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# Recursive identification with regularization and on-line hyperparameters estimation

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**Abstract:** This paper presents a regularized recursive identification algorithm with simultaneous on-line estimation of both the model parameters and the algorithms hyperparameters. A new kernel is proposed to facilitate the algorithm development. The performance of this novel scheme is compared with that of the recursive least-squares algorithm in simulation.

*Keywords:* Recursive identification, Regularized identification, Bayesian identification.

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## 1. INTRODUCTION

Recursive identification is unavoidable whenever one needs a model estimation performed on-line (real-time). The principle dates back to (Whitacker (1959)) and is almost as old as the concept of identification itself. Recursive identification algorithms have been designed for the three classes of classical identification schemes: Prediction Error Methods (PEM) (Ljung and Söderström (1983)), Instrumental Variable (Young (2011)), and Pseudo-Linear Regression (PLR) (Landau et al. (2011)) (for which some improvements have been made recently (Vau and Bourlès (2021))). For several decades since the famous paper of (Aström and Bohlin (1965)), identification has been dominated, directly or indirectly, by the Maximum Likelihood (ML) approach, and this is also the case for recursive schemes. On the contrary, Bayesian concepts consisting in introducing a prior knowledge in estimation, did not receive a significant attention, in spite of earlier attempts to estimate dynamical systems (Leamer (1972)), (Akaike (1979)).

It is only since the beginning of the 2010s, that the Bayesian approach has attracted a considerable interest in the identification field with the emergence of kernel-based identification schemes, coming from Machine Learning (Pillonetto and de Nicolau (2010)). By applying some regularization techniques, namely modifying a least squares criterion, the bias-variance trade-off can be significantly improved, especially if the initial problem is ill conditioned (Pillonetto et al. (2014)). The most significant kernels embedding a prior knowledge of intrinsic systems properties (such as stability and smoothness of the impulse response) are the Stable-Spline (SS) Kernel (Pillonetto and de Nicolau (2010)), the Diagonal Correlated (DC) and the Tuned-Correlated (TC) kernels (Chen et al. (2012)). These kernels are tuned with a number of hyperparameters whose values can be determined beforehand from data, using a Hierarchical Bayes approach (see Pillonetto et al. (2022) p. 107). So far, to the authors knowledge, this kernel approach has been employed in a recursive context in (Prando et al. (2016)), (Romeres et al. (2016)) and (Illg

et al. (2022)).

In the sequel one presents a novel recursive and regularized identification scheme for the estimation of the impulse response of a discrete Linear Time Invariant (LTI) system, and where the model is a Finite Impulse Response (FIR) filter. The proposed scheme is more easily implementable online than the one disclosed in (Romeres et al. (2016)). Like in this last reference it is assumed that the prior is not known at initialization, which requires to estimate online the prior hyperparameters at the same time as the parameters. In general, in regularized schemes, the number of Kernel hyperparameters is reduced (see Pillonetto et al. (2022)). Here, in order to make the algorithm implementable on line, one proposes unusually a prior with a number of hyperparameters equal to the number of parameters, the prior structure (smoothness of the impulse response and stability) being imposed by putting some constraints to those hyperparameters. This prior is finally very close to the TC prior (Chen et al. (2012)). The online estimation of the hyperparameters aims at minimizing the cost of the marginal likelihood criterion (Empirical Bayes approach) (Pillonetto et al. (2022), chap.4), by using a recursive gradient with projection, while the parameters are estimated with an algorithm quite similar to RLS.

The paper is structured as follows: Section 2 shows how the recursive least-squares can be implemented in case of regularized criterion, following a Bayesian approach, Section 3 proposes a prior structure inspired from the TC prior, Section 4 details the algorithm mixing on-line parameters and hyperparameters estimation. At last Section 5 displays some simulations showing the performances of this novel algorithm.

## 2. PROBLEM STATEMENT

### 2.1 Classical propagation equations for recursive estimation

Let us consider the “true” discrete-time system  $G_0(q)$ , which is assumed to be stable and proper, ( $q$  being the forward shift operator). The sequences  $\{u(t)\}$ ,  $\{y(t)\}$ ,  $\{e(t)\}$  correspond respectively to the input, output and

a centred gaussian white noise. This noise has a variance  $\sigma^2$  assumed to be known. One has

$$y(t) = G_0(q)u(t) + e(t) \quad (1)$$

The model is a finite impulse filter  $G$  with

$$G(q) = \sum_{k=1}^n b_k q^{-k} \quad (2)$$

Our purpose is to estimate the vector  $\theta$  where

$$\theta^T = [b_1 \ b_2 \ \dots \ b_n]$$

The regularized least-squares criterion with data up to time  $t$  denoted as  $J_0(t)$  is

$$J_0(t) = \sum_{i=1}^t \left( y(i) - \phi^T(i-1)\hat{\theta}(i) \right)^2 + \hat{\theta}^T(t)\sigma^2\Pi_0^{-1}\hat{\theta}(t), \quad (3)$$

$\hat{\theta}(t)$  being the estimation of  $\theta$  at time  $t$ . The positive definite matrix  $\Pi_0$  is the regularization matrix (corresponding to the kernel functions in a Machine Learning approach). By cancelling out the criterion gradient with respect to the vector  $\hat{\theta}$ , one finds immediately

$$\hat{\theta}(t) = \left( \sum_{i=0}^{t-1} \phi(i)\phi(i)^T + \sigma^2\Pi_0^{-1} \right)^{-1} \sum_{i=1}^t \phi(i-1)y(i) \quad (4)$$

Let us introduce the positive definite matrix  $F$  as

$$F^{-1}(t) = \sum_{i=0}^{t-1} \phi(i)\phi(i)^T + \sigma^2\Pi_0^{-1} \quad (5)$$

Its propagation equations are

$$F^{-1}(t+1) = F^{-1}(t) + \phi(t)\phi^T(t) \quad (6a)$$

$$F^{-1}(0) = \sigma^2\Pi_0^{-1} \quad (6b)$$

By using the matrix inversion lemma, one has classically (Landau et al. (2011), p.64)

$$F(t+1) = F(t) - \frac{F(t)\phi(t)\phi^T(t)F(t)}{1 + \phi^T(t)F(t)\phi(t)} \quad (7)$$

On the other hand, it is well known (see Landau et al. (2011), p.63) that the propagation of  $\hat{\theta}$  is given by

$$\hat{\theta}(t+1) = \hat{\theta}(t) + F(t)\phi(t) \frac{\varepsilon^o(t+1)}{1 + \phi^T(t)F(t)\phi(t)} \quad (8a)$$

$$\varepsilon^o(t+1) = y(t+1) - \phi^T(t)\hat{\theta}(t) \quad (8b)$$

In a Bayesian perspective, the prior knowledge is incorporated in  $\Pi_0$  (which corresponds to the covariance matrix of this prior). Therefore, in a recursive perspective, incorporating this prior is equivalent to specify  $F(0)$ . However, when one performs an identification in real-time, in general one cannot use some available data so as to specify this prior (by applying for example the Hierarchical Bayes approach) allowing to determine the optimal prior hyperparameters, by maximizing a marginal likelihood function. It is the reason why, we consider in this paper the issue of computing on-line the regularisation matrix  $\Pi(t)$  in parallel with the computation of  $\hat{\theta}(t)$ .

## 2.2 Case where the regularization matrix is no longer invariant

Let us assume that by a specific estimation algorithm (which will be detailed below), the regularization matrix  $\Pi$  is updated at each sample time, yielding  $\Pi(t)$ . We assume that  $\Pi$  is expressed from a vector of hyperparameters  $\eta$  which is no longer constant. From (5) and (6), one can write now

$$F^{-1}(0, \eta(0)) = \sigma^2\Pi^{-1}(0) \quad (9)$$

$$F^{-1}(t, \eta(t)) = \sum_{i=0}^{t-1} \phi(i)\phi(i)^T + \sigma^2\Pi^{-1}(t) \quad (10)$$

$$F^{-1}(t+1, \eta(t+1)) = \sum_{i=0}^t \phi(i)\phi(i)^T + \sigma^2\Pi^{-1}(t+1) \quad (11)$$

The propagation equations of  $F^{-1}$  becomes

$$F^{-1}(t+1, \eta(t)) = F^{-1}(t, \eta(t)) + \phi(t)\phi^T(t) \quad (12)$$

On the other hand, from (8), one has

$$\hat{\theta}(t+1, \eta(t)) = \hat{\theta}(t, \eta(t)) + \frac{F(t, \eta(t))\phi(t)\varepsilon^o(t+1)}{1 + \phi^T(t)F(t, \eta(t))\phi(t)} \quad (13)$$

## 2.3 Marginal Likelihood in a recursive context

Set

$$Y^T(t+1) = [y(1) \ y(2) \ \dots \ y(t+1)]$$

In the hierarchical Bayes approach, the vector of hyperparameters  $\eta$  is estimated from data at first, by maximizing the Marginal Likelihood function  $L(\eta|Y)$ . Afterwards the *maximum a posteriori (MAP) estimate*  $\hat{\theta} = \operatorname{argmax} \mathbf{p}(\theta|Y)$  is computed,  $\mathbf{p}$  being the probability density function (pdf).

In a recursive context, where data are obtained on-line, and available up to time  $t+1$ , the maximization of the Marginal Likelihood function  $L(\eta(t+1)|Y(t+1))$  and the computation of the MAP  $\hat{\theta}(t+1) = \operatorname{argmax} \mathbf{p}(\theta(t+1)|Y(t+1), \eta(t+1))$  must be computed alternatively, so as to take into account only available data at time  $t+1$ . For this purpose set

$$\Phi(t) = [\phi(0) \ \phi(1) \ \dots \ \phi(t)]^T$$

$$\Sigma(t+1) = \Phi(t)\Pi(t+1)\Phi^T(t) + \sigma^2 I_{t+1}$$

assuming that the prior expectation of  $\theta$  is null, and that  $\theta, \eta$  are normally distributed, the marginal LogLikelihood function  $\log L(\eta(t+1), Y(t+1))$  is (Pillonetto et al. (2022), p.108)

$$\begin{aligned} \log L(\eta(t+1)|Y(t+1)) &= -\frac{1}{2} \log(2\pi |\Sigma(t+1)|) \\ &\quad - \frac{1}{2} Y^T(t+1)\Sigma^{-1}(t+1)Y(t+1) \end{aligned} \quad (14)$$

One has the following result:

*Lemma 1.* The derivative of  $\log L(\eta(t+1)|Y(t+1))$  with respect to the  $k$  entry of  $\eta$  (denoted as  $\eta_k$ ) is

$$\begin{aligned} \frac{\partial \log(L(t+1))}{\partial \eta_k} &= \frac{1}{2} \operatorname{Tr} \left( [\sigma^{-2} (\Pi(t+1) - \right. \\ &\quad \left. - \hat{\theta}(t+1)\hat{\theta}^T(t+1)) - F(t+1)] \frac{\partial \sigma^2 \Pi^{-1}(t+1)}{\partial \eta_k} \right) \end{aligned} \quad (15)$$

Proof: See Appendix A.

### 3. PRIOR STRUCTURE AND HYPERPARAMETERS ESTIMATION

#### 3.1 Prior structure

One proposes a matrix regularization structure inspired from the TC Kernel (Chen et al. (2012)) but where  $\eta$  has  $n$  entries so as to facilitate the on-line estimation. This is contrary to the usual method where the amount of hyperparameters is generally very reduced (see for example Pillonetto et al. (2022)). Afterwards some constraints on the hyperparameters are introduced so as to incorporate the prior knowledge associated to the system impulse response (stability, exponential decay). The regularization matrix is defined as

$$\Pi(t, \eta) = UW(t, \eta)U^T \quad (16)$$

where

$$U = \begin{bmatrix} 1 & 1 & \dots & 1 \\ 0 & 1 & \dots & 1 \\ 0 & 0 & \ddots & \vdots \\ 0 & \dots & 0 & 1 \end{bmatrix} \quad (17)$$

and

$$W(t, \eta) = \begin{bmatrix} e^{\eta_1(t)} & 0 & \dots & 0 \\ 0 & e^{\eta_2(t)} & \dots & 0 \\ \vdots & \dots & \ddots & \vdots \\ 0 & 0 & \dots & e^{\eta_n(t)} \end{bmatrix} \quad (18)$$

with  $\eta^T(t) = [\eta_1(t) \ \eta_2(t) \ \dots \ \eta_n(t)]$ .

Now, some constraints are imposed to  $\eta$ .

- Constraint C1:  $\eta_2(t) < \eta_1(t) \ \forall t$
- Constraint C2:  $\eta_k(t) - 2\eta_{k+1}(t) + \eta_{k+2}(t) = 0$

Note that the vector subspace of  $\mathbb{R}^n$  subject to constraints C1 and C2 is convex, moreover these constraints are equivalent to imposing  $\eta_k(t) = \eta_1(t) - (k-1)\alpha(t)$ , where  $\alpha(t) > 0$ . One has also

$$W_{i,i}(t, \eta) = \lambda(t)W_{i-1,i-1}(t, \eta)$$

with

$$0 < \lambda(t) < 1$$

This prior structure differs from the TC kernel only because of the term  $W_{n,n}$  (see Carli et al. (2017)). One can now perform a change of basis where the parameters estimate is  $\hat{\theta}'$  and the regressor  $\phi'(t)$  such that

$$\hat{\theta}'(t) = U^{-1}\hat{\theta}(t) \quad (19)$$

$$\phi'(t) = U^T\phi(t) \quad (20)$$

$$F'(t+1, \eta(t+1)) = U^{-1}F(t+1, \eta(t+1))U^{-T} = \sum_{i=0}^t \phi'(i)\phi'^T(i) + \sigma^2W(t+1, \eta(t+1)) \quad (21)$$

and in this basis from (4), one can write

$$\hat{\theta}'(t+1, \eta(t+1)) = \left( \sum_{i=1}^t \phi'(i)\phi'(i)^T + \sigma^2W^{-1}(\eta(t+1)) \right)^{-1} \sum_{i=1}^t \phi'(i)y(i) \quad (22)$$

Thus, in the basis where  $\hat{\theta}'$  is the estimated parameters vector and  $\phi'$  the regressor, the associated prior  $W$  is nothing but a kernel matrix  $K'$  corresponding to the DI Kernel (see Pillonetto et al. (2022), p.145), with

$$K'_{i,j} = \beta\lambda^i\delta(i-j) \quad (23)$$

where  $\delta(i) = 1$  if  $i = 0$  and  $\delta(i) = 0$  otherwise, and  $\beta > 0$ .

*Property 1.* The prior  $\Pi(t, \eta)$  as defined in (16), (17), and (18) under the constraints C1 and C2 is stable.

Proof: One verifies immediately that  $\Pi_{k,k}^{1/2}(t, \eta) \leq \frac{W_{1,1}}{\sqrt{1-\lambda}}\lambda^{\frac{k-1}{2}}$ ,

hence  $\lim_{n \rightarrow \infty} \sum_{k=1}^n \Pi_{k,k}^{1/2}(t, \eta) \leq \frac{W_{1,1}}{\sqrt{1-\lambda}} \frac{1}{1-\lambda^{1/2}}$ ,

thus  $\lim_{n \rightarrow \infty} \sum_{k=1}^n \Pi_{k,k}^{1/2}(t, \eta) < \infty$ , which is the condition for the prior to be stable (see Lemma 5.1 of Pillonetto et al. (2022)).  $\square$

*Property 2.* The prior  $\Pi(t, \eta)$  as defined in (16), (17), and (18) has the property of maximum entropy.

Proof: See Appendix B.

#### 3.2 On-line hyperparameters estimation

The estimation procedure of the vector  $\eta$  consists in finding  $\eta$  that minimizes  $J_h(t) = \log(L(\eta(t)|Y(t)))$ . By combining (15), (19)–(21), one can write (the dependence with respect to  $t$  being omitted for sake of simplicity)

$$\frac{\partial J_h}{\partial \eta_k} = \frac{1}{2}Tr \left( U \left( W - \hat{\theta}'\hat{\theta}'^T - \sigma^2F' \right) U^T U^{-T} \frac{\partial W^{-1}}{\partial \eta_k} U^{-1} \right) \quad (24)$$

$$\frac{\partial J_h}{\partial \eta_k} = \frac{1}{2}Tr \left( \left( W - \hat{\theta}'\hat{\theta}'^T - \sigma^2F' \right) \frac{\partial W^{-1}}{\partial \eta_k} \right) \quad (25)$$

and finally

$$\frac{\partial J_h}{\partial \eta_k} = \frac{1}{2} \left( 1 - \frac{\hat{\theta}'_k'^2 + \sigma^2F'_{kk}}{e^{\eta_k}} \right). \quad (26)$$

Set  $\Psi(t)$  such that

$$\Psi(t) = \begin{bmatrix} 1 - \frac{\hat{\theta}'_1'^2(t) + \sigma^2F'_{11}(t)}{e^{\eta_1(t)}} \\ 1 - \frac{\hat{\theta}'_2'^2(t) + \sigma^2F'_{22}(t)}{e^{\eta_2(t)}} \\ \vdots \\ 1 - \frac{\hat{\theta}'_n'^2(t) + \sigma^2F'_{nn}(t)}{e^{\eta_n(t)}} \end{bmatrix} \quad (27)$$

One defines  $\Psi_p(t)$  as the projection of  $\Psi(t)$  onto the vector subspace subject to constraints C1 and C2 with

$$\Psi_p(t) = Proj(\Psi(t)),$$

where the function  $\Psi \mapsto Proj(\Psi(t))$  is such that

$$\Psi_p^* = \left( I_n - C^T (CC^T)^{-1} C \right) \Psi(t) \quad (28a)$$

$$\text{if } \Psi_{p(1)}^*(t) < \Psi_{p(2)}^*(t) \quad \Psi_p(t) = \Psi_p^*(t) \quad (28b)$$

$$\text{else } \Psi_{p(k)} = \Psi_{p(1)}^* - (k-1)\epsilon, \quad (28c)$$

$\epsilon > 0$  is close to 0, and

$$C = \begin{bmatrix} 1 & -2 & 1 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 1 & -2 & 1 & 0 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & \cdots & 1 & -2 & 1 \end{bmatrix}$$

the size of  $C$  being  $(n-2 \times n)$ . The estimation on-line of  $\eta$  is therefore performed using a gradient descent with projection ( $\gamma > 0$  being the corresponding adaptation gain)

$$\eta(t+1) = \eta(t) - \gamma \Psi_p(t). \quad (29)$$

#### 4. OVERALL ALGORITHM

##### 4.1 Update of the adaptation gain an estimated parameters vector

By combining (11) and (12), one obtains

$$F^{-1}(t+1, \eta(t+1)) = F^{-1}(t+1, \eta(t)) + \sigma^2 (\Pi^{-1}(\eta(t+1)) - \Pi^{-1}(\eta(t))). \quad (30)$$

Introducing (29), one gets

$$F'^{-1}(t+1, \eta(t+1)) - F'^{-1}(t+1, \eta(t)) = -\sigma^2 \gamma (\text{diag}(\Psi_p(t)))^{-1}. \quad (31)$$

For a square matrix  $X$  one has  $\partial X = -X \partial X^{-1} X$ . Consequently,

$$F'^{-1}(t+1, \eta(t+1)) - F'^{-1}(t+1, \eta(t)) = \sigma^2 \gamma F'(t+1, \eta(t)) (\text{diag}(\Psi_p(t)))^{-1} F'(t+1, \eta(t)) \quad (32)$$

On the other hand, from (22)

$$\hat{\theta}'(t+1, \eta(t+1)) = F'(t+1, \eta(t+1)) \sum_{i=0}^t \phi'(t)y(i+1) \quad (33a)$$

$$\hat{\theta}'(t+1, \eta(t)) = F'(t+1, \eta(t)) \sum_{i=0}^t \phi'(t)y(i+1) \quad (33b)$$

and by combining (31) and (33), one gets

$$\hat{\theta}'(t+1, \eta(t+1)) = (I_n + \sigma^2 \gamma F'(t+1, \eta(t+1)) (\text{diag}(\Psi_p(t)))^{-1}) \hat{\theta}'(t+1, \eta(t)) \quad (34)$$

##### 4.2 Summary

The algorithm computing alternatively the estimation of parameters and hyperparameters is given by

$$\hat{\theta}'(t+1, \eta(t)) = \hat{\theta}'(t, \eta(t)) + F'(t, \eta(t)) \phi'(t) \frac{\epsilon^o(t+1)}{1 + \phi'^T(t) F'(t, \eta(t)) \phi'(t)} \quad (35)$$

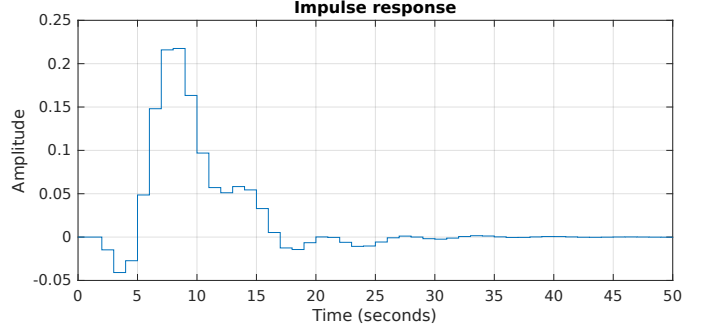


Fig. 1. Impulse response of  $G_o$ .

$$F'(t+1, \eta(t)) = F'(t, \eta(t)) - \frac{F'(t, \eta(t)) \phi'(t) \phi'^T(t) F'(t, \eta(t))}{1 + \phi'^T F'(t, \eta(t)) \phi'(t)} \quad (36)$$

$$\eta(t+1) = \eta(t) - \gamma \Psi_p(t) \quad (37)$$

$$F'^{-1}(t+1, \eta(t+1)) = F'^{-1}(t+1, \eta(t)) + \sigma^2 \gamma F'(t+1, \eta(t)) (\text{diag}(\Psi_p(t)))^{-1} F'(t+1, \eta(t)) \quad (38)$$

$$\hat{\theta}'(t+1, \eta(t+1)) = (I_n + \sigma^2 \gamma F'(t+1, \eta(t+1)) (\text{diag}(\Psi_p(t)))^{-1}) \hat{\theta}'(t+1, \eta(t)) \quad (39)$$

#### 5. SIMULATION RESULTS

In this section, the proposed regularization based recursive identification algorithm is compared with the classic recursive least squares (RLS) method (see (Landau et al., 2011, Chapter 5)). The identification data is generated using the nominal model:

$$G_o(z) = \frac{0.02008z^{-1} + 0.04017z^{-2} + 0.02008z^{-3}}{1 - 1.561z^{-1} + 0.6414z^{-2}} \cdot \frac{-0.7334z^{-1} + 1.516z^{-2} - 0.7591z^{-3} + 0.6941z^{-4}}{1 - 1.282z^{-1} + 1.298z^{-2} - 0.4757z^{-3} + 0.1775z^{-4}}$$

The impulse response of the transfer function  $G_o$  is shown in Fig. 1. The input is 250 samples from a zero mean random gaussian signal with 0.5 standard deviation. Measurement noise is added to the output of the model in the form of zero mean gaussian signal with standard deviation 0.05. The number of parameters is  $n = 50$ . The initial gain matrix  $F$  of the RLS algorithm is chosen diagonal, with identical diagonal values. Four RLS identifications have been done, with diagonal element given by 0.1, 1, 10, or 100. Figure 2 shows the comparison with the regularized RLS algorithm proposed in this paper. For each algorithm, the curves are obtained by averaging over 10 simulations. The average signal-to-noise ratio for these 10 simulations is 12.3 dB (computed using the standard deviation). The identification results depend on the initialization of the hyperparameters. The following initialization of  $\eta(t)$  is used here:

$$\eta_k(0) = \begin{cases} \log(0.001), & \text{if } k = 1, \\ \log(0.9) + \eta_{k-1}(0), & \text{for } k > 1. \end{cases} \quad (40)$$

The estimation gain  $\gamma$  is equal to 1. At each time-step when a new input-output data pair is available, the identified model of each algorithm is updated and the impulse response is computed. The mean square error (MSE) between the impulse response of the identified and the nominal models are shown in the upper plot of Figure 2. Similarly, after each new iteration of the recursive algorithms, the fit of the model to the identification data is computed and the result for the various algorithms are shown in the lower plot of Figure 2.

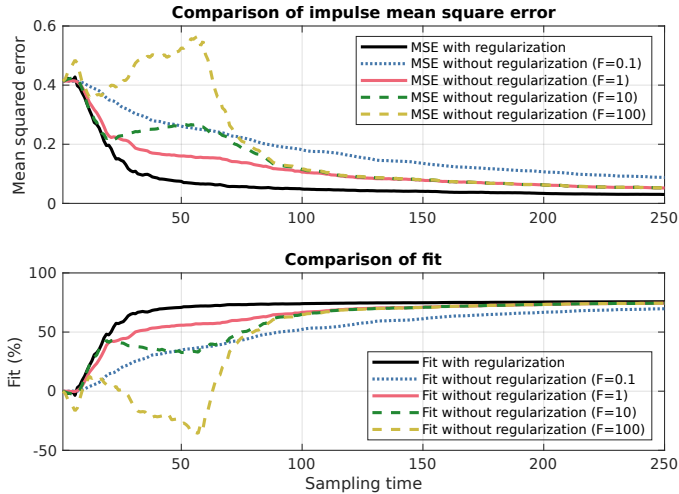


Fig. 2. Comparison with recursive least squares with and without regularization ( $\eta_1(0) = \log(0.001)$ ).

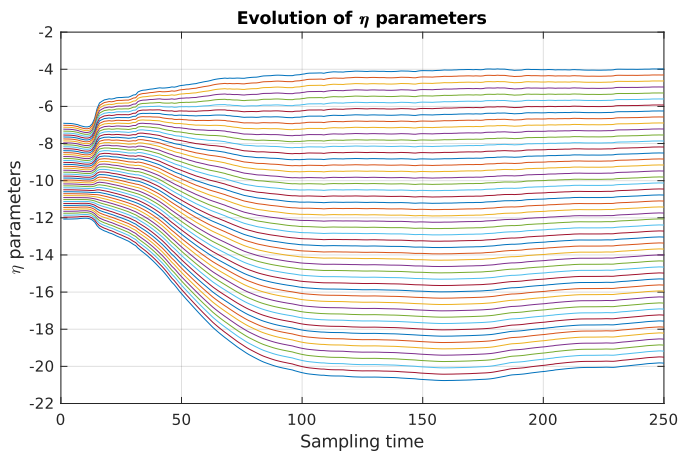


Fig. 3. Evolution of the hyperparameters ( $\eta_1(0) = \log(0.001)$ ).

For the computation of the fit, the full input-output data is used, even though the algorithms only use this data only one sample at a iteration time. Clearly, for both MSE and Fit, the regularized least squares algorithm yields better results. Figure 3 shows the evolution of the hyperparameters in one simulation. Figure 4 shows the comparison based on an average over 10 simulations for a different initialization of the hyperparameters using  $\eta_1(0) = \log(0.1)$ . The evolution of the hyperparameters in one of the simulations is displayed in Figure 5. Despite a degradation of the results during the first 15 s, the regularized least squares solution reaches the optimal solution more rapidly than the other algorithms. This is

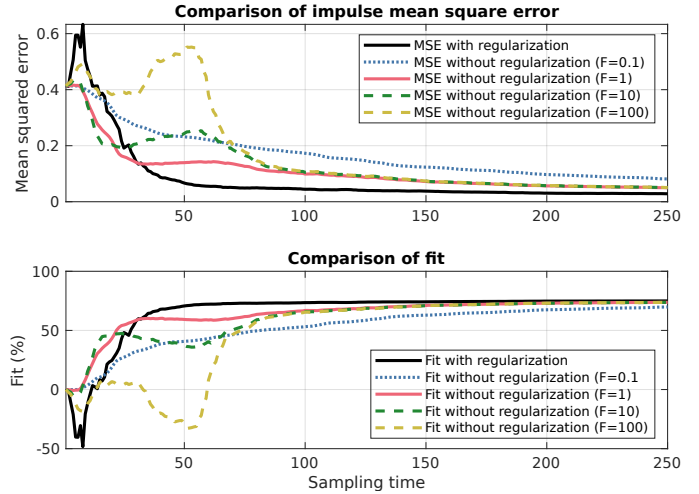


Fig. 4. Comparison with recursive least squares with and without regularization ( $\eta_1(0) = \log(0.1)$ ).

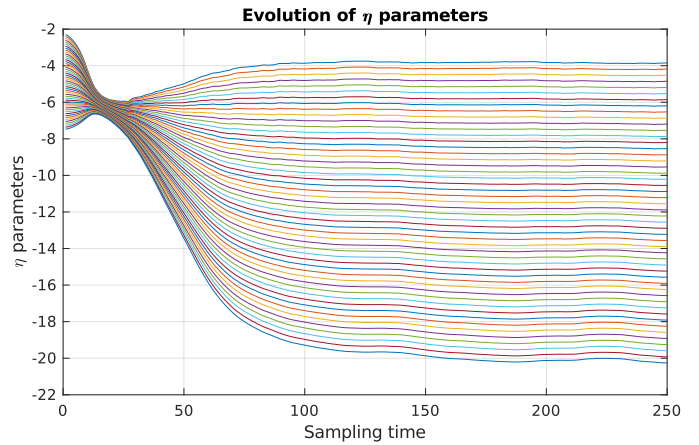


Fig. 5. Evolution of the hyperparameters ( $\eta_1(0) = \log(0.1)$ ).

related also to the reach of the optimal hyperparameter values.

## 6. CONCLUDING REMARKS

A novel recursive identification algorithm with online estimation of model parameters and regularization matrix hyperparameters has been presented. A simulation comparison with classical recursive least squares shows the superiority of the proposed method during the initial instants of the recursive algorithm, and the faster reduction of the prediction error squares. Future work will focus on the initialization and the convergence properties of the hyperparameters estimation.

## REFERENCES

- Akaike, H. (1979). Smoothness priors and the distributed lag estimators. *Technical report Department of statistics.*
- Aström, K. and Bohlin, T. (1965). Maximum entropy kernels for system identification. *Proc. IFAC symposium on Self-Adaptive Systems, Teddington.*
- Carli, F.P. (2014). On the maximum entropy property of the first-order stable spline kernel and its implica-

- tions. In *2014 IEEE Conference on Control Applications (CCA)*, 409–414. doi:10.1109/CCA.2014.6981380.
- Carli, F.P., Chen, T., and Ljung, L. (2017). Maximum entropy kernels for system identification. *IEEE Transactions on Automatic Control*, 62(3), 1471–1477. doi: 10.1109/TAC.2016.2582642.
- Chen, T., Ohlsson, H., and Ljung, L. (2012). On the estimation of transfer functions, regularizations and gaussian processes revisited. *Automatica*, 48(8), 1525–1535.
- Illg, S., Kösters, T., Kiroriwal, D., and Nelles, O. (2022). Adaptive system identification with regularized fir models. *Proc. 14th IFAC Workshop on adaptive and learning control systems*.
- Landau, I.D., Lozano, R., M'Saad, M., and Karimi, A. (2011). *Adaptive control*. Springer, London, 2nd edition.
- Leamer, E. (1972). A class of informative priors and distributed lag analysis. *Econometrica*, 40(6), 1059–1081.
- Ljung, L. and Söderström, T. (1983). *Theory and practice of recursive identification*. MIT Press, Cambridge.
- Pillonetto, G. and de Nicolau, G. (2010). A new kernel approach for linear system identification. *Automatica*, 46(1), 81–93.
- Pillonetto, G., Chen, T., Chiuso, A., De Nicolao, G., and Ljung, L. (2022). *Regularized System Identification: Learning Dynamic Models from Data*. Communications and Control Engineering. Springer International Publishing, Cham. doi:10.1007/978-3-030-95860-2.
- Pillonetto, G., Dinuzzo, F., Chen, T., De Nicolao, G., and Ljung, L. (2014). Kernel methods in system identification, machine learning and function estimation: A survey. *Automatica*, 50(3), 657–682. doi: 10.1016/j.automat.2014.01.001.
- Prando, G., Romeres, D., and Chuiso, A. (2016). Online identification of time-varying systems: A bayesian approach. *Proc. 55th Conference on Decision and Control (CDC)*.
- Romeres, D., Prando, G., Pillonetto, G., and Chuiso, A. (2016). Online identification system identification. *European Control Conference*.
- Vau, B. and Bourlès, H. (2021). Closed-loop output error identification algorithms with predictors based on generalized orthonormal transfer functions: Convergence conditions and bias distribution. *Automatica*, 125.
- Whitacker, H. (1959). Design of a model-reference adaptive control system for aircraft. In *Report R-164*. Instrumentation Lab MIT, Boston.
- Young, P. (2011). *Recursive and time series Analysis, An introduction for the student and practitioner*. Springer, London.

## Appendix A. PROOF OF LEMMA 1

By using the inversion lemma, one has

$$\begin{aligned} |\Sigma^{-1}(t+1)| &= \frac{1}{\sigma^2} I_{t+1} - \frac{1}{\sigma^2} \Phi(t) (\sigma^2 \Pi^{-1}(t+1) + \\ &+ \Phi^T(t) \Phi(t))^{-1} \Phi^T(t) \end{aligned} \quad (\text{A.1})$$

Since  $|\Sigma^{-1}| = 1/|\Sigma|$ , and by using the Sylvester theorem about determinants, one gets (the dependence with respect to  $t$  is omitted for sake of clarity)

$$|\Sigma^{-1}| = \left| (\sigma^2 \Pi^{-1} + \Phi^T \Phi)^{-1} \right| |\sigma^2 \Pi^{-1}| \left| \frac{I_N}{\sigma^2} \right| \quad (\text{A.2})$$

$N$  being the data number (This number doesn't play any role in the sequel), thus

$$\log |\Sigma^{-1}| = -N \log(\sigma^2) - \log |\sigma^2 \Pi^{-1} + \Phi^T \Phi| + \log |\sigma^2 \Pi^{-1}| \quad (\text{A.3})$$

Note that  $\frac{\partial \log |X(z)|}{\partial z} = \text{Tr} \left( X^{-1} \frac{\partial X}{\partial z} \right)$ , yielding ( $\eta_k$  is the  $k$ th entry of  $\eta$ )

$$\begin{aligned} \frac{\partial \log |\Sigma^{-1}|}{\partial \eta_k} &= -\text{Tr} \left[ (\sigma^2 \Pi^{-1} + \Phi^T \Phi)^{-1} \frac{\partial \sigma^2 \Pi^{-1}}{\partial \eta_k} \right] \\ &+ \text{Tr} \left[ \sigma^{-2} \Pi \frac{\partial \sigma^2 \Pi^{-1}}{\partial \eta_k} \right] \end{aligned} \quad (\text{A.4})$$

but  $\Phi^T(t) \Phi = \sum_{i=0}^t \phi(i) \phi^T(i)$ , and therefore  $(\sigma^2 \Pi^{-1}(t+1) + \Phi^T(t) \Phi(t))^{-1} = F(t+1)$ , therefore one gets

$$\begin{aligned} \frac{\partial \log |\Sigma^{-1}(t+1)|}{\partial \eta_k} &= \\ &= \text{Tr} \left[ (\sigma^{-2} \Pi(t+1) - F(t+1)) \frac{\sigma^2 \partial \Pi^{-1}(t+1)}{\partial \eta_k} \right] \end{aligned} \quad (\text{A.5})$$

Now let us consider the expression  $l(t+1) = Y^T(t+1) \Sigma^{-1}(t+1) Y(t+1)$ . One obtains

$$\begin{aligned} l &= \frac{1}{\sigma^2} Y^T(t+1) Y(t+1) - \frac{1}{\sigma^2} Y^T(t+1) \Phi(t) (\sigma^2 \Pi^{-1}(t+1) \\ &+ \Phi^T(t) \Phi(t))^{-1} \Phi^T(t) Y(t+1) \end{aligned} \quad (\text{A.6})$$

Consequently we get

$$\begin{aligned} \frac{\partial l(t+1)}{\partial \eta_k} &= -\sigma^{-2} Y^T(t+1) \cdot \\ &\cdot \Phi \frac{\partial (\sigma^2 \Pi^{-1}(t+1) + \Phi^T(t) \Phi(t))^{-1}}{\partial \eta_k} \Phi^T(t) Y(t+1) \end{aligned} \quad (\text{A.7})$$

Since  $\frac{\partial X^{-1}(z)}{\partial z} = -X^{-1} \frac{\partial X}{\partial z} X^{-1}$ , one has

$$\begin{aligned} \frac{\partial l(t+1)}{\partial \eta_k} &= \sigma^{-2} \sum_{i=1}^t y(i+1) \phi^T(i) F(t+1) \frac{\partial \sigma^2 \Pi^{-1}}{\partial \eta_k} F(t+1) \cdot \\ &\cdot \sum_{i=1}^t \phi(i) y(i+1) \end{aligned} \quad (\text{A.8})$$

now from (Landau et al., 2011, Eq. (3.36)),  $F(t+1) \sum_{i=1}^t \phi(i) y(i+1) = \hat{\theta}(t+1)$ , therefore

$$\frac{\partial l(t+1)}{\partial \eta_k} = \sigma^{-2} \hat{\theta}^T(t+1) \frac{\partial \sigma^2 \Pi^{-1}}{\partial \eta_k} \hat{\theta}(t+1) \quad (\text{A.9})$$

By combining (A.5) and (A.9), one obtains the gradient of  $\log L(\eta(t+1) | Y(t+1))$

$$\begin{aligned} \frac{\partial \log(L(t+1))}{\partial \eta_k} &= \frac{1}{2} \left( \text{Tr} \left[ (\sigma^{-2} \Pi(t+1) - F(t+1)) \right. \right. \\ &\left. \left. \frac{\partial \sigma^2 \Pi^{-1}(t+1)}{\partial \eta_k} \right] - \frac{1}{\sigma^2} \hat{\theta}^T(t+1) \frac{\partial \sigma^2 \Pi^{-1}(t+1)}{\partial \eta_k} \hat{\theta}(t+1) \right) \end{aligned} \quad (\text{A.10})$$

and therefore

$$\frac{\partial \log(L(t+1))}{\partial \eta_k} = \frac{1}{2} \text{Tr} \left[ [\sigma^{-2} (\Pi(t+1) - \hat{\theta}(t+1)\hat{\theta}^T(t+1)) - F(t+1)] \frac{\partial \sigma^2 \Pi^{-1}(t+1)}{\partial \eta_k} \right] \quad (\text{A.11})$$

## Appendix B. MAXIMUM ENTROPY PROPERTIES OF THE PROPOSED REGULARIZATION MATRIX

Notation: For a square matrix  $A$ ,  $A(\{a_1, \dots, a_k\})$  denotes the submatrix of  $A$  identified by row and column indexes

$$\Xi_n^{(1)}(x) = \begin{bmatrix} \sum_{k=1}^n e^{\eta_k} & \sum_{k=2}^n e^{\eta_k} & x_{13} & \dots & \dots & x_{1n} \\ \sum_{k=2}^n e^{\eta_k} & \sum_{k=2}^n e^{\eta_k} & \sum_{k=3}^n e^{\eta_k} & x_{24} & \dots & x_{2n} \\ x_{13} & \sum_{k=3}^n e^{\eta_k} & \sum_{k=3}^n e^{\eta_k} & \sum_{k=4}^n e^{\eta_k} & \dots & x_{3n} \\ \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & x_{n-2,1} \\ x_{1n} & \dots & \dots & \sum_{k=n-1}^n e^{\eta_k} & \sum_{k=n-1}^n e^{\eta_k} & \sum_{k=n}^n e^{\eta_k} \\ & & & x_{n-2,1} & \sum_{k=n}^n e^{\eta_k} & \sum_{k=n}^n e^{\eta_k} \end{bmatrix} \quad (\text{B.2})$$

Denote by  $x^\circ$  the optimal value. Then  $\Xi_n^{(1)}(x^\circ) = \Pi(t, \eta)$ , i.e. the solution of the Maximum Entropy Problem (Carli, 2014, (9)) coincides with the regularization matrix  $\Pi(t, \eta)$  in (16).

Proof: The statement is proved by induction as in (Carli, 2014, Proposition 4.1). Let  $m = 3$ , using (Carli, 2014, (13)–(14)), the central extension of

$$\begin{bmatrix} \sum_{k=1}^n e^{\eta_k} & \sum_{k=2}^n e^{\eta_k} & x_{13} \\ \sum_{k=2}^n e^{\eta_k} & \sum_{k=2}^n e^{\eta_k} & \sum_{k=3}^n e^{\eta_k} \\ x_{13} & \sum_{k=3}^n e^{\eta_k} & \sum_{k=3}^n e^{\eta_k} \end{bmatrix} \quad (\text{B.3})$$

is given by  $x_{13}^\circ = -\frac{1}{y_1} \xi_{32} y_2 = \xi_{32} = \sum_{k=3}^n e^{\eta_k}$ , since by (Carli, 2014, (14))

$$y_1 = e^{-\eta_1}, \quad y_2 = -e^{-\eta_1}. \quad (\text{B.4})$$

Assuming that the statement holds for  $m = l$ ,  $l \geq 3$ , we want to prove that it holds true also for  $m = l + 1$ . The proof is split into two parts:

- First, we show that for  $s = 2, \dots, l - 1$  the central one-step extension of

$$\begin{bmatrix} \sum_{k=s}^n e^{\eta_k} & \dots & \sum_{k=l}^n e^{\eta_k} & x_{s,l+1} \\ \sum_{k=s+1}^n e^{\eta_k} & \dots & \dots & \sum_{k=l+1}^n e^{\eta_k} \\ \vdots & \ddots & \ddots & \vdots \\ x_{s,l+1} & \sum_{k=l+1}^n e^{\eta_k} & \dots & \sum_{k=l+1}^n e^{\eta_k} \end{bmatrix} \quad (\text{B.5})$$

is the submatrix  $\Pi(\{s, \dots, l+1\})$ , namely the matrix formed by the lines and columns  $s, \dots, l+1$  from  $\Pi(t, \eta)$ . As in (Carli, 2014, (12)), consider the submatrix  $\Lambda = \Pi(\{s, \dots, l\})$ . Its cofactor matrix  $C$  is such that  $C_{11} = -C_{12}$  and  $C_{13} = \dots = C_{1,l} = 0$ . Using also that  $|\Lambda| = (\sum_{k=s}^n e^{\eta_k}) C_{11} + (\sum_{k=s+1}^n e^{\eta_k}) C_{12}$ , one obtains  $y_1 = \frac{1}{\eta_s}$ ,  $y_2 = -\frac{1}{\eta_s}$ , and  $y_3 = \dots = y_{s+l-2} = 0$ . Thus  $x_{s,l+1}^\circ = \xi_{s,l+1} = \sum_{k=l+1}^n e^{\eta_k}$  by (Carli, 2014, (13)–(14)).

- Finally, one has to show that  $\Pi(\{1, \dots, l+1\})$  is the central one-step extension of

given by  $a_1, \dots, a_k$ . Define also the set of indices  $\mathcal{I}_b^{(m)} := \{(i, j) \mid |i - j| \leq m\}$ .

Following the ideas of (Carli, 2014, Section IV), we prove the maximum entropy properties of the regularization matrix  $\Pi(t, \eta)$  in (16) of size  $n$  defined by  $\Pi_{ij}(t, \eta) = \sum_{k=\max(i,j)}^n e^{\eta_k(t)}$ . In the rest of this section, the parenthesis  $(t)$  is dropped to save space.

*Proposition 1.* Consider the maximum entropy band extension problem (Carli, 2014, (9)), with  $m = 1$  and

$$\xi_{ij} = \Pi_{ij}(t, \eta), \quad (i, j) \in \mathcal{I}_b^{(1)} \quad (\text{B.1})$$

i.e. consider the partially defined 1-band matrix

$$\begin{bmatrix} \sum_{k=1}^n e^{\eta_k} & \dots & \sum_{k=l}^n e^{\eta_k} & x_{1,l+1} \\ \vdots & \ddots & \ddots & \sum_{k=l+1}^n e^{\eta_k} \\ \sum_{k=l}^n e^{\eta_k} & \dots & \dots & \vdots \\ x_{1,l+1} & \sum_{k=l+1}^n e^{\eta_k} & \dots & \sum_{k=l+1}^n e^{\eta_k} \end{bmatrix} \quad (\text{B.6})$$

The proof is similar to the above and is omitted.  $\square$

The additional property that  $\forall k, e^{\eta_k} > 0$ , leads to the positive definiteness of  $\Pi(t, \eta)$  and  $\Xi_n^{(m)}$ , thus assuring also the feasibility of the problem (Carli, 2014, Theorem 3.1(i)).