous over the whole space.

We consider a singular integral operator T of even order $2l$ ($l \geqslant 1$):

$$T(g)(x) = p.v. \int_{\mathbb{R}^d} K(x-y)g(y)dy \tag{A.1}$$

where the kernel K is given by:

$$K(x) = \frac{P_{2l}(x)}{|x|^{d+2l}}$$

with P an harmonic homogeneous polynomial of degree $2l$. We also consider an open bounded simply connected subset $D \subset \mathbb{R}^d$ such that its boundary is an hyper-surface

$$\partial D =: \mathcal{C} = \left\{\gamma(s) \in \mathbb{R}^d \ : \ s \in \mathbb{T}^{d-1}\right\} \in \mathscr{C}^{1+\alpha} \tag{A.2}$$

verifying the following non self-intersection condition:

$$\exists\, c_\gamma > 0, \ | \ \forall \ (s,s') \in \mathbb{T}^{d-1}: \quad |\gamma(s) - \gamma(s')| \geqslant c_\gamma^{-1}|s-s'|. \tag{A.3}$$

We finally consider $\mathscr{C}^\alpha_{pw,\gamma}(\mathbb{R}^d)$, the space of piecewise Hölder continuous function. The first result by Gancedo and García-Juárez [15] reads as follows.

Theorem A.1. *Let $0 < \alpha < 1$, T be a singular integral operator in (A.1) and let $\mathcal{C}$ be an interface verifying (A.2) and (A.3). There exists a constant $C = C(d,\alpha)$ and a polynomial P depending on T such that the following estimate holds true:*

$$\|T(\mathbb{1}_D)\|_{\mathscr{C}^\alpha_{pw,\gamma}(\mathbb{R}^d)} \leqslant C(1+|\partial D|)\mathrm{P}(c_\gamma + \|\nabla\gamma\|_{L^\infty})(1+\|\nabla\gamma\|_{\dot{\mathscr{C}}^\alpha}). \tag{A.4}$$

The second result from [16, Step 4 & Step 5] reads as follows.

Proposition A.2. Assume that the hypotheses in Theorem A.1 hold true for singular integral operator T and interface $\mathcal{C}$ and let $1 \leqslant p < \infty$. There exists a constant $C = C(d,\alpha,p)$ such that for all $g \in L^p(\mathbb{R}^d) \cap \mathscr{C}^\alpha_{pw,\gamma}(\mathbb{R}^d)$, we have:

$$\|T(g)\|_{\mathscr{C}^\alpha_{pw,\gamma}(\mathbb{R}^d)} \leqslant C\left[\eta^{-\frac{d}{p}}\|g\|_{L^p(\mathbb{R}^d)} + \|g\|_{\mathscr{C}^\alpha_{pw,\gamma}(\mathbb{R}^d)} + \mathscr{P}(\mathcal{C})\|g\|_{L^\infty(\mathbb{R}^d)}\right],$$

where

$$\mathscr{P}(\partial D) = (1+|\partial D|)P(c_\gamma + \|\nabla\gamma\|_{L^\infty})\left(1+\|\nabla\gamma\|_{\dot{\mathscr{C}}^\alpha}\right) \quad \text{and} \quad \eta^\alpha := \min\left\{1, \ \frac{1}{16c_\gamma\|\nabla\gamma\|_{\dot{\mathscr{C}}^\alpha}}\right\}.$$

Due to some technical reasons [4], we gather Theorem A.1 and the following result from [21, Lemma 5.2] in order to obtain Theorem 2.7.

Lemma A.3. *Let $d = 2$ and $\mathcal{C}$ be an interface that verifies (A.2), (A.3). For all $g \in \mathscr{C}^{\alpha}_{pw,\gamma}(\mathbb{R}^2)$, there exists a function $g^e \in \mathscr{C}^{\alpha}(\mathbb{R}^2)$ such that $g^e = g$ on D (or D^c) with the following estimates:*

$$\begin{cases} \|g^e\|_{L^\infty(\mathbb{R}^2)} & \leqslant C\|g\|_{L^\infty(\mathbb{R}^2)}, \\ \|g^e\|_{\dot{\mathscr{C}}^{\alpha}(\mathbb{R}^2)} + \|g^e - g\|_{\dot{\mathscr{C}}^{\alpha}_{pw,\gamma}(\mathbb{R}^2)} & \leqslant C\left[\|g\|_{\mathscr{C}^{\alpha}_{pw,\gamma}(\mathbb{R}^2)} + \|[\![g]\!]\|_{L^\infty(\mathcal{C})} c_\gamma^2 \|\nabla\gamma\|^{1/\alpha}_{\dot{\mathscr{C}}^{\alpha}}\right], \\ \|g^e - g\|_{L^p(\mathbb{R}^2)} & \leqslant C\|[\![g]\!]\|_{L^p(\mathcal{C})}, \qquad 1 \leqslant p \leqslant \infty, \end{cases} \tag{A.5}$$

with a universal constant $C > 0$ independent of α and $\mathcal{C}$.

Remark A.4. The above result does not require the interface to be bounded; that is, D or D^c could be an unbounded domain.

Appendix B. Energy computations

In this section, we will provide details of the computations of two estimates for solution (ρ, u) of the system:

$$\begin{cases} \partial_t \rho + \operatorname{div}(\rho u) = 0, \\ \partial_t(\rho u) + \operatorname{div}(\rho u \otimes u) = \operatorname{div}\Pi, \\ [\![\Pi]\!] \cdot n = 0 \quad \text{in} \quad \mathcal{C} \end{cases} \tag{B.1}$$

where the stress tensor Π is given by:

$$\Pi = 2\mu(\rho)\mathbb{D}u + (\lambda(\rho)\operatorname{div} u - P(\rho) + \widetilde{P})I_d.$$

B.1. **Second Hoff energy.** We start by investigating the Rankine Hugoniot conditions for $\dot{u}$ at the surface of discontinuity $\mathcal{C}$. We first notice that after applying the derivative $\mathcal{A}$ to the momentum equation, we obtain that $\dot{u}$ solves the equation

$$\partial_t(\rho\dot{u}^j) + \operatorname{div}(\rho\dot{u}^j u) = \partial_k(\dot{\Pi}^{jk}) + \partial_k(\Pi^{jk}\operatorname{div} u) - \operatorname{div}(\partial_k u \Pi^{jk}). \tag{B.2}$$

So by Rankine Hugoniot conditions,

$$[\![\rho\dot{u}^j]\!]n_t + [\![\rho\dot{u}^j u^k]\!]n_x^k = [\![\dot{\Pi}^{jk}]\!]n_x^k + [\![\Pi^{jk}\operatorname{div} u]\!]n_x^k - [\![\partial_k u^l \Pi^{jk}]\!]n_x^l.$$

Also, thanks to the Rankine Hugoniot condition applied, this time, to the mass equation $(\text{B.1})_1$ the following jump condition holds true:

$$[\![\rho]\!]n_t + [\![\rho u^k]\!]n_x^k = 0$$

and since the material derivative of the velocity is continuous, we finally obtain that:

$$[\![\dot{\Pi}^{jk}]\!]n_x^k + [\![\Pi^{jk}\operatorname{div} u]\!]n_x^k - [\![\partial_k u^l \Pi^{jk}]\!]n_x^l = 0. \tag{B.3}$$

This relation will be used in the subsequent computations. We recall that the second Hoff estimate consists in multiplying (B.2) by the material derivative of the velocity before integrating in space. By doing so, we have:

$$\int_{\mathbb{R}^2} \dot{u}^j\{\partial_t(\rho\dot{u}^j) + \operatorname{div}(\rho\dot{u}^j u)\} = \int_{\mathbb{R}^2} \dot{u}^j\{\partial_k(\dot{\Pi}^{jk}) + \partial_k(\Pi^{jk}\operatorname{div} u) - \operatorname{div}(\partial_k u \Pi^{jk})\}. \tag{B.4}$$

The right hand side of the above equality, is:

$$\begin{aligned} \int_{\mathbb{R}^2} \dot{u}^j\{\partial_t(\rho\dot{u}^j) + \operatorname{div}(\rho\dot{u}^j u)\} &= \int_{\mathbb{R}^2} \partial_t(\rho|\dot{u}^j|^2) - \int_{\mathbb{R}^2} (\rho\dot{u}^j\partial_t\dot{u}^j) + \int_{\mathbb{R}^2} \operatorname{div}(\rho|\dot{u}^j|^2 u)(s,x) - \int_{\mathbb{R}^2} \rho\dot{u}^j u \cdot \nabla\dot{u}^j \\ &= \frac{1}{2}\int_{\mathbb{R}^2} \partial_t(\rho|\dot{u}^j|^2) + \frac{1}{2}\int_{\mathbb{R}^2} |\dot{u}^j|^2\partial_t\rho + \frac{1}{2}\int_{\mathbb{R}^2} \operatorname{div}(\rho|\dot{u}^j|^2 u) + \frac{1}{2}\int_{\mathbb{R}^2} |\dot{u}^j|^2 \operatorname{div}(\rho u) \\ &= \frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^2} \rho|\dot{u}^j|^2, \end{aligned} \tag{B.5}$$

[4] exponential growth of c_γ and $\|\nabla\gamma\|_{\mathscr{C}^{\alpha}}$ in time

where we have used the Liouville transport equation and the mass equation $(\text{B.1})_1$. We turn now to the computations of the right hand side of (B.4):

$$\int_{\mathbb{R}^2} \dot{u}^j \{\partial_k(\dot{\Pi}^{jk}) + \partial_k(\Pi^{jk} \operatorname{div} u) - \operatorname{div}(\partial_k u \Pi^{jk})\} = \int_{\mathbb{R}^2} \left\{\partial_k\{\dot{u}^j \dot{\Pi}^{jk}\} + \partial_k\{\dot{u}^j \Pi^{jk} \operatorname{div} u\} - \partial_l\{\dot{u}^j \Pi^{jk} \partial_k u^l\}\right\}$$
$$- \int_{\mathbb{R}^2} \partial_k \dot{u}^j \dot{\Pi}^{jk} - \int_{\mathbb{R}^2} \partial_k \dot{u}^j \Pi^{jk} \operatorname{div} u + \int_{\mathbb{R}^2} \partial_l \dot{u}^j \partial_k u^l \Pi^{jk}.$$

Since

$$\begin{aligned} \dot{\Pi}^{jk} &= 2\mu(\rho)\mathbb{D}^{jk}\dot{u} - \mu(\rho)\partial_j u^l \partial_l u^k - \mu(\rho)\partial_k u^l \partial_l u^j - 2\rho\mu'(\rho)\mathbb{D}^{jk} u \operatorname{div} u \\ &\quad + \left(\lambda(\rho)\operatorname{div}\dot{u} - \lambda(\rho)\nabla u^l \partial_l u - \rho\lambda'(\rho)(\operatorname{div} u)^2 + \rho P'(\rho)\operatorname{div} u\right)\delta^{jk}, \end{aligned} \tag{B.6}$$

then, the terms in the right hand side of (B.4) are:

$$\begin{aligned} &\int_{\mathbb{R}^2} \dot{u}^j \{\partial_k(\dot{\Pi}^{jk}) + \partial_k(\Pi^{jk} \operatorname{div} u) - \operatorname{div}(\partial_k u \Pi^{jk})\} \\ &\quad = -\int_{\mathbb{R}^2} 2\mu(\rho)|\mathbb{D}^{jk}\dot{u}|^2 + \int_{\mathbb{R}^2} \left\{\partial_k\{\dot{u}^j \dot{\Pi}^{jk}\} + \partial_k\{\dot{u}^j \Pi^{jk} \operatorname{div} u\} - \partial_l\{\dot{u}^j \Pi^{jk} \partial_k u^l\}\right\} \\ &\quad + \int_{\mathbb{R}^2} \partial_k \dot{u}^j \left\{\mu(\rho)\partial_j u^l \partial_l u^k + \mu(\rho)\partial_k u^l \partial_l u^j + 2\rho\mu'(\rho)\mathbb{D}^{jk} u \operatorname{div} u\right\} - \int_{\mathbb{R}^2} \lambda(\rho)|\operatorname{div}\dot{u}|^2 \\ &\quad + \int_{\mathbb{R}^2} \operatorname{div}\dot{u} \left\{\lambda(\rho)\nabla u^l \partial_l u + \rho\lambda'(\rho)(\operatorname{div} u)^2 - \rho P'(\rho)\operatorname{div} u\right\} - \int_{\mathbb{R}^2} \partial_k \dot{u}^j \Pi^{jk} \operatorname{div} u + \int_{\mathbb{R}^2} \partial_l \dot{u}^j \partial_k u^l \Pi^{jk}. \end{aligned} \tag{B.7}$$

We can combine (B.4), (B.5) and (B.7) and use the jump condition (B.3) and the continuity of $\dot{u}$ in order to obtain:

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^2} \rho|\dot{u}^j|^2 + \int_{\mathbb{R}^2} \left\{2\mu(\rho)|\mathbb{D}^{jk}\dot{u}|^2 + \lambda(\rho)|\operatorname{div}\dot{u}|^2\right\} &= \int_{\mathbb{R}^2} \partial_k \dot{u}^j \left\{\mu(\rho)\partial_j u^l \partial_l u^k + \mu(\rho)\partial_k u^l \partial_l u^j + 2\rho\mu'(\rho)\mathbb{D}^{jk} u \operatorname{div} u\right\} \\ &+ \int_{\mathbb{R}^2} \operatorname{div}\dot{u} \left\{\lambda(\rho)\nabla u^l \partial_l u + \rho\lambda'(\rho)(\operatorname{div} u)^2 - \rho P'(\rho)\operatorname{div} u\right\} - \int_{\mathbb{R}^2} \partial_k \dot{u}^j \Pi^{jk} \operatorname{div} u + \int_{\mathbb{R}^2} \partial_l \dot{u}^j \partial_k u^l \Pi^{jk}. \end{aligned}$$

B.2. **Third Hoff estimate.** While computing the second Hoff energy, one notices that the material derivative of the velocity solves a parabolic equation like the velocity. The goal is to perform the first Hoff energy to this equation (B.2) just by testing with the material derivative of $\dot{u}$. For this purpose, we write (B.2) as follows:

$$\rho\ddot{u}^j = \partial_k(\dot{\Pi}^{jk}) + \partial_k(\Pi^{jk} \operatorname{div} u) - \operatorname{div}(\partial_k u \Pi^{jk}) \tag{B.8}$$

where $\ddot{u}$ is the material derivative of $\dot{u}$, that is:

$$\ddot{u}^j = \partial_t \dot{u}^j + (u \cdot \nabla)\dot{u}^j.$$

One then multiplies the above by $\ddot{u}^j$ in order to obtain the following:

$$\begin{aligned} \int_{\mathbb{R}^2} \rho|\ddot{u}|^2 &= \int_{\mathbb{R}^d} \ddot{u}^j \partial_k(\dot{\Pi}^{jk}) + \int_{\mathbb{R}^d} \ddot{u}^j \partial_k(\Pi^{jk} \operatorname{div} u) - \int_{\mathbb{R}^d} \ddot{u}^j \operatorname{div}(\partial_k u \Pi^{jk}) \\ &= \int_{\Gamma} \left\{[\![\ddot{u}^j(\dot{\Pi}^{jk} + \Pi^{jk} \operatorname{div} u)]\!] n_x^k - [\![\ddot{u}^j \partial_k u^l \Pi^{jk}]\!] n_x^l\right\} \\ &\quad - \int_{\mathbb{R}^d} \partial_k \ddot{u}^j \dot{\Pi}^{jk} - \int_{\mathbb{R}^d} \partial_k \ddot{u}^j \Pi^{jk} \operatorname{div} u + \int_{\mathbb{R}^d} \partial_l \ddot{u}^j \partial_k u^l \Pi^{jk}. \end{aligned} \tag{B.9}$$

The first term in the right hand side above vanishes since $\ddot{u}$ is continuous through the interface and due to (B.3). Next, the second term is, thanks to (B.6):

$$\begin{aligned} -\int_{\mathbb{R}^d} \partial_k \ddot{u}^j \dot{\Pi}^{jk} = -\int_{\mathbb{R}^d} \partial_k \ddot{u}^j &\left\{2\mu(\rho)\mathbb{D}^{jk}\dot{u} - \mu(\rho)\partial_j u^l \partial_l u^k - \mu(\rho)\partial_k u^l \partial_l u^j - 2\rho\mu'(\rho)\mathbb{D}^{jk} u \operatorname{div} u\right. \\ &\left. + \left(\lambda(\rho)\operatorname{div}\dot{u} - \lambda(\rho)\nabla u^l \partial_l u - \rho\lambda'(\rho)(\operatorname{div} u)^2 + \rho P'(\rho)\operatorname{div} u\right)\delta^{jk}\right\}. \end{aligned} \tag{B.10}$$

The first term in the right hand side above is:

$$\begin{aligned}
-\int_{\mathbb{R}^d} 2\mu(\rho)\partial_k\ddot{u}^j\mathbb{D}^{jk}\dot{u} &= -2\int_{\mathbb{R}^d}\mu(\rho)\partial_{tk}\dot{u}^j\mathbb{D}^{jk}\dot{u} - 2\int_{\mathbb{R}^d}\mu(\rho)u^l\partial_{lk}\dot{u}^j\mathbb{D}^{jk}\dot{u} - 2\int_{\mathbb{R}^d}\mu(\rho)\partial_k u^l\partial_l\dot{u}^j\mathbb{D}^{jk}\dot{u}\\
&= -\int_{\mathbb{R}^d}\left[\partial_t\{\mu(\rho)|\mathbb{D}^{jk}\dot{u}|^2\} + \operatorname{div}\{\mu(\rho)u|\mathbb{D}^{jk}\dot{u}|^2\}\right] - 2\int_{\mathbb{R}^d}\mu(\rho)\partial_k u^l\partial_l\dot{u}^j\mathbb{D}^{jk}\dot{u}\\
&+ \int_{\mathbb{R}^d}|\mathbb{D}^{jk}\dot{u}|^2\{\partial_t\mu(\rho) + \operatorname{div}(\mu(\rho)u)\}\\
&= -\frac{d}{dt}\int_{\mathbb{R}^d}\mu(\rho)|\mathbb{D}^{jk}\dot{u}|^2 - 2\int_{\mathbb{R}^d}\mu(\rho)\partial_k u^l\partial_l\dot{u}^j\mathbb{D}^{jk}\dot{u} + \int_{\mathbb{R}^d}|\mathbb{D}^{jk}\dot{u}|^2\{\rho\mu'(\rho) - \mu(\rho)\}\operatorname{div} u.
\end{aligned}$$

As for the second term of the right hand side of (B.10), one has:

$$\begin{aligned}
\int_{\mathbb{R}^d}\mu(\rho)\partial_k\ddot{u}^j\partial_j u^l\partial_l u^k &= \int_{\mathbb{R}^2}\partial_t\{\mu(\rho)\partial_k\dot{u}^j\partial_j u^l\partial_l u^k\} + \int_{\mathbb{R}^2}\partial_m\{\mu(\rho)u^m\partial_k\dot{u}^j\partial_j u^l\partial_l u^k\}\\
&+ \int_{\mathbb{R}^2}\mu(\rho)\partial_k u^m\partial_m\dot{u}^j\partial_j u^l\partial_l u^k + \int_{\mathbb{R}^2}(\rho\mu'(\rho) - \mu(\rho))\partial_k\dot{u}^j\operatorname{div} u\partial_j u^l\partial_l u^k\\
&- \int_{\mathbb{R}^2}\mu(\rho)\partial_k\dot{u}^j\partial_j\dot{u}^l\partial_l u^k + \int_{\mathbb{R}^2}\mu(\rho)\partial_k\dot{u}^j\partial_j u^m\partial_m u^l\partial_l u^k\\
&- \int_{\mathbb{R}^2}\mu(\rho)\partial_k\dot{u}^j\partial_j u^l\partial_l\dot{u}^k + \int_{\mathbb{R}^2}\mu(\rho)\partial_k\dot{u}^j\partial_j u^l\partial_l u^m\partial_m u^k\\
&= \frac{d}{dt}\int_{\mathbb{R}^2}\mu(\rho)\partial_k\dot{u}^j\partial_j u^l\partial_l u^k + \int_{\mathbb{R}^2}\mu(\rho)\partial_k u^m\partial_m\dot{u}^j\partial_j u^l\partial_l u^k\\
&+ \int_{\mathbb{R}^2}(\rho\mu'(\rho) - \mu(\rho))\partial_k\dot{u}^j\operatorname{div} u\partial_j u^l\partial_l u^k - \int_{\mathbb{R}^2}\mu(\rho)\partial_k\dot{u}^j\partial_j\dot{u}^l\partial_l u^k\\
&- \int_{\mathbb{R}^2}\mu(\rho)\partial_k\dot{u}^j\partial_j u^l\partial_l\dot{u}^k + \int_{\mathbb{R}^2}\mu(\rho)\partial_k\dot{u}^j\partial_j u^m\partial_m u^l\partial_l u^k\\
&+ \int_{\mathbb{R}^2}\mu(\rho)\partial_k\dot{u}^j\partial_j u^l\partial_l u^m\partial_m u^k.
\end{aligned}$$

The third term in the right hand side of (B.10) can be deduced from the above computations just by interchanging j and k. On the other hand, the fourth term, is:

$$\begin{aligned}
2\int_{\mathbb{R}^d}\rho\mu'(\rho)\partial_k\ddot{u}^j\mathbb{D}^{jk}u\operatorname{div} u &= 2\frac{d}{dt}\int_{\mathbb{R}^2}\rho\mu'(\rho)\partial_k\dot{u}^j\mathbb{D}^{jk}u\operatorname{div} u + 2\int_{\mathbb{R}^d}\rho\mu'(\rho)\partial_k u^m\partial_m\dot{u}^j\mathbb{D}^{jk}u\operatorname{div} u\\
&+ 2\int_{\mathbb{R}^d}\rho^2\mu''(\rho)\partial_k\dot{u}^j\operatorname{div} u\mathbb{D}^{jk}u\operatorname{div} u - 2\int_{\mathbb{R}^2}\rho\mu'(\rho)\partial_k\dot{u}^j\mathbb{D}^{jk}\dot{u}\operatorname{div} u\\
&+ \int_{\mathbb{R}^2}\rho\mu'(\rho)\partial_k\dot{u}^j\left(\partial_j u^m\partial_m u^k + \partial_k u^m\partial_m u^j\right)\operatorname{div} u\\
&- 2\int_{\mathbb{R}^2}\rho\mu'(\rho)\partial_k\dot{u}^j\mathbb{D}^{jk}u\operatorname{div}\dot{u} + 2\int_{\mathbb{R}^2}\rho\mu'(\rho)\partial_k\dot{u}^j\mathbb{D}^{jk}u\nabla u^m\cdot\partial_m u.
\end{aligned}$$

Next, the fifth term of the right hand side of (B.10) can be computed as follows:

$$\begin{aligned}
-\int_{\mathbb{R}^d}\lambda(\rho)\operatorname{div}\ddot{u}\operatorname{div}\dot{u} &= -\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^2}\lambda(\rho)|\operatorname{div}\dot{u}|^2 - \int_{\mathbb{R}^2}\lambda(\rho)\nabla u^m\cdot\partial_m\dot{u}\operatorname{div}\dot{u}\\
&- \frac{1}{2}\int_{\mathbb{R}^2}(\rho\lambda'(\rho) - \lambda(\rho))|\operatorname{div}\dot{u}|^2\operatorname{div} u.
\end{aligned}$$

The sixth term is:

$$\begin{aligned}
\int_{\mathbb{R}^d}\lambda(\rho)\operatorname{div}\ddot{u}\partial_k u^l\partial_l u^k &= \frac{d}{dt}\int_{\mathbb{R}^2}\lambda(\rho)\operatorname{div}\dot{u}\partial_k u^l\partial_l u^k + \int_{\mathbb{R}^2}\lambda(\rho)\nabla u^m\partial_m\dot{u}\partial_k u^l\partial_l u^k\\
&+ \int_{\mathbb{R}^2}(\rho\lambda'(\rho) - \lambda(\rho))\operatorname{div} u\operatorname{div}\dot{u}\partial_k u^l\partial_l u^k - \int_{\mathbb{R}^2}\lambda(\rho)\operatorname{div}\dot{u}\partial_k\dot{u}^l\partial_l u^k\\
&+ \int_{\mathbb{R}^2}\lambda(\rho)\operatorname{div}\dot{u}\partial_k u^m\partial_m u^l\partial_l u^k - \int_{\mathbb{R}^2}\lambda(\rho)\operatorname{div}\dot{u}\partial_k u^l\partial_l\dot{u}^k\\
&+ \int_{\mathbb{R}^2}\lambda(\rho)\operatorname{div}\dot{u}\partial_k u^l\partial_l u^m\partial_m u^k.
\end{aligned}$$

The previous last term is:

$$\begin{aligned}\int_{\mathbb{R}^2} \operatorname{div} \ddot{u}\rho\lambda'(\rho)(\operatorname{div} u)^2 &= \frac{d}{dt}\int_{\mathbb{R}^2} \rho\lambda'(\rho)\operatorname{div}\dot{u}(\operatorname{div} u)^2 + \int_{\mathbb{R}^2} \rho\lambda'(\rho)\partial_j u^m \partial_m \dot{u}^j (\operatorname{div} u)^2 \\ &+ \int_{\mathbb{R}^2} \rho^2\lambda''(\rho)\operatorname{div}\dot{u}(\operatorname{div} u)^3 - 2\int_{\mathbb{R}^2} \rho\lambda'(\rho)\operatorname{div}\dot{u}\operatorname{div}\dot{u}\operatorname{div} u \\ &+ 2\int_{\mathbb{R}^2} \rho\lambda'(\rho)\operatorname{div}\dot{u}\partial_j u^m \partial_m u^j \operatorname{div} u.\end{aligned}$$

and finally the last term is:

$$\begin{aligned}-\int_{\mathbb{R}^2} \rho P'(\rho)\operatorname{div}\ddot{u}\operatorname{div} u &= -\frac{d}{dt}\int_{\mathbb{R}^2} \rho P'(\rho)\operatorname{div}\dot{u}\operatorname{div} u - \int_{\mathbb{R}^2} \rho P'(\rho)\nabla u^m \partial_m \dot{u}\operatorname{div} u \\ &- \int_{\mathbb{R}^2} \rho^2 P''(\rho)\operatorname{div}\dot{u}(\operatorname{div} u)^2 + \int_{\mathbb{R}^2} \rho P'(\rho)(\operatorname{div}\dot{u})^2 \\ &- \int_{\mathbb{R}^2} \rho P'(\rho)\operatorname{div}\dot{u}\nabla u^m \partial_m u.\end{aligned}$$

This ends the computations of the second term of the right hand side of (B.9). We now turn to the computations of the third term that we express as follows:

$$-\int_{\mathbb{R}^d} \partial_k \ddot{u}^j \Pi^{jk}\operatorname{div} u = -\int_{\mathbb{R}^d} \partial_k \ddot{u}^j \left(2\mu(\rho)\mathbb{D}^{jk}u + \{\lambda(\rho)\operatorname{div} u - P(\rho) + \widetilde{P}\}\delta^{jk}\right)\operatorname{div} u. \tag{B.11}$$

The first term of the right hand side above is:

$$\begin{aligned}-2\int_{\mathbb{R}^d} \partial_k \ddot{u}^j \mu(\rho)\mathbb{D}^{jk}u\operatorname{div} u &= -2\frac{d}{dt}\int_{\mathbb{R}^2} \mu(\rho)\partial_k \dot{u}^j \mathbb{D}^{jk}u\operatorname{div} u - 2\int_{\mathbb{R}^2} \mu(\rho)\partial_k u^m \partial_m \dot{u}^j \mathbb{D}^{jk}u\operatorname{div} u \\ &- 2\int_{\mathbb{R}^2} (\rho\mu'(\rho) - \mu(\rho))\partial_k \dot{u}^j \mathbb{D}^{jk}u(\operatorname{div} u)^2 + 2\int_{\mathbb{R}^2} \mu(\rho)\partial_k \dot{u}^j \mathbb{D}^{jk}\dot{u}\operatorname{div} u \\ &- \int_{\mathbb{R}^2} \mu(\rho)\partial_k \dot{u}^j \left(\partial_j u^m \partial_m u^k + \partial_k u^m \partial_m u^j\right)\operatorname{div} u \\ &+ 2\int_{\mathbb{R}^2} \mu(\rho)\partial_k \dot{u}^j \mathbb{D}^{jk}u\operatorname{div}\dot{u} - 2\int_{\mathbb{R}^2} \mu(\rho)\partial_k \dot{u}^j \mathbb{D}^{jk}u\nabla u^m \partial_m u.\end{aligned}$$

Regarding the second term of the right hand side of (B.11), one has

$$\begin{aligned}-\int_{\mathbb{R}^2} \lambda(\rho)\operatorname{div}\ddot{u}(\operatorname{div} u)^2 &= -\frac{d}{dt}\int_{\mathbb{R}^2} \lambda(\rho)\operatorname{div}\dot{u}(\operatorname{div} u)^2 - \int_{\mathbb{R}^2} \lambda(\rho)\nabla u^m \partial_m \dot{u}(\operatorname{div} u)^2 \\ &- \int_{\mathbb{R}^2} (\rho\lambda'(\rho) - \lambda(\rho))\operatorname{div}\dot{u}(\operatorname{div} u)^3 + 2\int_{\mathbb{R}^2} \lambda(\rho)(\operatorname{div}\dot{u})^2\operatorname{div} u \\ &- 2\int_{\mathbb{R}^2} \lambda(\rho)\operatorname{div}\dot{u}\nabla u^m \partial_m u\operatorname{div} u\end{aligned}$$

and finally, the last term is:

$$\begin{aligned}\int_{\mathbb{R}^2} \operatorname{div}\ddot{u}(P(\rho) - \widetilde{P})\operatorname{div} u &= \frac{d}{dt}\int_{\mathbb{R}^2} \operatorname{div}\dot{u}(P(\rho) - \widetilde{P})\operatorname{div} u + \int_{\mathbb{R}^2} \nabla u^m \partial_m \dot{u}(P(\rho) - \widetilde{P})\operatorname{div} u \\ &+ \int_{\mathbb{R}^2} \operatorname{div}\dot{u}(\operatorname{div} u)^2(\rho P'(\rho) - P(\rho) + \widetilde{P}) - \int_{\mathbb{R}^2} \operatorname{div}\dot{u}(P(\rho) - \widetilde{P})\operatorname{div}\dot{u} \\ &+ \int_{\mathbb{R}^2} \operatorname{div}\dot{u}(P(\rho) - \widetilde{P})\nabla u^m \partial_m u.\end{aligned}$$

These completes the computations of the terms in the expression (B.11), that are the third term of the right hand side of (B.9). We turn to the computations of the last term of (B.9) that we express as:

$$\int_{\mathbb{R}^d} \partial_l \ddot{u}^j \partial_k u^l \Pi^{jk} = \int_{\mathbb{R}^d} \partial_l \ddot{u}^j \partial_k u^l \left(2\mu(\rho)\mathbb{D}^{jk}u + \{\lambda(\rho)\operatorname{div} u - P(\rho) + \widetilde{P}\}\delta^{jk}\right). \tag{B.12}$$

The first term of the right hand side above is:

$$\begin{aligned}\int_{\mathbb{R}^d} 2\mu(\rho)\partial_l\ddot{u}^j\partial_k u^l\mathbb{D}^{jk}u &= 2\frac{d}{dt}\int_{\mathbb{R}^2}\mu(\rho)\partial_l\dot{u}^j\partial_k u^l\mathbb{D}^{jk}u + 2\int_{\mathbb{R}^2}\mu(\rho)\partial_l u^m\partial_m\dot{u}^j\partial_k u^l\mathbb{D}^{jk}u\\ &+ 2\int_{\mathbb{R}^2}(\rho\mu'(\rho)-\mu(\rho))\operatorname{div}u\partial_l\dot{u}^j\partial_k u^l\mathbb{D}^{jk}u - 2\int_{\mathbb{R}^2}\mu(\rho)\partial_l\dot{u}^j\partial_k\dot{u}^l\mathbb{D}^{jk}u\\ &+ 2\int_{\mathbb{R}^2}\mu(\rho)\partial_l\dot{u}^j\partial_k u^m\partial_m u^l\mathbb{D}^{jk}u - 2\int_{\mathbb{R}^2}\mu(\rho)\partial_l\dot{u}^j\mathbb{D}^{jk}\dot{u}\partial_k u^l\\ &+ \int_{\mathbb{R}^2}\mu(\rho)\partial_l\dot{u}^j\partial_k u^l\left(\partial_j u^m\partial_m u^k + \partial_k u^m\partial_m u^j\right)\end{aligned}$$

and the second term is:

$$\begin{aligned}\int_{\mathbb{R}^d}\lambda(\rho)\partial_l\ddot{u}^j\partial_j u^l\operatorname{div}u &= \frac{d}{dt}\int_{\mathbb{R}^2}\lambda(\rho)\partial_l\dot{u}^j\partial_j u^l\operatorname{div}u + \int_{\mathbb{R}^2}\lambda(\rho)\partial_l u^m\partial_m\dot{u}^j\partial_j u^l\operatorname{div}u\\ &+ \int_{\mathbb{R}^2}(\rho\lambda'(\rho)-\lambda(\rho))(\operatorname{div}u)^2\partial_l\dot{u}^j\partial_j u^l - \int_{\mathbb{R}^2}\lambda(\rho)\partial_l\dot{u}^j\partial_j\dot{u}^l\operatorname{div}u\\ &+ \int_{\mathbb{R}^2}\lambda(\rho)\partial_l\dot{u}^j\partial_j u^m\partial_m u^l\operatorname{div}u - \int_{\mathbb{R}^2}\lambda(\rho)\partial_l\dot{u}^j\operatorname{div}\dot{u}\partial_j u^l\\ &+ \int_{\mathbb{R}^2}\lambda(\rho)\partial_l\dot{u}^j\nabla u^m\partial_m u\partial_j u^l.\end{aligned}$$

Finally, the last term of (B.12) is:

$$\begin{aligned}-\int_{\mathbb{R}^d}\partial_l\ddot{u}^j\partial_j u^l(P(\rho)-\widetilde{P}) &= -\frac{d}{dt}\int_{\mathbb{R}^2}\partial_l\dot{u}^j\partial_j u^l(P(\rho)-\widetilde{P}) - \int_{\mathbb{R}^2}\partial_l u^m\partial_m\dot{u}^j\partial_j u^l(P(\rho)-\widetilde{P})\\ &- \int_{\mathbb{R}^2}\partial_l\dot{u}^j\operatorname{div}u\partial_j u^l(\rho P'(\rho)-P(\rho)+\widetilde{P}) + \int_{\mathbb{R}^2}\partial_l\dot{u}^j(P(\rho)-\widetilde{P})\partial_j\dot{u}^l\\ &- \int_{\mathbb{R}^2}\partial_l\dot{u}^j(P(\rho)-\widetilde{P})\partial_j u^m\partial_m u^l.\end{aligned}$$

Theses completes the computations of the third Hoff energy.

$$\begin{aligned}\int_{\mathbb{R}^2}\rho|\ddot{u}|^2 &+ \frac{d}{dt}\int_{\mathbb{R}^d}\mu(\rho)|\mathbb{D}^{jk}\dot{u}|^2 + \frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^2}\lambda(\rho)|\operatorname{div}\dot{u}|^2 = -2\int_{\mathbb{R}^d}\mu(\rho)\partial_k u^l\partial_l\dot{u}^j\mathbb{D}^{jk}\dot{u} + \int_{\mathbb{R}^d}|\mathbb{D}^{jk}\dot{u}|^2\{\rho\mu'(\rho)-\mu(\rho)\}\operatorname{div}u\\ &+ \frac{d}{dt}\int_{\mathbb{R}^2}\mu(\rho)\partial_k\dot{u}^j\partial_j u^l\partial_l u^k + \int_{\mathbb{R}^2}\mu(\rho)\partial_k u^m\partial_m\dot{u}^j\partial_j u^l\partial_l u^k + \int_{\mathbb{R}^2}(\rho\mu'(\rho)-\mu(\rho))\partial_k\dot{u}^j\operatorname{div}u\partial_j u^l\partial_l u^k\\ &- \int_{\mathbb{R}^2}\mu(\rho)\partial_k\dot{u}^j\partial_j\dot{u}^l\partial_l u^k - \int_{\mathbb{R}^2}\mu(\rho)\partial_k\dot{u}^j\partial_j u^l\partial_l\dot{u}^k + \int_{\mathbb{R}^2}\mu(\rho)\partial_k\dot{u}^j\partial_j u^m\partial_m u^l\partial_l u^k + \int_{\mathbb{R}^2}\mu(\rho)\partial_k\dot{u}^j\partial_j u^l\partial_l u^m\partial_m u^k\\ &+ \frac{d}{dt}\int_{\mathbb{R}^2}\mu(\rho)\partial_j\dot{u}^k\partial_k u^l\partial_l u^j + \int_{\mathbb{R}^2}\mu(\rho)\partial_j u^m\partial_m\dot{u}^k\partial_k u^l\partial_l u^j + \int_{\mathbb{R}^2}(\rho\mu'(\rho)-\mu(\rho))\partial_j\dot{u}^k\operatorname{div}u\partial_k u^l\partial_l u^j\\ &- \int_{\mathbb{R}^2}\mu(\rho)\partial_j\dot{u}^k\partial_k\dot{u}^l\partial_l u^j - \int_{\mathbb{R}^2}\mu(\rho)\partial_j\dot{u}^k\partial_k u^l\partial_l\dot{u}^j + \int_{\mathbb{R}^2}\mu(\rho)\partial_j\dot{u}^k\partial_k u^m\partial_m u^l\partial_l u^j + \int_{\mathbb{R}^2}\mu(\rho)\partial_j\dot{u}^k\partial_k u^l\partial_l u^m\partial_m u^j\\ &+ 2\frac{d}{dt}\int_{\mathbb{R}^2}\rho\mu'(\rho)\partial_k\dot{u}^j\mathbb{D}^{jk}u\operatorname{div}u + 2\int_{\mathbb{R}^d}\rho\mu'(\rho)\partial_k u^m\partial_m\dot{u}^j\mathbb{D}^{jk}u\operatorname{div}u + 2\int_{\mathbb{R}^d}\rho^2\mu''(\rho)\partial_k\dot{u}^j\operatorname{div}u\mathbb{D}^{jk}u\operatorname{div}u\\ &- 2\int_{\mathbb{R}^2}\rho\mu'(\rho)\partial_k\dot{u}^j\mathbb{D}^{jk}\dot{u}\operatorname{div}u + \int_{\mathbb{R}^2}\rho\mu'(\rho)\partial_k\dot{u}^j\left(\partial_j u^m\partial_m u^k + \partial_k u^m\partial_m u^j\right)\operatorname{div}u - 2\int_{\mathbb{R}^2}\rho\mu'(\rho)\partial_k\dot{u}^j\mathbb{D}^{jk}u\operatorname{div}\dot{u}\\ &+ 2\int_{\mathbb{R}^2}\rho\mu'(\rho)\partial_k\dot{u}^j\mathbb{D}^{jk}u\nabla u^m\cdot\partial_m u - \int_{\mathbb{R}^2}\lambda(\rho)\nabla u^m\cdot\partial_m\dot{u}\operatorname{div}\dot{u} - \frac{1}{2}\int_{\mathbb{R}^2}(\rho\lambda'(\rho)-\lambda(\rho))|\operatorname{div}\dot{u}|^2\operatorname{div}u\\ &+ \frac{d}{dt}\int_{\mathbb{R}^2}\lambda(\rho)\operatorname{div}\dot{u}\partial_k u^l\partial_l u^k + \int_{\mathbb{R}^2}\lambda(\rho)\nabla u^m\partial_m\dot{u}\partial_k u^l\partial_l u^k + \int_{\mathbb{R}^2}\left(\rho\lambda'(\rho)-\lambda(\rho)\right)\operatorname{div}u\operatorname{div}\dot{u}\partial_k u^l\partial_l u^k\end{aligned}$$

$$
\begin{aligned}
&-\int_{\mathbb{R}^2}\lambda(\rho)\operatorname{div}\dot{u}\partial_k\dot{u}^l\partial_l u^k+\int_{\mathbb{R}^2}\lambda(\rho)\operatorname{div}\dot{u}\partial_k u^m\partial_m u^l\partial_l u^k-\int_{\mathbb{R}^2}\lambda(\rho)\operatorname{div}\dot{u}\partial_k u^l\partial_l\dot{u}^k+\int_{\mathbb{R}^2}\lambda(\rho)\operatorname{div}\dot{u}\partial_k u^l\partial_l u^m\partial_m u^k\\
&+\frac{d}{dt}\int_{\mathbb{R}^2}\rho\lambda'(\rho)\operatorname{div}\dot{u}(\operatorname{div}u)^2+\int_{\mathbb{R}^2}\rho\lambda'(\rho)\partial_j u^m\partial_m\dot{u}^j(\operatorname{div}u)^2+\int_{\mathbb{R}^2}\rho^2\lambda''(\rho)\operatorname{div}\dot{u}(\operatorname{div}u)^3-2\int_{\mathbb{R}^2}\rho\lambda'(\rho)\operatorname{div}\dot{u}\operatorname{div}\dot{u}\operatorname{div}u\\
&+2\int_{\mathbb{R}^2}\rho\lambda'(\rho)\operatorname{div}\dot{u}\partial_j u^m\partial_m u^j\operatorname{div}u-\frac{d}{dt}\int_{\mathbb{R}^2}\rho P'(\rho)\operatorname{div}\dot{u}\operatorname{div}u-\int_{\mathbb{R}^2}\rho P'(\rho)\nabla u^m\partial_m\dot{u}\operatorname{div}u\\
&-\int_{\mathbb{R}^2}\rho^2P''(\rho)\operatorname{div}\dot{u}(\operatorname{div}u)^2+\int_{\mathbb{R}^2}\rho P'(\rho)(\operatorname{div}\dot{u})^2-\int_{\mathbb{R}^2}\rho P'(\rho)\operatorname{div}\dot{u}\nabla u^m\partial_m u-2\frac{d}{dt}\int_{\mathbb{R}^2}\mu(\rho)\partial_k\dot{u}^j\mathbb{D}^{jk}u\operatorname{div}u\\
&-2\int_{\mathbb{R}^2}\mu(\rho)\partial_k u^m\partial_m\dot{u}^j\mathbb{D}^{jk}u\operatorname{div}u-2\int_{\mathbb{R}^2}(\rho\mu'(\rho)-\mu(\rho))\partial_k\dot{u}^j\mathbb{D}^{jk}u(\operatorname{div}u)^2+2\int_{\mathbb{R}^2}\mu(\rho)\partial_k\dot{u}^j\mathbb{D}^{jk}\dot{u}\operatorname{div}u\\
&-\int_{\mathbb{R}^2}\mu(\rho)\partial_k\dot{u}^j\left(\partial_j u^m\partial_m u^k+\partial_k u^m\partial_m u^j\right)\operatorname{div}u+2\int_{\mathbb{R}^2}\mu(\rho)\partial_k\dot{u}^j\mathbb{D}^{jk}u\operatorname{div}\dot{u}-2\int_{\mathbb{R}^2}\mu(\rho)\partial_k\dot{u}^j\mathbb{D}^{jk}u\nabla u^m\partial_m u\\
&-\frac{d}{dt}\int_{\mathbb{R}^2}\lambda(\rho)\operatorname{div}\dot{u}(\operatorname{div}u)^2-\int_{\mathbb{R}^2}\lambda(\rho)\nabla u^m\partial_m\dot{u}(\operatorname{div}u)^2-\int_{\mathbb{R}^2}(\rho\lambda'(\rho)-\lambda(\rho))\operatorname{div}\dot{u}(\operatorname{div}u)^3+2\int_{\mathbb{R}^2}\lambda(\rho)(\operatorname{div}\dot{u})^2\operatorname{div}u\\
&-2\int_{\mathbb{R}^2}\lambda(\rho)\operatorname{div}\dot{u}\nabla u^m\partial_m u\operatorname{div}u+\frac{d}{dt}\int_{\mathbb{R}^2}\operatorname{div}\dot{u}(P(\rho)-\widetilde{P})\operatorname{div}u+\int_{\mathbb{R}^2}\nabla u^m\partial_m\dot{u}(P(\rho)-\widetilde{P})\operatorname{div}u\\
&+\int_{\mathbb{R}^2}\operatorname{div}\dot{u}(\operatorname{div}u)^2(\rho P'(\rho)-P(\rho)+\widetilde{P})-\int_{\mathbb{R}^2}\operatorname{div}\dot{u}(P(\rho)-\widetilde{P})\operatorname{div}\dot{u}+\int_{\mathbb{R}^2}\operatorname{div}\dot{u}(P(\rho)-\widetilde{P})\nabla u^m\partial_m u\\
&+2\frac{d}{dt}\int_{\mathbb{R}^2}\mu(\rho)\partial_l\dot{u}^j\partial_k u^l\mathbb{D}^{jk}u+2\int_{\mathbb{R}^2}\mu(\rho)\partial_l u^m\partial_m\dot{u}^j\partial_k u^l\mathbb{D}^{jk}u+2\int_{\mathbb{R}^2}(\rho\mu'(\rho)-\mu(\rho))\operatorname{div}u\partial_l\dot{u}^j\partial_k u^l\mathbb{D}^{jk}u\\
&-2\int_{\mathbb{R}^2}\mu(\rho)\partial_l\dot{u}^j\partial_k\dot{u}^l\mathbb{D}^{jk}u+2\int_{\mathbb{R}^2}\mu(\rho)\partial_l\dot{u}^j\partial_k u^m\partial_m u^l\mathbb{D}^{jk}u-2\int_{\mathbb{R}^2}\mu(\rho)\partial_l\dot{u}^j\mathbb{D}^{jk}\dot{u}\partial_k u^l\\
&+\int_{\mathbb{R}^2}\mu(\rho)\partial_l\dot{u}^j\partial_k u^l\left(\partial_j u^m\partial_m u^k+\partial_k u^m\partial_m u^j\right)+\frac{d}{dt}\int_{\mathbb{R}^2}\lambda(\rho)\partial_l\dot{u}^j\partial_j u^l\operatorname{div}u+\int_{\mathbb{R}^2}\lambda(\rho)\partial_l u^m\partial_m\dot{u}^j\partial_j u^l\operatorname{div}u\\
&+\int_{\mathbb{R}^2}(\rho\lambda'(\rho)-\lambda(\rho))(\operatorname{div}u)^2\partial_l\dot{u}^j\partial_j u^l-\int_{\mathbb{R}^2}\lambda(\rho)\partial_l\dot{u}^j\partial_j\dot{u}^l\operatorname{div}u+\int_{\mathbb{R}^2}\lambda(\rho)\partial_l\dot{u}^j\partial_j u^m\partial_m u^l\operatorname{div}u\\
&-\int_{\mathbb{R}^2}\lambda(\rho)\partial_l\dot{u}^j\operatorname{div}\dot{u}\partial_j u^l+\int_{\mathbb{R}^2}\lambda(\rho)\partial_l\dot{u}^j\nabla u^m\partial_m u\partial_j u^l-\frac{d}{dt}\int_{\mathbb{R}^2}\partial_l\dot{u}^j\partial_j u^l(P(\rho)-\widetilde{P})-\int_{\mathbb{R}^2}\partial_l u^m\partial_m\dot{u}^j\partial_j u^l(P(\rho)-\widetilde{P})\\
&-\int_{\mathbb{R}^2}\partial_l\dot{u}^j\operatorname{div}u\partial_j u^l(\rho P'(\rho)-P(\rho)+\widetilde{P})+\int_{\mathbb{R}^2}\partial_l\dot{u}^j(P(\rho)-\widetilde{P})\partial_j\dot{u}^l-\int_{\mathbb{R}^2}\partial_l\dot{u}^j(P(\rho)-\widetilde{P})\partial_j u^m\partial_m u^l.
\end{aligned}
$$

Many terms appearing on the left-hand side above can be grouped into three categories: I_1, I_2, and I_3, each having the respective forms (3.24), (3.25) and (3.26).

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