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# ON THE FOIAȘ AND STRATILA THEOREM

FRANÇOIS PARREAU.

ABSTRACT. We extend the Foiaș and Sratila theorem to the case of  $L^2$ -functions whose spectral measure is continuous and concentrated on an independent Helson set, and to ergodic actions of locally compact second countable abelian groups. We first prove it for functions satisfying Carleman’s condition for the Hamburger moment problem, without the assumption that the spectral measure is supported by a Helson set. Then we show independently that the spectral projector associated with a Helson set preserves each  $L^p$  space, with a appropriate bound of the corresponding norm.

## 1. INTRODUCTION

**1.1. Main results.** The Foiaș and Sratila theorem ([9]) asserts that, given an ergodic measure-preserving automorphism  $T$  on a standard probability space, if the spectral measure  $\sigma$  of a non zero square-integrable complex function  $f$  is continuous and supported by a *Kronecker* set, then the process  $(f \circ T^n)$  is Gaussian. The dynamical system generated by the process  $(f \circ T^n)$  is then determined up to a metric isomorphism by  $\sigma$ , the spectral measure of the process. Except for the case of discrete spectrum and not deep extensions (see [14]), this is the only result of spectral determination in ergodic theory.

A first motivation for this work was a better understanding of the Foiaș and Sratila theorem, which implies strong ergodic properties for *Gaussian-Kronecker* automorphisms. Factors and self-joinings of such systems can be completely described ([23]). In [15], we extend these properties and prove disjointness results for a wider class of Gaussian automorphisms, the “GAG” automorphisms, which include all Gaussian automorphisms with simple spectrum and thus mixing cases, and all the results there eventually rely on the Foiaș and Sratila theorem.

We shall here investigate possible extensions of this theorem. Our main result is its extension to the larger class of *algebraically independent Helson sets*, which allows us to get mildly mixing examples.

Let  $(X, \mathcal{B}, \mu)$  denote a standard probability space and let  $T = (T_g)_{g \in G}$  be an action of a locally compact second countable abelian group  $G$  on  $X$  by measure-preserving automorphisms (for sake of simplicity, we use the same notation as for a single measure-preserving automorphism).

The Fourier transform of a complex Borel measure  $\sigma$  on the dual group  $\Gamma = \widehat{G}$  is defined by  $\widehat{\sigma}(g) = \int_{\Gamma} \gamma(g) d\sigma(\gamma)$  for  $g \in G$ , and the spectral measure  $\sigma_f$  of  $f \in L^2(\mu)$  is the finite

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positive Borel measure on  $\Gamma$  given by

$$\widehat{\sigma}_f(g) = (f \circ T_g \mid f) \quad (g \in G).$$

A closed set  $K \subset \Gamma$  is a *Helson set* with constant  $\alpha$  ( $0 < \alpha \leq 1$ ), or a *Helson- $\alpha$  set* if, for every complex Borel measure  $\sigma$  on  $K$  ( $\|\cdot\|$  denotes the total variation norm),

$$\sup_{g \in G} |\widehat{\sigma}(g)| \geq \alpha \|\sigma\|.$$

An equivalent definition, maybe more usual ([16]), is that  $K$  is a Helson set if each continuous function on  $K$  vanishing at infinity is the restriction to  $K$  of some function in  $A(\Gamma)$ .

**Theorem 1.** *Assume that  $T$  is ergodic and let  $f$  be a non-zero function in  $L^2(\mu)$ . If the spectral measure of  $f$  is continuous and concentrated on an independent Helson set, then  $f$  has a Gaussian distribution.*

We shall firstly prove a result where the assumption that  $\sigma_f$  is supported by a Helson set is replaced by an additional hypothesis on the moments of  $f$ .

**Definition.** *Given a positive measure  $\mu$ , we denote  $\mathcal{C}(\mu)$  the class of all  $f \in \bigcap_{2 \leq p < +\infty} L^p(\mu)$  such that*

$$\sum_{p=2}^{\infty} (1/\|f\|_p) = +\infty.$$

For real random variables, this condition is known as Carleman's condition for the Hamburger moment problem.

**Theorem 2.** *Assume that  $T$  is ergodic and let  $f$  be a non-zero function in  $\mathcal{C}(\mu)$ . If the spectral measure of  $f$  is continuous and concentrated on an independent compact set, then  $f$  has a Gaussian distribution.*

**Corollary 1.** *Under the assumptions of Theorem 1 or of Theorem 2, the functions  $f \circ T_g$  ( $g \in G$ ) span a Gaussian space. In case of a single measure-preserving automorphism  $T$ , the process  $(f \circ T^n)_{n \in \mathbb{Z}}$  is Gaussian.*

Theorem 1 follows from Theorem 2 and from a result that may be of independent interest:

**Theorem 3.** *Let  $K$  be a compact Helson subset of  $\Gamma$ . Then, for every  $p \geq 2$ , the spectral projector  $\pi_K$  corresponding to  $K$  maps  $L^p(\mu)$  into itself with, for every  $f \in L^p(\mu)$ ,*

$$\|\pi_K f\|_p \leq Cp \|f\|_p$$

where  $C$  depends only on the Helson constant of  $K$ .

In particular  $\pi_K(L^\infty(\mu)) \subset \mathcal{C}(\mu)$ .

Under the only assumption that  $\sigma$  is supported by an independent compact set, the problem remains open, but there is little hope to get a complete answer. However, there are examples of measures for which the corresponding Gaussian automorphism has a simple spectrum, but which do not satisfy the spectral determination property.

## 1.2. Notation, definitions and preliminaries.

We refer to [2], [18], [19] for basic definitions and results in ergodic theory and spectral theory of dynamical systems. For harmonic analysis, we refer to [21], and to [16] for definitions and properties of thin sets.

The abelian group  $G$  acting on  $(X, \mathcal{B}, \mu)$  is supposed locally compact second countable, and non compact. For  $g \in G$  we also note  $T_g$  the unitary operator  $f \rightarrow f \circ T_g$  on  $L^2(\mu)$ . We shall note its dual group  $\Gamma$  multiplicatively and, for  $\gamma \in \Gamma$  and  $g \in G$ ,  $\gamma(g)$  is also denoted  $\langle \gamma, g \rangle$ . In the case  $G = \mathbb{Z}$  we identify  $\Gamma = \mathbb{T}$  with  $\mathbb{S}^1$ .

The group generated by  $K \subset \Gamma$  is denoted  $\text{Gp}(K)$ . We shall need algebraic independence through the property: if  $K_1, K_2$  are two disjoint subsets of  $K$ , then  $\text{Gp}(K_1)$  and  $\text{Gp}(K_2) \setminus \{1\}$  are still disjoint; thus the definition of independence can be taken here in the weaker sense: when  $\gamma_1, \dots, \gamma_k \in K$  and  $n_1, \dots, n_k \in \mathbb{Z}$ , then  $\gamma_1^{n_1} \times \dots \times \gamma_k^{n_k} = 1$  iff  $\gamma_1^{n_1} = \dots = \gamma_k^{n_k} = 1$  (so,  $n_1 = \dots = n_k = 0$  is not required, and this allows finite order elements).

Let  $\mathcal{B}(f)$  (resp.  $\mathcal{B}(H)$ ) denote the sub- $\sigma$ -algebra generated by a measurable function  $f$  (resp. a subset  $H$  of  $L^2(\mu)$ ). If  $f \in L^2(\mu)$ , the closed invariant subspace of  $L^2(\mu)$  generated by  $f$  is denoted  $Z(f)$ , so the factor generated by  $f$  is  $\mathcal{B}(Z(f))$  (factors will be taken as invariant sub- $\sigma$ -algebras).

By a measure  $\sigma$  on  $\Gamma$  we always mean a complex Borel measure on  $\Gamma$ . Then  $\tilde{\sigma}$  denotes the measure defined by  $\tilde{\sigma}(B) = \bar{\sigma}(B^{-1})$  for every Borel subset  $B$  of  $\Gamma$ ; given  $f \in L^2(\mu)$ , the spectral measure of  $\bar{f}$  is  $\tilde{\sigma}_f$  and it is concentrated on  $\text{supp}(\sigma_f)^{-1}$ . If  $\sigma$  is the spectral type of  $T$ , defined up to equivalence of measures, the spectral representation yields an isometry  $\varphi \rightarrow \varphi(T)$  from  $L^\infty(\sigma)$  onto a subalgebra of  $L^2(\mu)$  such that  $\varphi(T)$  corresponds to the multiplication by  $\varphi$ , and in particular each  $g \in G$ , taken as a character of  $\Gamma$ , corresponds to  $T_g$ .

Given a Borel subset  $K$  of  $\Gamma$ ,  $\mathbf{1}_K(T)$  is the spectral projector of  $L^2(\mu)$  corresponding to  $K$ , which we denote by  $\pi_K$ . For  $f \in L^2(\mu)$ , we have  $\overline{\pi_K f} = \pi_{K^{-1}} \bar{f}$ .

In our proofs, the operation of  $A(\Gamma)$  on the spaces  $L^p(\mu)$  plays a major role. Recall that  $A(\Gamma)$  denotes the Banach algebra of Fourier transforms of integrable functions, equipped with the norm inherited from the  $L^1(G)$  norm. That is, when  $G$  is discrete, the algebra of functions  $\gamma \rightarrow \varphi(\gamma) = \sum_{g \in G} a_g \langle \gamma, g \rangle$  where  $\sum_{g \in G} |a_g| < +\infty$ , equipped with the norm  $\|\varphi\|_{A(\Gamma)} = \sum_{g \in G} |a_g|$ . Then  $\varphi(T) : h \rightarrow \sum_{g \in G} a_g h \circ T_g$  defines an operator on each  $L^p(\mu)$  ( $1 \leq p \leq +\infty$ ), of norm  $\|\varphi(T)\|_{\mathcal{L}(L^p)} \leq \|\varphi\|_{A(\Gamma)}$ .

Since a  $L^2$ -limit of Gaussian functions still is Gaussian, we have the elementary lemma:

**Lemma 1.** *Let  $f \in L^2(\mu)$ . If  $(K_n)$  is a sequence of Borel subsets of  $\Gamma$  such that  $\sigma_f(\Gamma \setminus K_n) \rightarrow 0$  and each  $\pi_{K_n} f$  is Gaussian, then  $f$  is Gaussian.*

In particular, it is sufficient to prove Theorem 1 in the case when  $\sigma_f$  has compact support. Furthermore, for the proofs of Theorems 1 and 2, we can restrict ourselves to countable group actions. Indeed, let  $G_0$  be a countable dense subgroup of  $G$ , endowed with the discrete topology. Then  $\Gamma$  is continuously embedded in the compact group  $\Gamma_0 = \widehat{G_0}$ , the spectral measures of  $f$  for the actions of  $G$  and  $G_0$  being identified in this embedding. Clearly, if a set  $K$  in  $\Gamma$  is independent and compact in  $\Gamma$ , these properties still hold in  $\Gamma_0$ . The Fourier transform of a measure concentrated on  $K$  is continuous in the topology of  $G$ , hence by density of  $G_0$  in  $G$ , directly from the definition, if  $K$  is Helson in  $\Gamma$ , the same holds in  $\Gamma_0$ .

So, we henceforth assume that the abelian group  $G$  is countable and discrete, so that  $\Gamma$  is compact and metrizable.

Note also that, given  $\varepsilon > 0$ , we can choose a totally disconnected compact subset  $K$  with  $\sigma_f(\Gamma \setminus K) < \varepsilon$ ; indeed this is a standard fact for finite positive measures on  $\mathbb{T}^{\mathbb{N}}$  and the dual group of a countable abelian group is naturally embedded as a compact subgroup of  $\mathbb{T}^{\mathbb{N}}$ . By Lemma 1, if we obtain that every such  $\pi_K f$  is Gaussian, then  $f$  itself is Gaussian.

Concerning Gaussian automorphisms, we shall use the definitions and notation of [15]. In particular, if  $\sigma$  is a continuous symmetric measure on  $\mathbb{T}$ , we denote  $T_\sigma$  the Gaussian automorphism defined by the real Gaussian process of spectral measure  $\sigma$ . However, as Foias and Stratila in [9], we consider complex-valued Gaussian processes  $(f \circ T^n)$ , which will be more convenient. Then the corresponding Gaussian space  $H$  is the closed invariant subspace of  $L^2(\mu)$  spanned by  $f$  and  $\bar{f}$  and the spectral type  $\sigma_H$  of  $T$  on  $H$  is the symmetric measure  $\sigma_f + \bar{\sigma}_f$ .

## 2. ZSIDO'S THEOREM AND ABELIAN GROUP ACTIONS

### 2.1. Spectral process.

Let  $f$  be a non-zero function in  $L^2(\mu)$ . We consider firstly the case of a single measure-preserving automorphism  $T$ . Let  $\gamma(t) = \exp(2\pi it)$  for  $t \in [0, 1]$ . The spectral process corresponding to  $f$  is defined by

$$f_t = \pi_{\gamma([0,t])} f \quad (t \in [0, 1]).$$

As for the proof of C. Foias and S. Stratila, the main argument in the proof of Theorems 1 and 2 will be to prove that this process has independent increments. In order to obtain that it is a Gaussian process, it is then sufficient to know that it admits a version with a.s. continuous sample functions (see e.g. [4], chap. VII, Theorem 7.1). This latter fact has been proved independently by L. Zsido ([26]):

**Theorem (Zsido).** *Assume that  $T$  is ergodic and that  $\sigma_f$  is continuous. If the spectral process  $(f_t)$  has independent increments, then it is Gaussian.*

For a countable abelian group action  $(T_g)_{g \in G}$  we want to construct a process with similar properties.

We can assume that  $K = \text{supp } \sigma_f$  is totally disconnected. Then  $K$  is homeomorphic to a compact set  $L$  of  $[0, 1]$ . We can furthermore assume that  $L$  contains the endpoints 0 and 1. Choose a homeomorphism  $\gamma$  from  $L$  onto  $K$  and let

$$(1) \quad f_t = \pi_{\gamma([0,t] \cap L)} f \quad (t \in [0, 1]).$$

**Lemma 2.** *Assume that the action  $(T_g)$  is ergodic, that  $\sigma_f$  is continuous with compact support  $K$  homeomorphic to  $L \in [0, 1]$ , and let  $(f_t)$  be defined by (1). Then, for every  $t \in (0, 1)$ , the spectral measures  $\sigma_{f_t}$  and  $\sigma_{f_1 - f_t}$  are concentrated on disjoint open sets of  $K$ . Moreover, if this process has independent increments, then it admits a version with almost surely continuous sample functions and thus it is Gaussian.*

*Proof.* The first assertion is immediate from the facts that

$$f_1 - f_t = \pi_{\gamma([t,1] \cap L)} f = \pi_{\gamma((t,1] \cap L)} f$$

since  $\sigma_f$  is continuous, and that  $\gamma([0, t] \cap L)$  and  $\gamma((t, 1] \cap L)$  are disjoint open sets of  $K$ .

For the second assertion, we proceed by slight modifications of Zsido's proof [26]. By the same classical argument of probability theory (see [4]),  $(f_t)$  has a version whose sample functions  $t \mapsto (f_t(x))$  are a.e. right continuous with left-limits (*càdlàg*) and thus have a jump  $\Delta f_t(x)$  at every  $t \in [0, 1]$ . Since the complement of  $L$  consists of countably many open intervals  $(t'_n, t''_n)$ , we can moreover let  $f_t(x) = f_{t'_n}(x)$  everywhere on each of these intervals, so that the sample functions are continuous on  $[0, 1] \setminus L$ .

The main step is to show that, given  $g \in G$ , there is a set  $F$  of full measure in  $X$  such that for every  $x \in F$

$$(2) \quad \Delta f_t(T_g x) = \langle \gamma(t), g \rangle \Delta f_t(x) \quad \text{for every } t \in L$$

Let  $\eta > 0$ . Since the map  $t \mapsto \langle \gamma(t), g \rangle$  is continuous on  $L$ , we can find a finite subdivision  $t_0 = 0 < t_1 < \dots < t_k = 1$  of points in  $L$  such that  $|\langle \gamma(t), g \rangle - \langle \gamma(t_j), g \rangle| < \eta$  on each  $[t_{j-1}, t_j] \cap L$ . Let then  $f_j = f_{t_j} - f_{t_{j-1}} = \pi_{\gamma([t_{j-1}, t_j] \cap L)} f$ , ( $1 \leq j \leq k$ ). Since  $T_g$  corresponds in the spectral representation to the multiplication by  $\langle \cdot, g \rangle$ , it follows

$$\|f_j \circ T_g - \langle \gamma(t_j), g \rangle f_j\|_2 \leq \eta \|f_j\|_2,$$

hence the set  $N'_j$  of all  $x \in X$  where

$$|f_j(T_g x) - \langle \gamma(t_j), g \rangle f_j(x)| \geq \eta^{1/2}$$

has measure  $\leq \eta \|f_j\|_2^2$ .

Moreover, the set  $N''_j$  of all  $x$  where  $|f_j(x)| \geq \eta^{-1/2}$  also has measure  $\leq \eta \|f_j\|_2^2$  and outside  $N''_j$  we have, for all  $t \in [t_{j-1}, t_j]$ ,

$$|\langle \gamma(t_j), g \rangle f_j(x) - \langle \gamma(t), g \rangle f_j(x)| < \eta^{1/2}.$$

Let  $N_\eta$  be the union of all  $N'_j$  and  $N''_j$  ( $1 \leq j \leq k$ ). Since  $(f_t)$  has orthogonal increments,

$$\mu(N_\eta) \leq 2\eta \sum_{j=1}^k \|f_j\|_2^2 = 2\eta \|f\|_2^2,$$

and, for each  $x \notin N_\eta$ , we have for every  $t \in L$ , if  $t \in [t_{j-1}, t_j] \cap L$ ,

$$(3) \quad |f_j(T_g x) - \langle \gamma(t), g \rangle f_j(x)| < 2\eta^{1/2}.$$

Let  $(\eta_n)$  be a sequence of positive reals converging to 0 and let  $F$  be the set of  $x \in X$  where the sample functions are *càdlàg* and which belong to infinitely many sets  $[0, 1] \setminus N_{\eta_n}$ . Then  $F$  has full measure and, if  $x \in F$ , (3) holds for an infinite subsequence of  $(\eta_n)$  with the corresponding  $j = j_n(t)$ . As  $f_{j_n(t)}$  converges pointwise to the jump  $\Delta f_t$ , it follows that (2) holds for every  $x \in F$ .

Now, there is a set of full measure on which (2) holds for all  $g$  in  $G$ . Then the modulus  $|\Delta f_t(x)|$  of the jump is invariant under the action of  $G$ , for all  $t \in L$ . The rest of the proof is exactly as in [26]: by ergodicity, the number of jumps of modulus  $\geq \delta > 0$  in any given interval must be a.e. constant, whence non-zero jumps can only happen at fixed points of the interval, and thus correspond to eigenfunctions in the closed invariant subspace generated by  $f$ , which would contradict the hypothesis that  $\sigma_f$  is continuous.  $\square$

**2.2. A reduction.** The following lemma expresses a criterion for  $f$  to be Gaussian, independently of the construction of a spectral process.

**Lemma 3.** *Assume that  $(T_g)$  is ergodic and  $\sigma_f$  continuous. In order that  $f$  be Gaussian, it is sufficient that, for every disjoint open sets  $U$  and  $V$  in  $\Gamma$ , the factors generated by  $\pi_U f$  and  $\pi_V f$  be independent.*

*Proof.* Assume that this condition holds. By Lemma 2, it is enough to show that, for each totally disconnected compact set  $K$  in  $\Gamma$ , the corresponding process  $(f_t)$  has independent increments. Now, for every  $t \in (0, 1)$ , by the hypothesis and the first assertion of Lemma 2, the factors generated by  $f_t$  and  $f_1 - f_t$  are independent. A priori, we would have to show that every finite family  $(f_{t_j} - f_{t_{j-1}})$  is independent but, as given any  $t \in (0, 1)$ , all  $f_{t'}$  with  $t' < t$  belong to the closed invariant space  $Z(f_t)$ , the conclusion follows by induction.  $\square$

Of course, by regularity of  $\sigma_f$ , the same condition with disjoint compact sets instead of open sets is also sufficient, and in fact it will hold for any pair of disjoint Borel sets. Notice that the condition for symmetric Borel sets is also necessary in an invariant Gaussian space, since two orthogonal real functions in a Gaussian space are independent.

### 3. GROUP PROPERTY AND CARLEMAN'S CONDITION

We will now prove Theorem 2. We need the following result of Foiaş [7], which states that *supports* of spectral measures of locally compact group actions in ergodic theory satisfy a “group property” (see also [8]):

**Theorem (Foiaş).** *Let  $f_1, f_2 \in L^2(\mu)$ . If  $f_1 f_2 \in L^2(\mu)$  then*

$$\text{supp}(\sigma_{f_1 f_2}) \subset \text{supp}(f_1) \cdot \text{supp}(f_2).$$

**Lemma 4.** *Let  $f \in \mathcal{C}(\mu)$  and let  $U$  be an open set in  $\Gamma$ . The spectral type of  $T$  restricted to the factor generated by  $\pi_U f$  is concentrated on  $\text{Gp}(U \cap \text{supp}(\sigma_f))$ .*

*Proof.* Choose a sequence  $(\varphi_n)$  of functions in  $A(\Gamma)$ , with compact support contained in  $U$  and of norm  $\|\varphi_n\|_{A(\Gamma)} \leq 1$ , which span a dense subspace of  $L^2(\sigma_f|_U)$ . The spectral isomorphism yields that the  $\varphi_n(T)f$  span a dense subspace of  $Z(\pi_U f) = \pi_U Z(f)$ .

Let then  $(f_n)_{n \geq 1}$  be the sequence of all functions  $\text{Re } \varphi_n(T)f$  and  $\text{Im } \varphi_n(T)f$ , reordered. It is a sequence of real functions generating a dense subspace of  $Z(\pi_U f) + Z(\overline{\pi_U f})$ . For every  $n \geq 1$ , we have  $\|f_n\|_p \leq \|f\|_p$  for all  $p \geq 2$ ,  $f_n$  belongs to  $\mathcal{C}(\mu)$  and has a spectral measure  $\sigma_{f_n} \ll \sigma_f|_U + \tilde{\sigma}_f|_{U^{-1}}$ , so that  $\text{supp}(\sigma_{f_n}) \subset \text{Gp}(U \cap \text{supp}(\sigma_f))$ .

Foiaş' theorem applies to their finite products, which all belong to  $L^2(\mu)$ , so the spectral measure of any finite product of the  $f_n$  is concentrated on  $\text{Gp}(U \cap \text{supp}(\sigma_f))$ ; since the spectral measure of a sum is absolutely continuous with respect to the sum of the spectral measures of its terms, this remains true by linearity for all polynomials in the functions  $f_n$ .

The result will follow if we show that these polynomials are dense in  $L^2(\mathcal{B}(Z(\pi_U f)))$ . Moreover, as  $\mathcal{B}(Z(\pi_U f))$  is the sub- $\sigma$ -algebra generated by  $\{f_n\}_{n \geq 1}$ , the union of the subspaces  $L^2(\mathcal{B}(\{f_j\}_{1 \leq j \leq n}))$  is dense in  $L^2(\mathcal{B}(Z(\pi_U f)))$ . Thus, it is enough to show that for every given  $n \geq 1$  the polynomials in  $f_1, \dots, f_n$  are dense in  $L^2(\mathcal{B}(\{f_j\}_{1 \leq j \leq n}))$ .

This is a classical result for the case of a single function satisfying Carleman's condition, and we just need to extend it. As in this case, we shall use quasi-analytic classes, for which we refer to W. Rudin's book [22], Chap. 19.

Fix  $n \geq 1$  and let  $\nu$  be the joint distribution of  $f_1, \dots, f_n$ , so that for any positive measurable function  $h$  on  $\mathbb{R}^n$

$$\int_{\mathbb{R}^n} h(t_1, \dots, t_n) d\nu(t_1, \dots, t_n) = \int_X h(f_1(x), \dots, f_n(x)) d\mu(x).$$

Under the map  $h \rightarrow h \circ (f_1, \dots, f_n)$ ,  $L^2(\mathcal{B}(\{f_j\}_{1 \leq j \leq n}))$  is isomorphic to  $L^2(\nu)$ , the  $f_j$  corresponding to the coordinate functions, so we have to show that the polynomials are dense in  $L^2(\nu)$ .

Let  $h$  be a function of  $L^2(\nu)$  orthogonal to all polynomials, and consider the Fourier transform given on  $\mathbb{R}^n$  by

$$\Phi(s_1, \dots, s_n) = \int_{\mathbb{R}^n} \exp(i \sum_{j=1}^n s_j t_j) h(t_1, \dots, t_n) d\nu(t_1, \dots, t_n).$$

For every sequence  $k_1, \dots, k_n$  of natural integers,

$$\int_{\mathbb{R}^n} |t_1^{k_1} \dots t_n^{k_n}|^2 d\nu(t_1, \dots, t_n) = \int_X |f_1^{k_1} \dots f_n^{k_n}|^2 d\mu \leq \prod_{j=1}^n \|f_j^{2k_j}\|_n \leq \prod_{j=1}^n \|f\|_{2nk_j}^{2k_j},$$

and

$$\int_{\mathbb{R}^n} |t_1^{k_1} \dots t_n^{k_n}| h(t_1, \dots, t_n) d\nu(t_1, \dots, t_n) \leq \prod_{j=1}^n \|f\|_{2nk_j}^{k_j} \cdot \|h\|_2.$$

Hence  $\Phi$  is  $C^\infty$ . We denote by  $D^{k_1, \dots, k_n} \Phi$  the derivative  $\partial^{k_1 + \dots + k_n} \Phi / \partial^{k_1} s_1 \dots \partial^{k_n} s_n$ ,

$$D^{k_1, \dots, k_n} \Phi(s_1, \dots, s_n) = \int_{\mathbb{R}^n} \prod_{j=1}^n (it_j)^{k_j} \exp(i \sum_{j=1}^n s_j t_j) h(t_1, \dots, t_n) d\nu(t_1, \dots, t_n).$$

All these derivatives vanish at  $(0, \dots, 0)$  and we have the bound

$$(4) \quad |D^{k_1, \dots, k_n} \Phi(s_1, \dots, s_n)| \leq \prod_{j=1}^n \|f\|_{2nk_j}^{k_j} \cdot \|h\|_2 \quad \text{for all } s_1, \dots, s_n.$$

Given  $j$  ( $1 \leq j \leq n$ ), for every choice of  $k_\ell$  ( $\ell > j$ ) and  $s_\ell$  ( $\ell < j$ ), we consider the one-variable function

$$s \rightarrow D^{0, \dots, 0, k_{j+1}, \dots, k_n} \Phi(s_1, \dots, s_{j-1}, s, 0, \dots, 0),$$

and the sequence of its derivatives. By (4), the  $L^\infty$  norm of its  $k$ -th derivative is bounded by

$$\left( \prod_{\ell=j+1}^n \|f\|_{2nk_\ell}^{k_\ell} \|h\|_2 \right) \cdot M_k$$

where  $M_k = \|f\|_{2nk}^k$ , which means, according to the definition in [22], that these functions belong to the class  $C\{M_k\}$ .

Now, by standard application of the Hölder inequality, the sequence  $(\|f\|_p)$  is non-decreasing and  $(\|f\|_p^p)$  is logarithmically convex. Logarithmic convexity is inherited by the arithmetic subsequence  $(\|f\|_{2nk}^{2nk})$  and then by  $(\|f\|_{2nk}^k)$ , and the assumption that  $f$  belongs to  $\mathcal{C}(\mu)$  together with the monotony of  $(\|f\|_p)$  implies that

$$\sum_{k=1}^{\infty} (1/M_k)^{1/k} = \sum_{k=1}^{\infty} 1/\|f\|_{2nk} = +\infty.$$



It follows from the Denjoy-Carleman Theorem that the class  $C\{M_k\}$  is quasi-analytic, and thus that if one of these functions vanishes at 0 together with all its derivatives then it is identically 0.

We conclude by an easy induction. For  $j = 1$ , given any  $k_2, \dots, k_n$ , all the functions  $s \rightarrow D^{k, k_2, \dots, k_n} \Phi(s, 0, \dots, 0)$  for  $k \geq 0$  vanish at 0, whence  $D^{0, k_2, \dots, k_n} \Phi(s_1, 0, \dots, 0) = 0$  for all  $s_1$  and all  $k_2, \dots, k_n$ . Then, for a given  $j \geq 1$ , if we have that

$$D^{0, \dots, 0, k_{j+1}, \dots, k_n} \Phi(s_1, \dots, s_j, 0, \dots, 0) = 0 \quad \text{for all } s_1, \dots, s_j \text{ and all } k_{j+1}, \dots, k_n,$$

we get similarly that all functions  $s \rightarrow D^{0, \dots, 0, k_{j+2}, \dots, k_n} \Phi(s_1, \dots, s_j, s, 0, \dots, 0)$  are identically 0 and the induction hypothesis remains true for  $j + 1$  instead of  $j$ .

Finally,  $\Phi$  itself is identically 0 and this implies  $h = 0$   $\nu$ -a.e. So, the null function is the only function in  $L^2(\nu)$  orthogonal to all polynomials, which proves that polynomials are dense in  $L^2(\nu)$ , and the proof is complete. □

*Proof of Theorem 2.* Assume that  $T$  is ergodic, let  $f$  be a non-zero function in  $\mathcal{C}(\mu)$  whose spectral measure is continuous and concentrated on an independent compact set, and let  $U$  and  $V$  be two disjoint open sets in  $\Gamma$ . By Lemma 4, the spectral types of  $T$  on the factors generated by  $\pi|_U f$  and  $\pi|_V f$  are concentrated on  $\text{Gp}(U \cap \text{supp}(\sigma_f))$  and  $\text{Gp}(V \cap \text{supp}(\sigma_f))$  respectively. Since  $\text{supp}(\sigma_f)$  is independent these groups have no other common element than 1. So, these factors are spectrally disjoint and a fortiori independent. The conclusion follows then from Lemma 3. □

#### 4. THE SPECTRAL PROJECTOR ON A HELSON SET

To prove Theorem 3 we need to approximate the indicator function of a compact Helson set by functions in  $A(\Gamma)$  (here, we release the assumption that  $\Gamma$  itself is compact). The main tool is Drury's lemma, which was used to solve the problem of the union of two Sidon sets [6], and that of the union of two Helson sets by N. Varopoulos [24], [25]. We quote it in the version given by C. Herz [10] with a more convenient estimate of the norm of the functions obtained in  $A(\Gamma)$ .

**Definition.** Let, for  $0 < \varepsilon \leq 1$ ,

$$\omega(\varepsilon) = \sup_{n \geq 1} \omega_n(\varepsilon),$$

where, denoting by  $E_n$  the canonical basis of  $\mathbb{Z}^n$ ,

$$\omega_n(\varepsilon) = \inf_{\psi \in A(\mathbb{Z}^n)} \{ \|\psi\|_{A(\mathbb{Z}^n)} : \psi|_{E_n} = 1, |\psi| \leq \varepsilon \text{ on } \mathbb{Z}^n \setminus E_n \}.$$

**Theorem** (Drury, Varopoulos, Herz). *Let  $K$  be a compact Helson- $\alpha$  set of  $\Gamma$ . For each closed set  $F$  of  $\Gamma$  disjoint from  $K$ , for every  $\varepsilon \in (0, 1]$  and every  $\beta < \alpha^2$ , there exists a function  $\varphi$  in  $A(\Gamma)$  such that*

$$\varphi = 1 \text{ on } K, |\varphi| \leq \varepsilon \text{ on } F, \text{ and } \|\varphi\|_{A(\Gamma)} \leq \beta^{-1} \omega(\beta\varepsilon)$$

*Remark.* This is essentially Theorem 2 of [10], except that it only states that there exists a function  $\omega : (0, 1] \rightarrow [1, +\infty)$  with that property. C. Herz defines  $\omega$  later on (after the statement of Proposition 1), as above, and he shows that it is suitable for the theorem. Besides the Helson constant is inverted there.

However, the estimates of  $\omega(\varepsilon)$  given in [10] do not seem sufficient to show Theorem 3. A little later, in a paper [17] on a slightly different problem, J.-F. Méla gave indirectly a nearly optimal bound.

**Theorem (Méla).** *For all  $\varepsilon$  in  $(0, 1/2]$*

$$\omega(\varepsilon) \leq 2|\log \varepsilon| + 6.$$

*Proof.* As this result is not explicitly stated in [17], we explain briefly how to deduce it. Fix an arbitrary integer  $n \geq 1$ , and consider the construction in section 7 when the group  $G$  is  $\mathbb{T}^n$  and its dual is  $\mathbb{Z}^n$ . The measures noted  $\nu_s$  in [17] are then the finite Riesz products on  $\mathbb{T}^n \times \mathbb{T}$  admitting as density with respect to the Lebesgue measure the positive trigonometric polynomials

$$Q_s(z_1, \dots, z_n, z) = \prod_{j=1}^n (1 + s(z_j z + \bar{z}_j \bar{z})), \quad s \in (0, 1/2],$$

where  $\|Q_s\|_{L^1(\mathbb{T}^{n+1})} = 1$  for all  $s$ .

For  $s \in (0, 1/2]$ , the measure  $\mu_s$  on  $\mathbb{T}^n$  is then defined by  $\hat{\mu}_s(k_1, \dots, k_n) = \hat{\nu}_s(k_1, \dots, k_n, 1)$ . It admits as density the factor  $P_s$  of  $\bar{z}$  in the expansion of  $Q_s$ . On the canonical basis, its coefficients are all equal to  $s$ , all its other non-zero coefficients are odd powers  $s^{2k+1}$  of  $s$  with  $k \geq 1$ , and we still have  $\|P_s\|_{L^1(\mathbb{T}^n)} \leq 1$ .

Now, the main idea is to construct, given  $\varepsilon > 0$ , a measure  $\sigma$  on  $(0, 1/2]$  of norm as small as possible with  $\int s d\sigma = 1$  and  $|\int s^{2k+1} d\sigma| \leq \varepsilon$  for all  $k \geq 1$ . This is achieved by Lemma 3 of [17], where J.-F. Méla shows that we can obtain  $\|\sigma\| \leq 2|\log \varepsilon| + 6$  (taking  $a = \log 2 - 1/2$  in the bound given in [17]).

Then, integrating  $P_s$  with respect to  $\sigma$ , we get a trigonometric polynomial  $P$  on  $\mathbb{T}^n$  whose Fourier transform  $\psi$  on  $\mathbb{Z}^n$  satisfies  $\psi = 1$  on  $E_n$ ,  $|\psi| \leq \varepsilon$  elsewhere, and

$$\|\psi\|_{A(\mathbb{Z}^n)} = \|P\|_{L^1(\mathbb{T}^n)} \leq \int \|P_s\|_{L^1(\mathbb{T}^n)} d|\sigma| \leq 2|\log \varepsilon| + 6.$$

This proves that  $\omega_n(\varepsilon) \leq 2|\log \varepsilon| + 6$  for all  $n \geq 1$ . □

*Remark.* We also mention that this result by J.-F. Méla was used for a different problem in [13]. The bound is nearly optimal since it can similarly be deduced from [17] that  $\omega(\varepsilon)/|\log \varepsilon|$  is bounded from below by a positive constant.

*Proof of Theorem 3.* Let a compact Helson set  $K \subset \Gamma$  be given and denote by  $\sigma$  be the spectral type of  $T$ .

A remarkable fact in the results by S. Drury, N. Varopoulos and C. Herz is that the bound for  $\|\varphi\|_{A(\Gamma)}$  does not depend on the set  $F$  disjoint from  $K$ . Thanks to Méla's result, for  $\varepsilon \leq 1/2$ , this bound is less than  $c|\log \varepsilon|$ , where  $c$  is a constant depending only on the Helson constant of  $K$ .

If we fix  $\varepsilon \in (0, 1/2]$  and apply this to a non-decreasing sequence  $(F_n)$  of closed sets whose union is  $\Gamma \setminus K$ , we get a sequence  $(\varphi_{n,\varepsilon})$  of functions with  $A(\Gamma)$  norms  $\leq c|\log \varepsilon|$ , hence also bounded in  $L^\infty(\sigma)$ . By extracting a subsequence if needed, we may assume that it converges to some function  $\varphi_\varepsilon$  in the weak\* topology of the duality  $(L^1(\sigma), L^\infty(\sigma))$ . Then

$$\varphi_\varepsilon = 1 \text{ } (\sigma|_K)\text{-a.e.}, \quad |\varphi_\varepsilon| \leq \varepsilon \text{ } (\sigma|_{\Gamma \setminus K})\text{-a.e.},$$

and the operator  $\varphi_\varepsilon(T)$  on  $L^2(\mu)$  is the weak limit of the sequence  $(\varphi_{n,\varepsilon}(T))$ .

For  $2 \leq p \leq +\infty$ , as each  $\varphi_{n,\varepsilon}(T)$  maps  $L^p(\mu)$  into itself with operator norm  $\leq c |\log \varepsilon|$ , we still have that  $\varphi_\varepsilon(T)$  maps  $L^p(\mu)$  into itself, and

$$(5) \quad \|\varphi_\varepsilon(T)\|_{\mathcal{L}(L^p(\mu))} \leq c |\log \varepsilon|.$$

Let  $(\varepsilon_k)$  be a decreasing and summable sequence in  $(0, 1/2]$ . As  $|\mathbf{1}_K - \varphi_{\varepsilon_k}| \leq \varepsilon_k$   $\sigma$ -a.e., we may write

$$\mathbf{1}_K = \varphi_{\varepsilon_1} + \sum_{k=1}^{+\infty} (\varphi_{\varepsilon_{k+1}} - \varphi_{\varepsilon_k}) \quad \sigma\text{-a.e.},$$

where the series converges in the  $L^\infty(\sigma)$  norm, and the corresponding series

$$(6) \quad \varphi_{\varepsilon_1}(T) + \sum_{k=1}^{+\infty} (\varphi_{\varepsilon_{k+1}}(T) - \varphi_{\varepsilon_k}(T))$$

converges towards  $\pi_K$  in  $\mathcal{L}(L^2(\mu))$ .

Moreover, for all  $k \geq 1$ , as  $|\varphi_{\varepsilon_{k+1}} - \varphi_{\varepsilon_k}| \leq 2\varepsilon_k$   $\sigma$ -a.e.,

$$(7) \quad \|\varphi_{\varepsilon_{k+1}}(T) - \varphi_{\varepsilon_k}(T)\|_{\mathcal{L}(L^2(\mu))} = \|\varphi_{\varepsilon_{k+1}} - \varphi_{\varepsilon_k}\|_{L^\infty(\sigma)} \leq 2\varepsilon_k$$

and, by (5),

$$(8) \quad \|\varphi_{\varepsilon_{k+1}}(T) - \varphi_{\varepsilon_k}(T)\|_{\mathcal{L}(L^\infty(\mu))} \leq 2c |\log \varepsilon_{k+1}|,$$

Now, for  $p \in [2, +\infty)$ , we have, by (7), (8) and the Riesz-Thorin interpolation theorem,

$$\|\varphi_{\varepsilon_{k+1}}(T) - \varphi_{\varepsilon_k}(T)\|_{\mathcal{L}(L^p(\mu))} \leq (2\varepsilon_k)^{2/p} (2c |\log \varepsilon_{k+1}|)^{1-2/p}.$$

With  $\varepsilon_k = e^{-kp}$ , we get

$$\|\varphi_{\varepsilon_{k+1}}(T) - \varphi_{\varepsilon_k}(T)\|_{\mathcal{L}(L^p(\mu))} \leq 2^{2/p} e^{-2k} \cdot 2c(k+1)p \leq 4c(k+1)e^{-2k} p.$$

It follows that the series (6) converges in  $\mathcal{L}(L^p(\mu))$ . As  $L^p(\mu) \subset L^2(\mu)$ , this proves that  $\pi_K$  maps  $L^p(\mu)$  into itself, and that

$$\|\pi_K\|_{\mathcal{L}(L^p(\mu))} \leq cp + 4c \left( \sum_{k=1}^{+\infty} (k+1)e^{-2k} \right) p \leq Cp,$$

where  $C$  is a constant depending only on the Helson constant of  $K$ .

The last assertion of the theorem is an immediate consequence.  $\square$

*Proof of Theorem 1.* Assume again that  $G$  is discrete and  $\Gamma$  compact, and that  $T$  is ergodic. Let  $f$  be a non-zero function in  $L^2(\mu)$  whose spectral measure is continuous and concentrated on an independent Helson set  $K$  of  $\Gamma$ .

By theorem 3,  $\pi_K$  maps  $L^\infty(\mu)$  into  $\mathcal{C}(\mu)$  and thus there is a dense subspace of  $\pi_K(L^2(\mu))$  consisting of functions in  $\mathcal{C}(\mu)$ . In case when  $K$  does not contain any eigenvalue of  $T$ , these functions have continuous spectral measures and by Theorem 2 therefore have a Gaussian distribution. It then follows that every function in  $\pi_K(L^2(\mu))$  is Gaussian. Otherwise, as  $\sigma_f$  is continuous, we can choose a sequence  $(K_n)$  of closed subsets of  $\Gamma$  which do not contain any eigenvalue such that  $\sigma_f(\Gamma \setminus K_n) \rightarrow 0$ . Then, each  $\pi_{K_n} f$  is Gaussian and it follows again that  $f$  is Gaussian.  $\square$

*Proof of Corollary 1.* In general, a subspace  $H$  of  $L^2(\mu)$  is Gaussian if every non-zero real function in  $H$  has a Gaussian distribution. Under the assumptions of Theorem 1 or of Theorem 2, every function  $h$  in  $Z(f)$  has a continuous spectral measure concentrated on  $K = \text{supp}(\sigma_f)$ . If  $K$  is a Helson set, it follows directly from Theorem 1 that  $h$  has a Gaussian distribution. If  $f \in \mathcal{C}(\mu)$ , we can apply Theorem 2 to functions  $\varphi(T)f$  with  $\varphi \in A(\Gamma)$ , which all belong to  $\mathcal{C}(\mu)$ , and the result follows since they are dense in  $Z(f)$ .  $\square$

## 5. COMPLEMENTS

**5.1. Foiaş and Stratila measures and sets.** We say that a positive continuous measure  $\sigma$  on  $\Gamma$  is a *FS measure* if, whenever the measure-preserving action  $(T_g)_{g \in G}$  on  $(X, \mathcal{B}, \mu)$  is ergodic and  $f$  is a complex function in  $L^2(\mu)$  with  $\sigma_f = \sigma$ , then  $f$  is Gaussian, and that a Borel subset  $K$  of  $\Gamma$  is a *FS set* if every positive continuous measure concentrated on  $K$  is a *FS measure*. For symmetric sets or measures, these definitions match the definitions of [14] and [15].

Recall that a compact subset  $K$  of  $\Gamma$  is a *Kronecker set* if every continuous function of modulus one on  $K$  is a uniform limit of characters. By Lemma 1 (see also [14]), any positive measure  $\sigma$  on  $\Gamma$  concentrated on a non-decreasing union of *FS sets* is a *FS measure*. It follows that the Foiaş and Stratila theorem still holds for a *weak Kronecker set* in  $\mathbb{T}$ , that is a closed subset  $K$  such that every finite Borel measure  $\sigma$  on  $K$  is concentrated on a non-decreasing union of Kronecker sets. In the same way, the assumption on  $\sigma_f$  in Theorem 1 can be weakened: it is sufficient that  $\sigma_f$  be continuous and concentrated on a non-decreasing union of independent Helson sets.

Now, a weak Kronecker set is a Helson-1 set and conversely a Helson-1 set is the translate of a weak Kronecker set ([16], Chap. XIII). Moreover a weak Kronecker set is independent (in the strong sense). So, Theorem 1 extends the Foiaş and Stratila theorem, and the actual extension consists in the case of Helson- $\alpha$  sets with  $0 < \alpha < 1$ .

Let us also recall the following result of [14] and [15] for measures on  $\mathbb{T}$ , which shows that in this case  $\sigma_f$  need not be concentrated on an independent set:

**Proposition.** (1) *If  $\sigma$  is a FS measure and  $0 < \tau \ll \sigma$ , then  $\tau$  is a FS measure.*  
(2) *Assume that  $\sigma_1$  and  $\sigma_2$  are mutually singular symmetric FS measures. Then  $\sigma_1 + \sigma_2$  is a FS measure if and only if  $\sigma_2$  is singular with respect to each translate of  $\sigma_1$ .*

*Remark.* The only difficult point in this proposition is the “if” part of (2) ([15] Corollary 10): it is easy to see that  $\sigma_1 + \sigma_2$  is *FS* iff  $T_{\sigma_1}$  and  $T_{\sigma_2}$  are disjoint but the result then requires the characterization of disjointness for *GAG* automorphisms established in [15].

**5.2. Mildly mixing example.** The Helson hypothesis forbids mixing for the corresponding Gaussian systems, since the upper bound  $\sup_{g \in G} |\widehat{\sigma}(g)|$  in the definition of a Helson set can be replaced by the upper limit at infinity (with a different constant, [16], Chap. I, Prop. 5.2). When  $\alpha = 1$ , it is easy to see that, given any positive Borel measure  $\sigma$  on  $K$ , there are non-trivial sequences of characters converging to 1 in  $L^1(\sigma)$  (i.e.  $K$  is a *weak Dirichlet set*), which, in our context, implies rigidity: there are sequences  $(T_g)$  where  $g \rightarrow \infty$  converging to the identity on the factor generated by  $f$ . In [14] non rigid examples of *FS sets* are constructed by

considering independent unions of Kronecker sets, but then the Gaussian system is generated by rigid factors.

On the contrary, Theorem 1 allows mild mixing – absence of non-trivial rigid factors –, and even a spectral form of partial mixing (usual partial mixing never occurs for Gaussian automorphisms which are not strongly mixing). Indeed, by a result of T. Körner [11] (see [16], Chap. XIII, Theorem 3.14), for each  $\alpha$  in  $(0, 1)$ , there exists a finite positive measure  $\sigma$ , concentrated on an independent Helson- $\alpha$  set  $K$  of  $\mathbb{T}$ , with the property

$$\text{for every Borel set } B \subset \mathbb{T}, \limsup |\widehat{\sigma|_B}(n)| = \alpha \sigma(B).$$

Then  $\sigma$  is continuous, since  $\alpha < 1$ , and  $\tilde{\sigma}$  shares the same property. It follows that, for every positive integer  $k$  and every positive measure  $\tau \ll (\sigma + \tilde{\sigma})^{*k}$ ,  $\limsup |\widehat{\tau}(n)| \leq \alpha^k \|\tau\|$  (by density, it is sufficient to check this inequality for  $\tau$  of the form  $\sigma_1|_{B_1} * \cdots * \sigma_k|_{B_k}$  where  $\sigma_j = \sigma$  or  $\sigma_j = \tilde{\sigma}$  and  $B_j$  is a Borel subset of  $\mathbb{T}$ , for  $1 \leq j \leq k$ ).

Consider the Gaussian automorphism  $T_{\sigma+\tilde{\sigma}}$ , with spectral measure  $\sigma_H = \sigma + \tilde{\sigma}$  on its Gaussian space  $H$ . The spectral type of  $T$  on the  $n$ -th chaos  $H^{(n)}$  is the  $n$ -th convolution power  $\sigma_H^{*n}$ , and the spectral type on the factor  $\mathcal{B}(H)$  is the convolution exponential  $\exp(\sigma_H)$ , so we obtain:

**Corollary 2.** *For every  $\alpha$  in  $(0, 1)$  there exists a FS measure  $\sigma$  on  $\mathbb{T}$  such that the Gaussian automorphism  $T = T_{\sigma+\tilde{\sigma}}$  satisfies the property: for every square integrable zero-mean function  $h$ ,*

$$\limsup |(T^n h | h)| = \limsup |\widehat{\sigma}_h(n)| \leq \alpha \|h\|_2^2,$$

*and in particular  $T$  is mildly mixing.*

**5.3. Independence in measure.** Henceforth, we restrict ourselves to the action of a single automorphism, so  $G = \mathbb{Z}$ , and  $\Gamma = \mathbb{T}$  (identified with  $\mathbb{S}^1$ ).

A natural question is whether the assumption of support independence in Theorem 1 can be replaced by a notion of independence “in measure”. Such a property appears in the spectral analysis of Gaussian automorphisms. If  $\sigma$  is a continuous symmetric measure on  $\mathbb{T}$ , then  $T_\sigma$  has simple spectrum iff the following condition holds:

- (9) *For all  $n \geq 1$ , there exists a set of full  $\sigma^{\otimes n}$ -measure on which the product map  $(z_1, \dots, z_n) \rightarrow z_1 \cdots z_n$  is one-to-one modulo permutations of coordinates.*

Indeed, this is equivalent to saying that for every  $n \geq 1$  the cartesian power  $T_\sigma^{\otimes n}$  restricted to the sub- $\sigma$ -algebra of consisting of sets invariant under permutations of coordinates has simple spectrum, hence that  $T_\sigma$  restricted to the  $n$ -th chaos has simple spectrum, and it also implies that the convolution powers of  $\sigma$  are mutually singular (see e.g. [12]), i.e. that the chaos are spectrally disjoint.

In [1], O. Ageev proved property (9) for the reduced spectral type (i.e. the spectral type restricted to zero-mean functions) of a class of rank one transformations, including the classical Chacon transformation, and it is clear that not all zero-mean functions in  $L^2(\mu)$  can be Gaussian. So, Theorem 1 fails on the only assumptions that  $T$  be ergodic and  $\sigma_f$  be a continuous measure satisfying (9).

In the other direction, Theorem 1 implies that the spectral type of the Chacon transformation cannot be concentrated on an independent Helson set. But we don't know if it is concentrated on an independent set, and we leave open the question whether Theorem 1 is

valid if we only assume that  $\sigma_f$  is continuous and concentrated on an independent set. We do not know either if the result holds under the assumption that  $\sigma_f$  is continuous, satisfies (9) and is concentrated on a Helson set.

**5.4. Poisson suspensions.** Poisson suspensions are particularly interesting in our context, because they appear together with Gaussians systems in the theory of processes with independent increments and their spectral properties are similar. We recall briefly results already discussed by E. Roy in his article [20], to which we refer for a detailed exposition (see also [5]).

Let  $T$  be a measure-preserving automorphism of a standard space  $(X, \mathcal{B}, \mu)$  where  $\mu$  is  $\sigma$ -finite and infinite. Suppose that there is no invariant set  $E \in \mathcal{B}$  with  $0 < \mu(E) < +\infty$ . This is equivalent to saying that its spectral type  $\sigma$ , which can be assumed to be symmetric, is continuous. Then its Poisson suspension  $T_*$  is ergodic and spectrally isomorphic to the Gaussian automorphism  $T_\sigma$ .

More precisely, its  $L^2$ -space has a similar decomposition in an orthogonal sum  $\bigoplus_{n \geq 0} H^{(n)}$  of chaos where, up to a normalizing constant,  $H^{(n)}$  is isometric to the tensor product  $L^2(\mu)^{\otimes n}$  restricted to functions invariant under coordinate permutations, the action of  $T_*$  on  $H^{(n)}$  being conjugate to  $T^{\otimes n}$  and thus admitting the convolution power  $\sigma^{*n}$  as spectral type. The spectral isomorphism with  $T_\sigma$  sends each chaos  $H^{(n)}$  of the suspension onto the corresponding chaos of the Gaussian system.

Poisson suspensions may have simple spectrum. Ageev's proof can be extended to some infinite measure preserving transformations obtained by *cutting and stacking*, and such systems never have invariant sets of finite positive measure. In [3], A. Danilenko and V. Ryzhikov construct examples where moreover the Poisson suspension is mixing.

We thus obtain Poisson suspensions  $T_*$  with simple spectrum whose spectral type restricted to the chaos  $H^{(1)}$  is the spectral type  $\sigma$  of  $T$ . However, E. Roy proves a disjointness property between Gaussian systems and Poisson suspensions ([20], Prop. 4.11). In particular no function from the chaos  $H^{(1)}$  can have a Gaussian distribution, so  $\sigma$  cannot be a *FS* measure.

More generally the spectral type of a  $\sigma$ -finite measure preserving automorphism  $T$  must be singular to every positive *FS* measure whenever there is no invariant set  $E \in \mathcal{B}$  with  $0 < \mu(E) < +\infty$  ([20], Theorem 4.13). We have the following consequence:

**Corollary 3.** *Let  $T$  be a measure-preserving automorphism of a space  $(X, \mathcal{B}, \mu)$  of  $\sigma$ -infinite measure with no invariant set of non-zero finite measure, and let  $\sigma$  be its spectral type. Then  $\sigma(K) = 0$  for every independent Helson set  $K \subset \mathbb{T}$ .*

*Remark.* These latter results can easily be extended to actions of locally compact second countable groups.

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