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The practice of mathematics: cognitive resources and conceptual content

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1. The philosophy of mathematical practice: what is it philosophy of?

In the past ten years, contemporary philosophy of mathematics has seen the development of a trend that conceives mathematics as first and foremost a human activity and in particular a kind of practice, thus complementing other more standard approaches that insist on the nature of mathematics as knowledge (Carter, 2019). However, only recently the need for a general framework to account for the target of the so-called philosophy of mathematical practice (henceforth PMP) has emerged: What exactly should PMP be philosophy of? How should the notion of “mathematical practice” be intended? And can this approach be reconciled with more standard philosophical approaches to mathematics?

In truth, it is difficult to give a unified account of the so-called “practical turn” in philosophy of mathematics; such a program has unfolded relatively recently and was initially motivated by a urge to renew classical topics beyond the focus on mathematics as knowledge characterizing the philosophy of mathematics of the beginning of the 20th century. PMP grew out of a reaction to the “received view” of mathematics as the “exact science”; since then, the research exploring the more practical aspects of mathematics has been flourishing and nonetheless no clear account of how the term “mathematical practice” should be intended has been provided yet. PMP is indeed characterized by a pluralism of approaches and methodologies, in an interdisciplinary fashion¹.

Elsewhere, I mentioned four possible ways to reply to the question about the target of PMP (Giardino, 2017): according to the *situated* reply, the practice of mathematics is a human activity that has to be intended as historically situated; the *semiotic* reply focuses instead on the use in mathematics of several texts in most cases multi-modal, that is, including not only linguistic elements but also figures, sketches, diagrams and so on and so forth; the *epistemological* reply considers that

¹This emerges from the profiles of the members of the Association for the Philosophy of Mathematical Practice (APMP) created in 2009. For reference, see the APMP website: <http://www.philmathpractice.org>.

the main feature of the mathematical practice is the construction of theories, which do not have to correspond to formal systems; finally, the *pragmatist* reply looks at mathematical practice as the set of activities characterizing a community of mathematicians working from within particular theoretical and symbolic frameworks and having specific cognitive abilities, which gives to this view a “naturalistic” flavor (Ferreiros, 2016).

The purpose of the present article is to make progress towards the definition of a more precise general framework for PMP by exploring two strategies, starting from the assumption that philosophy of mathematics should expand beyond its boundaries into other regions of philosophy. A first strategy will be to take the situated and the semiotic replies seriously and turn to philosophy of mind in order to better understand the cognitive issues at play when considering a community of mathematicians; a second strategy will be to look into the pragmatist reply and refer to philosophy of language, by focusing on mathematical conceptual content.

I mention here another possibility, which would be to deepen the epistemological reply and consider the philosophy of the *scientific* practice; as it is well known, philosophy of science has indeed taken its practical turn already in the 70s, much earlier than philosophy of mathematics (Soler *et al.* (eds), 2014): Is there something that the philosopher of the mathematical practice can learn from the studies on the practice of science? What is their target? However, I will not consider here this strategy because of what I define the “practical turn *impasse*”. In fact, a similar tension is found in both studies on the practice of science and mathematics: if on the one hand philosophy should offer a general discourse about science or mathematics to account for their practice, on the other hand analyses of specific cases, no matter how deep, remain too local and its conclusions cannot be generalized over science or mathematics as global human enterprises. “Philosophy of” science or mathematics simply collapses into “science” or “mathematics”, as for example in some forms of Quine-inspired naturalisms according to which mathematics is the sole arbitrator of itself (see for reference Maddy, 1997). The focus on some scientific (or mathematical) practice from the past would instead replace philosophy with history; similar considerations arise for studies that look at the work of scientists (or mathematicians) in the lab, where philosophy risks collapsing into sociology of science (or mathematics). For these reasons, despite the fact that it seems very natural to rely on philosophy of science when talking about mathematics – mathematics is indeed a scientific discipline – I will exclude this strategy as too risky to be pursued. My aim will be to define the target of PMP in such a way that the first “P” of the acronym will not be dropped.

In Section 2, I will introduce some notions from philosophy of mind and cognitive anthropology, in particular the view the cognition is *distributed*; in Section 3, I will consider philosophy of language, that is, a possible extension to the practice of mathematics of Robert Brandom’s notion of *conceptual content*; in Section 4, I will hint at a general framework that combines the best features of both strategies; finally, in Section 5, I will draw some conclusions.

2. The practice of mathematics as distributed cognition

The first strategy to think in terms of a general framework for PMP will be to refer to both the situated and the semiotic replies above: mathematics is to be considered as a human *symbolic* activity, which is *distributed* both among the community of its practitioners and the texts and more in general the material tools that they rely on to do mathematics.

2.1 Mathematics in the wild

In the 90s, the cognitive anthropologist Edwin Hutchins famously devoted a book to the exploration of a case of cognition “in the wild”; in his words, cognition in the wild refers to “human cognition in its natural habitat – that is, to naturally occurring culturally constituted human activity” (Hutchins 1995, pp. xiii). Hutchins’s idea is to use this metaphor to approach thinking in an ecological framework, where human cognition interacts with an environment already equipped with many organizing resources. His case-study at the time was the very particular context of the functioning of a U.S. Navy ship; the aim was to show that when thinking of cognition as situated, it is not only a matter of the naturally situated cognition of the individual, since cognition is also *socially* distributed: a group of people, for example the crew of a ship, depending on the way they are organized and the environment and the tools they share, will be characterized by specific cognitive properties that are not predictable from the knowledge of the cognitive properties of each individual considered in isolation.

It is interesting to note that Hutchins thinks in terms of distributed cognition as the true motivation for the Cognitive revolution of the 50s: in his view, the aim of this revolution was to imagine a cognitive model for the individual manipulating symbols – and other cognitive resources in their environment – with his or her hands and eyes: “the mathematician or logician was visually and manually interacting with a *material* world” (Hutchins, 1995, p. 361). In Hutchins’ view, Alan Turing’s merit, which is not to be diminished, is that he realized that the embodied actions of the mathematicians as well as the environment in which they act could be *idealized* and *abstracted away* so that the agents could be in the end eliminated; the result was that the brain was removed and replaced with a computer, and “the surgery was a success; however, there was an apparently unintended side effect: the hands, the eyes, the ears, the nose, the mouth, and the emotions all fell away when the brain was replaced by a computer” (*ibidem*, p. 363). It is in fact important to note that Turing’s discovery did not concern the practice of the logicians or of the mathematicians *as it is*, but only its *computational* properties: the subsequent extension of human computational powers was of course a positive outcome, but this was not a model that was intended to describe a specific human activity. As Hutchins explains, “*the physical-symbol-system architecture is not a model of individual cognition. It is a model of the operation of a sociocultural system from which the human actor has been removed*” (*ibidem*).

What are then the characteristics of such sociocultural systems where individuals are organized in communities of practitioners and act together on the cognitive resources their environment provide them with?

2.2 Cultural cognitive ecosystems and stable practices

As discussed in the previous paragraph, Hutchins suggests that mathematics is a sociocultural and material culture; in more recent work, he introduces the notion of “cultural cognitive ecosystem”, where cognition is distributed among practitioners and their material resources (Hutchins 2013). This approach is a reaction to extended cognition views: according to the extended mind thesis, the mind is a container – this metaphor is fundamental – and there is only *one* cognition, which sometimes extends beyond the brain, some others not. However, distributed cognition is a broader perspective because it includes all aspects of cognition; in Hutchins’ view, any process that emerges from an interaction among elements in some system can be labeled as “cognitive”. It is important to note that this view does not imply any explicit stance on the *nature* of the mind, but chooses a specific way of looking at it. Moreover, for the extended mind thesis, the center of the cognitive system is the organism’s brain; for distributed cognition, it is an empirical question and not something to assume in advance whether a cognitive system has anything like a center or a boundary: instead of focusing on local examples of extensions of the mind, the real target should be cultural cognitive ecosystems within which human cognition is embedded.

A cultural cognitive ecosystem is thus a dynamical system where certain configurations of elements emerge preferentially by self-assembling, this becoming “stable practices”. Of course, there are constraints determining such practices, of three different sorts. The first level of constraints is merely *causal*, and amounts to neural mechanisms. The other two are *intentional* because they depend on the existence of a community of individuals: the second level of constraints is also *material*, because it can be implemented in a material tool, while the third is emergent in social processes of *collective intelligence*. As an example, Hutchins introduces the family of practices that he labeled as “*imagined-trajector-based*”: we see a line *as* a queue, and this is because a conceptual structure is mapped *onto* a physical array of people, in complex social and institutional contexts (see Figure 1); other practices belonging to the same family are the method of *loci*, fictive motion, numeracy and literacy, all exploiting the *linearity* of writing and reading.

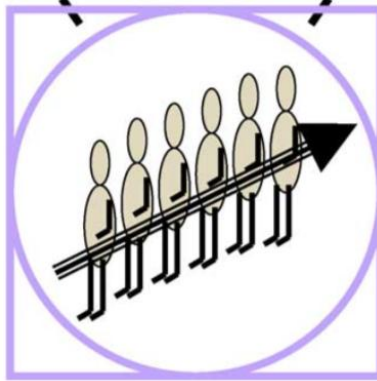


Figure 1. A conceptual linear structure as mapped *onto* a physical array of people

2.3 Applying distributed cognition to mathematics: pros and cons

Let us now take stock and consider the advantages of applying Hutchins' perspective to the practice of mathematics. First, assuming distributed cognition, a practice is intended as a set of organizing resources that a cultural cognitive ecosystem comes equipped with, and it is based on several levels of self-assembling along a spectrum going from very spontaneous cognitive capacities to the use of material cultural artifacts up to sophisticated social behaviors. Cognitive science research already shows that humans are born with a series of core knowledge systems that are ready to work in simple tasks such as comparing numerosities or discerning basic shapes (Carey & Spelke, 1996); this would be a first level of stable practice, based on neural assembling and depending on evolution. Then – as the semiotic reply argues for – the practice of mathematics implies the use of many different kinds of material tools – symbols of many different kinds, notations, diagrams; such tools are “material anchors”, in Hutchins' sense, that is, their material structure is intended to correspond to some conceptual structure, in such a way that performing some specific actions with them –constrained by the practice – would amount to carry out some operations on the conceptual models. Finally, another level of practices that are found in mathematics might have a more social nature; for example, the notion of “universal audience” in the background of the work of mathematicians (Ashton, 2020) or axiomatics as a “social institution” (Cantù, manuscript) can be intended as such third level-practices.

There are nonetheless two important limits to take into account in applying this framework to the practice of mathematics. First, Hutchins' view has the vocation of being merely *descriptive*, and does not aim to give any necessary and sufficient definition of what a cultural ecosystem is: cognitive anthropology may simply recognize something as a particular configuration emerging from self-assembling, call it a stable practice – possibly related to mathematics – and study it; however, no general characterization of a practice as mathematical will be given. Nonetheless, a mathematical practice should have a normative component: some material or social actions are of course possible from within a particular context, and they might correspond to some operations at a conceptual level, but norms should be given on

how to recognize a subset of these actions as allowed and the corresponding operations as correct. Second, assuming that the environment comes equipped with several resources that might constitute anchors for conceptual models,: How are these conceptual models constructed? Do they evolve in time? Can they have more than one anchor corresponding to them? Would this have any consequences on the way they are interpreted and used?

3. The practice of mathematics as building conceptual content

The second strategy that I will explore is a look at philosophy of language; in particular, following the pragmatist reply, I will consider the philosophy of Robert Brandom. There are two main reasons for this choice, in reaction to the limitations that were found in the distributed cognition approach to mathematics. First, Brandom's view presents an all-encompassing framework to talk about mathematical practice, which is *inferentialism*; second, *normativity* is at the core of inferentialism, which is defined on the basis of entitlements and commitments. In the rest of the section, I will consider in turn the features of this view that reveal themselves as promising in providing an account of a mathematical practice.

3.1 Inferentialism and the interplay of practices

It is important to clarify that Brandom's target is *linguistic* practice, whereas the practice of mathematics goes beyond language, as distributed cognition argues for. However, Brandom's main interest is conceptual content, which is important not only for linguistic but also for mathematical practice. According to the philosopher, conceptual content is determined by our *inferential* skills: when we hear or form a linguistic sentence, such a sentence has a meaning expressing the content of our thoughts or of those of our interlocutor. What does this amount to? In Brandom's view, such content is determined by our *inferential* abilities and by our *social* practices as based on such skills. For this reason, he endorses a form of semantic *inferentialism*.

Imagine one says: "This ball is red". By saying this, one is doing much more than simply uttering some sounds; in fact, what one makes is a potential *move* in what Brandom defines as the *game of giving and asking for reasons*. This is a form of pragmatism: language is acknowledged as *action*, and more importantly an action that is embedded in a larger context where interlocutors aim to make sense of what happens around them. Therefore, by uttering the previous sentence, one adopts a particular stance, related to the other claims that one will be *entitled* to endorse and *commit* to. To clarify, "This ball is red" engages oneself towards its consequences, for example that the object that one perceives is indeed colored; moreover, if one says that the object is red, then the object is *not* green; finally, to say that the object one perceives is scarlet would end up having the same entitlements. To sum up, "That's red": (i) *entails* "That's colored"; (ii) is *incompatible* with "That's green"; (iii) *follows from* "That's scarlet". Brandom thus proposes a semantic *holism* for conceptual

content, according to which “one cannot have any concepts unless one has many concepts. For the content of each concept is articulated by its inferential relations to other concepts. Concepts, then, must come in packages (though it does not yet follow that they must come in just one great big one).” (Brandom 2000, p. 15). A claim corresponds to an action, that is, a move in the game of giving and asking for reasons, because no claim can be considered in isolation from other claims related to it by a system of entitlements and commitments.

Inferentialism and semantic holism seem very pertinent also to define mathematical content. In order to argue for this point, I will refer to one passage from Ferreiros’ work on the interplay of practices in mathematics. In his book, Ferreiros defends the idea that modern mathematics, which is constituted by number systems, geometries, algebra, analysis and basic set theory, cannot be regarded as *a priori* or as a body of necessary truths, but “as a body of knowledge strongly interconnected, whose branches are linked systematically among themselves and ultimately with elementary mathematics” (Ferreiros 2016, p. 310). The conceptual content of mathematics resides in this interplay of practice, which seems to imply that considering one part of the edifice without the others would put its stability in danger. This is also coherent with the practice: when one makes some mathematical claim, there will be consequences one will be entitled to, commitments to some other incompatibilities, and finally, one knows similar entitlements concerning other statements – and when one is not sure about them, he or she can look for them. Moreover, the game of giving and asking for reasons, which might sound too restrictive for ordinary linguistic practice, seems instead very appropriate to the practice of mathematics intended as a human practice. The activity of proving seems in fact to be related not only to simple deduction from axioms but to *understanding*, that is, to finding reasons for some particular mathematical claim.

There is also another advantage that comes with inferentialism as related to the practice of mathematics: according to Brandom, *inference* – in the sense defined above – is privileged over *reference*. On this point, he mentions the work of the young Frege who suggested as well that it is inference and not truth that is primary in the order of explanation: formal inferences are *derivative* of material inferences, that is – in this meaning of “material” – inferences are endorsed or not depending on the conceptual content of their premises and conclusion and not on their formal properties. As an example, consider the following inferences: “Pittsburgh is to the west of Princeton” *then* “Princeton is to the east of Pittsburgh”; “Lightning is seen now” *then* “Thunder will be heard soon.” According to Brandom, first come materially correct inferences, and it is on their basis that the notion of formally valid inferences can be defined in a natural way. Would it then be possible to go from formally valid inferences to materially correct ones? Not really. A formal inference is such that some vocabulary is assumed in advance as privileged, and it can be treated as good in virtue of its form with respect to that vocabulary; however, it will be valid only if it is materially good from the start, so that it would not be possible to change it into a materially bad one by substituting *non-privileged* vocabulary for *non-privileged* one. Moreover, there is nothing special about *logical* form. One might for

example be interested in the *aesthetical* form of the inference, and assume aesthetic vocabulary as privileged. Then, it suffices to check for which substitutions of *non-privileged* vocabulary for *non-privileged* one the aesthetic form is still good. Therefore, formal validity is explained in terms of material correctness: logic does not give some special epistemic access to truth but has the *expressive* role of codifying *explicitly* conceptual content; its task is “in the first instance to help us *say* something about the conceptual contents expressed by the use of nonlogical vocabulary, not to *prove* something about the conceptual contents expressed by the use of logical vocabulary.” (Brandom, 2000, p. 30). The focus here is *expression* rather than on *representation*, and the view moves against the standard *representational* paradigm based on truth.

If applied to mathematics, this view has two interesting consequences. First, it argues against the *formalist* paradigm in the foundation of mathematics. As nicely put by Lakatos, “one of the most dangerous vagaries of formalist philosophy is the habit of (1) stating something – rightly – about formal systems; (2) then saying that this applies to ‘mathematics’ – this is again right if we accept the identification of mathematics and formal systems; (3) subsequently, with a surreptitious shift in meaning, using the term ‘mathematics’ in the ordinary sense.” (Lakatos, 1976, p. 4, footnote 1). If the focus is on material inference, in the sense defined above, then a formal proof is at most a useful tool to codify explicitly its conceptual content. Second, if the focus is on inference rather than on reference, then what matters is the relevant inferences and how they are intertwined and not the properties of the represented objects; by claiming something, we evoke, thanks to a system of entitlements and commitments, a list of other mathematical statements without strictly speaking endorsing the existence of the objects they refer to; this move would give us some leverage against Platonism and all the difficulties it brings about.

3.2 Mathematical reasoning as a kind of *knowing how*

Another element of Brandom’s view that might be of interest to define the practice of mathematics is the focus on *action*: concepts are *used* in order to *reason*, which goes back to both the situated and the pragmatic reply above. In Brandom’s view, semantics must answer to pragmatics, that is, the contents of conceptually explicit propositions or principles are addressed from the direction of what is *implicit* in practices of using expressions and acquiring and deploying beliefs; meaning and content have thus a role in reasoning: one *knows how* to use them in the game of giving and asking for reasons.

This view of conceptual competence as a kind of *knowing how* is in line once again with Ferreiros’ approach: as he explains, “much of the learning of any central mathematical concept has to do with learning to use it in the context of some particular subdiscipline or branch of mathematics” (Ferreiros, 2016, p. 297). This is not surprising since mathematics can be intended as first of all a *reasoning* practice, based on inferences and therefore on the activation of a network of entitlements and commitments starting from on statement, as a move in the game of giving and asking for reasons. Grasping a concept is mastering its inferential use, which means

“knowing (in the practical sense of being able to distinguish, a kind of knowing *how*) what else one would be committing oneself to by applying the concept, what would entitle one to do so, and what would preclude such entitlement.” (Brandom 2000, p. 11). Entitlements and commitments add a *normative* element to the picture, but the important point is that norms are not given explicitly but are determined by what we are responsible for and entitled to. Here the social element is crucial, but with no risk of making philosophy collapse into sociology. Norms come in a variety; for example, in a list inspired by Davidson, Brandom talks of *prudential* (or *instrumental*) norms, as in “Only opening my umbrella will keep me dry, so I shall open my umbrella”, *institutional* norms, as in “I am a bank employee going to work, so I shall wear a necktie”, or *unconditional* norms, as in “Repeating the gossip would harm someone, to no purpose, so I shall not repeat the gossip” (Brandom 2000, p. 100). Of course, it is debatable what kinds of norms are at play in the practice of mathematics. In any case, this kind of knowing how allows going beyond Hutchins’ distributed cognition, since it is not only descriptive but provides standard of correctness, without at same time neglecting the importance of actions. From this point of view, we are back to the epistemological reply that considers mathematics as knowledge, but in a renewed context. Mathematics is a normative practice, where norms are not explicitly stated but are implicitly at work in distinguishing which of the practitioners’ actions are legitimate or permissible.

An interesting consequence of this view is the way in which objectivity is intended, without objects. Normative statuses – entitlements and commitments – define meaningful moves in the game of giving and asking for reasons, in an ongoing articulation that does not simply correspond to an agreement of perspectives, despite the fact that these status are understood as *social* statuses, that is, as the production of individual and communal attitudes. As Brandom sums up, the contents of the declarative sentences that are used in this game can swing “free of the attitudes of the linguistic practitioners who deploy them in assertions”, and be considered as truth or false (Brandom 2000, p. 188).

3.3 Applying inferentialism to mathematics: pros and cons

Let us take stock and consider the advantages of applying Brandom’s view to the practice of mathematics. First, assuming inferentialism, mathematical practice is intended as a form of reasoning based on the conceptual content of the claims that are put forward, in a normative framework where one concept cannot be considered in isolation from the others but is embedded in a network of entailments and commitments. This has some nice consequences: first, formal justification can be seen as a way of making explicit implicit conceptual content involved in a claim, but it not necessary to prove that claim; second, the focus is on expression and not representation, thus avoiding both the primacy of truth and the risk of assuming Platonism; third, standard of correctness are defined by norms that are implicit in the actions of the practitioners, without implying any sort of social constructivism.

There are nonetheless two crucial limits to consider. The first is Brandom’s *anti-naturalism*, which is in opposition to the spirit of the pragmatic reply, to Hutchins’

view of cognition as well as to Ferreiros' insistence on cognitive constraints. Conceptual norms are the products of social interactions: that does not mean that they are *supernatural* but they are not studied by the natural sciences. Brandom explains this point by comparing a thermostat and a parrot to the case of someone uttering "This is red". Of course, the thermostat might turn a furnace on when the temperature drop below some particular value, and a parrot can be trained to say "This is red" in presence of red things; in all cases, a particular stimuli will be classified according to a general category, and this response will be repeatable. However, according to Brandom, it is only in the case of a speaker who reports "This is red" that the classification is specifically *conceptual*; this distinguishes a practice where a concept is used from a merely responsive classification. Here Brandom is in strong contrast with Hutchins, who, as discussed above, has a very broad idea of cognition as whatever emerges from a self-assembling configuration; however, a response triggered in some neural association will for sure not be considered by Brandom as a move in the game of giving and asking for reasons, while other more sophisticated practices might, as in the concept of "queue". The case of material anchors is more difficult to discuss, and I will go back to this issue in the next section. The second limit of applying Brandom to mathematics is his focus on *propositionality*. His inferentialism is a propositionalism: the fundamental form of the conceptual is in his view *propositional*, and the practices using concepts are *discursive*, despite the fact that they have in the background various other kinds of skillful doing. Formal correctness depends on material correctness, that is the case; however, at the center of the scene there are propositions, as moves in the game of giving and asking for reasons are determined by a practice that is based on language use.

4. Enhanced material inferentialism

In the sections above, I presented two possible strategies to account for the practice of mathematics as based on distributed cognition and inferentialism. The first view gives a promising picture of *stable practices* as emerging from *cognitive and social configurations*; however, it is mainly descriptive and does not give us instruments to explain what kind of specifically conceptual practice mathematics is, as subject to norms and standards of correctness. The second view provides instead a very interesting account of *conceptual content* as based on a network of *inferences*, but considers irrelevant the triggering of basic cognitive competences and the appeal to non-linguistic tools and possibly institutions. In this last section, I will discuss whether there is a way of reconciling these two positions and thus provide a more suitable account of the practice of mathematics. I will consider in particular Brandom's limitations, to see whether some of Hutchins's claims are useful to overcome them.

4.1 Against a "reductive" form of naturalism

Let us start with the first limit of Brandom's inferentialism, that is, his anti-naturalism. If Hutchins on the one hand seems to have a very broad view of cognition, by defining as cognitive any system where some self-assembling configurations emerge, Brandom considers that mere responsiveness has nothing to do with conceptual competence, as explained in the quotation above where a difference is made between a thermometer, a parrot and a genuine conceptual speaker. However, Brandom does seem to accept empiricism, even if only in a "platitudinous" version: perceptual practices still play a crucial role in our epistemology and semantics, for the reason that without perceptual experience there is no knowledge of contingent matters of fact. More deeply, in this view, conceptual content is unintelligible apart from its relation to perceptual experience. Nonetheless, as argued by Levine, Brandom differs from the pragmatic tradition in getting rid of the pragmatic conception of *experience* (Levine, 2012). On the one hand, Brandom is in fact in favor of the 'priority of the practical', that is, "the view that explicit beliefs and representations depend on and are somehow emergent from a background of implicit practical abilities" (Levine, 2012, p. 126): to know that something is the case amounts to *do* something, and is revealed in the practical mastery of being able to handle the inferences that can be made in the game of giving and asking for reasons. However, and unfortunately in our view, as Levine very well sums up, "his pragmatism remains *immanent to the realm of the discursive*. But this makes it impossible for him to understand a richer sense of practice that is often at work in the classical pragmatists, namely, the *bodily* practices, habits, and skills through which subjects inhabit and cope with the environment, physical and social" (Levine, 2012, p. 128). For many pragmatists, a subject engages with a physical and social environment and these engagements may already have sense before linguistic codifications come onto the scene. We "are-in-the-world", and our practices emerge from the world and us in it. Brandom insists that "experience" is not one of his words, for the reason that in his view this word is used to create intermediaries between perceptible facts and the linguistic reports for them that are elicited by reliable responsive dispositions – for example, the one of the parrot. From a causal point of view, there are of course such intermediaries, some of them neurophysiological – think of Hutchins first level – but "any of these has any particular conceptual or (therefore) cognitive or semantic significance" (Brandom, 2000, p. 206, footnote 7).² Beliefs that are acquired perceptually, or from memory and testimony, constitute knowledge only to the extent that the believer can offer reasons for his or her belief, by relying on linguistic claims.

In the rest of the section, I will argue that inferentialism is in the end compatible with a more liberal view of the practices that are recognized as having conceptual content. Consider again Hutchins' first level of self-assembling configurations, that is, neural associations. Let us side with Brandom and accept that despite being of interest because they give us information about the mechanisms that underline our mental life, they do not involve concepts. This is important also in relation to what

² In the same footnote, Brandom claims that McDowell (1994) with his conceptualism about perceptual experience presents the strongest argument against his view.

has been done in cognitive science in relation to mathematics: there exist some core knowledge systems that are apparently very spontaneous and allow us – as well as other non-human animals – comparing between different numerosities and discriminating among different shapes. It is perfectly fine to consider these the cognitive bases or foundations for mathematics and still believe that they do not constitute knowledge, as they relate to the mere responsiveness of our perceptual systems. This does not go against Hutchins' view, which is expressed in terms of "cognitive" systems and introduces the term "conceptual model" only starting from the second level of external resources. It is also important to note that such responses grew out of evolution, and they are always available, even once other more sophisticated competences have been developed on top of them.

Let us also agree with Brandom that with language we have conceptual content, but disagree that conceptual knowledge reserved to language and language only. He identifies language as a cultural practice that is crucial precisely because it allows overcoming our nature of merely responsive animals; however, there are cases in which our physical and social environments whose contours are defined by our culture "make sense" and have conceptual content, without involving strictly speaking language: material anchors. This is in line with what other pragmatists such as C. S. Peirce argued for. Moreover, material anchors are still objects that are perceived and on which we act; for this reason, the spontaneous cognitive systems that are at play at the first level may continue having an influence in this new specifically cultural cognitive context.

The view that I am sketching here is a form of naturalism to some extent, but not a "radical" or "reductive" one: it is in fact not possible to claim that spontaneous competences already constitute mathematical knowledge and nonetheless mathematical knowledge might be modulated by the cognitive systems that are triggered by our experience in our environment, both physical and social. That is why instead of thinking in terms of a discursive practice only, I will propose to think in terms of Hutchins' second and third level of "artifacts" emerging from our implicit everyday practices: material tools are used as anchors to reason about conceptual models, and institutions may direct our choices in view of some specific cognitive task. This is particularly important in the case of mathematics, whose practice is constellated by many cognitive tools of different sorts (see Giardino, 2018), and evolves in a community expressing particular epistemic values along the norms and customs of the several available institutions.

In the next paragraph, I will explore the possibility of including discursive practice among a broader category of conceptual reasoning practices incorporating also other formats.

4.2 Claims and manipulations as (inferential) actions

Inferentialism as defined by Brandom is based on propositions, which are at the core of our linguistic reasoning practice. However, in approaches such as distributed cognition, a cognitive task can be solved also thanks to the familiarity with material

objects that serve as “anchors” for some particular conceptual models, and that can be manipulated and changed based on interpretation so as to apply some operations on the conceptual model. These tools are omnipresent in mathematics, and the rules to work with them are in most cases not explicitly stated but implicitly shown in the experts’ behavior.

In a more recent passage, Brandom gives the example of a non-linguistic sign, and makes the following remark. He imagines his young son in a park, seeing a sign where the silhouette of a dog – a Scottish terrier – is depicted. A red circle with a slash through it surrounds the silhouette. Brandom says that his son is “familiar with the force of prohibition associated with signs of this general form” (Brandom 2009); however, the boy is still not sure: Would that mean that no Scottish terriers are allowed in the park? Or simply that no dogs are allowed? The sign might even mean that no animals at all are allowed in the park. The point is that according to Brandom it is only by referring to concepts that his son will free himself “from the bonds of the here- and-now”; to use a concept, he will have to make some claim and therefore a move in the game, for example by justifying his own claim or the others’ claim about that sign. This particular game – the game of giving and asking for reasons – is “not a par with other “games” one can play but is precisely *“what in the first place make possible talking, and therefore thinking: sapience in general.”* (Brandom 2000, p. 15, *emphasis mine*). The application of a concept is a node in a network of such broadly inferential involvements; propositionally contentful commitments can both serve as and stand in need of reasons.

However, consider the case of a move in the game in mathematics in analogy to the one presented above. Let us imagine I say: “This figure is a triangle”. If I do that, what I make is indeed a potential move in a game of giving and asking for reasons, by adopting a stance that: (i) involves further consequential commitments – for instance, if I claim that the figure is a triangle, then I also claim that the object perceived is a polygon; (ii) is incompatible with other commitments – for instance, the object perceived is not a square; (iii) I can show one’s entitlements in terms of other commitments (for instance, to the object perceived as being an equilateral triangle). Of course, in order to be able to make this move I have to be a practitioner; the level of expertise I need to use a concept in the appropriate way in relation to a figure depends of course by the concept in question, that can be familiar also to novices – as in the case of “triangle”. But one does not have to stop here. Moves in the game can be made also by applying an action on social objects other than claims, for example a figure; what is important is the inferential articulation that a modification of them would make available, in a system of entitlements and commitments analogous to that for propositions. If a practitioner makes this move and therefore shows he or she possesses the concept of triangle *then* – and this is a crucial point to give an account of mathematics – he or she can make other moves as well, and some of them will not be directly linguistic, but will be based on a modification of the original figure.

Consider the Pythagorean theorem. Once a practitioner has the concept of right triangle and of square in the sense defined by inferentialism, he or she has the conceptual resources to construct the figure corresponding to the Pythagorean theorem and consequently to apply some modifications on it, for example as in the proof proposed by Euclid in the *Elements* (Proposition I,47), where lines are added and new figures have to be recognized. The point is that inferentialism, when it comes to mathematics, does not have to be restricted to the use that one can make of linguistic claims precisely to disambiguate the meaning of some visual or spatial tools, as in the case of the park sign; my proposal is to think in terms of inferential competences that are at play also when actions are applied to a cognitive tool such as the figure of a triangle. This is not far from some of Hutchins' observations. When he presents the example of the line, he claims that seeing it *as a queue* is a cognitive practice because "it makes possible a set of inferences. Who is next in line? Who arrived before whom? How far am I in space from (and how long must I wait before) getting service?" (Hutchins 2013, p. 6). Why would this be different from the use an expert makes of tools such as a figure or more in general a symbol? Actions on such tools can be *inferential*: they make emerge new properties and new conceptual content, in the sense defined above; this is an implicit practice that might in some cases need to be disambiguated by making a linguistic claim. Think again of the inferential "moves" Euclid makes in his proof of the Pythagorean theorem in Proposition I, 47: starting from the original figure of a right triangle where the squares on its sides have been drawn, a first line is added (CL), then other lines (CE and CD) and others again (BF and AK), so as to turn to the original figure into what is commonly called "the Windmill" because of its shape (see Figure 2). Each of these steps is an action that is made in order to create new conceptual content thus making new inferences available.

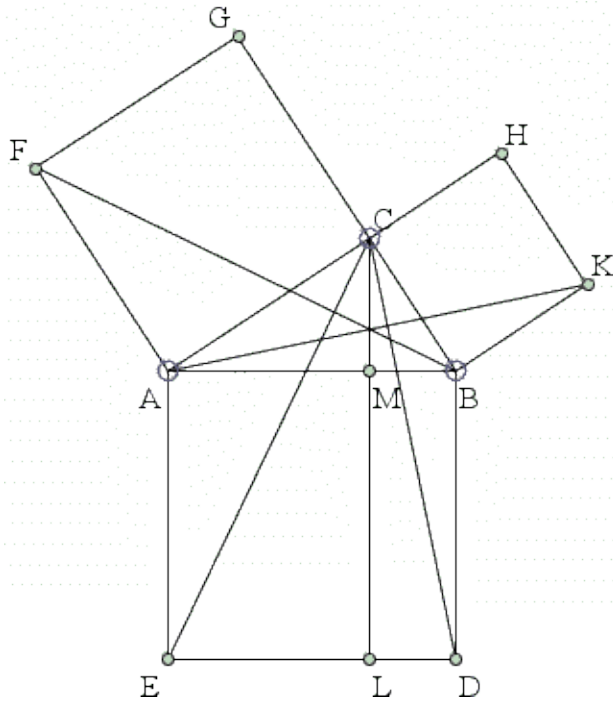


Figure 2. Euclid's "windmill" (Proposition I, 47)

Therefore, an action on a figure can be a move in the game of giving and asking for reasons, precisely because it is inferential. And this also might solve the *vexata quaestio* of how a geometrical figure that is inherently particular – in this case, representing a particular right triangle – can be considered as general. The primacy is in fact, also in this format, of the expression, that is, of the inferences that become available, and not of the reference. Precisely as in the case of a purely linguistic practice, that is, making use of linguistic claims as moves, here also there are norms at play, depending on the practice of Euclidean geometry, which is related to a network of both interrelated statements and interrelated manipulations as entitlements and commitments that can be applied to the figures.³

This explains for example why it is not possible to really talk of proofs based on figures as "proofs without words" (see for reference Nelsen, 1993) since every non linguistic proof starts from a figure that refers to some statement, which supports the acknowledging of the conceptual content of the figure; then, manipulations can indeed maintain a degree of independence from language as actions making some inferences available; in the conclusion, we might end up with another linguistic claim. After having gone through all the relevant steps, in this network of entitlements and commitments, in some cases we might even obtain a genuine proof. Given this framework, a formal proof is not needed, unless for some – mathematical – reasons, one wants to make all the reasoning passages explicit. Therefore, the

³ It has to be noted here that this is another point where Brandom and the view I propose here move away from Hutchins' framework, which is representational.

application of a concept is a node in a network of such broadly inferential involvements; *propositionally* contentful commitments and *materially* contentful commitments can both serve as and stand in need of reasons.⁴ This is in the end in line with Brandom, that is, “an account of knowing (or believing, or saying) *that* such and such is the case in terms of knowing *how* (being able) to *do* something. It approaches the contents of conceptually *explicit* propositions or principles from the direction of what is *implicit* in practices of using expressions and acquiring and deploying beliefs.” (Brandom 2000, p. 4).

4.3 Inferential actions on material anchors

In the previous sections, I explored two strategies to reply to the question about the practice of mathematics. The first embraces a very broad perspective on cognition as being wherever a self-assembling system constitutes a stable practice, that is, an organizational source for our reasoning, in our biological, physical or social environment. The second assumes instead a very narrow perspective on knowledge and conceptual content as defined inferentially in discursive practice.

However, I proposed a possible reconciliation of these two views into a third view that I will call *enhanced material inferentialism*. On the one hand, as discussed above, Brandom inferentialism is material: any formalism is secondary to the material nature of inferences and will at most be a useful tool to make the implicit reasoning practices explicit; in his view, logic may have this expressive task but is not needed as a mean of proof. On the other hand, Hutchins talks in terms of “material” anchors, using of course the same word with another meaning, as related to the materiality of the cognitive tools that despite being concrete objects can be used, changed, modified, and manipulated in order to extend the network of inferences that are activated by them.⁵ I will therefore exploit the ambiguity of the term “material” in this label, to define a form of inferentialism *à la* Brandom that integrates elements from Hutchins’ perspective.

According to enhanced material inferentialism as applied to the practice of mathematics:

1. Mathematical conceptual content is given by entitlements and commitments that are at work when an expert makes a particular action in the game of giving and asking for reasons
2. Mathematical actions can be of different sorts: a practitioner can (i) make a statement; (ii) recognize some particular property of a material mathematical cognitive tool, for example a figure or a notation – or that is used; (iii) apply some particular modification to a tool; in all these cases, the

⁴ Note that here I deliberately use “materially” and not “visually”, because I want to focus on the *dynamical* features of the use of these tools and the *invariances* that emerge and not so much on their visual properties, that are in general subject to irrelevant variations (at the exception of tools that are designed with visual features conveying relevant meaning, as for example in Byrne 1847).

⁵ See Derry 2017 for a discussion of inferentialism in mathematics education.

- practitioner adopts a particular stance, related to the other actions that he or she will be entitled to endorse and commit to;
3. All actions have an expressive nature, that is, they are inscribed in this network of commitments and entitlements;

The framework is based on entitlements and commitments related to the inferential competence that is subscribed by a concept, and therefore this is a normative practice; at the same time, it goes beyond language use only and extends itself towards the consideration of other kinds of cognitive material social tools.

5. Conclusions

In recent years, many studies have been devoted to the practice of mathematics. However, no general definition of what a mathematical practice is has been given up to today and it is time to work towards more comprehensive accounts.

In the present paper, I addressed two possible strategies. First, I considered the claim that mathematics is a form of distributed cognition, where stable practices emerge from self-assembling, in a spectrum going from more basic and spontaneous competences, to more sophisticated social or institutional practices. Second, I presented Brandom's inferentialism and discuss the interest and the limits in applying it to the practice of mathematics. Finally, I proposed a possible combination of the two views that I call enhanced material inferentialism, according to which the practice of mathematics is first and foremost an inferential reasoning practice, involving different – linguistic but also material – cognitive tools in order to make moves in the game of giving and asking for reasons.

References

- Z. Ashton (2020), "Audience role in mathematical proof development", *Synthese* 198 (Suppl 26): 6251-6275.
- R. Brandom (2010), "How Analytic Philosophy Has Failed Cognitive Science", *Critique and Humanism* 31 (1), pp. 151-174.
– (2000), *Articulating Reasons: An Introduction to Inferentialism*, Harvard University Press.
- O. Byrne (1847), *The Elements of Euclid*, William Pickering, London.
- P. Cantù (manuscript), 'Philosophy of Axiomatics'.
- S. Carey, S. and E. S. Spelke (1996), Science and core knowledge. *Philosophy of Science*, 63(4): 515–533.
- J. Carter (2019), Philosophy of Mathematical Practice — Motivations, Themes and Prospects, *Philosophia Mathematica* 27 (1): 1-32.
- J. Derry (2017), An introduction to inferentialism in mathematics education, *Mathematics Education Research Journal* 29, 403–418 .

J. Ferreiros (2016), *Mathematical Knowledge and the Interplay of Practices*, Princeton University Press.

V. Giardino (2018), 'Tools for thought: the case of mathematics', *Endeavour* 42, 172 - 179.

- (2017), 'The Practical Turn in Philosophy of Mathematics : A Portrait of a Young Discipline', *Phenomenology and Mind*, 12, pp. 18 - 28.

E. Hutchins (2013), 'The cultural ecosystem of human cognition', *Philosophical Psychology* 27(1), 34 - 49.

- (1995), *Cognition in the Wild*, ???

I. Lakatos (1976), *Proofs and Refutations*, Cambridge University Press, Cambridge.

S. Levine (2012), Brandom's Pragmatism, *Transactions of the Charles S. Peirce Society*, 48 (2), 125- 140.

P. Maddy (1997). *Naturalism in Mathematics*, Oxford: Oxford University Press.

R. B. Nelsen (1997), *Proofs without Words: Exercises in Visual Thinking*, Mathematical Association of America.

L. Soler, S. Zwart, M. Lynch & V. Israel-Jost (eds) (2014). *Science after the Practice Turn in the Philosophy, History, and Social Studies of Science*. New York-London: Routledge.