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Singularity Removal for 3D Elliptic Problems with Variable Coefficients and Line Sources

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Abstract

Three dimensional elliptic problems with variable coefficients and line Dirac sources arise in a number of fields. The lack of regularity on the solution prompts users to turn towards alternative variational formulations. Rather than using weighted Sobolev spaces, we prefer the dual variational formulation written in the Hilbertian Lebesgue space, the one used by G. Stampacchia [Séminaire Jean Leray, 1964]. The key work is to show a singular/regular expansion where the singularity of the potential is fully expressed by a convolution formula, based on the Green kernel of the Laplacian. The correction term restores the boundary condition and fits with the standard variational formulation of Poisson equation (in the Sobolev space H^1). We intend to develop a thorough analysis of the proposed expansion while avoiding stringent assumptions on the conductivities. Sharp technical tools, as those developed in [E. Di-Giorgi, Mem. Accad. Sci. Torino. 1957] and [N. G. Meyers Ann. Scuo. Norm. Sup. Pisa, 1963], are necessary in the proofs.

KEYWORDS: Dirac line sources, variable conductivity, weak variational formulation, singular-regular expansion.

1 Introduction

Let Ω be a bounded connected domain in \mathbb{R}^3 , with a Lipschitz-continuous boundary $\Gamma = \partial\Omega$. The generic point in Ω is denoted by \mathbf{x} and the generic point on Γ by $\boldsymbol{\tau}$, while \mathbf{n} stands for the unit normal vector to Γ which is outward to Ω .

The elliptic problem to deal with consists in finding a potential field $\varphi(\cdot)$ created by a linear Dirac source $f(\cdot)$: *find $\varphi \in ?$ such that*

$$\begin{aligned} -\operatorname{div}(a\nabla\varphi) &= f && \text{in } \Omega, \\ \varphi &= 0 && \text{on } \Gamma. \end{aligned} \tag{1}$$

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The conduction parameter $a(\cdot)$ is a bounded space-varying positive coefficient. To ensure the coerciveness of the problem (possibly in $H_0^1(\Omega)$), it is taken positively bounded from below away from zero, that is (α is a given real-number),

$$0 < \alpha \leq a(\mathbf{x}) \leq \alpha^* (= \|a\|_{L^\infty(\Omega)}) < \infty, \quad \forall \mathbf{x} \in \Omega. \quad (2)$$

The line Dirac source is supported by $\gamma = (\gamma(t) = \mathbf{s}(t))_{t \in I=[0,\ell]}$, a parameterized compact curve strongly contained in Ω . This curve is simple, in the sense that the map $\gamma(\cdot)$ is injective. It can be a closed line, the parameterization is hence periodic. We have $\gamma(0) = \gamma(\ell)$ and so to preserve injectivity, the extreme-point ℓ is excluded from I and then $I = [0, \ell[$. We consider that it is a Lipschitz manifold: the function $t \mapsto \gamma(t)$ is a lipeomorphism mapping I into the curve $\gamma \subset \mathbb{R}^3$, meaning that $\gamma(\cdot)$ as a function and its inverse $\gamma^{-1}(\cdot)$ are both Lipschitzian. This class of curves accept angular points. Finally, we choose t as the curvilinear abscissa, then the tangent vector $\gamma'(\cdot)$ is hence unitary that is $|\gamma'(t)| = 1$ and ℓ is the length of γ (see [29]). The linear Dirac data is denoted as $f(\cdot) = \lambda(\cdot)\delta_\gamma$, it is defined as the measure : $\forall \psi \in \mathcal{C}(\Omega)$,

$$\langle f, \psi \rangle_\gamma := \int_I \lambda(t)\psi(\mathbf{s}(t)) dt = \int_\gamma \lambda(\mathbf{s})\psi(\mathbf{s}) ds. \quad (3)$$

The first integral stands for the rigorous mathematical definition of the curvilinear integral (see [29]). However, writing the second form will be preferred for its commodity.

The support of the Dirac measure is the curve γ , representing for instance a loaded fiber or a wire, and the density is $\lambda(\cdot)$. That we consider a single ‘connected’ curve for the source is by no means restrictive. The overall study is readily extended to more than a single linear source.

Models such as (1) may be encountered in growing number of fields : solid and fluid mechanics, physics or biomathematics. Fractures in three dimensional porous media can mathematically be driven by linear sources (see [23, 14]). Computing electric and magnetic fields from power transmission lines is of a common concern [27], the source are the electric wires. Is also of a great importance the noise propagation that may be radiated by pointwise or linear sources in traffic acoustic models [20] (roads, railways . . .). Optimal control of elliptic/parabolic equations with some particular observations, supported by curves, are another example for line sources [7]. Coupled 1D-3D equations governed by problems similar to (1) are recurrent in biomathematics : coupling between blood flow and tissue perfusion [9], drug administration by micro-circulation [28]. We shall stop here. A quite rich bibliography, though certainly not exhaustive, is provided in the introduction of [17]. Readers interested in the subject are recommended to browse that paper. Therein, additional references can be found on the computational approximation methods applied to models similar to (1) so as their numerical analysis. We limit ourselves to the most recent references (see [24, 16, 21, 15]).

At last, the analysis carried out here has an important mathematical interest of its own. In fact, it is a compelling intermediary step for the study of non-linear equations when the conductivity is

also dependent on the potential $\varphi(\cdot)$ itself, that is $a = a((\cdot), \varphi(\cdot))$.

A singular/regular expansion of the solution φ is the purpose. The singular contribution is expressed by a mathematical formula while the correction is solution to a regular elliptic problem set in $H^1(\Omega)$. The particular feature and source difficulty is that the conductivity $a(\cdot)$ is space-dependent. This is in a way the continuation of [3], where the data was a point-wise Dirac sources. As will be seen later on, results in there are the starting block of the analysis undertaken here. The paper is outlined as follows. Section 2 lays the ‘weakened’ variational framework, the foundation of the analysis of problem (1). The solution is sought for in the Lebesgue space $L^2(\Omega)$. Di-Giorgi continuity result is central to the well posedness (see [11, 1957]). The singular/regular expansion, the key result, is provided and a first assessment is presented in Section 3. The mathematical full justification of that expansion is the subject of Section 4. Bochner’s integrals is the chief tool we use (see [5, 1933]). In Section 5, assumptions on the conductivity are softened to include a larger class of conductivities for which the expansion is still valid. Another regularity result by N. G. Meyers is at the chief tool of this generalization (see [25, 1963]). At last, Section 7 presents some analytical examples and a numerical experimentation.

Notations — The Lebesgue space $L^2(\Omega)$ of square integrable functions is endowed with the natural norm $\|\cdot\|_{L^2(\Omega)}$. We need some Sobolev spaces, $H^1(\Omega)$ involves all the functions that are in $L^2(\Omega)$ so as their partial derivatives. The subspace containing the functions in $H^1(\Omega)$ that vanish on Γ is denoted by $H_0^1(\Omega)$. The set of the traces over Γ of all the functions of $H^1(\Omega)$ is denoted $H^{1/2}(\Gamma)$ and $H^{-1/2}(\Gamma)$ is its dual (see [2]). For a positive real-number $\nu < 1$, we need the Sobolev space

$$W^{\nu,\infty}(\Omega) = \left\{ \chi \in L^\infty(\Omega), \quad |\chi|_{W^{\nu,\infty}} = \sup_{\mathbf{x}, \mathbf{y} \in \Omega} \frac{|\chi(\mathbf{x}) - \chi(\mathbf{y})|}{|\mathbf{x} - \mathbf{y}|^\nu} < \infty \right\}.$$

It is denoted $\text{Lip}_\nu(\Omega)$ the set of ν -Lipschitzian functions (see [32], especially for the particular density properties). It coincides also with the Hölder space generally denoted $\mathcal{C}^{0,\nu}(\Omega)$. If endowed with the natural norm $\|\cdot\|_{W^{\nu,\infty}} = \|\cdot\|_{L^\infty(\Omega)} + |\cdot|_{W^{\nu,\infty}}$, it has a Banach structure.

The symbol $[\cdot]$ stands for the jump across a given boundary. Notice that for $\nu \in]0, 1[$, we use the symbol $(\nu)_-$ to point out any of the real numbers $< \nu$, generally close to ν . The symbol $(\nu)_+$ is a given real number $> \nu$.

2 Dual Variational Formulation

Functions in $H^1(\Omega)$ may not have traces on the curve $\gamma^{(1)}$. An important fact is that the data f in (3) switches off the $H^1(\Omega)$ -Sobolev functional framework. An alternative to study (not to solve!) problem (1) is a weaker framework. The regularity to prescribe to the solution $\varphi(\cdot)$ is lower and the dual variational formulation is obtained by transposition (see, eg, [30]). Before coming to the details, we recall that if the Dirac measure $f(\cdot)$ is replaced by a data $F \in L^2(\Omega)$, the unique solution denoted now r_F , belongs to $H^1(\Omega)$, with $(r_F)|_\Gamma = 0$ and is such that

$$\int_{\Omega} a \nabla r_F \nabla \psi \, d\mathbf{x} = \int_{\Omega} F \psi \, d\mathbf{x}, \quad \forall \psi \in H_0^1(\Omega). \quad (4)$$

If no further regularity is introduced on $a(\cdot)$, that is it simply belongs to $L^\infty(\Omega)$, the pioneering work by Di-Giorgi in [11] (see also [26]) establishes that $r_F(\cdot)$ enjoys a Hölderian continuity. We refer to [6, Theorem 9.34] for the following lemma.

Lemma 2.1 (Di Giorgi, 1957) *Assume $F \in L^2(\Omega)$. Then, the solution $r_F(\cdot)$ to Problem (4) is Hölderian. There exists $\nu \in]0, 1[$ such that*

$$\|r_F\|_{\mathcal{C}^{0,\nu}(\Omega)} \leq C \|F\|_{L^2(\Omega)}. \quad (5)$$

The real ν and the constant C depend both upon the parameter $a(\cdot)$.

In complement, we need another useful result established by N. G. Meyers (see [25]). Therein is stated that $r_F(\cdot)$ belongs in fact to more regular Sobolev spaces. The lemma below is found in [13, Theorem 1].

Lemma 2.2 (Meyers, 1963) *Assume $F \in L^2(\Omega)$. Then, there exists $p_* > 2$ such that for all p with $2 < p < p_*$, the solution $r_F(\cdot)$ belongs to $W^{1,p}(\Omega)$ and*

$$\|r_F\|_{W^{1,p}(\Omega)} \leq C \|F\|_{L^2(\Omega)}. \quad (6)$$

The threshold number p_ depends upon the ellipticity parameter $a(\cdot)$. The constant C is dependent on p and on α .*

The derivation of the variational formulation for (1) relies on a duality argument (see, eg, [22]). Let $F(\cdot)$ be a test function in $L^2(\Omega)$ and $r_F(\cdot)$ be the unique solution to problem (4). The transposition implies the following equation : *find φ in $L^2(\Omega)$ such that*

$$\int_{\Omega} \varphi(\mathbf{x}) F(\mathbf{x}) \, d\mathbf{x} = \int_{\gamma} \lambda(\mathbf{s}) r_F(\mathbf{s}) \, d\mathbf{s}, \quad \forall F \in L^2(\Omega). \quad (7)$$

By Lemma 2.1, $r_F(\cdot)$ is continuous and the duality pairing in the right hand side makes sense.

¹Despite the fact that the function $\ln(\ln(r))$, where $r = \sqrt{(x_1)^2 + (x_2)^2}$, belongs to $H_{loc}^1(\mathbb{R}^3)$, it has not a trace along the x_3 -axis.

Lemma 2.3 *Assume that $\lambda \in L^2(\gamma)$. Then, problem (7) has a unique solution $\varphi \in L^2(\Omega)$ with*

$$\|\varphi\|_{L^2(\Omega)} \leq C\|\lambda\|_{L^2(\gamma)}.$$

Proof: The linear form

$$F \mapsto \langle f, r_F \rangle = \int_{\gamma} \lambda(\mathbf{s}) r_F(\mathbf{s}) d\mathbf{s},$$

is continuous on $L^2(\Omega)$. Indeed, according to the Di-Giorgi stability (5), we have that

$$\langle f, r_F \rangle \leq C\|\lambda\|_{L^2(\gamma)} \|r_F\|_{\mathcal{C}(\gamma)} \leq C\|\lambda\|_{L^2(\gamma)} \|F\|_{L^2(\Omega)}.$$

Existence and uniqueness are ensured by the Riesz representation Theorem. The proof is complete.

■

Remark 2.1 *The data $f(\cdot)$ may be a Borel measure on the curve γ . It belongs then to the dual space of $\mathcal{C}(\gamma)$ and may be even in $(\mathcal{C}^{0,\nu}(\gamma))'$, according to Lemma 2.1.*

The elliptic regularity shows that the solution $\varphi(\cdot)$ falls short to be in $H^1(\Omega)$. Given that $f(\cdot)$ is in $H^{(-1)-}(\Omega)$, the proof of the following result can be obtained by Hilbertian interpolation, as in [18].

Lemma 2.4 *The solution $\varphi(\cdot)$ belongs to $H^{(1)-}(\Omega)$ with*

$$\|\varphi\|_{H^{(1)-}(\Omega)} \leq C\|\lambda\|_{L^2(\gamma)}.$$

The constant C blows up if $(1)_- (= 1 - \varrho, \text{ for arbitrarily small } \varrho > 0)$ tends to 1.

When numerical approximations of problem (1) are intended, formulation (7) is of no practical use. Indeed, the linear form involved in there can hardly be implemented, problem (4) would be repeatedly solved to get $r_F(\cdot)$. This is highly expensive. A possible argument to circumvent this defect is to proceed following [3] (see also [30]). Extract the singular part in the potential $\varphi(\cdot)$ and then compute the correction as the solution of a discretized Poisson problem set in $H^1(\Omega)$. Of course, the process is operative and successful if we provide an explicit formula for the singular contribution to the solution.

To get a first insight on how to work out the expansion, consider a conductivity $a(\cdot)$ defined on the open space \mathbb{R}^3 , while still fulfilling (2). Denote by $G_a((\cdot), \mathbf{s})$ the fundamental solution of the operator $A_a = -\text{div}(a\nabla(\cdot))$ in \mathbb{R}^3 . Then, setting,

$$K_a(\cdot) = \int_{\gamma} \lambda(\mathbf{s}) G_a((\cdot), \mathbf{s}) d\mathbf{s},$$

it is readily checked out that $A_a(K_a) = f$, in \mathbb{R}^3 (see [22]). The potential $\varphi(\cdot)$ solution of (7) can hence be split up into singular/regular contributions according to

$$\varphi(\cdot) = K_a(\cdot) + \chi_a(\cdot). \tag{8}$$

The correction $\chi_a(\cdot)$ lies in $H^1(\Omega)$, takes care of the boundary condition through $\chi|_{\Gamma} = -K_a(\cdot)$ and is solution to the Laplace equation (4) (with $f = 0$). Doing so requires a closed form of $G_a((\cdot), \mathbf{s})$, most often out of access. More, it is a bivariate Green function, not convolutional $G_a((\cdot), \mathbf{s}) \neq G_a((\cdot) - \mathbf{s})$, when $a(\cdot)$ is space-dependent. Its computation will be consequently too expensive.

Earlier research have been engaged to derive expansions similar to (8), for pointwise Dirac sources in [3] (see [33, 4, 10]) and for linear sources in three dimensions [17, 15] and references therein. For the line sources, assumptions on the smoothness of the conductivity, more or less restrictive, have been introduced in most of those papers. The generality we have in mind needs to release some of them and reduces the others. We pursue the extension of the analysis achieved in [3], and to write down an alternative expansion to (8) by means of the more advantageous convolutional Green function $G_1((\cdot), \mathbf{s}) = G((\cdot) - \mathbf{s})$ of the Laplace operator ($A_1 = -\Delta(\cdot)$) (instead of $G_a((\cdot), \mathbf{s})$).

3 The Singular-Regular Splitting

In spite of its mathematical value, formula (8) does not bring any numerical advantage. An affordable remedy consists in operating some changes to it. This is why we extend to our problem the ideas developed in [3] and undertake the analysis to come up with the desired new expansion. To be specific, we aim to check out the following decomposition

$$\varphi(\mathbf{x}) = \int_{\gamma} \frac{\lambda(\mathbf{s})}{a(\mathbf{s})} G(\mathbf{x} - \mathbf{s}) d\mathbf{s} + \chi(\mathbf{x}) = \int_{\gamma} K(\mathbf{s}, \mathbf{x}) d\mathbf{s} + \chi(\mathbf{x}) = K(\mathbf{x}) + \chi(\mathbf{x}), \quad \forall \mathbf{x} \in \Omega. \quad (9)$$

The notation $K(\mathbf{s}, (\cdot))$ and $K(\cdot)$ are obvious. The correction is there to account for the boundary condition and will be hopefully solution to the 'regular' Poisson problem : *find* $\chi \in H^1(\Omega)$ *such that*

$$\begin{aligned} -\operatorname{div}(a\nabla\chi) &= \operatorname{div} \mathbf{g} && \text{in } \Omega, \\ \chi &= -K && \text{on } \Gamma. \end{aligned} \quad (10)$$

The data \mathbf{g} in (10) is provided by the integral

$$\mathbf{g}(\mathbf{x}) = \int_{\gamma} \mathbf{g}(\mathbf{s}, \mathbf{x}) d\mathbf{s} = \int_{\gamma} \frac{\lambda(\mathbf{s})}{a(\mathbf{s})} (a(\mathbf{x}) - a(\mathbf{s})) \nabla G(\mathbf{x} - \mathbf{s}) d\mathbf{s}, \quad \forall \mathbf{x} \in \Omega. \quad (11)$$

The preliminary action is to show, under minimal assumptions on the parameter $a(\cdot)$, that the correction $\chi(\cdot)$ lies in $H^1(\Omega)$ and is spared from any singularity.

Accordingly, and from now on, the terminology regular/singular will qualify functions that are/are not in $H^1(\Omega)$.

The objects introduced above have to make a mathematical sense. The singular function $K(\cdot)$ is the opening step. For $K(\cdot)$ to be defined, the least to expect is that the trace of $a(\cdot)$ on γ be determined. We therefore assume in the subsequent that $a(\cdot)$ is continuous at a vicinity ω_{γ} of the curve γ .

Lemma 3.1 *Assume that $a \in \mathcal{C}(\omega_{\gamma})$ and $\lambda \in L^2(\gamma)$. The singularity $K(\cdot)$ in formula (9) is well defined, in the sense that*

$$K(\cdot) = \int_{\gamma} \frac{\lambda(\mathbf{s})}{a(\mathbf{s})} G((\cdot) - \mathbf{s}) d\mathbf{s} \in L^2(\Omega).$$

Proof: The integral can be understood either in the sense of Bochner integrals or equivalently in the almost everywhere sense (as in Fubini's theorem) (see [31]).

The function to integrate

$$\mathbf{s} \mapsto K(\mathbf{s}, (\cdot)) = \frac{\lambda(\mathbf{s})}{a(\mathbf{s})} G((\cdot) - \mathbf{s}),$$

can be viewed as a function with values in $L^2(\Omega)$. In [31, Chapter 5], appropriate tools are proposed to assess the existence of $K(\cdot)$ as an integral in $L^2(\Omega)$ (see also [12]). One needs simply to show that $K(\mathbf{s}, (\cdot))$ is absolutely integrable on γ or stronger (more useful!), that it is square integrable

(in $L^2(\gamma, L^2(\Omega))$). To do so, let R be the diameter of Ω , meaning that $R = \sup_{\mathbf{x}, \mathbf{y} \in \Omega} |\mathbf{x} - \mathbf{y}|$. Then for all \mathbf{s} , the inclusion $\Omega \subset B(\mathbf{s}, R)$ implies the bound

$$\|G((\cdot) - \mathbf{s})\|_{L^2(\Omega)}^2 = \int_{\Omega} (G(\mathbf{x} - \mathbf{s}))^2 d\mathbf{x} \leq \frac{1}{(4\pi)^2} \int_{B(\mathbf{s}, R)} \frac{1}{|\mathbf{x} - \mathbf{s}|^2} d\mathbf{x} = \frac{R}{4\pi}.$$

Switching to the spherical coordinates system with the origin at point \mathbf{s} , the calculations become straightforward. Back to the integral of interest, we get that

$$\int_{\gamma} \|K(\mathbf{s}, (\cdot))\|_{L^2(\Omega)}^2 d\mathbf{s} \leq \frac{R}{\alpha^2} \|\lambda\|_{L^2(\gamma)}^2.$$

According to [31, Lemma 5.21], we conclude that the integral over γ is well defined in $L^2(\Omega)$ with

$$\|K\|_{L^2(\Omega)} \leq \int_{\gamma} \|K(\mathbf{s}, (\cdot))\|_{L^2(\Omega)} d\mathbf{s} \leq C \|\lambda\|_{L^2(\gamma)}.$$

The constant can be fixed to $C = \frac{\sqrt{R\ell}}{\alpha}$. The proof is complete. \blacksquare

Remark 3.1 *The continuity assumption on $a(\cdot)$ is made for simplicity. It may be weakened to a piecewise continuity for instance. It may be changed into $a \in W^{1,p}(\omega_\gamma)$ with $p > 2$ so that the trace $a|_\gamma \in L^p(\gamma)$.*

The study of the elliptic problem (10), for the corrector $\chi(\cdot)$, is subordinated to the statement $\chi = -K|_\Gamma$ for the Dirichlet condition. We need thus to corroborate that $K|_\Gamma$ belongs to $H^{1/2}(\Gamma)$.

Lemma 3.2 *Assume that $a \in \mathcal{C}(\omega_\gamma)$ and $\lambda \in L^2(\gamma)$. There holds that*

$$\|K\|_{H^{1/2}(\Gamma)} \leq C \|\lambda\|_{L^2(\gamma)}.$$

Proof: The kernel $K(\cdot)$ belongs to $H_{loc}^1(\Omega')$ provided that $\overline{\Omega'} \subset \mathbb{R}^3 \setminus \gamma$. The proof is thus straightforward by means of the trace theorem. \blacksquare

We turn to the data $\mathbf{g}(\cdot)$. Further smoothness is mandatory on the conductivity. Assume (only for a while) that $a \in W^{\varrho, \infty}(\omega_\gamma)$ with $\varrho > 1/2$. Define the distance $d > 0$ that separates γ from $\Omega \setminus \omega_\gamma$, then we set

$$\|a\|_{\varrho, \infty} = \max(\|a\|_{W^{\varrho, \infty}(\omega_\gamma)}, d^{-\varrho} \|a\|_{L^\infty(\Omega)}).$$

Therefore, it is easily seen that

$$|a(\mathbf{x}) - a(\mathbf{s})| \leq \|a\|_{\varrho, \infty} |\mathbf{x} - \mathbf{s}|^\varrho, \quad \forall (\mathbf{x}, \mathbf{s}) \in \Omega \times \gamma. \quad (12)$$

Lemma 3.3 *Assume that $a \in W^{\varrho, \infty}(\omega_\gamma)$, for $\varrho > 1/2$ and $\lambda \in L^2(\gamma)$. Then $\mathbf{g}(\cdot)$ defined in (11) belongs to $(L^2(\Omega))^3$ and*

$$\|\mathbf{g}\|_{(L^2(\Omega))^3} \leq C \|\lambda\|_{L^2(\gamma)}.$$

Proof: We need to know whether $\mathbf{s} \mapsto \|g(\mathbf{s}, \cdot)\|_{(L^2(\Omega))^3}$ belongs to $L^2(\gamma)$. We start by observing that

$$\|(a(\cdot) - a(\mathbf{s}))\nabla G((\cdot) - \mathbf{s})\|_{(L^2(\Omega))^3}^2 \leq \frac{1}{(4\pi)^2} \int_{B(\mathbf{s}, R)} \frac{|a(\mathbf{x}) - a(\mathbf{s})|^2}{|\mathbf{x} - \mathbf{s}|^4} d\mathbf{x}.$$

Owing to (12) we get

$$\|(a(\cdot) - a(\mathbf{s}))\nabla G((\cdot) - \mathbf{s})\|_{(L^2(\Omega))^3}^2 \leq \frac{\|a\|_{\varrho, \infty}^2}{(4\pi)^2} \int_{B(\mathbf{s}, R)} |\mathbf{x} - \mathbf{s}|^{-4+2\varrho} d\mathbf{x}.$$

Resorting again to the spherical coordinates we state

$$\|(a(\cdot) - a(\mathbf{s}))\nabla G((\cdot) - \mathbf{s})\|_{(L^2(\Omega))^3} \leq \frac{\|a\|_{\varrho, \infty} R^{\varrho-1/2}}{\sqrt{\varrho-1/2}}.$$

Processing following the line of the previous proof, we conclude to

$$\|g(\cdot)\|_{(L^2(\Omega))^3} \leq \sqrt{\ell} \|g(\mathbf{s}, \cdot)\|_{L^2(\gamma, L^2(\Omega)^3)} \leq \frac{\sqrt{\ell}}{\alpha} \left(\frac{R^{\varrho-1/2}}{\sqrt{\varrho-1/2}} \right) \|a\|_{\varrho, \infty} \|\lambda\|_{L^2(\gamma)} \leq C(\varrho) \|\lambda\|_{L^2(\gamma)}.$$

The dependence upon (R, ϱ) is kept there on purpose. Accordingly, ϱ cannot be $1/2$ nor can R be infinity. The result is stated. \blacksquare

Lemma 3.4 *Assume that $a \in W^{\varrho, \infty}(\omega_\gamma)$, for $\varrho > 1/2$ and $\lambda \in L^2(\gamma)$. Problem (10) has a unique solution $\chi \in H^1(\Omega)$ with*

$$\|\chi\|_{H^1(\Omega)} \leq C \|\lambda\|_{L^2(\gamma)}.$$

Proof: The variational formulation of (10) reads as: *find $\chi \in H^1(\Omega)$ with $\chi|_\Gamma = -K$ and such that*

$$\int_{\Omega} a \nabla \chi \nabla \psi \, d\mathbf{x} = - \int_{\Omega} \mathbf{g} \nabla \psi \, d\mathbf{x}, \quad \forall \psi \in H_0^1(\Omega). \quad (13)$$

The result is directly deduced from the estimates in Lemmas 3.2 and 3.3. The proof is complete. \blacksquare

All ingredients are at our disposal to assess the expansion (9) for some regular conductivities. The result will be extended later on to a wider class of conductivities.

Proposition 3.5 *Assume that $a \in W^{\varrho, \infty}(\omega_\gamma)$, for $\varrho > 1/2$ and $\lambda \in L^2(\gamma)$. Let $\varphi \in L^2(\Omega)$ be the solution of problem (7). Then, expansion (9) holds true.*

4 From Pointwise to Linear Sources

The mathematical proof of Proposition 3.5 and then the expansion (9) is the purpose. As suggested earlier, the basic tool comes from the representation of the linear source as a dense cloud of point-wise Dirac sources. For each point source, we use the singular-regular expansion shown in [3, Proposition 3.2], before going back to the linear sources, by practicing a summation. In a way we start from the following integration (in the distributional sense)

$$\langle f, r_F \rangle_\gamma = \int_\gamma \lambda(\mathbf{s}) \langle \delta_{\mathbf{x}-\mathbf{s}}, r_F \rangle d\mathbf{s} = \int_\gamma \lambda(\mathbf{s}) r_F(\mathbf{s}) d\mathbf{s}, \quad \forall F \in L^2(\Omega).$$

Before stepping any further, let $(\mathbf{s}, \lambda(\mathbf{s})) \in \gamma \times \mathbb{R}$ be given. We consider then problem (1), where the data is radiated by the pointwise Dirac source $(\lambda(\mathbf{s})\delta_{\mathbf{x}-\mathbf{s}})$. The variational problem can be expressed analogously to (7) : *find $\varphi(\mathbf{s})$ in $L^2(\Omega)$ such that*

$$\int_\Omega \varphi(\mathbf{s}, \mathbf{x}) F(\mathbf{x}) d\mathbf{x} = \lambda(\mathbf{s}) r_F(\mathbf{s}), \quad \forall F \in L^2(\Omega). \quad (14)$$

That (14) has only one solution is established in [3, Lemma 2.2] and we have that

$$\|\varphi(\mathbf{s})\|_{L^2(\Omega)} \leq C|\lambda(\mathbf{s})|. \quad (15)$$

It is of prime importance to notice that *the constant C is the very constant of Lemma 2.1, is universal and does not depend neither on \mathbf{s} nor on $\lambda(\mathbf{s})$* . The process to be permanently used shows finally how to derive, by an integration argument, the solution of (7) from the one of (14).

Lemma 4.1 *Assume that $\lambda \in L^2(\gamma)$. Then, the solution $\varphi(\cdot)$ of problem (7) is given by the Bochner integral*

$$\varphi(\cdot) = \int_\gamma \varphi(\mathbf{s}, (\cdot)) d\mathbf{s} \in L^2(\Omega).$$

Proof: For the existence of that integral almost everywhere, showing that the bivariate function $(\mathbf{s}, \mathbf{x}) \mapsto \varphi(\mathbf{s}, \mathbf{x})$ is square integrable (and therefore absolutely integrable) in $\gamma \times \Omega$ is enough. The stability identity (15) implies that

$$\int_\gamma \|\varphi(\mathbf{s})\|_{L^2(\Omega)}^2 d\mathbf{s} \leq C\|\lambda\|_{L^2(\gamma)}^2.$$

Denote for a while $\psi \in L^2(\Omega)$ that integral. Integrating the variational problem (14), with respect to \mathbf{s} , and using Fubini theorem yields

$$\int_\Omega \psi F d\mathbf{x} = \int_\gamma \lambda(\mathbf{s}) r_F(\mathbf{s}) d\mathbf{s} = \langle f, r_F \rangle_\gamma, \quad \forall F \in L^2(\Omega).$$

This tells that $\psi(\cdot)$ satisfies problem (7). The uniqueness concludes to the equality $\psi(\cdot) = \varphi(\cdot)$. The proof is complete. ■

According to [3, Proposition 3.2], provided that $a \in W^{\varrho, \infty}(\omega_\gamma)$ with $\varrho > 1/2$, we have the following

$$\varphi(\mathbf{s}, (\cdot)) = \frac{\lambda(\mathbf{s})}{a(\mathbf{s})} G((\cdot) - \mathbf{s}) + \chi(\mathbf{s}, (\cdot)) = K(\mathbf{s}, (\cdot)) + \chi(\mathbf{s}, (\cdot)), \quad \forall \mathbf{s} \in \gamma. \quad (16)$$

The correction term $\chi(\mathbf{s}, (\cdot))$ belongs to $H^1(\Omega)$, to which the Dirichlet boundary condition $\chi(\mathbf{s}, (\cdot)) = -K(\mathbf{s}, (\cdot))|_\Gamma$ is prescribed and fulfills the regular variational formulation,

$$\int_{\Omega} a \nabla \chi(\mathbf{s}, (\cdot)) \nabla \psi \, d\mathbf{x} = - \int_{\Omega} \mathbf{g}(\mathbf{s}, (\cdot)) \nabla \psi \, d\mathbf{x}, \quad \forall \psi \in H_0^1(\Omega). \quad (17)$$

The data $\mathbf{g}(\mathbf{s}, (\cdot))$ is defined as in (11).

The aim is twofold, the derivation of (9) from (16) and then the discussion about the conductivity assumption to hopefully weaken it as far as possible.

Proof of Proposition 3.5: Integrate (16) over γ , and using results of Lemmas 3.1 and 4.1, we obtain

$$\varphi(\cdot) = K(\cdot) + \int_{\gamma} \chi(\mathbf{s}, (\cdot)) \, d\mathbf{s}.$$

To establish that the correction integral coincides with $\chi(\cdot)$, we integrate problem (17), proceed according the proof of Lemmas 4.1, after invoking the results of Lemma 3.3 and 3.2. The proof is complete. \blacksquare

Remark 4.1 *Be aware that, for almost every $\mathbf{s} \in \gamma$, the singular part $K(\mathbf{s}, (\cdot))$ belongs to $H^{(1/2)-}(\Omega)$, while $K(\cdot)$ has a higher smoothness, as it belongs to $H^{(1)-}(\Omega)$.*

5 Conductivities with Limited Smoothness

The smoothness required on the conductivity $a(\cdot)$, for the expansion (9) to still hold true, is the issue we tackle. Can it be reduced? To what extent? The answer to the first question is : Yes! In a large extent, to answer the other question.

Theorem 5.1 *Assume $a \in W^{\varrho, \infty}(\omega_\gamma)$ for $\varrho > 0$ and $\lambda \in L^2(\gamma)$. The singular/regular expansion (9) is still valid.*

Basically the proof uses a density argument for the spaces the conductivities belong to. So, we need some continuity of the non linear map $a \mapsto (\varphi, K, \chi)$. The target space for (φ, K, χ) being still $L^2(\Omega) \times L^2(\Omega) \times H^1(\Omega)$, the starting space cannot be simply $L^\infty(\Omega)$. Neither can it be $L^\infty(\Omega) \cap W^{\varrho, \infty}(\omega_\gamma)$, as a Banach space, because of the specific density of the canonical embeddings in Hölder spaces (see [32]). As will be seen, endowing it with the norm of $L^\infty(\Omega) \cap W^{\beta, \infty}(\omega_\gamma)$, with $\beta < \varrho$, will meet our purposes. Be aware that this norm fails to provide $L^\infty(\Omega) \cap W^{\varrho, \infty}(\omega_\gamma)$ with a Banach structure. This will not harm the proof we are elaborating. Most of the sequel is dedicated to studying this operator. Some adaptations of the notations are mandatory. They will be fixed as we go along.

5.1 Preliminary Lemmas

We consider the closed convex subset of $L^\infty(\Omega)$ of conductivities fulfilling condition (2),

$$D_\alpha = \{a(\cdot) \in L^\infty(\Omega), \quad a(\cdot) \geq \alpha\}.$$

For a given $a \in D_\alpha$, we set the notation φ_a for the solution φ of (7). Moreover, if $F \in L^2(\Omega)$, the solution to (4) will be called $r_a(F)$ instead of r_F . In some places, when no confusion is feared we simply write r_a (idem for K_a, \mathbf{g}_a and χ_a).

Lemma 5.2 *The operator $a \mapsto \varphi_a$ is a Lipschitz mapping from D_α into $L^2(\Omega)$, that is*

$$\|\varphi_a - \varphi_b\|_{L^2(\Omega)} \leq C \|a - b\|_{L^\infty(\Omega)} \|\lambda\|_{L^2(\gamma)}, \quad \forall a, b \in D_\alpha.$$

The constant C depends on α and ℓ the length of γ .

Proof: Observe that in view of formulation (7), we obtain that

$$\int_{\Omega} (\varphi_a - \varphi_b)(\mathbf{x}) F(\mathbf{x}) d\mathbf{x} = \int_{\gamma} \lambda(\mathbf{s})(r_a - r_b)(\mathbf{s}) d\mathbf{s}, \quad \forall F \in L^2(\Omega). \quad (18)$$

The key result is therefore to bound appropriately the norm $\|r_a - r_b\|_{L^2(\gamma)}$. Let $p \in]2, p_*[$ be as in Lemma 2.2 and $q < 2$ its conjugate. We have $(r_a - r_b) \in W_0^{1,p}(\Omega)$ obeys the following variational

identity

$$\int_{\Omega} a \nabla(r_a - r_b) \nabla \psi \, d\mathbf{x} = \int_{\Omega} (b - a) \nabla r_b \nabla \psi \, d\mathbf{x}, \quad \forall \psi \in W_0^{1,q}(\Omega). \quad (19)$$

Following the proof of [25, Theorem 1] —*in particular the inf-sup condition stated in there*—, it comes out that

$$\|r_a - r_b\|_{W^{1,p}(\Omega)} \leq C \|a - b\|_{L^\infty(\Omega)} \|r_b\|_{W^{1,p}(\Omega)}.$$

The constant C depends upon α . Applying Lemma 2.2 on $r_b(\cdot)$ yields that

$$\|r_a - r_b\|_{W^{1,p}(\Omega)} \leq C \|a - b\|_{L^\infty(\Omega)} \|F\|_{L^2(\Omega)}.$$

Given that that $p > 2$, by the trace theorem functions in $W^{1,p}(\Omega)$ have traces on the curve γ (those in $H^1(\Omega)$ don't) (see [2]). Hence, $(r_a - r_b)|_\gamma \in W^{1-\frac{2}{p},p}(\gamma)$ and we get in particular

$$\|r_b - r_a\|_{L^2(\gamma)} \leq C \|r_b - r_a\|_{W^{1,p}(\Omega)} \leq C \|a - b\|_{L^\infty(\Omega)} \|F\|_{L^2(\Omega)}.$$

Back to (18), the above bound is enough to deduce that

$$\|\varphi_a - \varphi_b\|_{L^2(\Omega)} \leq C \|a - b\|_{L^\infty(\Omega)} \|\lambda\|_{L^2(\gamma)}.$$

The proof is complete. ■

The continuity of the operator $a \mapsto K_a$ determines the integral over the curve γ , in (9). Here again, the Lipschitz-continuity can be obtained.

Lemma 5.3 *The operator $a \mapsto K_a$ is Lipschitzian $D_\alpha \cap \mathcal{C}(\omega_\gamma)$ into $L^2(\Omega)$, that is*

$$\|K_a - K_b\|_{L^2(\Omega)} \leq C \|a - b\|_{L^\infty(\Omega)} \|\lambda\|_{L^2(\gamma)}, \quad \forall a, b \in D_\alpha \cap \mathcal{C}(\omega_\gamma).$$

The constant C depends on α and ℓ .

Proof: The proof follows the spirit of Lemma 3.1. Using Bochner Theorem we have

$$\|K_a - K_b\|_{L^2(\Omega)} \leq \int_{\gamma} |\lambda(\mathbf{s})| \left| \frac{1}{a(\mathbf{s})} - \frac{1}{b(\mathbf{s})} \right| \|G((\cdot) - \mathbf{s})\|_{L^2(\Omega)} \, d\mathbf{s}.$$

Reproducing the computations in the proof of Lemma 3.1, we deduce that —recall that R is for diam Ω .

$$\|K_a - K_b\|_{L^2(\Omega)} \leq \frac{\sqrt{R\ell}}{\alpha^2} \|\lambda\|_{L^2(\gamma)} \|a - b\|_{L^\infty(\Omega)}.$$

The proof is complete. ■

We come now to the continuity of the map $a \mapsto K_a(\cdot)|_\Gamma$. This will be useful for the Dirichlet condition enforced on the correction function χ_a .

Lemma 5.4 *The operator $a \mapsto (K_a)|_\Gamma$ is of Lipschitz type from $D_\alpha \cap \mathcal{C}(\omega_\gamma)$ into $H^{1/2}(\Gamma)$,*

$$\|K_a - K_b\|_{H^{1/2}(\Gamma)} \leq C \|a - b\|_{L^\infty(\Omega)} \|\lambda\|_{L^2(\gamma)}, \quad \forall a, b \in D_\alpha \cap \mathcal{C}(\omega_\gamma).$$

The constant is so that $C = C(\alpha)$.

Proof: Proceeding as in Lemma 5.4 and using the fact that $\sup_{s \in \gamma} \|G((\cdot) - s)\|_{H^{1/2}(\Gamma)}$ is finite (stated in the proof of Lemma 3.2) gives the result. \blacksquare

Lemma 5.5 *The operator $a \mapsto \mathbf{g}_a$, is a locally Lipschitz mapping from $D_\alpha \cap W^{\varrho, \infty}(\omega_\gamma)$ into $L^2(\Omega)^3$,*

$$\|\mathbf{g}_a - \mathbf{g}_b\|_{L^2(\Omega)^3} \leq C \|a - b\|_{\varrho, \infty} \|\lambda\|_{L^2(\gamma)}, \quad \forall a, b \in D_\alpha \cap W^{\varrho, \infty}(\omega_\gamma).$$

The constant is such that $C = C(\alpha, \varrho, \|a\|_{\varrho, \infty})$.

Proof: Denote $\kappa = (a - b)$. Hence, the difference $(\mathbf{g}_b - \mathbf{g}_a)$ is the sum of two contributions

$$\begin{aligned} (\mathbf{g}_b - \mathbf{g}_a)(\mathbf{x}) &= \int_\gamma \frac{\lambda \kappa}{ab}(\mathbf{s})(a(\mathbf{x}) - a(\mathbf{s})) \nabla G(\mathbf{x} - \mathbf{s}) d\mathbf{s} \\ &\quad - \int_\gamma \frac{\lambda}{b}(\mathbf{s})(\kappa(\mathbf{x}) - \kappa(\mathbf{s})) \nabla G(\mathbf{x} - \mathbf{s}) d\mathbf{s} = \mathbf{g}_1(\mathbf{x}) + \mathbf{g}_2(\mathbf{x}). \end{aligned} \quad (20)$$

Each integral function $\mathbf{g}_1(\cdot)$ and $\mathbf{g}_2(\cdot)$ is handled apart.

(i.) The L^2 -norm of $\mathbf{g}_1(\cdot)$ can be expressed by Fubini's theorem

$$\begin{aligned} \|\mathbf{g}_1\|_{(L^2)^3}^2 &= \int_{\gamma \times \gamma} \frac{\lambda \kappa}{ab}(\mathbf{s}) \frac{\lambda \kappa}{ab}(\boldsymbol{\sigma}) \left(\int_\Omega (a(\mathbf{x}) - a(\mathbf{s}))(a(\mathbf{x}) - a(\boldsymbol{\sigma})) \nabla G(\mathbf{x} - \mathbf{s}) \nabla G(\mathbf{x} - \boldsymbol{\sigma}) d\mathbf{x} \right) d\mathbf{s} d\boldsymbol{\sigma} \\ &\leq \frac{1}{\alpha^4} \|\kappa\|_{L^\infty(\Omega)}^2 \int_{\gamma \times \gamma} |\lambda(\mathbf{s}) \lambda(\boldsymbol{\sigma})| \left(\int_\Omega \frac{|a(\mathbf{x}) - a(\mathbf{s})| |a(\mathbf{x}) - a(\boldsymbol{\sigma})|}{|\mathbf{x} - \mathbf{s}|^2 |\mathbf{x} - \boldsymbol{\sigma}|^2} d\mathbf{x} \right) d\mathbf{s} d\boldsymbol{\sigma}. \end{aligned}$$

Owing to Lemma 8.1 (in the appendix) we get

$$\|\mathbf{g}_1\|_{(L^2)^3}^2 \leq \frac{C}{\alpha^4} \|\kappa\|_{L^\infty(\Omega)}^2 \|a\|_{\varrho, \infty}^2 \int_{\gamma \times \gamma} \frac{|\lambda(\mathbf{s}) \lambda(\boldsymbol{\sigma})|}{|\mathbf{s} - \boldsymbol{\sigma}|^{1-2\varrho}} d\mathbf{s} d\boldsymbol{\sigma}.$$

Applying Lemma 8.2 ends to

$$\|\mathbf{g}_1\|_{(L^2)^3} \leq C \|b - a\|_{L^\infty(\Omega)} \|\lambda\|_{L^2(\gamma)}.$$

The constant is such that $C = C(\varrho, \alpha, \|a\|_{\varrho, \infty})$.

(ii.) Switching to the second contribution $\mathbf{g}_2(\cdot)$, we start from

$$\begin{aligned} \|\mathbf{g}_2\|_{(L^2)^3}^2 &= \int_{\gamma \times \gamma} \frac{\lambda}{b}(\mathbf{s}) \frac{\lambda}{b}(\boldsymbol{\sigma}) \left(\int_\Omega (\kappa(\mathbf{x}) - \kappa(\mathbf{s}))(\kappa(\mathbf{x}) - \kappa(\boldsymbol{\sigma})) \nabla G(\mathbf{x} - \mathbf{s}) \nabla G(\mathbf{x} - \boldsymbol{\sigma}) d\mathbf{x} \right) d\mathbf{s} d\boldsymbol{\sigma} \\ &\leq \frac{1}{\alpha^2} \int_{\gamma \times \gamma} |\lambda(\mathbf{s}) \lambda(\boldsymbol{\sigma})| \left(\int_\Omega \frac{|\kappa(\mathbf{x}) - \kappa(\mathbf{s})| |\kappa(\mathbf{x}) - \kappa(\boldsymbol{\sigma})|}{|\mathbf{x} - \mathbf{s}|^2 |\mathbf{x} - \boldsymbol{\sigma}|^2} d\mathbf{x} \right) d\mathbf{s} d\boldsymbol{\sigma}. \end{aligned}$$

Invoking once again Lemmas 8.1 and 8.2 as above yields that

$$\|\mathbf{g}_2\|_{(L^2)^3} \leq C \|b - a\|_{\varrho, \infty} \|\lambda\|_{L^2(\gamma)}.$$

The constant is such that $C = C(\varrho, \alpha)$.

Notice that the operator $a \mapsto \mathbf{g}_a(\cdot)$ is locally Lipschitz because the constant in step (i.) depends upon $\|a\|_{\varrho, \infty}$. The proof is complete. \blacksquare

Lemma 5.6 *The operator $a \mapsto \chi_a(\cdot)$, is a locally Lipschitz mapping from $D_\alpha \cap W^{\varrho, \infty}(\omega_\gamma)$ into $H^1(\Omega)^3$,*

$$\|\chi_a - \chi_b\|_{H^1(\Omega)} \leq C \|a - b\|_{\varrho, \infty} \|\lambda\|_{L^2(\gamma)}, \quad \forall a, b \in D_\alpha \cap W^{\varrho, \infty}(\omega_\gamma).$$

The constant C is as follows $C = C(\alpha, \varrho, \|a\|_{\varrho, \infty}, \|b\|_{\varrho, \infty})$. The dependence with respect to the two last arguments is linear.

Proof: The Dirichlet condition $(\chi_a - \chi_b)|_\Gamma = -(K_a - K_b)|_\Gamma$ should be imposed before writing down the elliptic variational equation

$$\int_{\Omega} a \nabla(\chi_a - \chi_b) \nabla \psi \, d\mathbf{x} = \int_{\Omega} (b - a) \nabla \chi_b \nabla \psi \, d\mathbf{x} - \int_{\Omega} (\mathbf{g}_a - \mathbf{g}_b) \nabla \psi \, d\mathbf{x}, \quad \forall \psi \in H_0^1(\Omega).$$

A direct consequence is the following stability

$$\begin{aligned} \|\chi_a - \chi_b\|_{H^1(\Omega)} &\leq C (\|\chi_b\|_{H^1(\Omega)} \|a - b\|_{L^\infty(\Omega)} + \|K_a - K_b\|_{H^{1/2}(\Gamma)} + \|\mathbf{g}_a - \mathbf{g}_b\|_{(L^2(\Omega))^3}) \\ &\leq C (\|b\|_{\varrho, \infty} \|a - b\|_{L^\infty(\Omega)} + \|K_a - K_b\|_{H^{1/2}(\Gamma)} + \|\mathbf{g}_a - \mathbf{g}_b\|_{(L^2(\Omega))^3}). \end{aligned}$$

The proof is complete owing to Lemmas 5.4 and 5.5. \blacksquare

5.2 Proof of Theorem 5.1

Let $a(\cdot)$ be in $W^{\varrho, \infty}(\omega_\gamma) \cap D_\alpha$ with $\varrho > 0$. The Sobolev (Hölder) space $W^{1/2+\varrho, \infty}(\omega_\gamma)$ is more regular and is dense in $W^{\varrho, \infty}(\omega_\gamma)$ with respect to the norm of $W^{\beta, \infty}(\omega_\gamma)$ ($\beta < \varrho$) (see [32]) —*It is important to know that the density is lost if $W^{\varrho, \infty}(\omega_\gamma)$ is endowed with its own norm and then with its own Banach structure.*

One can find a sequence $(a_n(\cdot))_{n \geq 0} \subset W^{1/2+\varrho, \infty}(\omega_\gamma) \cap D_\alpha$ that converges towards $a(\cdot)$ in $W^{\beta, \infty}(\omega_\gamma) \cap L^\infty(\Omega)$. It is easy to manage so that $(a_n(\cdot))_{n \geq 0}$ belongs to $D_{\alpha/2}$. Working in the larger set $W^{\beta, \infty}(\Omega) \cap D_{\alpha/2}$ enables the whole preparatory Lemmas 5.3, 5.4, 5.5 and 5.6 stated above. We have the following convergence in the product space $L^2(\Omega) \times L^2(\Omega) \times (L^2(\Omega))^3 \times H^{1/2}(\Gamma) \times H^1(\Omega)$

$$\lim_{n \rightarrow \infty} (\varphi_{a_n}, K_{a_n}, \mathbf{g}_{a_n}, K_{a_n}|_\Gamma, \chi_{a_n}) = (\varphi_a, K_a, \mathbf{g}_a, K_a|_\Gamma, \chi_a).$$

Based on Proposition 3.5, expansion (9) takes place for the regular conductivity $a_n(\cdot)$. Going thus to the limit makes the same expansion to be true for the less regular conductivity $a(\cdot)$. The proof is complete. \blacksquare

6 Miscellaneous

The purpose is to add few comments on the expansion (9) and discuss some possible extensions so as an alternative.

6.1 The Neumann Condition

The Dirichlet condition on the potential $\varphi(\cdot)$ may be replaced by a Neumann condition, that is

$$(a\partial_{\mathbf{n}})\varphi = 0 \quad \text{on } \Gamma.$$

We consider here that the full boundary is subjected to the given flux condition. The non-uniqueness caused by spurious constants is a classical issue, successfully handled in a well mastered way.

It is of an equal interest to look at this case carefully. The analysis carried out here can be undertaken following the same technical process. The singular contribution $K(\cdot)$ is the same as in (9). Conversely, a small modification is introduced in the variational problem determining the correction function $\chi(\cdot)$, to account for the right boundary condition. One can check out that the new formulation reads as

$$\int_{\Omega} a \nabla \chi \nabla \psi \, d\mathbf{x} = - \int_{\Omega} \mathbf{g} \cdot \nabla \psi \, d\mathbf{x} - \int_{\Gamma} g_N \psi \, d\boldsymbol{\tau}, \quad \forall \psi \in H^1(\Omega).$$

The vector data \mathbf{g} is the one provided in (11). The Neumann condition g_N is defined by

$$g_N(\mathbf{x}) = \int_{\gamma} \lambda(\mathbf{s}) \partial_{\mathbf{n}} G(\mathbf{x} - \mathbf{s}) \, d\mathbf{s}, \quad \text{in } \Gamma_N. \quad (21)$$

The overall results elaborated can be stated in this new context. To show the validity of expansion (9), only few adaptations are necessary, they are all related to the Neumann condition $g_N(\cdot)$.

6.2 An Alternative Method

Another way to proceed is possible for smooth conductivities, when it comes to the computation of the right-hand-side of problem (10) (or (13)). This alternative may be practical and spare users from long integral computations of (11), when exactness is sought for.

First, let us set that the singular function $K(\cdot)$ is in fact solution to the following Poisson problem

$$-\Delta K = \frac{\lambda}{a} \delta_{\gamma} (= \frac{1}{a} f), \quad \text{in } \Omega. \quad (22)$$

The framework where to check out this claim is provided by the dual variational formulation (7), when the conductivity is unity, $a(\cdot) = 1$. Let then $F \in L^2(\Omega)$ be given and $r_F(\cdot)$ be defined as in (4). Then

$$\int_{\Omega} K(\mathbf{x}) F(\mathbf{x}) \, d\mathbf{x} = \int_{\gamma} \frac{\lambda(\mathbf{s})}{a(\mathbf{s})} \left(\int_{\Omega} G(\mathbf{x} - \mathbf{s}) (-\Delta r_F(\mathbf{x})) \, d\mathbf{x} \right) d\mathbf{s} = \int_{\gamma} \frac{\lambda(\mathbf{s})}{a(\mathbf{s})} r_F(\mathbf{s}) \, d\mathbf{s}.$$

The next step is conducted formally (can be made variationally!). A special care should be paid to the boundary conditions. Plugging expansion (9), we have

$$-\operatorname{div}(a\nabla\varphi) = -\operatorname{div}(a\nabla K) - \operatorname{div}(a\nabla\chi) = a(-\Delta K) - (\nabla a)(\nabla K) - \operatorname{div}(a\nabla\chi),$$

completed by the boundary condition $\chi|_{\Gamma} = -K$. Then, on account of (22), we derive that

$$-\operatorname{div}(a\nabla\chi) = (\nabla a)(\nabla K), \quad \text{in } \Omega. \quad (23)$$

A variational formulation written in $H^1(\Omega)$ makes sense as soon as the data $(\nabla a)(\nabla K)$ belongs to $H^{-1}(\Omega)$.

Be aware that for non-smooth conductivities the correct formulation of the correction problem is the one given in (13). For example, if $a(\cdot)$ has jumps, problem (23) fail to provide the right $\chi(\cdot)$.

6.3 Two Dimensional Domains

Things are different in two dimensions. Changes are driven by the properties of the Sobolev spaces which are known to be closely related to the dimension. The line source belongs in fact to $H^{-1}(\Omega)$. Functions in $H^1(\Omega)$ have indeed traces on the Lipschitz curve γ . They belong to $H^{1/2}(\gamma)$. As a result, the regular variational formulation (4) works and the potential φ lies in $H_0^1(\Omega)$. There is no need of the dual formulation (7). In spite of this statement, the expansion (9) remains valid, though probably less useful. In fact, its interpretation has not the same effects. The smoothness of both $K(\cdot)$ and $\chi(\cdot)$ are modified dramatically.

Observe first that, in two dimensions, the ‘‘supposed’’ singular part $K(\cdot)$ belongs actually to $H^1(\Omega)$. Moreover, it is easy to see, that the source term $\frac{f}{a}(\cdot)$ is in $H^{-(1/2)+}(\Omega) = H^{(-1/2)-}(\Omega)$. The elliptic regularity (for the Laplacian) proves that $K \in H^{(3/2)-}(\Omega)$. This is tightly linked to the structure and the nature of the line source distribution. Now, we turn to the potential $\varphi(\cdot)$, if no more regularity is put on $a(\cdot)$, it only belongs to $H^1(\Omega)$. This says in particular that the correction $\chi(\cdot)$ cannot be more regular than $\varphi(\cdot)$ while $K(\cdot)$ is really smoother.

Assume now that $a \in W^{\varrho, \infty}(\Omega)$. Then, after a careful investigation of the data \mathbf{g} in equation (13), we conclude, again in the frame of the elliptic regularity, that the smoothness of the correction function $\chi(\cdot)$, solution to equation (13), cannot exceed $H^{1+\varrho}(\Omega)$. So, in two dimensions again the allegedly singularity $K(\cdot)$ is smoother than the (expected to be) regular correction $\chi(\cdot)$, when ϱ is small. That exponent ϱ needs to grow larger than 1/2 to see the regularity properties of both contributions inverted. To summarize, $K(\cdot)$ is smoother for low regular conductivities and $\chi(\cdot)$ becomes smoother if the conductivities are regular enough, when $\varrho \geq 1/2$.

7 Examples

The numerical assessment of expansion (9) is the purpose, for some selected conductivities. Our computational results may be viewed as a generalization of the work in [17]. Recall the fact that conductivities in there are assumed to be in $W^{2,\infty}(\Omega)$, too stringent from our point of view. We choose not to confront the singularity extraction process to the standard finite element method applied to the H^1 -variational formulation⁽²⁾ of problem (1). This task has been successfully carried out in [17] and in [3], highlighting the numerical advantages brought by the expansion method.

The domain of computation is the vertical cylinder with radius $R = 0.5$ and height $H = 0.5$, it is the solid of revolution around the z -axis, defined by $(r, z) \in [0, R] \times [0, H]$. The segment $\gamma = \{0\} \times [0, H]$, support of the Dirac source, is the axis of the cylinder. The cylindrical coordinates system (r, θ, z) is naturally retained here.

Regular conductivity — Let α, ϵ be a given couple of real numbers with $0 < \epsilon < \alpha$ and we fix a real-number $q \in \mathbb{R}$. The variable conductivity is therefore defined by

$$a(\mathbf{x}) = \frac{1+z}{\alpha + \epsilon \exp(qr)}, \quad \text{in } \Omega.$$

The solution has a symmetry of revolution; it is set to be ⁽³⁾

$$\varphi(\mathbf{x}) = (1+z)\eta(r) = (1+z) \left(-\frac{\alpha}{2\pi} \ln r + \frac{\epsilon}{2\pi} (\text{Ei}(q) - \text{Ei}(qr)) \right).$$

This potential is generated by a source term $f(\cdot)$ to be plugged in equation (1) —that can be computed to

$$f(\mathbf{x}) = (1+z)^2 \delta_\gamma - \frac{\eta(r)}{\alpha + \epsilon \exp(qr)}, \quad \text{in } \Omega.$$

As predicted, the solution fails to be in $H^1(\Omega)$ because of the logarithmic term.

The first step is to illustrate that the contribution $K(\cdot)$ in (9) succeeds in fully capturing the singular behavior of the solution $\varphi(\cdot)$, caused by the linear Dirac contribution to the data $f(\cdot)$. Prior to compute $K(\cdot)$, we mention that the points of γ are characterized by their own cylindrical coordinates $\mathbf{s} = (\rho = 0, t)$ while \mathbf{x} is specified by (r, z) , as above. We start from

$$K(\mathbf{x}) = \frac{1}{4\pi} \int_\gamma \frac{(1+t)^2}{a(\mathbf{s})} \frac{1}{|\mathbf{x} - \mathbf{s}|} d\mathbf{s} = \frac{\alpha + \epsilon}{4\pi} \int_{[0,1]} \frac{1+t}{\sqrt{r^2 + (z-t)^2}} dt.$$

All calculations carried out produce the expression

$$\begin{aligned} K(\mathbf{x}) = & \frac{\alpha + \epsilon}{4\pi} (\sqrt{r^2 + (1-z)^2} - \sqrt{r^2 + z^2}) + \frac{\alpha + \epsilon}{4\pi} (1+z) \ln(\sqrt{r^2 + (z-1)^2} + (1-z)) \\ & + \frac{\alpha + \epsilon}{4\pi} (1+z) \ln(\sqrt{r^2 + z^2} + z) - \frac{\alpha + \epsilon}{2\pi} (1+z) \ln r. \end{aligned} \quad (24)$$

²Can possibly be used in the numerical ground but it is not valid because not respectful of the mathematical results. The potential $\varphi(\cdot)$ fails to belong to $H^1(\Omega)$.

³ $\text{Ei}(\cdot)$ is the special function of exponential integral (see [1]). It has a logarithmic singularity: $\text{Ei}(r) = \gamma + \ln r + A(r)$, $A(\cdot)$ is an analytic function. γ is the Euler-Mascheroni constant

The last logarithmic term in $(\ln r)$ is exhibited on purpose. According to the footnote ⁽³⁾, it is the very singularity of the potential $\varphi(\cdot)$. This (singular) term excepted, all the others belong to $H^{(3/2)-}(\Omega)$. The correction $\chi(= \varphi - K)$ will be affected by those terms. All in all, the extraction process retrieves the strong logarithmic singularity and leaves there moderately smooth terms. The effect is therefore a moderate smoothness of $\chi(\cdot)$, though it stands a H^1 -type variational formulation which is the ultimate target.

The singular contribution being explicitly isolated, we pass to the finite approximation of the correction term $\chi(\cdot)$. The computations are realized in `Freefem++`, the code developed by F. Hecht and his team (see [19]).

A Dirichlet condition is first prescribed on the whole boundary of Ω , immediately available from the knowledge of $\varphi(\cdot)$. The data $\mathbf{g}(\cdot)$ in (11) is evaluated by hands. The computed correction $\chi_h(\cdot)$ is hence solution to the affine finite element discretization of the variational problem (13). The approximate potential $\varphi_h(\cdot)$ is obtained by adding *exact* $K(\cdot)$ (in reality its interpolation). The gap $(\varphi - \varphi_h) = (\chi - \chi_h)$ is therefore evaluated in the H^1 - and L^2 -norms and the convergence curves are depicted in Figure 1. The parameters are fixed to $\alpha = \epsilon = 1$ and $q = 2\pi$. Logarithmic scale is adopted on both axis.

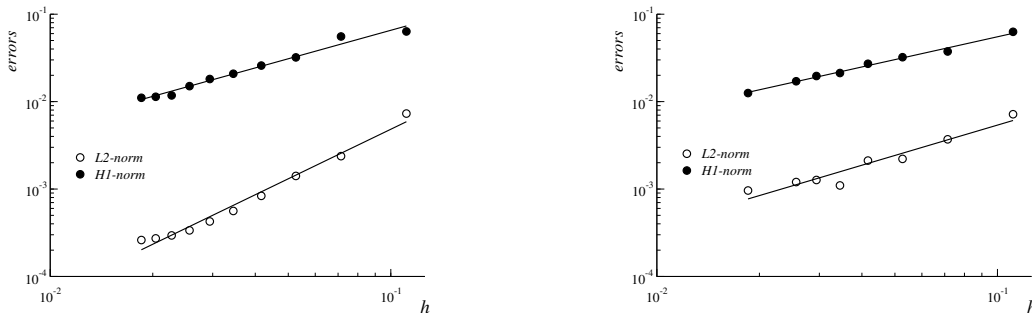


Figure 1: Convergence : Dirichlet boundary (left), Dirichlet-Neumann boundary (right).

As notified above, the correction function $\chi(= \varphi - K)$ is in $H^{(3/2)-}$. The theory predicts that the gap $e(\chi) = (\chi - \chi_h)$ is bounded by $C(h^{(1/2)-}, h^{(3/2)-})$ with respect to the (H^1, L^2) -norms respectively (see [8]). The linear regressions of the errors plots (left panel in Figure 1) bring an interesting insight on the finite element accuracy. The errors may be evaluated ; results indicate the following trends $e_{H^1}(\varphi) \approx 0.8 \times h$ and $e_{L^2}(\varphi) \approx 0.36 \times h^{1.88}$. The convergence rates seem to be better than expected. Let us underline the fact that, as currently followed, the errors are obtained by confronting the computed finite element potential φ_h to the Lagrange interpolate $i_h\varphi(\cdot)$ rather than $\varphi(\cdot)$ itself. If precision were demanded, we should write that $e_{H^1}(i_h\varphi) \approx 0.8 \times h$. Notice also that we use a refined uniform mesh (not the one used to compute $\varphi_h(\cdot)$) to evaluate the errors. The

refined mesh size is $h_F = 0.0082$, while the finite element solutions are obtained on meshes with size $h \in [0.018, 0.12]$. Notice that using the alternative method exposed in Subsection 6.2 gives almost exactly the same result; the conductivity enjoys the necessary smoothness.

The curves in the left panel of Figure 1 are when the boundary conditions are changed. Neumann conditions are applied on both ‘up and down’ faces of the cylinder, while the lateral surface of the boundary is still subjected to a Dirichlet condition. The flux data is picked up from formula (21). The errors are given by $e_{H^1}(i_h\varphi) \approx 0.4 \times h^{0.8}$ for the H^1 -norm while $e_{L^2}(i_h\varphi) \approx 0.07 \times h^{1.15}$ for the L^2 -norm. The convergence rate for the L^2 is quite less than expected, it suffers from the substantial loss of 0.35. A likely explanation comes from the regression constant 0.07 which is small, compared to the range of mesh sizes $[0.018, 0.12]$. For this range of meshes, it may be checked out, for instance, that $e_{L^2}(i_h\varphi) = 0.07 \times h^{1.15} \leq 0.3 \times h^{1.5}$. And this would in a perfect agreement with the theoretical findings. In fact, typically only a bound on the error is given by the mathematical estimates.

Moderately regular Conductivities — The subsequent examples deal with a set of conductivities with reduced smoothness,

$$a(\mathbf{x}) = \frac{(1+z)}{\alpha + \epsilon r^\varrho}, \quad \text{in } \Omega.$$

The exponent $\varrho > 0$ is given. The parameter $a(\cdot)$ belongs then to $W^{e,\infty}(\Omega)$. The new solution is selected to be

$$\varphi(\mathbf{x}) = (1+z)\eta(r) = (1+z) \left(-\frac{\alpha}{2\pi} \ln r + \frac{\epsilon}{2\pi\varrho} (1-r^\varrho) \right),$$

and the related source data in equation (1) reads as

$$f(\mathbf{x}) = (1+z)^2 \delta_\gamma - \frac{\eta(r)}{\alpha + \epsilon r^\varrho}, \quad \text{in } \Omega.$$

The singular contribution $K(\cdot)$ looks like the one in (24), simply change $\alpha + \epsilon$ into α .

Looking closely at the correction function $\chi(\cdot)$ shows that it belongs to $H^{(1+\varrho)-}$ for small $\varrho < 1/2$. Actually, the regularity is driven by the term r^ϱ . The gap $(\varphi - \varphi_h)$ is expected to decay like $Ch^{\varrho-}$ with respect to the H^1 -norm. The slopes of the curves of the H^1 -error (full circles in Figure 2) are $(0.38, 0.63)$ for both conductivities, related to the exponents $\varrho = (0.15, 0.5)$ respectively. The observations are quite close to the predictions with a slightly better behavior (than expected). Exploring the convergence rates with respect to the L^2 -norm shows an agreement with the predicted ones. The regression slopes, a worthy indication of the effective convergence rate provide $(1.33, 1.51)$. The constant (α, ϵ) are chosen equal to $(1, \pi)$.

The last computations are dedicated to the case where Neumann conditions are applied on upper and lower faces of the cylinder. Here again, the error-curves are depicted in Figure 3. We observe the same trends as for the previous example. The slopes of the regressions of the H^1 -curves are

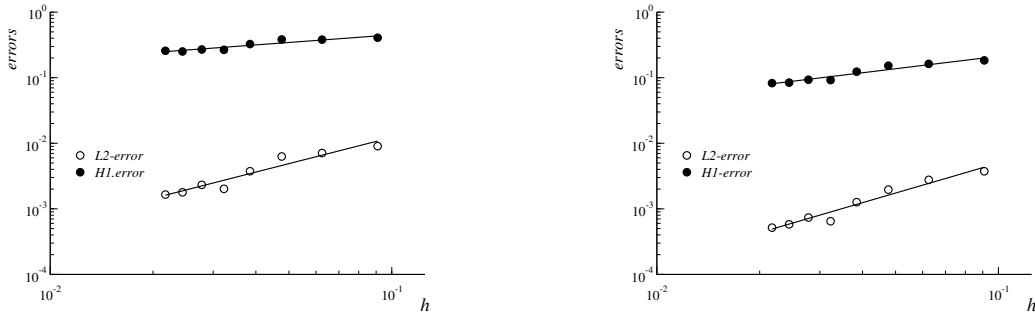


Figure 2: Errors with $\varrho = 0.15$ (left) and $\varrho = 0.5$ (right).

computed to $(0.24, 0.45)$ for the exponents $\varrho = (0.15, 0.5)$. The slopes for the discretization error $\|\varphi - \varphi_h\|_{L^2}$ is almost 1 for both cases. This hardly meets the theoretical predictions and sounds a little bit disappointing, at least for $\varrho = 0.5$. Indeed, for the latter case, the predicted convergence predicted should be rather of order $h^{1.5}$. This prompts us to look at the regression constant in $\|\chi - \chi_h\|_{L^2} \approx 0.04 \times h$, for a mesh size h ranging in $[0.02, 0.15]$. The likely explanation is the one provided above for the first example.

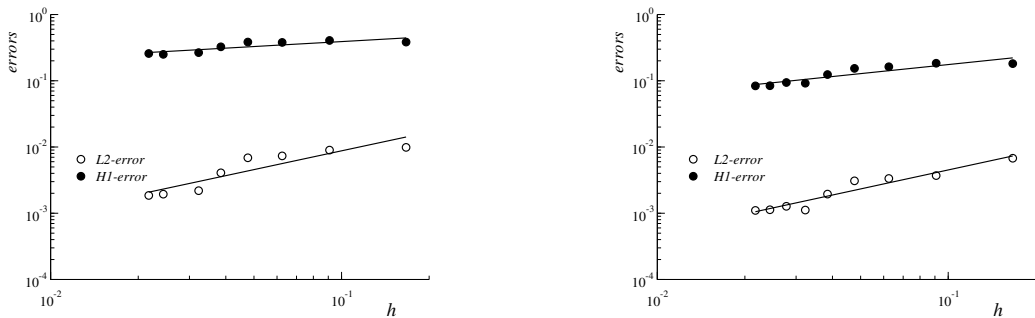


Figure 3: Convergence, Neumann conditions: ($\varrho = 0.15$ (left), $\varrho = 0.5$ (right)).

8 Conclusion

Potentials created by Dirac sources are singular. Their singular behavior prevent them from fulfilling the standard variational formulation (in the Sobolev space H^1), often fit for second order boundary value problems. In three dimensions, we provide a *generic mathematical formula* that fully captures the singular behavior of the potential when the source is a line Dirac measure. The conduction parameter is space dependent: identified as the main difficulty which is specifically dealt with in this paper. After removing the singularity, the residual part (full potential minus singularity) brings the correction to the boundary conditions, enjoys enough regularity to fulfill a tractable variational problem (the one practiced in $H^1(\Omega)$). It can be computed by many of

the available numerical methods. Before closing, although the proof is limited to the continuous conductivities here, we see no reason why it cannot cover the case of parameters with jumps. Hopefully, the process detailed in [3], for pointwise sources, applies as well to line Dirac sources, with some improvements. Research on this issue is underway and the forthcoming numerical study is devoted to a deeper validation of the extraction process either for the diffusion or elastic problems.

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Appendix

We prove two technical lemmas that are the centerpiece of Section 5.

Lemma 8.1 *Assume $a \in L^\infty(\mathbb{R}^3) \cap W^{\varrho, \infty}(\omega_\gamma)$ with $0 < \varrho < 1/2$. Let $(\mathbf{s}, \boldsymbol{\sigma})$ be a couple of distinct points in γ . There holds that*

$$\int_{\mathbb{R}^3} \frac{|(a(\mathbf{x}) - a(\mathbf{s}))(a(\mathbf{x}) - a(\boldsymbol{\sigma}))|}{|\mathbf{x} - \mathbf{s}|^2 |\mathbf{x} - \boldsymbol{\sigma}|^2} d\mathbf{x} \leq C(\varrho) \frac{\|a\|_{\varrho, \infty}^2}{|\mathbf{s} - \boldsymbol{\sigma}|^{1-2\varrho}}.$$

Proof: Choose the Cartesian coordinate system (x_1, x_2, z) so that the z -axis is directed along the vector $\overrightarrow{\mathbf{s}\boldsymbol{\sigma}} = (\mathbf{s} - \boldsymbol{\sigma})$. The origin is located at the mid point of the segment $[\mathbf{s}, \boldsymbol{\sigma}]$ so that $\mathbf{s}(0, 0, t)$ with $t > 0$, and $\boldsymbol{\sigma} = (0, 0, -t)$ and then $|\mathbf{s} - \boldsymbol{\sigma}| = 2t$. To alleviate the presentation, we denote $\mathcal{I} = \mathcal{I}(\mathbf{s}, \boldsymbol{\sigma})$ that integral and $\eta = \|a\|_{\varrho, \infty}$. The use the cylindrical system (r, θ, z) is appropriate. In view of (12), we derive that

$$\begin{aligned} \mathcal{I} &\leq \eta^2 \int_{\mathbb{R}^3} \frac{d\mathbf{x}}{|\mathbf{x} - \mathbf{s}|^{2-\varrho} |\mathbf{x} - \boldsymbol{\sigma}|^{2-\varrho}} \\ &\leq 2\pi\eta^2 \int_{[0, \infty[\times \mathbb{R}} \frac{r dr dz}{(r^2 + (z-t)^2)^{1-\varrho/2} (r^2 + (z+t)^2)^{1-\varrho/2}}. \end{aligned}$$

Notice that the integrand is even with respect to z , the integration is hence reduced to the $z \geq 0$. After the variable change $r = (z+t)u$, we obtain that

$$\mathcal{I} \leq 4\pi\eta^2 \int_{[0, \infty[} \frac{1}{(z+t)^{2-2\varrho}} \left(\int_{[0, \infty[} \frac{udu}{(u^2 + \zeta^2)^{1-\varrho/2} (u^2 + 1)^{1-\varrho/2}} \right) dz.$$

We have set $\zeta = \frac{|z-t|}{z+t}$ for all $t \geq 0, z \geq 0$. Now, accounting for $\zeta \leq 1$ we derive the bound

$$\begin{aligned} \mathcal{I} &\leq 2\pi\eta^2 \int_{[0, \infty[} \frac{1}{(z+t)^{2-2\varrho}} \left(\int_{[0, \infty[} \frac{2udu}{(u^2 + \zeta^2)^{(3-\varrho)/2}} \right) dz \\ &\leq \frac{4\pi\eta^2}{1-\varrho} \int_{[0, \infty[} \frac{1}{(z+t)^{2-2\varrho}} \frac{1}{\zeta^{1-\varrho}} dz = \frac{4\pi\eta^2}{1-\varrho} \int_{[0, \infty[} \frac{1}{|z^2 - t^2|^{1-\varrho}} dz. \end{aligned}$$

The second changing of variable $z = vt$ gives

$$\mathcal{I} \leq \frac{4\pi\eta^2}{1-\varrho} \frac{1}{t^{1-2\varrho}} \int_{[0, \infty[} \frac{1}{|v^2 - 1|^{1-\varrho}} dv.$$

The fact that the integral is finite ends to

$$\mathcal{I} \leq \frac{C(\varrho)\eta^2}{t^{1-2\varrho}} \leq \frac{C\eta^2}{|\mathbf{s} - \boldsymbol{\sigma}|^{1-2\varrho}}.$$

The result is complete. ■

Lemma 8.2 *Assume that the simple curve γ is of class $\mathcal{C}^{1, \alpha}$, with $\alpha > 0$. Let $\varrho > 0$ be given, then*

$$\int_{\gamma} \int_{\gamma} \frac{|\lambda(\mathbf{s})\lambda(\boldsymbol{\sigma})|}{|\mathbf{s} - \boldsymbol{\sigma}|^{1-\varrho}} d\mathbf{s} d\boldsymbol{\sigma} \leq C \|\lambda\|_{L^2(\gamma)}^2, \quad \forall \lambda \in L^2(\gamma).$$

Proof: The stability in the lemma is derived from the properties of the (ϱ) -Riesz potential defined on the curve γ ,

$$(R_\varrho\lambda)(\mathbf{s}) = \int_\gamma \frac{\lambda(\boldsymbol{\sigma})}{|\mathbf{s} - \boldsymbol{\sigma}|^{1-\varrho}} d\boldsymbol{\sigma}.$$

This aim is to show that it is a Calderon-Zygmund operator.

According to [34, Theorem 1.1], checking out the bondedness on the lines is enough to extend it to $\mathcal{C}^{1,\alpha}$ -curves. Following the terminology of the curves theory, a line is a curve γ that is a scaled isometric copy of a segment in the sense that

$$c_*|t - \tau| \leq |\gamma(t) - \gamma(\tau)| \leq c^*|t - \tau|, \quad \forall t, \tau \in I.$$

The constants c_* and c^* are positive.

It is thus sufficient to prove the result for the operator R_ϱ defined as

$$(R_\varrho\lambda)(t) = \int_I \frac{\lambda(\tau)}{|t - \tau|^{1-\varrho}} d\tau.$$

The proof starts as follows

$$(R_\varrho\lambda)^2(t) \leq \int_I \frac{\lambda^2(\tau)}{|t - \tau|^{1-\varrho}} d\tau \int_I \frac{1}{|t - \tau|^{1-\varrho}} d\tau \leq \frac{2\ell^\varrho}{\varrho} \int_I \frac{\lambda^2(\tau)}{|t - \tau|^{1-\varrho}} d\tau.$$

Integrating with respect to t yields

$$\|R_\varrho\lambda\|_{L^2(I)}^2 \leq \frac{2\ell^\varrho}{\varrho} \int_I \left(\int_I \frac{1}{|t - \tau|^{1-\varrho}} dt \right) \lambda^2(\tau) d\tau \leq \frac{4\ell^{2\varrho}}{\varrho^2} \|\lambda\|_{L^2(I)}^2.$$

The proof is complete. ■

Remark 8.1 *Actually, a sufficient condition, for the proof to still work, can be expressed directly on arbitrary curves, as follows*

$$\sup_{\mathbf{s} \in \gamma} \int_\gamma \frac{1}{|\mathbf{s} - \boldsymbol{\sigma}|^{1-\varrho}} d\boldsymbol{\sigma} < \infty.$$

Using the atlas of local cards, together with the compactness of the curve, extends the proof for any Lipschitz curves. This is easy to check out, for instance when the curves γ are parameterized by $\mathbf{s}(t) = (t, \xi(t), \zeta(t))$, where ξ and ζ are Lipschitz functions.

Be careful that if $\varrho = 0$, then the operator R_0 is not a Riesz transformation. The convolution kernel should be odd. It is not, it is the contrary, it is even! Actually, R_0 fails to be bounded.

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