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Diagrammatic proofs in mathematics: (almost) 20 years of research

Valeria Giardino

Abstract. The objective of this chapter is to present some of the views that have been put forward in almost 20 years of research on diagrammatic proofs in the philosophy of mathematical practice. In Section 1, some contextual elements will be presented on the reasons why diagrammatic proofs have attracted so much philosophical attention in the past years. In Section 2, the “first wave” in the research on diagrammatic proofs based on the analysis of case studies will be described: this is how it started. In Section 3, the “second wave”, which is calling for “big pictures” and divides into two different strategies, will be discussed: this is where it is going. In Section 4, some conclusions will be drawn.

Keywords. Diagrammatic proofs, Greek (Euclidean) geometry, enhanced manipulative imagination, fruitful representations, informal and formal proofs.

1. Introduction

It is undeniable that diagrams are part and parcel of the practice of mathematics: mathematicians draw them on blackboards and occasionally on napkins at conference lunches; they use them to teach students and to show to colleagues that some relations obtain, or more simply to share ideas with them; in some cases – but not as often compared to their heavy presence in more informal exchanges – diagrams end up being published in handbooks, encyclopedias, or research articles. How can the pervasiveness of diagrams in the practice of mathematics be accounted for? To clarify, this phenomenon can be investigated in two related but conceptually distinct ways.

First, very general questions can be asked. Why do diagrams seem so effective in mathematics (and beyond)? How are they used? Do they help understanding and explanation? If so, do they in all cases? What cognitive advantages do they offer? What are their cognitive costs? What are the cognitive foundations of the reasoning based on them? Are diagrams typical of Western mathematics or are they also found in other mathematical cultures? And how are they related to other forms of visualizations? All these questions are of crucial interest not only for philosophy, but also for other disciplines in its vicinity, such as history, sociology, anthropology, and education.

Additionally, a second, more specifically philosophical, question can be raised about the role that diagrams might – or might not – play in mathematical *proofs*. Of course, answers to the first set of questions are relevant to discuss diagrammatic proofs; nonetheless, it is important to note that the second question is relatively independent from the first set of questions: it is indeed possible to accept that diagrams are omnipresent and useful tools in the practice of mathematics, that they are important heuristic tools used for discovering or even for proving some mathematical results, and yet to consider that they are not and cannot be parts of proofs or necessary for them.

In this chapter, I will focus on the second question, that is, on the possibility of finding proofs in the practice of mathematics that make use of diagrams. My objective will be to survey some of the views that have been put forward so far in the literature in the philosophy of mathematical practice. In order to do that, at times I will raise some of the more general questions; however, this will be done only to the extent that some of the replies can be used as evidence for the presence of diagrams in proofs. I will proceed by presenting some of the available proposals and by describing how the discussion in the philosophy and partially in the history of mathematical practice has taken shape in the last 20 years or so.

In the following, I will introduce some contextual elements that are essential to understand the reasons why diagrammatic proofs have attracted so much philosophical attention in the past years. As already said, it is undeniable that diagrams appear in some proofs; however, the nature of their contribution is controversial, that is, there is no agreement on whether diagrams are necessary for the proof to go through or whether they are just useful illustrations that can be in principle removed from the proof without epistemic loss. Before proceeding, a word of caution is necessary regarding the meaning of the term “diagram”. It is evident from the literature that there is no consensus on how to define diagrams. “Diagram” has indeed been used for a long time in a loose way, in order to refer to different objects, spanning from Euclidean diagrams – figures? – to diagrams – schemas? – in category theory. As I will show, some scholars consider two-dimensionality as a necessary feature for something to be considered as a diagram, while others focus on the manipulations that can be performed on the diagrams to be their characteristic feature, thus opening the way for linear notations to be included into the class of diagrams. In this chapter, I will keep using “diagram” in a general fashion, as a sort of umbrella term encompassing several kinds of representations, without taking any stance on the debate on how to better define them.

The relation between diagrams and proofs is as old as mathematics. Famously in his *Parallel Lives*, in the section entitled *Marcellus* (2nd century AD), Plutarch describes the death of Archimedes, which deeply afflicted Marcellus. In this account, Archimedes happened to be by himself, “working out some problem *with the aid of a diagram*, and having fixed his thoughts and his eyes as well upon the matter of his study, he was not aware of the incursion of the Romans or of the capture of the city” (*emphasis mine*). A soldier ordered him to go with him to see Marcellus, but Archimedes refused because he wanted to first establish his proof. Hearing these words, the soldier got angry, drew his sword, and killed him. Archimedes was thus attempting to prove a result by fixing *not only his eyes but also his thoughts* upon the matter of his study – the diagram itself or what it represents? Whatever the reply to this latter question, Plutarch’s story tells us that nothing at the time prevented a mathematician to use a diagram in a proof – at the exception of a Roman intruder.

As is well known, things had changed by the late 19th century, when the urge to make mathematics rigorous had grown stronger and stronger, to the point of pressing Pasch to claim that “the theorem is only truly demonstrated if the proof is completely independent of the figure” (see Pasch, 1882). Dedekind was also

very much dissatisfied with the appeal to diagrams that exploited geometric intuition in analysis – notwithstanding the fact that his manuscripts are replete of tables, doodles, and sketches (see Haffner, forthcoming) – and Russell (1901) agreed that the best (mathematical) books do not contain figures.¹

These attitudes contributed to the definition of what has been labeled the “received view” of mathematics (see van Bendegem, 1993), according to which only reasoning that is expressed in formulas can bear epistemological weight, while the rest of the representations that are found in the practice of mathematics – geometric figures, for example – are just illustrations or at best a form of heuristics that would facilitate the grasp of the content of their linguistic counterparts, but nothing more. This received view, inherited from the 19th century, became the standard in the philosophy of mathematics of the 20th century, as summed up in Tennant (1986, p. 304)’s claim: “[the diagram] has no proper place in the proof as such. For the proof is a syntactic object consisting only of sentences arranged in a finite and inspectable array”². Despite the fact that diagrams can be important for understanding or explanation, they have no place in proofs.

The context changed again at the beginning of the present century (for a survey, see Giardino, 2017a and Carter, 2019a): in 2005, Mancosu, Jørgensen, and Pedersen edited a volume of essays that were devoted to “new” – compared to the received view – research subjects: *Visualization, explanations and reasoning styles in mathematics* (Mancosu *et al.*, Eds., 2005). In the introduction to this volume, the editors explain how the focus of the research in philosophy of mathematics should be on “what mathematicians are actually doing when they produce mathematics”, thus posing the foundations for a philosophy of the practice of mathematics. Indeed, three years later, Mancosu edited a new collection of articles on *The Philosophy of the Mathematical Practice*, to show that several topics – concept-formation, understanding, heuristics, changes in style of reasoning, analogies, and diagrams – still belong in the domain of philosophy, despite the fact that philosophers had neglected them for a century or so (Mancosu, Ed., 2008). The essays collected in this volume constitute evidence for a restored interest in these subjects that, according to him, were complementary to more traditional and ‘received’ questions, for instance the one about the access to mathematical objects. A challenge for research was to put these questions in relation with each other, for example by exploring how mathematical concepts and objects are generated and how the process of justification works. In his words, epistemology of mathematics could not be reduced to “a torso” (Mancosu (Ed.), 2008, p. 1), and it was time to return to issues like fruitfulness, evidence, visualization, diagrammatic reasoning, understanding, and explanation.

¹ For the details of this historical background and a general discussion of the epistemology of visual thinking in mathematics, see Giaquinto (2020).

² This passage from Tennant (1986) is quoted in Barwise and Etchemendy (1996, p. 3) as expressing the “dogma of logocentricity” that the authors want to challenge. Despite the fact that the context here is logic, the quote is still relevant for mathematics. I will briefly refer to the relation between the question about diagrammatic proofs in logic and in mathematics later in the article.

In this renewed philosophical panorama, studies on diagrams ended up occupying a preeminent position. The aim of this chapter is to trace back the literature about the role of diagrams in proofs starting from the publication of Mancosu's 2008 volume, that is, 15 years ago. However, given this long period of time, it will be impossible to cover all of the details – other people, including myself, have done it already (see Giardino, 2017b). What I will try to do is rather to propose a narrative of how this research started and where it is going today. Given the distinction I made between two different sets of questions concerning diagrams, I will focus on the second one and therefore on a selection of studies devoted to giving an account of cases in which diagrams are part and parcel of mathematical proofs. In order to do that, it is necessary to specify which features they possess and how they are intended to be used, and furthermore which relations they entertain with other elements of the practice of mathematics.

On this last point, let us recall an important methodological issue. To say that a proof is “diagrammatic” does not mean that it is “without words”, as three celebrated volumes edited by Nelsen seem to suggest (Nelsen, 1997, 2001, 2015). The proofs making use of diagrams that scholars are interested in are *always* proofs that also include some form of *text*, be it the labels accompanying the diagrams or the formulas obtained by considering some of the diagram's features. The point about diagrammatic proofs is precisely to understand how diagrams and text –two crucial elements in the practice of mathematics – are related, and how the contribution of the diagram to the information presented in the text can be (or not) essential for the proof to go through. In epistemological jargon, the interesting question is about how a mathematical belief that is obtained by looking at the spatial features of the diagram and by manipulating it can be raised to the status of knowledge in the absence of independent textual grounds, as well as be related to other pieces of knowledge expressed in the text.

The narrative that I propose is the following. In Section 2, I will quickly describe the “first wave” in the research on diagrammatic proofs. As I mentioned above, at the beginning of the 2000s, old topics were brought back to the front by raising questions that had been almost forgotten. This is how it started. Since then, many articles have been published that present case studies of diagrammatic proofs, with Greek mathematics taking center stage; as we will see, we have to wait until the beginning of the 2010s for case studies in contemporary mathematics. In Section 3, I will describe what I call the “second wave” in the research on diagrammatic proofs. This is where it is going. Now that the richness and the variety of diagrammatic proofs have been demonstrated, it is time to draw big pictures, which is what some scholars are presently trying to do. In order to introduce this second wave, I will first focus on pioneering work on the richness of mathematics and then on two strategies that are emerging today, aiming to specify what is meant by “proofs” – first strategy – and by “diagrammatic” – second strategy – in “diagrammatic proofs”. In Section 4, I will briefly sum up the topics of the chapter.

Two last methodological notes. First, nothing of what I will write about in this chapter will refer to research on diagrammatic proofs *in logic*, despite the fact that some of the questions and resources in the literature are common to the two disciplines. The reason for this is that I consider that the practice of logic has its

own specificities and thus deserves a separate treatment.³ However, logic will be evoked at times in discussing rigor in contemporary mathematics. Second, I will mainly focus on studies in the *philosophy* of mathematical practice. I am aware that enormous work has been done in history – and I would say also archeology – of mathematics on diagrammatic proofs from the past, not only in the West but also in places such as old China and Mesopotamia (see for reference Chemla, 2005 and Høyrup, 2005). Many of the chapters of this handbook are devoted to the history of the practice of mathematics and will tackle the subject of diagrammatic proofs among others; therefore, for pragmatic reasons, I will not go into that research here. However, integration between the historical and philosophical points of view on diagrammatic proofs is desirable: as testified by the first of Mancosu’s collections in 2005, historians and mathematicians started working together on these new-old topics. Since then, the two disciplines took their own research paths; it is time to foster again collaboration between the two to give an account of the practice of mathematics.⁴

2. The first wave of research on diagrammatic proofs: looking for case studies

2.1 The philosophers’ darling: Greek mathematics

The Greek diagram is to some extent the paradigm of diagrammatic reasoning in mathematics, which explains why it has been a privileged target of research for philosophy for a long time. There is no need to be a philosopher or a mathematician to be familiar with Euclidean geometry, and in Euclidean geometry the recourse to diagrams is natural and spontaneous: starting from a very early age, these geometrical figures become part of our visual repertoire. From this, can we take their presence and effectiveness in proofs for granted?

As Netz (1999) pointed out, diagrams are interdisciplinary research objects, because they are of interest both for the science of cognition – that is looking at knowledge from the point of view of its practices – and for history of science – that studies cultural artifacts. For this reason, diagrams are the perfect subjects for what he calls a “cognitive history” and can be approached from many different points of view. I agree with this and I consider philosophy as the approach targeting more specifically their role in proofs. In his book on deduction in Greek geometry – going also beyond Euclid’s *Elements* – Netz defends the idea that the “lettered diagram” is an effective geometric *tool* because of the richness of the different levels at which it can be accessed, which is what makes it appropriate to promote and justify deduction and, as a consequence, to demonstrate mathematical propositions.

The lettered diagram can in fact be considered from several points of view: logical, cognitive, semiotic, and historical. At all of these levels, it combines two

³ There are several studies that are relevant in this respect, for example Shin (1995), Allwein & Barwise (1996), or Stenning (2002). More recently, Dutilh-Novaes (2012) discussed the role of formal language and notation in logic.

⁴ Another subject I will not touch upon in this section is computational approaches to diagrammatic proofs. Some relevant references can be found in Giardino, 2017b.

elements that would normally be considered to be in contrast to each another, if one follows the standard opposition between image and language: lines and letters at the logical level, visual features and models at the cognitive level, icons and indices at the semiotic level, and a kind of know-how together with a very sophisticated reflexivity at the historical one. According to Netz, a proof in Greek geometry is based on the lettered diagram, and as a consequence is an *event* that occurs on a papyrus or in a given oral communication. Despite its singularity, it is *felt* to be valid: the materiality of the proof is important as well as its immateriality, since the proof is considered as an *invariant* under the variability of the single action of drawing one diagram on the papyrus or of presenting the particular proof orally. The proof can be *repeated*, which is what matters for it to be recognized as a proof; contrary to what we are used to think today, the focus is not on the possibility of generalizing the result. For Netz, in order to give an account of how the lettered diagram makes the proof go through, it is necessary to consider its cognitive appeal. The diagram is in fact a static object that is nonetheless related to the text via the letters, which makes it *kinesthetic*: a diagram gets constructed and can be manipulated.

In a very renowned paper, which has been circulating since the 1990s but was published only in 2008, Manders (1995/2008) investigates Euclidean geometry specifically. He reminds us that up to the 19th century no one would have denied that the practice of Euclidean geometry was *rigorous*; rather, such a practice was considered to be the *most* rigorous practice among the various human ways of accessing knowledge. In Manders' view, such a practice is based on a division of labor between two artifact types – the diagram and the text sequence – that must be considered together. As humans, we have cognitive limitations, and therefore we cannot control the interpretation of a diagram so as to avoid any case that is alternative to the correct one; for this reason, the text is introduced. In the practice, diagram and text share the responsibility of allowing the practitioners to respond to physical artifacts in a “stable and stably shared fashion” (p. 83). In this framework, proofs have a verbal and a graphical part – the *diagram* and the *discursive text*: the diagram makes the very objects of traditional geometry appear, while the text ascribes particular features to the diagram, the so-called diagram *attributions*. Letters are introduced so as to facilitate cross-references between the text and the diagram. In this reconstruction of the Euclidean practice, Manders introduces a distinction between *co-exact* and *exact* features of a diagram that became very influential.⁵ A co-exact feature is a feature that can be directly attributed to the diagram, having certain perceptual cues that are fairly stable across a range of variations; such a feature cannot be readily eliminated, because of what Manders calls the diagram *discipline*, that is, the proper exercise of skill in producing diagrams that is required by the practice. An exact feature is instead in general affected by deformation, and it is licensed by the discursive text: for example, the fact that in a diagrams two segments have to be considered as – “seen as” – equal. Note that in both Netz' and Manders' proposals the diagram is part and parcel of the proof *because* of the combination of diagram and text and thanks to its *dynamic* features: invariances or co-exact features – not varying under deformations.

⁵ Mander's work was influential to the point of inspiring cognitive research. See Hamami & Mumma (2013) and the more recent Hamami *et al.* (2021).

Macbeth (2012, 2014) proposed another account for the use of diagrams in Euclidean proofs, that emphasizes their nature of “signs”. She considers Euclid’s *Elements* as a system of *natural deduction*, where common notions, postulates, and definitions are not intended as premises, but as *rules* or *principles* according to which to reason. Along these lines, she suggests that in order to understand and use a diagram one has to recognize the *intention* that is behind its construction. Moreover, in her view, a diagram is fruitful because it is an *icon*, that is, it “resembles” what it signifies. The Euclidean diagram has to be considered at different levels: first, it has primitive parts – points, lines, angles and areas; then, thanks to the concepts of geometry, one gets the elements that someone doing geometry will be interested in – for example, an equilateral triangle is “seen as” composed of three equal lines; finally, one has to consider the diagrams as a *whole*, which shows at a glance the elements in the other two levels. In this sense, a Euclidean diagram is not different from other kinds of representations such as the famous Wittgensteinian duck-rabbit.

Consider for example the diagram of an equilateral triangle obtained from a segment AB by drawing two circles with center A and B respectively intersecting in C. Draw the two segments from A to C and from B to C – this is Euclid I, 1: the “same” segments – AB, AC and BC – are “seen as” the sides of the triangle *as well as* the radii of the same circle (see Figure 1). Thanks to the combination of all these different elements in the same icon, one reasons *in* the diagram and obtains the proof.⁶

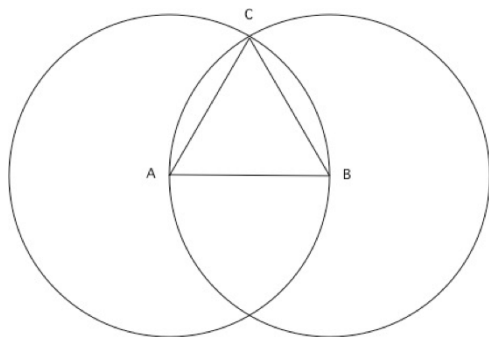


Figure 1. Euclid, Book I, Proposition 1. *To construct an equilateral triangle on a given finite straight line.*

Another interesting notion to apply to Euclidean proofs was proposed by Azzouni (2005). In his view, traditional mathematical proofs imply the appeal to a kind of know-how and tacit knowledge involving “inference packages” applied to the proof as “black box units”. The elements encapsulated by an inference package are in principle not visible by introspection as a list of assumptions, for the reason that these assumptions have been “knit together” in the representations of the objects. In traditional proofs – compared to derivations in algorithmic systems – it is not the case that each step follows from some

⁶ Actually, there is a difference between the duck-rabbit and the triangle in Euclid, since the second is not an instable percept; on the contrary, someone who is competent in geometry, will be able to see the two aspects and acknowledge that they are not incompatible. For a discussion of this point, see Giardino & Wöpking (2019).

sentences appearing earlier in the proof: traditional proofs work “semantically” rather than “syntactically”, that is, mathematicians recognize properties of the objects their inferences are *about* by operating with the available representations. The details of the inferences provided by inference packages are thus not only left out because everyone is familiar with them, but because they are not introspectively accessible to practitioners. In this semantic view of proofs, representations – for example some particular diagram – have precisely the role of allowing for a form of intuition to apply the inference package that comes with them, and for this reason they should be at the core of the philosophical investigation (Azzouni, 2005, pp. 23-24).⁷

I have discussed elsewhere these views and others focusing on the case of Greek diagrams and diagrammatic proofs (Giardino, 2017b), and therefore I will not go here into further details. In the following, I will simply sum up some of the important points that have been proposed for the case of Euclidean (or Greek) diagrammatic proofs. It is crucial to note that all these studies show that in order to give an account of diagrammatic proofs it is necessary to go beyond the insistence on the alleged opposition between a “visual” and a “linguistic” part of mathematical proofs, for two reasons. First, diagrams have to be considered together with the text, and therefore the focus of the research should be on the way in which the two relate. Second, diagrams can be interpreted as *dynamic*: for example, co-exact properties are not subject to modifications – which suggests that the diagram has to be manipulated in order to “test” our correct interpretation of it; the text clarifies what the intentions in the construction of the diagrams are in relation to the properties that may vary under the same manipulations. Such manipulations allow for particular inferences that are not accessible through the text only. Seeing the same configuration “differently”, by identifying several levels that have to be considered separately but also as a whole, may be considered as a very basic way of making the diagram “dynamic” without modifying it. Moreover, the proof is an “event” that can be repeated, once the diagrams and the text are correctly put together.

In the following subsection, we will look at more recent case studies taken from contemporary mathematics and see how they relate to the features that emerged in the analyses of Greek mathematics.

2.2 Contemporary mathematics enters the scene

If Greek diagrams have been at the center of philosophical discussion for a long time, case studies of diagrammatic proofs taken from contemporary mathematics have started becoming of interest to philosophers in most recent years, approximately after 2010. Why so late? Of course, a strong background in the practice of mathematics is necessary to get the gist of the examples taken from contemporary mathematics. Only recently, philosophers who were previously trained in mathematics started tackling specific questions about diagrammatic proof and collaborations between philosophers and mathematicians have developed – a good thing, given the interdisciplinary nature of the topic. It should also be noted that the support of interdisciplinarity

⁷ To learn more about Azzouni’s view, see Azzouni (this volume).

in education today is encouraging more than before – when the training was more rigidly disciplinary – the creation of a new generation of interdisciplinary scholars. I have already presented some of these more recent case studies in detail (Giardino, 2017b); in the rest of this section, I will sum up some of these results.

Carter (2010) was one of the first scholars to introduce case studies from today’s mathematics – as well as one of the first who are now trying to think in terms of more comprehensive pictures, as we will see below. In her contributions, she focused on the role that diagrams have in a proof to define its *strategy*. She analyses a case from free probability theory, where diagrams are introduced in informal discussions among mathematicians to represent particular mathematical entities, for example permutations (see Figure 2).

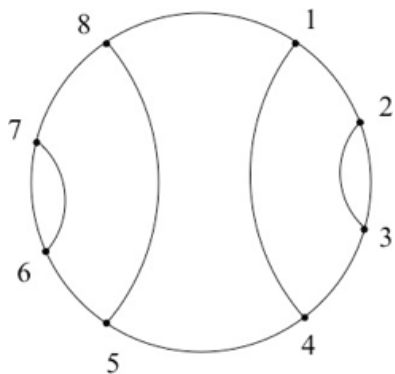


Figure 2. An example of a diagram of a permutation, in this case the permutation $\pi (13)(24)$.

Carter argues that diagrams make it easier to evaluate relevant properties, e.g. whether the permutation is crossing or not, or whether neighboring pairs can be canceled. Despite the fact that they are *not* published in the final article, it can be shown that they still suggest definitions and proof strategies, thus inspiring at least a part of the formal version of the proofs. Diagrams here function as a *framework* for the proof, and help breaking it down into parts that are more manageable than the whole proof. We mentioned above the importance of the relationship between diagrams and text in Greek diagrams: here, some properties of the diagrams *correspond* to formal definitions, but *experiments* are performed on the diagrams.⁸ However, the relations used in the proof based on the diagrams refer to relations that *also* hold in the algebraic setting, which allows going back from the diagram to the algebraic setting. Moreover, some features of the diagrams – for example crossings and neighboring pairs – are indeed co-exact properties, since they do not change under a series of variations. Carter’s inspiration is C. S. Peirce’s philosophy, according to which all mathematical reasoning is *diagrammatic*: in her case studies, diagrams are considered as icons in relation to the algebraic setting. One recognizes in them “indices” – letters – as an element that captures attention and, as Peirce writes, tells the user to look “There!” (CP 3.361). Another case study Carter focuses on is directed graphs (Carter, 2018). Also here diagrams are icons, that is, mediating objects, which link – metaphorically – two types of mathematical objects. This is possible because they constitute a “faithful” representation, which is able to put

⁸ On experiments and diagrams, see Schlimm & Fernández González (this volume).

together the metaphorical aspects of using a mathematical diagram with the control that it is necessary to exercise on its manipulation. In fact, manipulations that are performed on the diagram respect the relations between the concepts that they represent, and this leads to the definition of new relations between parts of those concepts that were not appearing at the beginning.⁹

In Carter's view, diagrams are fruitful frameworks in parts of proofs because they trigger imagination. She refers to the notion of "manipulative imagination" as introduced by De Toffoli and Giardino (2014) in the context of an analysis of the role of knot diagrams in topology (see Figure 3). In our reconstruction, experts in knot theory have developed a form of imagination that allows them to manipulate these diagrams – by redrawing them or by figuring out in their mind which kinds of actions can be performed on them.¹⁰ Manipulative imagination is very effective in low-dimensional topology, where diagrams are used to explore the properties of mathematical objects and eventually to prove theorems about them, for example in Rolfsen's demonstration of the equivalence of two presentations of the Poincaré homology sphere (see De Toffoli and Giardino, 2015). In our view, the actual practice of proving in low-dimensional topology involves a kind of reasoning that requires intuition: the manipulations that are allowed for and on the representations as well as the representations themselves are *epistemologically relevant*, because they are integral parts both of the reasoning and of the justification provided by the proofs.

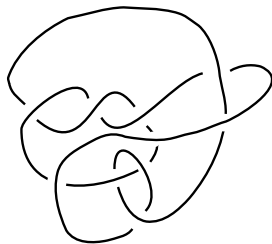


Figure 3. An example of a knot diagram. By applying the appropriate manipulations, one finds out that this is a knot diagram of the unknot.

In a more recent paper, De Toffoli (2017) describes commutative diagrams and the "diagram-chasing" technique in homological algebra, where a new feature of diagrams in proofs emerges. Diagrams indeed can correspond to an abstract counterpart of geographic maps.¹¹ As she explains, the *Five Lemma* and the *Fundamental Theorem of Homological Algebra* are both based on chasing an element in different places *in* the diagram, which in the second case is potentially infinite (see Figure 4).

⁹ In a more recent paper, Carter (2021) reviews some of her previous examples by referring to the notion of "free ride" as proposed by Shimojima (2015). In the paper, she shows that "free rides" are not completely "for free": in most cases, some interpretation is required to "read-off" some properties from a diagram, and therefore a free ride is not always a free sight.

¹⁰ In previous work, I defined diagrams as *props* – by referring to Walton's theory of imagination – for some mathematical object: working with them would correspond to test ways in which the mathematical objects they allude to might "behave" (Giardino, 2018a). On the notion of manipulative imagination, see also Giardino (2018b).

¹¹ For case studies in category theory, see Halimi (2012) and Weber (2013).

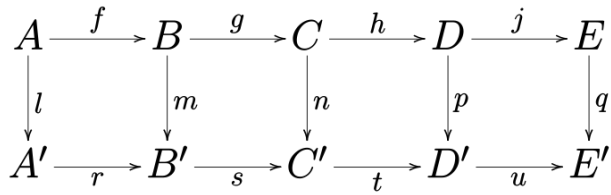


Figure 4. The commutative diagram used in the proof of the *Five Lemma*.

The chase aims to identify paths in the diagram by exploiting its commutativity in order to verify some algebraic properties. The basic idea is to imagine moving an element around the diagram by following the relations that are there displayed – perhaps the analogues of the “co-exact” properties of Euclidean diagrams – according to some definitions – the “exact” attributions of the text. De Toffoli explains that in the case of homological algebra, commutative diagrams play the two roles already identified by Carter, that is, they inspire the proof strategy and they offer a framework that allows getting to it. However, they also have two additional roles: first, they display algebraic content in a way that is meaningful and *mathematically tractable*; second, they allow for *calculations* that can be integral parts of the proof. In De Toffoli (forthcoming) she argues that commutative diagrams (as well as certain topological diagrams) not only can be used within rigorous proofs but are even essential to the proofs in which they figure. This would mean that non-diagrammatic conversions of such proofs would plausibly be classified not as different presentations of the same proof, but as different proofs.

To sum up, a look at case studies from contemporary mathematics shows that diagrams happen to be used effectively in proofs, for the reason that they can make new properties or relations appear. Of course, this is done in reference to some mathematical context. However, there are some caveats. First, not all diagrams are used specifically in proofs: some diagrams support intuition in the context of discovery.¹² What emerges clearly from the literature is in fact the variety of diagrams: diagrams to display permutations have very different features compared to, say, commutative diagrams. This said, a common feature seems to be their way of triggering some form of intuition or imagination – whether they can be ultimately part and parcel of a proof depends on the control that may be applied to them by fixing their interpretation. To use an expression from Larvor (2012), within a certain diagrammatic practice, some actions on the diagrams are recognized by the experts as “permissible” or not. There is something more to note – to which I will return when I will discuss where we are now in the research on diagrammatic proofs: cases in contemporary mathematics show that diagrams and notations can be brought together. In her paper on homological algebra, for example, De Toffoli explicitly argues for diagrams as a subset of notations and gives three criteria to evaluate them as such: *expressiveness*, *calculability*, and *transparency*. For Carter as well, a piece of notation such as $2k + 1$ for some natural number k is also a case of a faithful iconic representation of the concept of odd number. I will return to this issue below.

¹² For an interesting case on Caley graphs and their use for discovery, see Starikova (2010).

3. The second wave: Calling for big pictures

3.1 Pioneering work on the richness of the practice of mathematics

In the narrative that I propose in this chapter about these past 20 years or so of studies on diagrammatic proofs, I maintain that the urge for providing a big picture about their role in proofs has appeared only very recently. However, I will focus here on two scholars who were undoubtedly the first to try to offer a comprehensive account of the richness of the mathematical practice related to diagrammatic proofs.¹³

Giaquinto (2007/2011) was maybe the first to consider *visual* thought in mathematics in an interdisciplinary fashion, by relying on some explanations coming from cognitive science, which was on the rise at the time. The general aim of his book was to give an account of the complexity of mathematical thinking. To do so, he explored and presented some results in cognitive science about our capacity for visualizing in an attempt to integrate them with other more theoretical reconstructions of high-level processing of information. The book is interestingly devoted to discovery in mathematics, which, according to Giaquinto and in line with the revival for the attention to the practice of mathematics, is a subject that had been neglected by the most recent philosophy of mathematics as legitimately philosophical. However, there are parts of the book where Giaquinto discusses the possible presence of diagrams in proofs. In his view, there is no need to give a uniform story about visual thinking in mathematics, which comes in a variety of ways: depending on the specific mathematical context, visual operations can be different and consequently it may be possible to evaluate differently their epistemic value.

One point that seems important in the present context is Giaquinto's focus on *aspect shifting* in mathematics. Consider, for example, a possible proof of the Pythagorean theorem. In order to understand that it is valid, here is one possibility (see Figure 5).

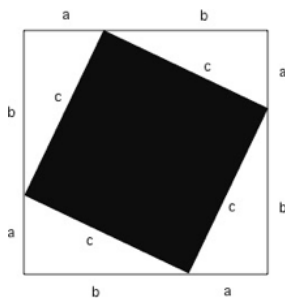


Figure 5. One of the many so-called “visual proofs” of the Pythagorean theorem.

First, one reasons geometrically and arranges a small square and four right triangles in a bigger square, so that each of the longer of the two sides of the

¹³ Another important early publication on diagrams in mathematics is Brown (1999), where several cases of diagrammatic reasoning are discussed. Brown's view has strong Platonic assumptions: his “bold conjecture”, as the author famously put it, is that (some) pictures “*are not really pictures, but rather are windows to Plato's heaven*” (p. 40, emphasis in the original).

triangles corresponds to one of the sides of the bigger square and their hypotenuses form the smaller square. Then, one reasons algebraically and calculates the area of the smaller square plus the areas of the four triangles and equates it to the area of the bigger square. Some more calculations, and the claim is established. This procedure is accessible because we shift from one aspect – what we “see” in the diagram – to the other – what the rules on how to obtain the area of a square and of a triangle tell us. If this is the case, would this reasoning be primary geometric or primary algebraic? In the last chapter of the book, Giaquinto suggests that neither of these two categories would be fully appropriate to capture it. Is this a general feature of mathematical reasoning?

Another interesting early work is Grosholz (2007, 2011), which discusses at depth the *heterogeneity* of mathematical reasoning as something that has been overlooked by what she identifies as the Russell-Carnap tradition – the received view? Also in this case, the practice of mathematics went back to the center of the philosophical investigation. In Grosholz’ reconstruction, the Russell-Carnap tradition represents an epistemological ideal whose objective is to unify mathematics, or mathematics and the sciences, in one formalized theory. With this objective in the background, this tradition demands that logic is the homogeneous idiom constituting the only vehicle for deductive inference. Macbeth’s view that was mentioned in the previous section on Euclidean diagrams recalled Grosholz’ notion of “productive ambiguity” of diagrams, which is precisely what allows us to see different things in the same configuration – the sides of an equilateral triangle and the radii of the same circles – and to put the two together in order for the proof to go through. However, productive ambiguity and heterogeneous reasoning go beyond geometry: one of Grosholz’ examples is Gödel’s procedure in his incompleteness theorem. In her reconstruction, by his use of numbering as a strategic bridge between formalized arithmetic qua logical system and arithmetic, Gödel exploits a “carefully controlled and fruitful ambiguity” (Grosholz, 2011, p. 421), which multiplies the information that is available to a mathematician. As she puts it, “it is hard for a logician to admit that he is *trafficking in heterogeneity*” (*ibid.*, p. 425, *emphasis mine*). As we see, the richness and the variety of the representations involved in the reasoning process were extended to other parts of mathematics and even to logic. This opened up a new research horizon that will constitute a heritage for the views that I will present in the next subsection.

3.2 Today’s views on diagrammatic proofs

In Section 2, I summarized some of the case studies of diagrammatic proofs that were subject to philosophical scrutiny, starting from Greek mathematics and growing into cases from contemporary mathematics. This variety called for some kind of comprehensive framework that could account for them. However, how is it possible to talk in terms of a “big picture” for a phenomenon that is so diverse and that also depends – as Giaquinto suggests – on the specificity of the mathematical context considered in turn?

Let us focus on the term “diagrammatic proofs” and the way in which it is commonly intended. On the one hand, “proofs” is here meant narrowly, mostly in the sense of the received view: the question about diagrammatic proofs is

whether forms of reasoning that make use of diagrams reach the standards of what is commonly considered as a proof – to the point that “diagrammatic proofs” may sound to some as an oxymoron. On the other hand, “diagrammatic” is interpreted as having a very wide scope so as to include many case studies and thus has the advantage of accounting for their richness. However, the fact that the term is used very loosely has also its disadvantages. What exactly is a diagram? Are geometrical figures diagrams? Are they analogous to a matrix? What is the relationship between diagrams and notations? Are they one and the same thing?

In my view, the strategies to give a comprehensive account of diagrammatic proofs can be divided into two groups. First, one might argue for a notion of proof that moves away from definitions *à la* Tennant and easily includes diagrammatic proofs as perfectly acceptable. Second, one might try to put forward some criteria for different categories of diagrams in mathematics, thus providing a taxonomy and more importantly drawing a line between diagrams that can be reliably used in a proof and others that play the role of useful illustrations or good heuristics but do not possess the necessary features to be considered part and parcel of a proof.

In the following sections, I will briefly present some of the views that correspond to these two strategies, which would give on my view a fair picture of where the research on diagrammatic proofs is going today.¹⁴

3.2.1 First strategy: what is a “proof”?

The first strategy that has recently emerged in the literature is the focus on mathematical proofs in the practice. Such an investigation does not target diagrams in particular, but aims to move away from the narrow definition of proof typical of the received view. I will present here some frameworks that would allow for diagrams to have a constitutive role in proofs: I will start from the recipe model of proofs introduced in Tanswell (forthcoming) and then briefly discuss recent work on informal proofs. It is important to note that the topic of informal proof in particular would demand a chapter of its own, and for reasons of space I will not be able to go into the details; my discussion will be limited to the aspects of this debate that are relevant for diagrammatic proofs.

In Tanswell (forthcoming)’s “recipe model”, proofs are to be intended as analogous to cooking recipes since, as cooking recipes, they give instructions on how to carry out actions – in the case of a proof, “mathematical” actions – in order to obtain the desired result. Of course, this does not conform with the received view, according to which the proof is a deductive syntactic argument that moves from some axioms to a conclusion by following some pre-established inference rules. In Tanswell’s view, despite the success and fruitfulness of such a definition of proof, it is time to provide a new definition that would be more faithful to the practice of contemporary mathematics: written proofs are written records of how the theorem is obtained, and their main objective is to explain to the reader how to actually do the reasoning that is required in order for the

¹⁴ Given that the studies I will refer to are newly published, I might happen not to know yet about other studies that could easily be included in this survey.

proof of the theorem to go through. As in the case of recipes, proof has a communicative intention and there might be different styles of presenting the same proof.

The evidence that Tanswell brings about to argue for this analogy is first of all linguistic: the texts of proofs include instructions in the imperative tense (see Tanswell & Inglis, in this volume). However, such a view of proof allows including diagrams and figures interacting with the text as alternative means of giving instructions. The opposition between assertions on the one hand and pictures on the other is ill posed: both of them can convey instructions, and for both of them some competence for interpretation is required. However, in Tanswell's view, pictures give instructions more directly than text, as is the case for LEGO games or IKEA furniture: there is no reason to require that a picture corresponds to a particular sequence of assertions, since what matters are the actions that it guides the reader through (Tanswell, 2017). It is important to note that such a view stresses the role of *action* in proving, which is in line with what we have seen so far in interpretations of diagrammatic reasoning as based on conceiving the legitimate operations and manipulations they make accessible. Moreover, such a conception is also related to wider debates; in the following, I will focus on three of them.

First, as Tanswell himself explains, the focus on action is inspired by other work on informal proofs, in particular Azzouni (2004) and Larvor (2012). The topic of informal proofs is very much discussed in the literature and clearly goes beyond the scope of this chapter. However, some of the elements of this debate are relevant for proofs using diagrams. I will refer to Burgess and De Toffoli (2022), who claim that the standard view, according to which a mathematical argument is a rigorous proof if and only if it can in principle be converted into a formal derivation, is a very general claim – a family of views rather than a single view – that can be specified in many different ways. However, the two authors themselves assume different positions in relation to the role of diagrams in proofs. According to Burgess (2015), there is a correspondence between the formalized version of the steps of an informal proof and the components of the formal proof. Formalization is thus a process of *filling in* the details, and the analysis of the role of diagrams is thus excluded from this account of rigor. However, De Toffoli (2021) argues that this is difficult to accept if one looks at the practice of mathematics: a definition of rigor such as this one would exclude a considerable number of proofs that mathematicians believe to be rigorous, which is not a desideratum if one moves from within a perspective that is interested in the practice.¹⁵ As a consequence, diagrams, they claim, are relevant to the debate. This is because if the connection between a proof and one of its formal counterparts is simply the one of filling in, then diagrams cannot be part of rigorous proofs. However, the relation between informal and formal proofs might be more complicated – and still consistent with some versions of the standard view.¹⁶

¹⁵ As I will explain in the next section, the way to go according to De Toffoli is to focus on what kinds of diagrams one refers to when asking questions about their being part and parcel of rigorous proofs.

¹⁶ On the standard view, see also Hamami (2022).

Second, the recipe model of proofs can be seen as an element of a larger picture where the focus is on the *activity* of proof (Hamami and Morris, 2021). If a mathematical proof is to be intended as a sequence of deductive steps, then there is an isomorphism between such steps and the sequence of deductive *inferences* constituting the proof activity. The analogy here is between proof activity and other more ordinary activities such as travelling, where actions need to be planned in advance in order to be able to move from the current location to the desired one.¹⁷

Third, alternative models of proofs might have important consequences in other related research domains such as mathematics education. For example, if students are taught the received view according to which proofs are sequences of assertions, they will miss important elements of the practice, for example they will not be aware of the fact that there are imperatives in the written reports of a proof and that these are what would guide them through the steps of the proof (see Weber & Tanswell, 2022). Why should students be taught definitions of proofs that do not correspond to what is really going on in the practice of mathematics and in the very practice of learning? More in general, the results of a philosophical approach to the practice of mathematics are opening up a potentially very fruitful space for discussion with education (see for reference Hamami & Morris 2020). Giaquinto, in his pioneering work, was one of the first to recognize the potential of such a collaboration and to foster it.

Once again, these topics go beyond diagrammatic proofs, and therefore for the details I would encourage the reader to refer to the articles in the bibliography. However, from the narrative that I offer in the present chapter, it emerges that the questions that were raised in the context of diagrammatic proofs have at least in part contributed to the definition of these new research topics.

3.3 Second strategy: what is a “diagram”?

The second strategy to formulate a big picture concerning diagrammatic proofs is based on a focus on the variety of diagrams and the different roles that they might play in the practice of mathematics, some of them being relevant for proofs.

One way to go is to “zoom out” and to consider diagrams as “representational cognitive tools”, that is, as a kind of multi-recruiting systems constituting an “interface” on which information coming from perception or action, already at play in ordinary pragmatic contexts, and other more cognitive resources such as conceptual knowledge gets integrated (Giardino, 2018b). However, this more cognitive approach will not suffice to respond to the specific question about the possibility of including diagrams as part and parcel of proofs. In order to do that, one useful notion would be the one of “material anchor”, as introduced in cognitive anthropology (Hutchins, 2005). Mathematics would offer physical structures – diagrams and other representations – that result from a cultural process of “crystallization” of conceptual models into external and material objects. Such objects, when correctly interpreted, can be manipulated

¹⁷ For a survey on the recent studies about agency in mathematical practice, see Hamami (in this volume).

meaningfully; in some particular contexts, physical manipulations can correspond to mathematical operations so as to reduce memory and processing loads. In some cases, the particular physical structure would have an influence on the way in which the mathematical content is understood or conceptualized. Some other notions coming from cognitive science can in this respect be applied to such tools, so as for example the notion of *affordance*: diagrams afford some epistemic actions on them, given their physical features and also and more importantly their correct interpretation – the consideration of physical features, if not controlled, can lead us astray. As Rabouin (2018) has pointed out, the notion of material anchor might also be relevant to account for Netz’ or Manders’ studies of Greek mathematics, which consider how external resources are manipulated in order to operate on imagined structures. In his view, the historical evolution of mathematics has brought to a progressive “internalization” of these structures: representational tools such as figures still remained important, but new symbols were introduced in order to better control these operations on the imagined mathematical structures.

Another possibility to work on a big picture is to go in the opposite direction and to “zoom in”, by considering again the specifically mathematical case studies that have been produced so far in order to give a more precise characterization of what can be recognized in it as a diagram.

In a recent paper, Carter (2019b) goes back to her notion of *fruitful diagrams* to put together the several examples of diagrammatic proofs that were discussed in the literature. She refers to Macbeth’s work on diagrammatic reasoning and her claim that diagrams and, more in general, other forms of notations are fruitful because they make mathematical objects “pop out”, thanks to the way in which the representation is constructed. However, if this were the case, what would the peculiarity of diagrams compared to other forms of representation? Carter’s proposal is to define the appropriate class of objects to think in, which is the one of “faithful representations”. Her background is once again Peirce’s semiotics, according to which icons – a type of sign – divide into (i) *images*, resembling what they represent, (ii) *diagrams*, displaying relations, and (iii) *metaphors*, depending on conventional rules or structural similarities. Faithful representations are defined as icons, that is, as images, diagrams or metaphors in Peirce’s sense; moreover, manipulations made on them would respect manipulations made on “the object of study”, which may be properties, relations, concepts or more simply objects – we are not far from the idea that they are material anchors. These representations are fruitful precisely because manipulations performed on them can make new relations appear that. It is important to note that the border between diagrams and notations in general gets blurred; Carter is aware of this, and argues for two peculiarities of diagrams. First, diagrams show that a relation holds between objects by joining their respective representations by a line; in some cases, the nature of the relation is given by the figure. Second, thanks to two-dimensionality, diagrams can be used to show that an object is related to multiple objects, as shown in Macbeth’s examples.

In some of her recent work, De Toffoli (2022) tries to give a more precise definition to the vague notion of diagrams that has been considered in the

philosophy of mathematical practice. The taxonomy that she proposes is flexible enough to allow for borderline cases but has the advantage of offering a tool that is useful to navigate the literature by pointing to the relevant differences. In her reconstruction, and in line with her previous work, the starting point is notational systems in general, some of which are geometric-topological (GT), that is, they possess geometric-topological features that are relevant to enable the use of a form of intuition. GT-figures can be diagrams or illustrations, but only the first are used in a systematic way, so as to allow them to eventually be part and parcel of a proof. Moreover, GT-diagrams – it is the case of knot diagrams briefly mentioned above – trigger enhanced manipulative imagination. However, other more precise distinctions have to be introduced. In fact, it is only thanks to further elements that the class of mathematical diagrams can be defined: mathematical diagrams are indeed systematic notational items that are *either* GT or two-dimensional *or* both.

There are two important points to note: first, the diagrams that constitute a challenge for the received view of proof – and the ones that we have mostly discussed so far in this chapter – are specifically GT. In fact, non-GT diagrams are generally accepted in proofs – this is for example the case of commutative diagrams. However, it is only after the middle of the 20th century that non-GT mathematical diagrams entered the scene.¹⁸ To sum up, according to De Toffoli's big picture, notational systems can be distinguished across three orthogonal dimensions: (i) systematic vs. non systematic use; (ii) geometric-topological vs. non-geometric-topological; (iii) two-dimensional vs. non-two-dimensional. Borderline cases are accepted. We now have a map to situate the different case studies, also in relation to the questions about which of these dimensions are decisive to allow for diagrams to be part and parcel of diagrammatic proofs.

This second strategy also opens up new research territories. On the one hand, diagrams can be considered as externalizations for thought, and therefore they can be inquired from the point of view of cognitive science – as Netz and Giaquinto had already been suggesting. Would that be some piece of evidence in favor of an “extension” of the mind? On the other hand, diagrams can be intended as signs and so they have to be studied in relation to other kinds of notations; as a consequence – as already proposed by scholars like Carter – diagrammatic proofs are a subject also for semiotics.¹⁹

4. Conclusions

In this chapter, I tried to build a narrative about how the views on diagrammatic proofs have developed in the philosophy of mathematical practice, starting from the first pioneering works that appeared at the beginning of the 2000s and getting to the most recent publications. I have shown that the interest in diagrammatic proofs was promoted by renewed attention to topics that were

¹⁸ It would be interesting to discuss the relation between the process of internalization as mentioned above by Rabouin and the return to diagrams in a very different formal and formally controlled context.

¹⁹ This latter research direction is developed in the Section of this Handbook on the *Semiology of Mathematics*. See for example Waszek (this volume) that, as others papers in the same section, partially overlaps with some of the topics of the present chapter.

excluded from the philosophical research by the so-called received view of mathematics. The question about the role of diagrams in proofs is specifically philosophical despite the fact that it relates to other more general questions about the effectiveness of diagrams and their omnipresence in mathematics.

The research on diagrammatic proofs is characterized by two waves. The first wave focused on case studies, in particular from Greek geometry and more recently from contemporary mathematics. The second wave, to which we assist today, is instead aimed to build comprehensive frameworks on how to account for diagrammatic proofs. In my view, this wave is characterized by two strategies. First, it is possible to stretch the notion of proof as inherited from the received view and to consider alternative models for it, which would allow including diagrammatic representations in proofs. Second, it is desirable to focus on the term “diagram” and to focus on its cognitive features or to distinguish several taxa for mathematical representations. For both strategies, it is evident that the research on diagrammatic reasoning has to widen its scope, either by relating to broader contexts – such as the reflection on formal and informal proofs and rigor in mathematics – or by opening up to new research domains, such as the consideration of the role of cognitive tools for thought and the philosophical study of notation in mathematics, which are both crucial issues for education and cognitive science. This sums up how the interest in diagrammatic proofs started – almost 20 years ago – and where it is going today.

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