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# New Perspectives: An Introduction

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## Abstract

In this short Introduction to the present section, I will first briefly point out the reasons why the chapters collected here present original research in the context of the philosophy of the practice of mathematics, and open even newer perspectives. It is important to note that one crucial issue for future research will be to explore the connections within these chapters and with other chapters included in other sections – in particular, but not exclusively, the sections on Proof, on “Experimental” mathematics, and on the Semiology of Mathematical Practice. Second, I will present a short – and not exhaustive – summary of the themes that are tackled in the chapters. There will be overlaps and common threads: These chapters are indeed the final product of several discussions and exchanges within the community of the philosophers of the mathematical practice, which is extending toward other regions of philosophy thus blurring standard boundaries.

**Keywords** New perspectives -Interplay of practice -Experiments in mathematics -Social ontology - (Social) Epistemology -Philosophy of action -Mathematics education -Corpus linguistics

## 1 New Perspectives: Where Do They Come From?

As it is well known – and the number of pages of this Handbook proves it – the attention to the practice of mathematics has extensively grown in the past years, thus becoming a trend that has gained more and more momentum in philosophy as well as in other neighboring disciplines.

Now that the philosophical interest in considering the more dynamic aspects of mathematics has been settled, it is time to ask for more details and in particular to clarify the implications for the study of mathematics, possibly opening new research perspectives. It should be noted that the interdisciplinary nature of the attention to the practice of mathematics has been emphasized at many occasions, from the beginning; however, there is a sense in which, at least in the case of philosophy, such an approach can also be seen as “intradisciplinary”: As it is evident from the chapters collected in this section, the notions that have emerged in the context of the philosophy of the practice of mathematics can be connected to and enriched by the consideration of lively debates in other

territories of philosophy, such as philosophy of science, social ontology, epistemology, and philosophy of action. Indeed, the aim of this section is to see whether examining the research in other fields of philosophical inquiry can reply to questions emerged in the philosophy of the practice of mathematics and open even newer perspectives, which might also imply the application of new methods.

All chapters present very recent and original work, as testified in the lists of suggestions for further research that are found in the conclusions of each of them. Moreover, despite the fact that the chapters seem to go in different directions, they are not necessarily incompatible and many of the topics that are touched upon can be related to one another. Before summing up the contents of the chapters, I will highlight three themes that seem to me to be recurrent in the characterization of new perspectives for the philosophical approach to the practice of mathematics. I will leave to the reader the challenge of finishing “connecting the dots” and finding these and other relevant links among the chapters of the section.

The first theme is the relation between a form of knowledge that is recognized as *knowing-how* and the traditional definition of knowledge as *knowing-that*. If the philosophy of mathematical practice has been stressing the “procedural” aspect of mathematical expertise, some authors – for example, Ferreiros and De Toffoli in this section – propose a framework where mathematical knowledge is still intended as a form of knowing-that, where beliefs do not lose their central place; however, this is done in a renovated fashion, so as to emphasize as well the processes – the knowing-how – that have brought to it.

A second theme is the focus on the social aspects of mathematics: What does it mean precisely that mathematics is a collective enterprise? This social element comes to the fore in all chapters, for example, in connection with social ontology or social epistemology. In particular, Cantù discusses at length the growth and the development of mathematical knowledge in a social context, with its repercussions on traditional notions such as objectivity, while Hamami stresses the role for mathematical knowledge of the actions that are learnt and performed – collectively – by the mathematical subjects. An approach focused on the social aspects of the practice of mathematics can also make a difference in considering mathematics education, as shown in Tanswell and Inglis’ chapter.

A third theme that appears in the background of the chapters is the tension between formal and informal methods in mathematics, in particular in relation to the notion of proofs.

## 2 Summary of the Chapters in the Section

The first two chapters in the section, Ferreiros and van Bendegem’s, present innovative and more general frameworks to account for the practice of mathematics. It is important to note that both Ferreiros and van Bendegem are among the founding fathers of the *Association for the Philosophy of Mathematical Practice*,<sup>1</sup> which was created in 2009 – the first conference took place in Brussels in 2010.

In his chapter, Ferreiros comes back to his very influential book from [2016](#), and to his picture of mathematics as based on an interplay of practices. At the time, Ferreiros’ book had surely the merit of putting forward an exhaustive account of mathematical knowledge as requiring a “web of practices.” In this chapter, he outlines some of the ideas presented in the book and makes them even more explicit. First, he clarifies that his framework presupposes that mathematical knowledge is

knowing-that: Mathematics has for him a specific theoretical orientation besides mathematical research. However, it is clear that an approach directed to the practice of mathematics will attend to the process of mathematics and look for epistemic values and goals. At the forefront, there are indeed the dynamics of knowledge; for example, we might look at the prehistory of concepts and therefore go also beyond mathematics. However, mathematics is not a natural product of the brain but involves the reference to external representations: writing, notations, and geometrical figures; this symbolic part of mathematics is important not only for reasoning but also for communication. It should be pointed out that mathematics is not a symbolic practice as any other, but a specific one where the contents involved are the notions of number and form; in general, it is the study of patterns – an idea that we will find also in van Bendegem's chapter. Ferreiros also points out that his interplay of practice does not entail a form of social constructivism. Such interplay is in fact characterized by many constraints at different levels, our biology for a start but also the several symbols that are introduced in the practice; moreover, mathematics has cultural preconditions and requires social institutions. We will come back to this point when presenting Cantu's paper.

Another interesting element in Ferreiros' approach is that it allows for the study of other proper practices that are interested in knowing-that beside Western mathematics: This is the case of Chinese or Indian mathematics, even if they do not have a notion of "theorem proving." For this reason, Ferreiros distinguishes his view from ethnomathematics. Interestingly, according to him, cognitive science and ethnomathematics share the inclination to identify something as innate or panhuman in mathematics; in contrast to this idea, Ferreiros reminds that there is a danger in retroprojection, that is, in looking with modern eyes at practices that are in fact not the same as ours. This may happen, for example, in the case of the concept of "function," which he discusses in detail, that cannot be thought as a simple extension of previous number concepts. In presenting the case of function, Ferreiros introduces Cavaillès' *thematization* and Piaget's *reflecting abstraction* as having an important role in accounting for the historical emergence of this notion, but also for the way students grasp this concept. It looks like there is some kind of "meta" element that is very important for Ferreiros in considering the development and the stabilization of mathematical concepts, and their emancipation from other previous more technical notions. This opens a new perspective in the history of mathematics.

van Bendegem's chapter considers mathematics as an experimental science and refers to some of the studies presented in the section of the Handbook on "experimental" mathematics. According to van Bendegem, experiments in mathematics are far from being a fiction, because they are a fact, and a fact that will keep being a part of the practice of mathematics for long; for this reason, they deserve philosophical attention. He invites the reader to think "within" his framework, which he calls a "scheme," and possibly to individuate new questions and solutions, hoping for what he calls a "*Mendelejev-effect*." The framework presents a list of elements that characterize what can be called an experiment and is a sort of working grid that is discussed and modified throughout the chapter, following the analysis of concrete examples and the comparison of them to what is found in the empirical sciences. van Bendegem's strategy is in fact to start from a scheme that can be proposed for scientific experiments, and proceed by analogy to formulate the corresponding one to talk about experiments in mathematics. Of course, the notion of experiment is a matter of debate also in philosophy of science; however, for the sake of the argument, van Bendegem does not go into details of these discussions and simplifies them in order to focus on the case of mathematics. He specifies that his schema is just the beginning of the story, as it is plainly first order; new research might be able to provide a second-order version of it.

The connection with philosophy of science is thus made and presents frictions. In fact, if in science to make a statement and to make an observation seem to imply different things, in mathematics to make a conjecture and to acknowledge some specific instance of it look like they are of the same kind, that is, they give two mathematical statements as a result. A second problem is that while an experiment in science needs to be performed, how can we perform a mathematical experiment? In fact, it seems that in mathematics we simply prove (or disprove) a statement. Another difference is that there seem to be far more surprises in the universe that science is concerned with than in the world of mathematics. For van Bendegem, the only way of resolving these tensions is to open a new perspective, which will not focus on confirmation in mathematics but on insights and on the generation of ideas and concepts, something similar to what happens when we explore new territories. One possibility is to come back to the old notion of “probing” instead of proving, where the mathematician looks for patterns to show up, and to consider tacit knowledge – knowing-how – as well.

van Bendegem offers several examples of experiments and introduces possible extensions starting from his schema. Among others, one possible new direction of research would be to ask whether given a mathematical theory and a scientific theory, there exist experiments that are related to the scientific theory but are also relevant for the mathematical theory. If this were the case, then such an experiment should be considered at least in part a mathematical experiment. Further research concerns the role of thought experiments in mathematics, and their relation to mathematical proofs.<sup>2</sup> In van Bendegem’s view, a “grander” scheme should be envisaged so as to account for empiricism in the philosophy of mathematics – getting rid of the Lakatosian “quasi.” He concludes his chapter by claiming that mathematical experiments are an essential ingredient of the activity of mathematicians to the same extent that understanding the path that brings to a proof is as important as obtaining the proof.

The next three chapters in the section discuss more specifically debates in three particular areas of philosophy – social ontology, epistemology, and philosophy of action – in relation to notions that have emerged in the philosophy of the practice of mathematics. In all three cases, the aim is to capitalize on these discussions to clarify notions that are still understudied in the philosophy of the mathematical practice, and conversely to evaluate whether the research about the practice of mathematics can shed a new light on them.

Cantù’s very rich chapter opens up the new perspective of considering concepts from social ontology as relevant to explain the social constitution of mathematics. In her view, a philosophical-normative framework is necessary to guide the discussion on the objectivity of mathematics and on the kinds of semantics that are introduced when the attention is focused on the practice. It is only once such a framework is specified that we can look back at episodes in the history and philosophy of mathematics and see what an approach based on the practice of mathematics can teach us that we did not know already – starting from more standard pictures of mathematics – about them. As an example, Cantù takes the case of the axiomatic practice; if mathematics is assumed as having crucial informal elements, what does it remain to say about its more formal aspects, such as axiomatics or formal derivations? Is it the case that a sharp distinction between formal and informal methods in mathematics cannot be drawn? In a sense, Cantù’s chapter resonates with Ferreiros’, as she insists on the need to specify more what is intended by practice or practices and why they should still be subject of philosophy.

The chapter starts with a brief introduction to social ontology, which is undoubtedly very useful for the reader of this Handbook who is not necessarily acquainted with it. She discusses at length the philosophy of John Searle, who was not interested in mathematical phenomena as social facts but in

social phenomena such as money, property, or marriage; however, a lot of social ontology developed in reaction to his views. Francesco Guala criticizes Searle and claims that the focus should shift from theories about the mind to the sociohistorical study of institutions: Social ontology can be naturalized, and the dependence of social phenomena on representation can be explained in purely causal terms, without the need to introduce collective intentionality.

The point is then to compare social facts and mathematical objects, and to do that Cantù considers the views of Feferman, Ferreiros, and Cole. Feferman considered mathematical objectivity as a special case of intersubjective social objectivity, and his view, according to Cantù, can be integrated by looking at Lorenzen's research. Ferreiros – as we know it from his book and from his chapter in this section – bases his notion of practices on the presence of agents and of symbolic frameworks; however, he does not agree with Cole, who claims that mathematical objects are institutional facts, since he believes that mathematical objects are more stable than institutions. Carter, Giardino, and Pantsar's proposals on how to intend mathematical objectivity are then introduced; Cantù highlights that three different guiding principles emerge from the literature: naturalization, historicization, and normatization of mathematical practices and mathematical agents. For what regards the forms of semantics that are introduced when considering mathematical objects in this new perspective, according to Cantù's reconstruction, the semantics is representational when the focus is on mental states, and nonrepresentational when the focus is on the practice. Finally, as mentioned above, Cantù takes the example of axiomatics conceived as an institution along the lines of Guala's social theory: Axiomatics can be considered as a coordination tool and not necessarily as having a justificatory role.

Some of the elements already discussed in Cantù's chapter will be taken up again in the next two chapters, opening other complementary new perspectives. De Toffoli's objective is to reconcile analytic epistemology and the philosophy of the mathematical practice, and to this aim she introduces a fallible conception of mathematical justification.<sup>3</sup> In her view, beliefs, which are the main topic of epistemology, are still central to the consideration of the practice of mathematics, no matter if mathematicians are recognized as embodied, fallible and embedded in a specific context. In line with Ferreiros' insistence on mathematics as theoretical knowledge, she argues in favor of a *human* epistemology of mathematics, where subjects should not be considered in isolation. She also offers some points of contact between the mathematical subject and the characterization of subjects in social and feminist epistemologies: Knowledge can in fact be indexed to groups rather than to individuals.

Furthermore, De Toffoli shows how traditional topics in epistemology can be applied in mathematics – notably, there are Gettier cases also in mathematics – provided that we substitute the traditional infallible notion of mathematical justification that is standardly associated with proofs with its fallible counterpart; at the center of our epistemological inquiry, there are fallible human subjects and not anymore highly idealized ones. In her view, justification is fallible and therefore there is a gap between justification and truth: Whether a subject is justified is not only a matter of logic but depends also on social considerations, which does not mean that we have to endorse a form of relativism. De Toffoli's knowing subject is obviously related to Ferreiros and Hamami's agents; however, she defends the idea that it is crucial to keep separate the subjects of epistemology from the agents of ethics and opts for the use of "subject" as the appropriate term.

Hamami's chapter aims at making sense of the claim that mathematics is an activity, which has been one of the "mottos" of the philosophy of the mathematical practice from the start. It is evident that such a claim goes against standard philosophy of mathematics, which is agentless; however, what does it mean really? What is agency in mathematics? According to Hamami, it is not enough to

acknowledge that mathematics has an active dimension, but we should figure out how to study it. Ferreiros is always on the background, since also for Hamami there is no practice without practitioners; closely to De Toffoli's project, he wants to account for what a subject needs to do to acquire knowledge.

Hamami surveys several themes that have emerged in the literature concerning mathematical activities and their relation to artifacts, proofs, and texts. Mathematical knowledge has been considered as rooted or grounded into elementary human activities: Kitcher has followed this line of thought as based on an ideal agent, while Ferreiros has chosen to talk about the interplay of practices. Several new perspectives are now open: the consideration of what it means to do things with artifacts such as diagrams, symbols, or graphs – referring to the notion of epistemic action, or of mathematical cognition. An important point of the chapter is the interaction of an agent with mathematical informal proofs. In his reconstruction based on previous work (see Hamami and Morris [2021](#)), proof activity refers to the sequence of deductive inferences that correspond to a mathematical proof, which is in turn the result of the execution of a plan that is produced by a mathematical agent – what Poincaré described as the proof's *architecture* or *unity*. Moreover, proof plans can be related to proof understanding.

Hamami discusses also the question of how mathematical agents can realize mathematical actions based on mathematical texts, which directly brings to the subject of the last chapter in the section. Finally, he stresses how the most obvious areas for future research are social agency, computer agency, and extended agency. This new perspective raises questions that are both theoretical and methodological in an interdisciplinary fashion, in line with the characterization of the philosophy of the mathematical practice.

In the last chapter in the section, Tanswell and Inglis apply methods from corpus linguistics to look for empirical evidence for the prevalence of imperatives and instructions in mathematical preprints. Their focus, as highlighted by Hamami, is on the relation between agency and texts. In previous publications, Tanswell has argued for a *recipe model* of proofs (Tanswell [forthcoming](#)): In his view, proofs are like recipes, that is, they give instructions for mathematical actions to be carried out. In this chapter, together with Inglis, he offers a systematic, large-scale, computational analysis of mathematical texts, so as to focus on the relationship between the language used in written, textual mathematics, and the way in which we should understand what proofs are. Another merit of the chapter is that it surveys interesting studies in mathematics education on the use of instructions. Methodologically, the chapter is very careful in making the hypotheses explicit before presenting the analysis in detail, and in pointing out the limits of the particular study, thus paving the way for new quantitative analyses.

As evoked above, all the chapters in this section end with a list of suggestions for further research, so as to prepare the ground for even newer perspectives. From the discussions in the chapters, we see that the interdisciplinary nature of the philosophy of the mathematical practice is confirmed today, in line with the pioneering works published at the beginning of the 2000s; furthermore, the chapters collected here also show that assuming an “intradisciplinary” approach within the regions of philosophy can enrich the debates and raise new questions. I cannot but encourage the reader to dive into them.

### 3 Cross-References



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## Footnotes

[1](#)

For more information about the Association, see <http://www.philmathpractice.org>

[2](#)

See Starikova’s article in this volume and Giardino ([2022](#)), where I distinguish between thought experiments, real experiments, and quasi-experiments in mathematics.

[3](#)



On this point, she draws on previous work. See De Toffoli ([2021](#)).