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Abstract

This paper looks at the challenge of making maintenance decisions for deteriorating systems when the degradation process leading to failure cannot be directly observed or measured. In this scenario, the system's health is monitored by observing the progression of a degradation-related marker index, which can be obtained through inspections. To model this configuration, a bivariate gamma process is employed. One component represents the marker process, while the other represents the degradation process, which dictates the time of failure. Two condition-based maintenance (CBM) policies are proposed and analyzed. The first policy is based on a conventional decision structure, utilizing a fixed preventive threshold directly applied to the measured process. The second policy relies on monitoring data related to the marker process to estimate the level of latent degradation at inspections. We demonstrate that the second policy is equivalent to a policy employing an adaptive preventive threshold that sequentially evolves. We provide insights into some key properties associated with this approach. The expected cost rate is calculated and employed for policy optimization. Additionally, a numerical study is presented that showcases the practical implementation of the method and highlights the effectiveness of the second approach, even when the correlation between degradation and the marker process is low.

Keywords: Bivariate gamma process, condition-based maintenance, degradation process, marker process.

1. Introduction

Industrial organizations are actively seeking new strategies to enhance the efficiency of their operations. In this pursuit, maintenance optimization has emerged as a pivotal factor, simultaneously minimizing production costs while providing high-quality products. It involves deciding which action to take at the appropriate moment based on available information. Corrective actions refer to maintenance measures necessary when a system experiences a failure, while preventive actions are maintenance activities designed to prevent or postpone future failures. These preventive actions are carried out while the system is still operational. There are three primary categories of preventive

maintenance strategies: (1) age-based maintenance, (2) time-based maintenance, and (3) condition-based maintenance, as mentioned in reference [36]. Due to its advantages, condition-based maintenance (CBM) has garnered increasing attention in maintenance research. CBM relies on one or more variables that measure the system's condition, with the primary objective of reducing unnecessary maintenance interventions and eliminating the risks associated with preventive maintenance actions.

The application of CBM in industrial areas relies on the development of stochastic degradation models. The choice of the stochastic process that best describes the degradation dramatically influences the CBM decision-making strategy. Research on CBM has two main streams, one focusing on discrete and the other on continuous degradation processes. Markov chains are commonly employed to model discrete deterioration processes [2, 9, 22, 30, 31], while continuous time and space stochastic processes such as the Wiener, gamma, and inverse Gaussian processes are widely used in continuous degradation processes. The Wiener process is suitable in describing non-monotonic degradation over time; see for example, [15, 18, 35, 55]. On the other hand, the gamma and inverse Gaussian (IG) processes are applied to model the monotonic degradation of a system. These processes can be regarded as the limit of a compound Poisson process and are thus appropriate models when degradation takes the form of cumulative damage. There is limited literature available on utilizing a CBM-based strategy in the context of the IG process, as seen in references such as [12] for a maintenance policy in the presence of heterogeneity among a product population, [29] for dynamic auto-adaptive predictive maintenance with successive Bayesian updates, [28] for predictive maintenance with imperfect repairs, [45] for CMB considering a two-stage inverse Gaussian process with random effects, and [24] for a mission-oriented maintenance policy.

On the contrary, the gamma process has been studied extensively in CBM models, see [43] for major works prior to 2007. Without attempting exhaustiveness, some illustrative works are quoted hereafter. Meier-Hirmer et al. [26] developed a model of maintenance optimization for a system with a gamma deterioration process with intervention delay and studied its application in railway track maintenance. Caballé et al. [10] proposed a periodic inspection/replacement strategy for a system subject to an internal gamma degradation process and an external non-homogeneous Poisson sudden shocks process. Yuan et al. [52] presented a model to quantify the economic value gained by the implementation of maintenance actions. Zhang et al. [53] studied how heterogeneity in degradation influences condition-based maintenance. Mercier and Castro [27] introduced and compared two imperfect repair models for a degrading system, with deterioration level modeled by a non-homogeneous gamma process. Han [19] developed an optimal CBM policy with the optimal inspection points under the gamma degradation process with random effects to account for potential population/environmental heterogeneities.

Early research in CBM modeling for gradually deteriorating systems primarily concentrated on single-component systems. Maintenance policies and models for multi-component systems were sub-

sequently proposed. A wide range of configurations has been envisaged [21]. For multi-component systems, several degradation indicators are monitored, each associated with a component that requires maintenance. In the context of multi-component systems with gradual degradation phenomena, degradation models based on Lévy stochastic processes, specifically gamma and Wiener processes, are frequently employed. A multi-component configuration is relevant when it comes to explicitly exploiting dependencies between components or subsystems. The most commonly explored dependencies are economic, structural, and stochastic.

Some researchers ([7, 8, 11]) are investigating CBM strategies that incorporate economic dependencies. Their primary motivation is to achieve cost reduction when maintaining multiple components jointly. In the context of degradation modeling, the particular interest is in stochastic dependencies. Interactions between components can be state-state or state-rate interactions (see e.g. [6, 49]). Various types of stochastic dependencies are being explored and modeled using copulas [1, 23], multivariate processes [25], load sharing [32], or the common effect of environmental factors [54]. It's important to note that CBM studies on multi-component systems that take structural dependence into account are somewhat limited, as they require more complex analytical formulations. Furthermore, it's essential to clarify that the terminology "single-componen" or "multi-component" does not necessarily correspond to the real configuration of the system but rather to the model being used. For instance, a CBM policy for a single-unit system may refer to a multi-component system considered as a whole, characterized by a one-dimensional "health index" rather than its complete multi-dimensional state [17].

In the majority of the aforementioned studies, failure mechanisms can be attributed to an underlying, observable physical or chemical degradation process, such as fatigue crack growth, corrosion, and wear, among others. However, there are situations in which failure cannot be linked to an observable degradation process. Thus, it becomes challenging to directly and accurately measure or observe the degradation process itself, as noted by [44]. In other words, the degradation process is latent. In such cases, besides the latent degradation, we can measure other indices that are related to the performance characteristics. Such indices are referred to as markers [20]. These markers are statistically associated with the latent degradation state, and hence, contain valuable information about the unobservable degradation process.

This configuration has the potential to be utilized in a wide range of real-life scenarios. It is applicable to systems where failure is linked to a specific component or a subsystem that is inaccessible. Although the degradation indicator is well-defined, measuring its level may necessitate complex or cost-prohibitive operations that are impractical. Consequently, non-destructive or remote testing processes are being explored to indirectly assess the degree of degradation in an alternative manner. Clearly, the performance of the measurement system can impact the accuracy of the marker and its correlation with actual degradation. A prime example of this configuration is the degradation of mechanical

transmissions. Directly accessing the gears for measuring degradation levels poses considerable challenges. However, the observation of oil wear debris can serve as a valuable marker [13]. Another illustrative example is the corrosion-induced degradation of aluminum alloys, a significant factor leading to structural failure in various safety-critical and mission-critical engineering components/systems, such as aircraft wings and nuclear batteries. Therefore, degradation performance remains latent, but non-destructive testing techniques like electrochemical impedance spectroscopy (EIS) can be employed to evaluate the corrosion-induced degradation process [40].

Another scenario arises in systems where a global degradation phenomenon is identified, but a definitive evaluation indicator is lacking. In such cases, information about the level of degradation is derived from precise measurements of a correlated indicator. For instance, this can be observed in studies related to AIDS, where CD4 cell count serves as an indicator of residual lifetime [14, 20, 39], in monitoring disease progression in neurological disorders (e.g., amyotrophic lateral sclerosis) using serum and cerebrospinal fluid neurofilament light chain protein as an indicator, and in tracking gradual seal degradation through the dynamic response of hydraulic actuators for blade-pitch control in wind turbines.

Additional examples can be found in various situations. The performance of lithium-ion batteries deteriorates with decreasing capacity and increasing impedance, potentially leading to equipment and system failures or even catastrophic losses. Health indicators (HI), such as the charge quantity stored in the battery during constant current charging or the discharge voltage over time, can be employed to represent changes in capacity degradation, which is otherwise unmeasurable [48]. The accumulation of undesirable materials on solid surfaces can significantly increase resistance to heat transfer in subsea heat exchangers. Failure results from fouling and can be detected while the fouling process remains latent. The marker process is associated with changes in certain physical properties of the fluid, which can be periodically checked.

Different mathematical models have been developed to describe the relationship between latent degradation and marker processes. Whitmore et al. [47] presented a model based on a bivariate Wiener process, where one component represents the marker and the second, latent component determines the failure time. Ting Lee et al. [41] extended the model proposed in [47] and applied it in a clinical trial for AIDS. Ting Lee et al. [42] also explored Bayesian inference for the bivariate Wiener process introduced in [47]. Peng and Zhou [34] assumed that the degradation process follows a Wiener process, and the marker is the integral of the performance degradation. Shemehsavar [38] employed a bivariate gamma process to model monotone-increasing latent degradation and marker processes. Zhou et al. [56] introduced a model where latent degradation follows a gamma process, and the marker process is a function of this latent degradation in the presence of an error term. They used a Monte Carlo-based algorithm to estimate and predict crack depth on gear teeth in a spur gearbox based on vibration

signals. Xu et al. [50] introduced a dynamic system characterized by a hidden degradation process identified through particle filtering based on measurable outputs from the considered dynamic system.

This paper addresses the challenge of making optimal condition-based maintenance decisions in case of latent degradation. System failure occurs when the latent degradation level reaches a predefined threshold. The marker process serves as an indirect source of information regarding latent degradation. The paper focuses on systems where both marker and latent processes exhibit similar behaviors, featuring non-decreasing trajectories and continuous evolution over time. The paper's central concern lies in exploiting the connection between latent degradation and the marker. To maintain clarity and avoid introducing additional complexities stemming from interdependencies between components, we concentrate our efforts on scenarios associated with one degradation phenomenon linked to a single indicator. This indicator may pertain to a specific component or may encapsulate information from multiple components that can not be observed individually.

As discussed above, gamma processes are commonly used for degradation modeling and maintenance modeling in engineering systems. Hence, we select a modeling framework based on Kibble's bivariate gamma process. This process possesses the essential characteristic of monotonically increasing sample paths and has a well-known conditional infinitely divisible distribution (namely, randomized gamma distribution) for the latent degradation given the marker. The Kibble model includes a single parameter that indicates the Pearson correlation, providing an explicit and easily interpretable measure of the dependency between two processes.

We suggest two condition-based maintenance (CBM) strategies involving periodic inspections, and we develop analytical expressions for the long-term expected maintenance cost rate for both strategies. The first strategy follows a conventional structure reliant solely on the marker process. The second and primary strategy introduces an advanced maintenance decision rule utilizing predictive inferences to estimate the degradation level based on observed data. An alternative representation of the primary policy is provided, which relies on an adaptive threshold. To derive the cost function analytically rather than resorting to Monte Carlo simulations, we explore and establish the properties of the adaptive threshold.

Our objective is to explore whether it is more advantageous to base decisions directly on the observed marker indicator or if it would be beneficial to make predictive inferences about the level of latent degradation. Thus, the final contribution involves a comparative analysis of the two maintenance strategies against an ideal scenario where the degradation process is perfectly understood. A series of representative numerical experiments highlight the value of an advanced maintenance policy and demonstrate the advantages of inferring the degradation level from the marker indicator.

The paper is structured as follows: Section 2 introduces the bivariate gamma process and reviews key probabilities. Section 3 presents, analyzes, and discusses the maintenance policies. Section 4

illustrates and compares the proposed policies through a simulation study. Finally, Section 5 provides the conclusion.

2. General Framework

2.1. Stochastic Degradation Model

Two random variables, X and Y, are said to follow Kibble's bivariate gamma distribution (Balakrishnan and Lai [3]) with shape parameter α, rate parameters λ_1, λ_2, and correlation ρ, if their joint distribution is as follows:

$$f(x, y|\alpha, \rho) = \frac{(\lambda_1\lambda_2)^\alpha}{(1-\rho)\Gamma(\alpha)} \left(\frac{xy}{\rho\lambda_1\lambda_2}\right)^{\frac{\alpha-1}{2}} \exp\left(-\frac{\lambda_1 x + \lambda_2 y}{1-\rho}\right) I_{\alpha-1}\left(\frac{2\sqrt{\rho\lambda_1\lambda_2 xy}}{1-\rho}\right), \tag{1}$$

where $x, y > 0, 0 < \rho < 1, \alpha, \lambda_1, \lambda_2 > 0$ and $I_\alpha(\cdot)$ is the modified Bessel function of the first kind of order α defined as:

$$I_\alpha(z) = \sum_{k=0}^{\infty} \frac{(\frac{z}{2})^{2k+\alpha}}{\Gamma(k+\alpha+1)k!}, \qquad z > 0.$$

The marginal distributions of X and Y are gamma with the same shape parameter α and rate parameters λ_1 and λ_2, respectively. The parameter ρ is also Pearson's product-moment correlation coefficient between X and Y.

Consider a two-dimensional process, $\{(X_t, Y_t), t \geq 0\}$ with the initial value $(X_0, Y_0) = (0, 0)$, such that the vector (X_t, Y_t) has Kibble's bivariate gamma distribution with shape parameter αt, rate parameters λ_1, λ_2, and correlation ρ. We consider the process $\{X_t, t \geq 0\}$ to be the latent degradation process which represents the level of degradation of the system at time t. Hence, X_t has a gamma distribution with shape parameter αt and rate parameter λ_1.

The system fails when the degradation process $\{X_t, t \geq 0\}$ reaches a failure threshold $L > 0$ for the first time. We denote this first-hitting time by the random variable S. The failure threshold L is given and related to the characteristics of the system under study. Here, we assume that the failure of the system is self-announced.

The other process, $\{Y_t, t \geq 0\}$, represents a marker process that is correlated with the degradation process $\{X_t, t \geq 0\}$ and tracks its progress. Y_t has a gamma distribution with shape parameter αt and rate parameter λ_2. The degradation process is latent and cannot be measured but inference about the degradation process can be performed using the marker process. The maintenance decision-making procedure is based on the marker process observations. The correlation coefficient ρ describes the strength of the association between the two processes.

2.2. Key Probability Distributions

For a system surviving at time t, two types of predictive inference that exploit marker information can be considered:

- Prediction of the degradation level, X_t which is latent, given its marker level Y_t at that time;
- Prediction at time t of the future system failure time S given that its marker level at that time is $Y_t = y$.

Our main concern is to employ the first. For parametric inference and maintenance optimization with this model, we need different probability distributions which are mentioned hereafter.

Consider a time interval $[0, t]$ partitioned into time points $0 = T_0 < T_1 < \cdots < T_k = t$, where k is the total number of inspection points up to time t. Let $\Delta X_i = X_{T_i} - X_{T_{i-1}}$ be the increment for process X over the time interval $(T_{i-1}, T_i]$ for $i = 1, ..., k$. Moreover, $\Delta Y_i = Y_{T_i} - Y_{T_{i-1}}$ denotes the increment for the process Y in the same time interval. Then, the distribution of ΔY_i conditional on $\Delta X_i = \Delta x_i$ has a randomized gamma distribution (RGD) (see [16] and [51]); i.e,

$$(\Delta Y_i | \Delta X_i = \Delta x_i) \sim \text{RGD}\left(\alpha \Delta T_i + \eta_i, \frac{\lambda_2}{1-\rho}\right), \tag{2}$$

where $\eta_i \sim \text{Poisson}(\frac{\rho\lambda_1 \Delta x_i}{1-\rho})$ and $\Delta T_i = T_i - T_{i-1}$. We know that the bivariate gamma process has independent increments and hence we can extend the conditional event from (2) to the entire σ-field generated by $\{\Delta X_1, \Delta X_2, \cdots, \Delta X_k\}$. So we have

$$(\Delta Y_i | \Delta X_j = \Delta x_j,\ 1 \le j \le k) \sim \text{RGD}\left(\alpha \Delta T_i + \eta_i, \frac{\lambda_2}{1-\rho}\right).$$

Since the $(\Delta Y_i)_{1 \le j \le k}$ are independent we can write their joint distribution conditional on $(\Delta X_i)_{1 \le i \le k}$ and thus obtain the conditional density of $Y_t = \sum_{i=1}^{k} \Delta Y_i$. We also have

$$\sum_{i=1}^{k} \Delta x_i = x_t, \text{ and } \sum_{i=1}^{k} \Delta T_i = t.$$

So, conditional on $(\Delta X_i)_{1 \le i \le k}$, the random variable Y_t has randomized gamma distribution

$$(Y_t | \Delta X_i,\ 1 \le i \le k) \sim \text{RGD}\left(\alpha t + \eta, \frac{\lambda_2}{1-\rho}\right),$$

where $\eta = \sum_{i=1}^{k} \eta_i \sim \text{Poisson}(\frac{\rho\lambda_1 x_t}{1-\rho})$ depends on the partitioned sample path only through its terminal point x_t and not on the structure of the partition. Therefore, the conditional distribution of Y_t given $(\Delta X_i = \Delta x_i,\ 1 \le i \le k)$ is the same as the conditional distribution of Y_t given X_t,

$$(Y_t | X_t = x_t) \sim \text{RGD}\left(\alpha t + \eta, \frac{\lambda_2}{1-\rho}\right), \tag{3}$$

where $\eta \sim \text{Poisson}(\frac{\rho\lambda_1 x_t}{1-\rho})$ (see [38]).

Now suppose a system has survived until time t when a marker level $Y_t = y$ is recorded. This occurrence constitutes a censored observation on the failure time because we know $S > t$. In this case,

the distribution of $(Y_t|X_t = x)$, for $S > t$, follows (3) and we can write its pdf as follows:

$$\begin{aligned} f_1(y|x) &= P(Y_t \in dy|X_t = x,\ S > t) \\ &= \frac{y^{\frac{\alpha t-1}{2}} \lambda_2^{\frac{\alpha t+1}{2}} \exp\left(-\frac{\rho\lambda_1 x + \lambda_2 y}{1-\rho}\right)}{(1-\rho)(\rho x\lambda_1)^{\frac{\alpha t-1}{2}}} I_{\alpha t-1}\left(\frac{2\sqrt{\rho\lambda_1\lambda_2 xy}}{1-\rho}\right), \qquad y > 0. \end{aligned} \tag{4}$$

Also, the probability of a system surviving beyond time t such that the terminal point X_t does not exceed the threshold L is as follows (see [33]):

$$f_2(x) = P(X_t \in dx,\ S > t) = \frac{\lambda_1^{\alpha t}}{\Gamma(\alpha t)} x^{\alpha t-1} e^{-\lambda_1 x}, \qquad 0 < x < L. \tag{5}$$

Now the joint pdf for a system surviving beyond time t, and having marker level Y_t and degradation level X_t at time t, is given by $f_1(y|x)f_2(x)$. Since x is not observed, we integrate it out of the joint density and we obtain:

$$f_3(y) = P(Y_t \in dy, S > t) = \int_0^L f_1(y|x)f_2(x)dx. \tag{6}$$

As already stated, we are interested in the prediction of the degradation level of the system surviving at time t from its current marker value. The related conditional distribution function is as follows:

$$P(X_t \in dx|Y_t = y, S > t) = \frac{f_1(y|x)f_2(x)}{f_3(y)}, \quad 0 < x < L. \tag{7}$$

We also need the distribution of the failure time S. Since X_t is strictly increasing in t, we have:

$$P(S > s) = P(X_s < L) = \int_0^L \frac{\lambda_1^{\alpha s}}{\Gamma(\alpha s)} x^{\alpha s-1} e^{-\lambda_1 x} \mathrm{d}x = \frac{\Gamma(\alpha s, \lambda_1 L)}{\Gamma(\alpha s)}$$

where $\Gamma(a, z)$ is the incomplete gamma function defined by $\Gamma(a, z) = \int_0^z u^{a-1}e^{-u}\mathrm{d}u$.

3. Maintenance Policies

The degradation leading to system failure remains latent, and there is no direct access to information about the actual degradation or the remaining useful life. Consequently, the degradation level cannot be directly used for making maintenance decisions. However, monitoring information can be accessed through inspections, which reveals the status of the marker process before the system's failure. To address the challenge of optimal maintenance decision-making, we introduce two condition-based maintenance (CBM) policies. The first policy involves making maintenance decisions directly based on the marker process. Since CBM decision parameters typically include inspection frequency and preventive maintenance threshold related to degradation level, we adopt a conventional CBM decision rule with a threshold on the marker level for initiating preventive actions. The second policy aims to leverage the marker process to infer information about the current latent degradation level of the

system. In this scenario, we propose the use of predictive probabilities to determine the probability of surpassing a threshold on the latent degradation level, which then guides preventive maintenance actions.

In the subsequent subsections, we will provide more detailed descriptions of these policies and explore their characteristics.

3.1. Maintenance decision rules

Suppose that the time interval between two inspections is fixed and equal to T, so the i-th inspection time during one cycle is $T_k = kT,\ k \in \mathbb{N}$, while $T_0 = 0$. At the time T_k, if the system has not failed yet, we can measure the marker value Y_{T_k}. The two decision rules are as follows:

Classical Maintenance Policy (CMP): This policy simply uses a preventive limit on the observable marker process. The reason behind this is that the marker can somehow reflect the situation of the latent process due to the relationship between them. Consequently, traditional decision rules can be employed in the marker process. Let us denote the critical level of the marker process by M_Y where the subscript Y is to emphasize that this threshold is for the marker process. Therefore, if $Y_{T_k} > M_Y$, we replace the system preventatively; otherwise, the system is properly working, and no maintenance action is needed. In this case, maintenance decision-making is postponed to the next inspection time. The time interval between two inspections, T, and the preventive threshold for the marker process, M_Y, are decision variables and must be optimized.

Intelligent Maintenance Policy (IMP): Inspired by Russel et al. [37], an intelligent agent is defined as a system that perceives its environment and takes actions to maximize its chances of success. In the context of latent degradation, the term "intelligent" may refer to the use of predictive inference to formulate a maintenance policy designed to adeptly adjust to the system's behavior. To this aim, this policy is defined based on the conditional distribution of the latent degradation X_t given the observed marker variable Y_t introduced in equation (7).

For a system with observable degradation, preventative replacement usually occurs whenever it is in a critical situation which means its degradation level has exceeded a threshold $M_X < L$. The latent nature of the degradation X_t implies that we cannot be aware if this event occurs. However, we can assess the probability of this incident given the marker information. Thus, if the chance of having a system in a critical situation is high or in other words if $P(X_{T_k} > M_X | Y_{T_k} = y_{T_k}, S > T_k) > p$, we replace the system. Otherwise, the system is considered properly working. No maintenance is needed, and maintenance decision-making is postponed to the next inspection time. It is worth pointing out that the subscript X in M_X denotes that this threshold is for the latent degradation process. Here the decision variables are T, M_X, and the tolerance probability, p.

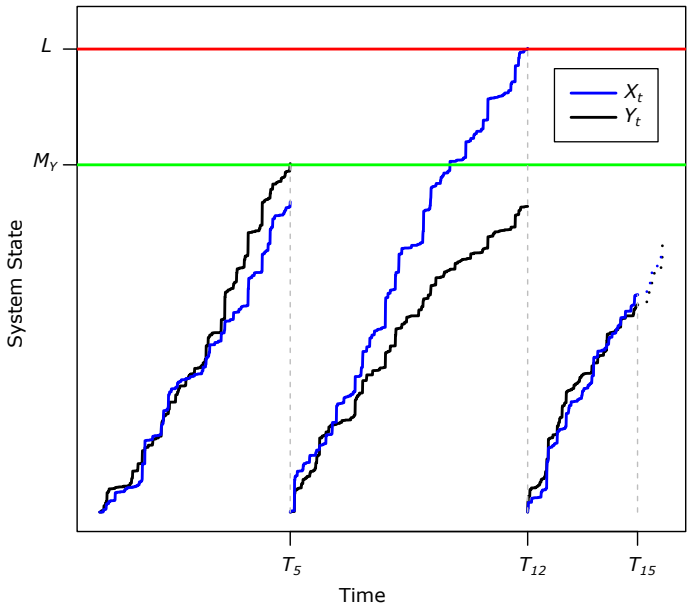


Figure 1: Schematic evolution of a maintained system state under CMP

For both CMP and IMP, preventive replacements are considered to be perfect. So for both decision rules after inspection time and possible maintenance, we have:

$$(X_{T_k}, Y_{T_k}) = \begin{cases} (0,0) & \text{if preventive replacement has been performed;} \\ (X_{T_k^-}, Y_{T_k^-}) & \text{otherwise.} \end{cases}$$

Here T_k^- refers to the time just before the maintenance action. Since the system failure is self-announcing, whenever the system fails, it is perfectly and immediately replaced by a new system, in accordance with the corrective replacement action.

Figure 1 illustrates possible sample paths of the bivariate gamma degradation process for a system subjected to CMP. The interval between 0 and T_5 is the first cycle of a system that terminates with a perfect preventive replacement. The second cycle is between T_5 and T_{12}. As the latent degradation of the system crosses the threshold L, it finishes with a corrective replacement.

3.2. IMP as a policy with an adaptive threshold

Let's focus on the Intelligent Maintenance Policy proposed in the previous section. Here, we will introduce an alternative definition for IMP that is equivalent, yet more intuitive for comparison with CMP. For this policy, decision-making at a given time t is based on the conditional probability $P(X_t > M_X | Y_t = y, S > t)$, which is compared with decision variable p. The considered conditional probability is obviously a function of the time t and of the observed value y of the marker process. One first characterization of its behavior is given by the following theorem:

Theorem 1. *For a given M_X, the following statements are true.*

(i) *Considering a fixed t, the function $y \to P(X_t > M_X | Y_t = y, S > t)$ is strictly increasing on $(0, \infty)$.*

(ii) *Considering a fixed y, the function $t \to P(X_t > M_X | Y_t = y, S > t)$ is increasing on $(\frac{1}{\alpha}, \infty)$.*

The proof of Theorem 1 is given in Appendix A. The constraint $t > \frac{1}{\alpha}$ is necessary for the mathematical proof, but extensive numerical experiments show that statement (ii) remains valid for $t \in [\frac{0.4}{\alpha}, \frac{1}{\alpha}]$. Let us now state some corollaries of Theorem 1. At inspection time T_k, we know that the system must be preventively replaced if $P(X_{T_k} > M_X | Y_{T_k} = y, S > T_k)$ is greater than p. Theorem 1-(i) indicates that, if the value of the marker process is higher than a level, the probability that the system is at risk is greater, which is logical because of the positive correlation between latent and marker processes. The question is: what is the appropriate level of the marker process that ensures that the sufficient condition to perform a preventive action is met? Based on Theorem 1, the conditional probability $P(X_{T_k} > M_X | Y_{T_k} = y, S > T_k)$ is a bounded monotonic function of y which has a unique value ξ_k with the following definition

$$\xi_k = \inf\{y : P(X_{T_k} > M_X | Y_{T_k} = y, S > T_k) > p\}. \tag{8}$$

Therefore, Theorem 1-(i) implies that $P(X_{T_k} > M_X | Y_{T_k} = y_{T_k}, S > T_k) > p$ is equivalent to $y_{T_k} > \xi_k$. Moreover, based on Theorem 1-(ii), for $T_k > \frac{1}{\alpha}$ we have:

$$P(X_{T_k} > M_X | Y_{T_k} = y, S > T_k) < P(X_{T_{k+1}} > M_X | Y_{T_{k+1}} = y, S > T_{k+1}), \qquad \text{for all } y > 0.$$

Hence, $\xi_{k+1} \leq \xi_k$. Altogether, from Theorem 1 we can deduce that despite the different structures of decision rules in CMP and IMP, these two policies consider a threshold on the marker level. The difference is the nature of the proposed threshold in these two policies. The threshold in CMP is fixed while IMP suggests an adaptive threshold that dynamically evolves itself as time passes. More properties of this adaptive threshold are also investigated in Subsection 4.1.

It is worth noting that the definition of ξ_k, which gives us an equivalent condition to $P(X_{T_k} > M_X | Y_{T_k} = y, S > T_k) > p$, as well as the decreasing behavior of ξ_k over time are essential to evaluate the corrective and preventive replacement probabilities and to obtain the analytical expression of the cost function in the following sections. For IMP, we hereafter assume $\alpha T > 1$ to evaluate the cost function in a closed form. Otherwise, we can use a simulation-based evaluation of the cost function.

3.3. Corrective and preventive probabilities at inspection times

Depending on the values of their respective decision variables, the two policies CMP and IMP can evolve between a purely preventive and a purely corrective behavior. Their performance, which

depends on the balance between these two limit configurations, has to be optimized. For this purpose, it is necessary to determine the probabilities of preventive and corrective actions. Maintenance actions are perfect. Hence the degradation process of the maintained system is a regenerative process and maintenance times define regeneration epochs. The intervals of time between successive regeneration epochs are renewal cycles. Let the notations P_p and P_c represent respectively the probability that a renewal cycle ends with preventive and corrective replacement. As a consequence, we have:

$$P_p = \sum_{k=1}^{\infty} P_p(k), \qquad \text{and} \qquad P_c = \sum_{k=0}^{\infty} P_c(k)$$

where $P_p(k),\ k = 1, 2, \cdots$ is the probability that preventive maintenance is performed at time T_k and $P_c(k),\ k = 0, 1, \cdots$ denotes the probability that corrective maintenance has been performed between times T_k and T_{k+1}. This section is devoted to the derivation of preventive action and corrective action probabilities. For this aim, according to the properties of the bivariate stochastic process introduced in Subsection 2.1, we denote:

$$f_{(X_{T_{i_2}}-X_{T_{i_1}},Y_{T_{i_2}}-Y_{T_{i_1}})}(x,y) = \frac{(\lambda_1\lambda_2)^{\alpha(T_{i_2}-T_{i_1})}}{(1-\rho)\Gamma\left(\alpha(T_{i_2}-T_{i_1})\right)}\left(\frac{xy}{\rho\lambda_1\lambda_2}\right)^{\frac{\alpha(T_{i_2}-T_{i_1})-1}{2}} \times \exp\left(-\frac{\lambda_1 x+\lambda_2 y}{1-\rho}\right) I_{\alpha(T_{i_2}-T_{i_1})-1}\left(\frac{2\sqrt{\rho\lambda_1\lambda_2 xy}}{1-\rho}\right), \quad x,y \geq 0,$$

which is the Kibble's bivariate gamma pdf for $(X_{T_{i_2}}-X_{T_{i_1}}, Y_{T_{i_2}}-Y_{T_{i_1}})$ based on (1). Also, its marginal is represented by

$$f_{(X_{T_{i_2}}-X_{T_{i_1}})}(x) = \frac{{\lambda_1}^{\alpha(T_{i_2}-T_{i_1})}}{\Gamma\left(\alpha(T_{i_2}-T_{i_1})\right)} x^{\alpha(T_{i_2}-T_{i_1})-1} e^{-\lambda_1 x}, \quad x \geq 0,$$

which is the gamma pdf for the degradation increment $X_{T_{i_2}} - X_{T_{i_1}}$. We recall that when $T_{i_1} = 0$, $X_{T_{i_1}} = Y_{T_{i_1}} = 0$ and then they can be omitted in the notations.

3.3.1. Classical Maintenance Policy

• **Corrective probabilities for CMP:** A corrective action occurs between inspection times T_k and T_{k+1} if no failure and no preventive action have occurred before T_k. For CMP this means that the marker process has not reached the maintenance threshold at inspection times before system failure. Consequently,

$$P_c(0) = P(X_{T_1} > L) = \int_L^{\infty} f_{(X_{T_1})}(x)\mathrm{d}x,$$

and for $k \geq 1$,

$$\begin{aligned} P_c(k) &= P(X_{T_1} < L,\ \ldots, X_{T_k} < L, X_{T_{k+1}} > L, Y_{T_1} < M_Y,\ \ldots, Y_{T_k} < M_Y) \\ &= P(X_{T_k} < L, X_{T_{k+1}} > L, Y_{T_k} < M_Y) \\ &= \int_0^L \int_0^{M_Y} P(X_{T_{k+1}} > L | X_{T_k} = x, Y_{T_k} = y) f_{(X_{T_k}, Y_{T_k})}(x,y) \mathrm{d}y \mathrm{d}x \\ &= \int_0^L \int_0^{M_Y} \int_{L-x}^{\infty} f_{(X_{T_{k+1}} - X_{T_k})}(z) f_{(X_{T_k}, Y_{T_k})}(x,y) \mathrm{d}z \mathrm{d}y \mathrm{d}x. \end{aligned} \tag{9}$$

The second equation holds because $(X_t)_{t\geq 0}$ and $(Y_t)_{t\geq 0}$ are increasing processes and the last equation holds because $(X_t, Y_t)_{t\geq 0}$ has independent increments.

• **Preventive probabilities for CMP:** A preventive action occurs at an inspection time T_k if the marker process crosses the threshold M_Y for the first time and no corrective action has been performed before, i.e. the latent process remains below the failure level. Hence:

$$P_p(1) = P(X_{T_1} < L, Y_{T_1} > M_Y) = \int_0^L \int_{M_Y}^{\infty} f_{(X_{T_1}, Y_{T_1})}(x,y) \mathrm{d}y \mathrm{d}x,$$

and for $k \geq 2$, using a similar calculation as in (9), we have:

$$\begin{aligned} P_p(k) &= P(X_{T_1} < L,\ \ldots, X_{T_k} < L, Y_{T_1} < M_Y,\ \ldots, Y_{T_{k-1}} < M_Y, Y_{T_k} > M_Y) \\ &= P(X_{T_k} < L, Y_{T_{k-1}} < M_Y, Y_{T_k} > M_Y) \\ &= \int_0^L \int_0^{M_Y} \int_0^{L-x} \int_{M_Y - y}^{\infty} f_{(X_{T_k} - X_{T_{k-1}}, Y_{T_k} - Y_{T_{k-1}})}(z,w) f_{(X_{T_{k-1}}, Y_{T_{k-1}})}(x,y) \mathrm{d}w \mathrm{d}z \mathrm{d}y \mathrm{d}x. \end{aligned}$$

3.3.2. Intelligent Maintenance Policy

Recall that for IMP, preventive action is performed at inspection time T_k if $P(X_{T_k} > M_X | Y_{T_k} = y_{T_k}, S > T_k) > p$, or equivalently if $Y_{T_k} > \xi_k$, where ξ_k is defined by equation (8). Then with some modification in the probability calculation for CMP, we can derive the probabilities for IMP.

• **Corrective probabilities for IMP:** A corrective action occurs between times T_k and T_{k+1} if no preventive action has been performed before. For IMP this means that the marker process remains below the adaptive maintenance threshold ξ_k at each inspection time T_k before system failure. Therefore,

$$P_c(0) = P(X_{T_1} > L) = \int_L^{\infty} f_{(X_{T_1})}(x) \mathrm{d}x,$$

and for $k > 0$, using similar calculations,

$$\begin{aligned} P_c(k) &= P(X_{T_1} < L,\ \ldots, X_{T_k} < L, X_{T_{k+1}} > L, Y_{T_1} < \xi_1,\ \ldots, Y_{T_k} < \xi_k) \\ &= P(X_{T_k} < L, X_{T_{k+1}} > L, Y_{T_k} < \xi_k) \\ &= \int_0^L \int_0^{\xi_k} \int_{L-x}^{\infty} f_{(X_{T_{k+1}} - X_{T_k})}(z) f_{(X_{T_k}, Y_{T_k})}(x, y) \mathrm{d}z\mathrm{d}y\mathrm{d}x. \end{aligned}$$

The second equation holds due to the monotonicity of the gamma process and also the fact that increasing k results in smaller values of ξ_k. That means the event $Y_1 < \xi_1,\ \ldots, Y_k < \xi_k$ is equivalent to the event $Y_k < \xi_k$ since $Y_{T_i} < Y_{T_k} < \xi_k \leq \xi_i$ for $1 \leq i < k$.

• **Preventive probabilities for IMP:** A preventive action occurs at inspection time T_k if the marker process overpasses its preventive threshold ξ_k for the first time and no corrective action has been performed before in the cycle. Thus

$$P_p(1) = P(X_{T_1} < L, Y_{T_1} > \xi_1) = \int_0^L \int_{\xi_1}^{\infty} f_{(X_{T_1}, Y_{T_1})}(x, y) \mathrm{d}y\mathrm{d}x,$$

and for $k > 1$

$$\begin{aligned} P_p(k) &= P(X_{T_1} < L,\ \ldots, X_{T_k} < L, Y_{T_1} < \xi_1,\ \ldots, Y_{T_{k-1}} < \xi_{k-1}, Y_{T_k} > \xi_k) \\ &= P(X_{T_k} < L, Y_{T_{k-1}} < \xi_{k-1}, Y_{T_k} > \xi_k) \\ &= \int_0^L \int_{\xi_{k-1}}^{\xi_k} \int_0^{L-x} \int_{\xi_k - y}^{\infty} f_{(X_{T_k} - X_{T_{k-1}}, Y_{T_k} - Y_{T_{k-1}})}(z, w) f_{(X_{T_{k-1}}, Y_{T_{k-1}})}(x, y) \mathrm{d}w\mathrm{d}z\mathrm{d}y\mathrm{d}x \\ &\qquad + \int_0^L \int_0^{\xi_k} \int_0^{L-x} f_{(X_{T_k} - X_{T_{k-1}})}(z) f_{(X_{T_{k-1}}, Y_{T_{k-1}})}(x, y) \mathrm{d}z\mathrm{d}y\mathrm{d}x \\ &= \int_0^L \int_0^{\xi_{k-1}} \int_0^{L-x} \int_{\max(\xi_k - y, 0)}^{\infty} f_{(X_{T_k} - X_{T_{k-1}}, Y_{T_k} - Y_{T_{k-1}})}(z, w) f_{(X_{T_{k-1}}, Y_{T_{k-1}})}(x, y) \mathrm{d}w\mathrm{d}z\mathrm{d}y\mathrm{d}x. \end{aligned}$$

The second part of the third equality refers to the following situation where $Y_{T_{k-1}}$ is less than the adaptive threshold ξ_{k-1} but is greater than the next threshold ξ_k. In this case, preventive action is not required at T_{k-1}. But even without measuring the degradation level Y_{T_k}, we already know that preventive action must be taken at T_k provided that the system does not face a failure before that time.

3.4. Maintenance Cost

Maintenance actions incur a burden of costs. Inspections take place at times T_k, each with cost c_i. At each inspection, preventive action may be performed, costing c_p. Also, a corrective replacement with cost c_c is needed whenever the system fails. Hence, the cumulative cost function on $[0, t]$ is:

$$C(t) = c_i N_i(t) + c_p N_p(t) + c_c N_c(t),$$

where $N_i(t)$, $N_p(t)$, and $N_c(t)$ are respectively the number of inspections, the number of preventive replacements, and the number of corrective replacements in [0,t]. This cost depends on the decision variables. We employ the long-run average cost per unit of time EC as the objective function to be minimized. From classical renewal theory, it is well known that for any degradation process with the regenerative property, we have the following equation.

$$EC = \lim_{t\to\infty} \frac{E\left(C(t)\right)}{t} = \frac{E\left(C(L_1)\right)}{E(L_1)},$$

where L_1 is the length of one renewal cycle, i.e., the time between two successive replacements. So we proceed to minimize EC with the following formula to find optimal values of the decision variables:

$$EC = \frac{c_i E(N_i) + c_c E(N_c) + c_p E(N_p)}{E(L_1)}. \tag{10}$$

We know that a cycle ends either with a corrective or a preventive replacement. Thus,

$$E(N_p) = P_p = \sum_{k=1}^{\infty} P_p(k),$$

and

$$E(N_c) = P_c = \sum_{k=0}^{\infty} P_c(k).$$

Moreover, N_i is a discrete random variable with non-negative support and the following probabilities:

$$P(N_i = 0) = P_c(0), \qquad P(N_i = k) = P_c(k) + P_p(k), \qquad \text{for } \; k = 1, 2, \cdots .$$

We also know that the length of a cycle is equal to $N_i T$ or S in cases of ending the cycle with a preventive or a corrective replacement, respectively. Hence,

$$E(L_1) = P_p E(N_i) T + P_c E(S),$$

where

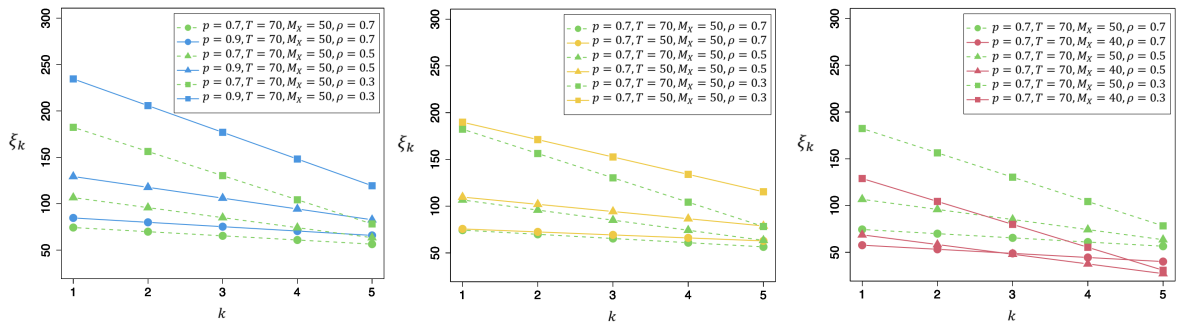
$$E(N_i) = \sum_{k=0}^{\infty} k P(N_i = k),$$

and

$$E(S) = \int_0^{\infty} P(S > s)\mathrm{d}s = \int_0^{\infty} \frac{\Gamma(\alpha s, \lambda_1 L)}{\Gamma(\alpha s)}\mathrm{d}s.$$

4. Numerical Study

In this section, we illustrate our proposed maintenance policies through a simulation study. First of all, we try to discuss the characteristics of the adaptive threshold ξ_k and then the optimal decision rule is obtained for CMP and IMP.



(a) Effect of changing p (b) Effect of changing T (c) Effect of changing M_X

Figure 2: Effect of changing decision variables on ξ_k for three different values of $\rho = 0.3, 0.5, 0.7$. The parameter set considered here is $\alpha = 0.1, \lambda_1 = \lambda_2 = 0.7$ and the failure threshold is $L = 60$.

4.1. The effect of decision variables on the IMP adaptive threshold

The adaptive threshold ξ_k at inspection time T_k is defined by equation (8). Its value depends on the decision variables p, T, and M_X. Various remarks can be given regarding the evolution of ξ_k in case of modifying these decision variables. These remarks, which are related to the facts declared in Theorem 1, are given in Appendix B. Some numerical experiments are conducted to complement the above-mentioned remarks. Figure 2 shows for ξ_k as a function of k, and for different values of ρ, the effect of changes in decision variables. The main points are as follows:

- For each k, increasing p generates an increase of ξ_k, see Remark 1 for proof. A higher value of p means that we want preventive action to be taken only when the system is close to failure. To ensure that only highly deteriorated systems are replaced, the IMP decision rule chooses a higher adaptive threshold value. Otherwise, a preventive replacement is performed even when the risk of failure is low.

- For each k, decreasing T i.e. reducing the value of $T_k = kT$ results in increasing values of ξ_k, see Remark 2 for proof. With frequent inspections, the marker value can be measured more often. The conditional probability (7), and thus the risk of failure, is updated at a higher frequency. Therefore, it is possible to increase the values of ξ_k without the system being in a vulnerable state during the transition from ξ_k to L.

- For each k, the value of ξ_k increases with M_X, see Remark 3 for proof. For a given value of failure threshold L, choosing higher values of M_X results in a smaller preventive replacement zone. This corresponds to choosing to replace only the systems that are really deteriorated, i.e. choosing a higher value of ξ_k. We see that the impact of the decision variable M_X is similar to that of the decision variable p. Choosing smaller values of p and M_X leads to a more conservative decision

rule. Indeed, at each inspection time T_k, the lower the value of ξ_k, the higher the chances of taking preventive action and avoiding an unexpected failure.

Given the above remarks, it is clear that the self-adaptation dynamic of the IMP policy is a function of the choice of decision variables. The evolution of the threshold ξ_k does not require a new online optimization at each time step. Structurally, this policy can change the preventive threshold on the marker process in a rational way. The evolution of the dynamic maintenance threshold ξ_k depends on the model parameters as well. Among all the parameters of the degradation model, the correlation parameter ρ is particularly important as it stands for the relationship between the latent process and the marker process. Three values of ρ are considered in Figure 2 to show the effect of changing ρ on the adaptive preventive threshold. It can be seen that all configurations result in a decrease of ξ_k when ρ increases.

In all, as already stated, the IMP policy is a more sophisticated version than CMP with an adaptive decreasing threshold instead of constant fixed M_Y. Figure 2 confirms that ξ_k decreases when k increases. It is apparent that the adaptive threshold decreases with an approximately linear slope. The slope clearly depends on the decision variable configuration and also the parameter set. It can be seen that, for smaller values of ρ the decreasing slope is steeper. That means with a stronger correlation, we have more reliable information about the state of the latent degradation process given the observed state of the marker process; hence there is no need for a large decrease in the values of the adaptive threshold for future inspections.

4.2. Numerical study and performance comparison

Unless otherwise stated we assume that the parameters of the bivariate gamma process are $\alpha = 0.1, \lambda_1 = 0.7, \lambda_2 = 0.7$, while the failure threshold is set at $L = 60$. The selected parameters are intended to depict a scenario featuring the degradation indicator of a system with an expert-estimated mean lifespan of 14 months (equivalent to 420 days), and a variance of the marker indicator around 50 after 250 days of usage. Furthermore, in the scenario where both the latent and marker processes exhibit identical characteristics, their marginal distributions share the same parameter values. The correlation coefficient ρ ranges over values $0.1, 0.3, 0.5, 0.7$ and 0.9 to show the strength of the relation between the marker and latent process. These specific parameter selections have been made to showcase how the proposed policies behave and to facilitate numerical comparisons. It's worth noting that different sets of parameters could be employed in a similar fashion for further analysis.

Let $\theta_1 = (M_Y, T)$ and $\theta_2 = (M_X, T, p)$ be the decision variables for CMP and IMP respectively. We aim to find the optimal values θ_i^*, $i = 1, 2$ of these decision variable, such that:

$$EC_i^* = EC(\theta_i^*) \leq EC(\theta_i), \qquad \text{for } \theta_i \in \Theta_i$$

where $\Theta_1 = [0, +\infty) \times [0, +\infty)$ and $\Theta_2 = [0, L] \times [0, +\infty) \times [0, 1]$ for CMP and IMP decision rules, respectively.

By considering the following cost configurations, we conducted a numerical study.

$$c_i = 5, c_p = 140, c_c = 240. \tag{11}$$

In this numerical study, we use, as a benchmark for comparison, the hypothetical ideal case where the latent degradation process X_t is observable. In this case, it is no longer necessary to use a marker process at all. Adapting the procedure defined in [17] to periodic inspections, the related long-run optimal maintenance cost rate can be obtained. It is then compared to the ones obtained with IMP and CMP policies.

With the unit costs configuration and parameter set given previously, the optimal cost for the hypothetical ideal case is $EC^*_{\text{Ideal}} = 0.4590998$, while the optimal time interval between inspections and the optimal maintenance threshold are 110.6426 and 43.63159, respectively. Thus, the value of EC^*_{Ideal} will serve as a reference for comparison to assess the cost increase incurred by the loss of direct observation.

The percentage relative cost increase is considered for comparison. It is defined as:

$$\text{PRC} = \frac{EC^*_i - EC^*_{\text{Ideal}}}{EC^*_{\text{Ideal}}} \times 100 \tag{12}$$

where EC^*_i is defined as previously for CMP and IMP.

Tables 1 and 2 depict the optimal values of the decision variables and the cost for various correlations between the marker and latent process, respectively, for CMP and IMP. The relative increase in cost which illustrates the effect of ρ on the maintenance cost is given in these tables. The results reveal that with the decrease of ρ, the cost increases and deviates from EC^*_{Ideal}. From the tables, it can be seen that the decision rule IMP is better than CMP in all cases. Notably, the highest percentage of the relative increase in cost for IMP occurs in $\rho = 0.1$. It is approximately equal to the lowest percentage of the relative increase in cost for CMP, which corresponds to $\rho = 0.9$.

4.3. Robustness in case of differences between the latent and marker evolution rates

In the previous numerical experiments, the rates of evolution of the marker and latent processes are identical, i.e. $\lambda_1 = \lambda_2$. From properties of the bivariate gamma process introduced in Section 2, we know that for a given value of ρ, if $\lambda_1 < \lambda_2$ the sample path of the marker process tends to be lower than the sample path of the latent process. In such a situation and with identical maintenance thresholds, the use of the marker process instead of the latent process for decision-making may lead to more failures. Indeed the marker process indicates that the system is far from failure which is not the case. On the contrary, when $\lambda_1 > \lambda_2$, the marker process values are higher than the values of the

Table 1: Optimal cost and decision variables for different values of ρ for CMP

	EC_1^*	M_Y^*	T^*	PRC
$\rho = 0.9$	0.4745916	43.66953	108.4251	3.38%
$\rho = 0.7$	0.4871396	43.53404	112.3639	6.11%
$\rho = 0.5$	0.4974453	43.31415	118.9446	8.35%
$\rho = 0.3$	0.5048235	42.86665	120.4265	10.00%
$\rho = 0.1$	0.5107319	42.71846	122.7395	11.25%

Table 2: Optimal cost and decision variables for different values of ρ for IMP

	EC_2^*	M_X^*	T^*	p^*	PRC
$\rho = 0.9$	0.4680314	41.67136	114.1677	0.7148207	1.94%
$\rho = 0.7$	0.473598	40.4905	119.2963	0.6851776	3.16%
$\rho = 0.5$	0.4749485	39.47189	120.6514	0.6575764	3.45%
$\rho = 0.3$	0.4755291	37.97544	123.1166	0.5910648	3.58%
$\rho = 0.1$	0.4760132	36.88819	125.1819	0.5173126	3.68%

latent process. Hence, it may result in more conservative overly cautious maintenance actions. To go further, we investigate how moderate differences between the mean rates of the marker and the latent processes affect the optimal decision variables and the optimal cost. To this aim, we consider that $\lambda_2 = \lambda_1 \pm \epsilon$. Numerical results are given for $\epsilon = 0.2$ as an example of a moderate difference between λ_1 and λ_2. The values of the other parameters are the same as in the previous section. The results are shown in Tables 3-6 which can be considered in addition to Tables 1 and 2 where $\epsilon = 0$. A first analysis of optimal cost rate EC^* shows that it remains very stable for the two policies when the value of λ_2 is slightly changing and the decision variables are updated accordingly.

Nevertheless, for CMP, the decision variable M_Y^* is sensitive to changes in the value of λ_2. It decreases significantly as λ_2 increases. Indeed, as the rate parameter λ_2 increases, the mean evolution rate of the marker process decreases. This means that system failure arises when the values of the marker process are lower. The preventive threshold M_Y is following this trend to keep the same probability of preventive actions. In parallel, T^* is slightly decreasing to ensure optimal tuning. For IMP, the decision variables are stable. Specifically, M_X^* and T^* do not change, and p^* only slightly fluctuates as λ_2 changes. As a consequence, small changes in the value of λ_2 have very little impact on the location of the optimal point of the cost function, as well as the optimal decision variables.

This stability of optimal cost rate and optimal decision variables characterizes strong robustness to variations in the λ_2 parameter for IMP, which is not the case for CMP.

Table 3: Optimal cost and decision variables for different values of ρ for CMP: Case $\lambda_2 < \lambda_1$ ($\lambda_2 = 0.5$)

	EC_1^*	M_Y^*	T^*
$\rho = 0.9$	0.4747293	61.64174	110.9064
$\rho = 0.7$	0.487961	60.32118	115.6872
$\rho = 0.5$	0.4972741	59.62128	120.7024
$\rho = 0.3$	0.5061086	59.0271	121.4659
$\rho = 0.1$	0.5124461	58.77783	123.9657

4.4. Correlation as a Decision Variable

Suppose that the cost of the inspection is closely related to the amount of correlation between the latent degradation and the marker process, i.e., $c_i = h(\rho)$ for a given function h. This situation can arise when the correlation is associated with monitoring the performance of the marker process. Several options of monitoring devices for a given marker process can be available, or several marker processes can be considered, such that each of them has a specific correlation with the latent process and its associated cost of measurement. Then, it is relevant to ask for the best value of ρ to get the optimal value of maintenance cost and decide jointly on the best-associated monitoring strategy. Suppose we have a system with a latent degradation process and several options for marker processes, such that each of them has a different correlation with the latent process and hence a different cost of measurement. It might be thought that the best marker process is the one most strongly correlated with the latent process. However, it may impose a higher cost when the marker and latent processes are strongly correlated. It is reasonable to choose the correlation that minimizes the cost function,

Table 4: Optimal cost and decision variables for different values of ρ for CMP: Case $\lambda_2 > \lambda_1$ ($\lambda_2 = 0.9$)

	EC_1^*	M_Y^*	T^*
$\rho = 0.9$	0.4750954	34.8642	106.0671
$\rho = 0.7$	0.4893817	34.01058	111.3013
$\rho = 0.5$	04998618	33.82293	116.8621
$\rho = 0.3$	0.5051535	33.32315	118.2206
$\rho = 0.1$	0.513101	32.84561	119.9563

Table 5: Optimal cost and decision variables for different values of ρ for IMP: Case $\lambda_2 < \lambda_1$ ($\lambda_2 = 0.5$)

	EC_2^*	M_X^*	T^*	p^*
$\rho = 0.9$	0.4674119	41.89745	114.4464	0.7115131
$\rho = 0.7$	0.4708573	40.05201	119.6824	0.6995851
$\rho = 0.5$	0.4752868	39.13543	120.5517	0.6847704
$\rho = 0.3$	0.4768508	37.98851	123.5777	0.5652202
$\rho = 0.1$	0.4782173	36.97921	125.3537	0.5324252

Table 6: Optimal cost and decision variables for different values of ρ for IMP: Case $\lambda_2 > \lambda_1$ ($\lambda_2 = 0.9$)

	EC_2^*	M_X^*	T^*	p^*
$\rho = 0.9$	0.4693209	41.8424	114.4749	0.7495639
$\rho = 0.7$	0.4734863	40.86702	119.3662	0.670277
$\rho = 0.5$	0.475567	39.98011	120.5209	0.6234719
$\rho = 0.3$	0.4775137	37.14151	123.4667	0.5888839
$\rho = 0.1$	0.4796114	36.85059	125.013	0.5665818

which means that ρ also becomes a decision variable.

Here, we assume that $c_i = a + b\rho^\gamma$ where a and b are positive constants and γ is a non-negative real number. Considering this, different kinds of inspection cost functions can be defined depending on the value of γ. More precisely, the inspection cost is:

- constant when $\gamma = 0$; which is a trivial case and we will not address this case any further.
- a concave increasing function when $0 < \gamma < 1$. That means as a function of ρ, the rate of increase of c_i is higher at the beginning of the interval $[0, 1]$, while it reduces afterward.
- a linear increasing function when $\gamma = 1$.
- a convex increasing function when $\gamma > 1$. For small values of ρ, inspection cost c_i evolves slowly and then grows more rapidly. Obviously, this growth's speed depends on the magnitude of γ.

A numerical study was conducted with $a = 5$ and $b = 7$ while γ takes values $0.5, 1, 2, 5, 10$, and 15. Corresponding shapes of the inspection cost function are depicted in Figure 3. Figure 4 shows how changing ρ can affect the optimal cost when the inspection cost is a function of the correlation between the marker and latent processes. This figure shows the evolution of optimal cost with respect to ρ for various values of γ in CMP. In each case, the minimum value of optimal cost is obtained and

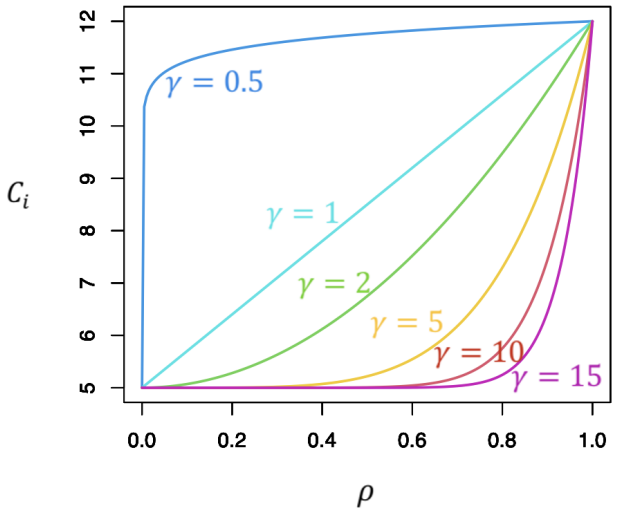


Figure 3: Various shapes of c_i as a function of ρ

depicted by a "$*$" sign.

Under the same cost configuration of c_p and c_c as in (11) and with parameters $\alpha = 0.1$, $\lambda_1 = \lambda_2 = 0.7$, and $L = 60$, the optimal values of decision variables are given in Tables 7 and 8 for CMP and IMP, respectively. It can be seen that, for a given value of γ, the optimal cost for IMP is always lower than for CMP. Moreover, the optimal value of ρ^* is much lower in IMP than in CMP. From Tables 7 and 8, we see that the optimal value of ρ^* increases with γ, which is reasonable given the shape of the inspection cost function in Figure 3.

Hence, if there are two marker processes and the CMP decision rule has to be employed, the decision-maker should choose the marker process which is more closely related to the latent process, i.e. with the higher correlation. This choice may result in a higher maintenance cost than with the IMP decision rule.

5. Conclusion

In this paper, we investigated CBM policies for a system with an unobservable degradation process. Instead of directly observing this degradation process, we have access to a marker process that exhibits a correlation with the underlying latent degradation process. The stochastic degradation model follows a bivariate gamma distribution. Within this context, we have introduced two periodic maintenance

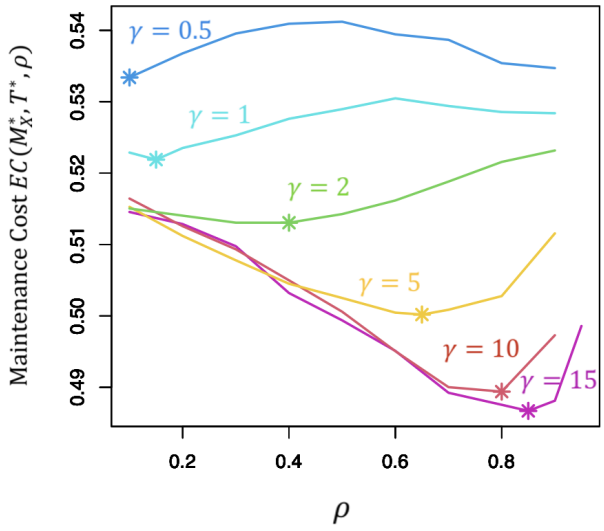


Figure 4: CMP Maintenance cost as a function of ρ for different values of γ

Table 7: Optimal cost and decision variables for different values of γ for CMP

	EC_1^*	M_Y^*	T^*	ρ^*
$\gamma = 0.5$	0.5325465	42.40107	122.5668	0.1019388
$\gamma = 1$	0.5212979	44.94092	123.9586	0.1517832
$\gamma = 2$	0.5111996	43.63356	119.5199	0.3972142
$\gamma = 5$	0.4976918	42.68946	116.9143	0.6492435
$\gamma = 10$	0.4887491	42.10388	121.7721	0.7849923
$\gamma = 15$	0.4854622	41.87729	124.5724	0.8493222

Table 8: Optimal cost and decision variables for different values of γ for IMP

	EC_2^*	M_X^*	T^*	p^*	ρ^*
$\gamma = 0.5$	0.4970541	44.53968	120.3403	0.6866454	0.1123388
$\gamma = 1$	0.4912671	41.61062	120.1753	0.7129884	0.1939604
$\gamma = 2$	0.4782309	39.87464	119.8381	0.6578021	0.2234431
$\gamma = 5$	0.4762208	38.06256	118.9271	0.6426162	0.3943600
$\gamma = 10$	0.4751424	38.81985	117.4567	0.6527617	0.5102471
$\gamma = 15$	0.4746997	40.45746	115.1368	0.6726548	0.5943782

policies with the objective of minimizing the long-term cost rate function. The first policy, referred to as CMP, traditionally employs a fixed preventive maintenance threshold based on the marker process. On the other hand, the second policy, known as IMP, utilizes a prediction of the latent degradation level of the system derived from its current marker level for decision-making. We demonstrate that IMP can be described as a policy with an adaptive "intelligent" preventive threshold based on the marker process. This threshold is updated automatically at each inspection time without any requirement for an online optimization process. We outline the properties of this adaptive threshold and provide analytical expressions for the maintenance cost rate for both policies. Additionally, we present a numerical study to evaluate the performance of these policies, comparing them and underscoring the effectiveness and robustness of IMP, even in cases where the correlation between the marker and latent processes is weak. We also investigated the case where this correlation is related to the monitoring cost. This can help the decision-maker decide on the optimal investment to be made in the monitoring technique. Furthermore, we delve into the scenario where this correlation is linked to monitoring costs. This information can be valuable to decision-makers in determining the optimal investment in the monitoring technique. In future studies, it is imperative to study the influence of various parameters, particularly the correlation coefficient, which characterizes the strength of the association between the degradation and marker processes. We can extend the concept of periodic inspections to encompass a condition-based inspection policy, scheduling the timing of the next inspection based on the currently observed degradation level. Moreover, we can explore more intricate models for the lifetime of systems operating in environments with two competing risks. Finally, natural extensions of the present work could be other types of bivariate processes with imperfect maintenance actions.

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Appendix A: Proof of Theorem 1

This section focuses on the proof of the Theorem 1. To this aim, we need to prove some lemmas.

First of all, we mention some Turán-type inequalities for the modified Bessel function of the first kind. Barciz [4] showed that for $\kappa > -1$ and $x > 0$, we have:

$$I_\kappa^2(x) - I_{\kappa-1}(x)I_{\kappa+1}(x) > 0, \tag{A.1}$$

Moreover, Barciz [5] also showed that for $\mu > \kappa > -1$ and $x > 0$, the following inequality holds.

$$I_{\kappa+1}(x)I_\mu(x) - I_{\mu+1}(x)I_\kappa(x) \geq 0. \tag{A.2}$$

Lemma 1. *Let g_1 and g_2 be two positive differentiable functions on $[0,+\infty)$, such that $g_1(0) = g_2(0) = 0$ and the function $x \to \dfrac{g_1(x)}{g_2(x)}$ is a continuous increasing function of x on $[0,+\infty)$. Then, $h(z) = \dfrac{\int_0^z g_1(u)\mathrm{d}u}{\int_0^z g_2(u)\mathrm{d}u}$ is an increasing function of z on $[0,+\infty)$.*

Proof. Calculate as follows:

$$\begin{aligned}
h'(z) &= \frac{g_1(z)\int_0^z g_2(u)\mathrm{d}u - g_2(z)\int_0^z g_1(u)\mathrm{d}u}{\left(\int_0^z g_2(u)\mathrm{d}u\right)^2} \\
&= \frac{\int_0^z g_1'(v)\mathrm{d}v \int_0^z g_2(u)\mathrm{d}u - \int_0^z g_2'(v)\mathrm{d}v \int_0^z g_1(u)\mathrm{d}u}{\left(\int_0^z g_2(u)\mathrm{d}u\right)^2} \\
&= \frac{\int_0^z \int_0^z \left[g_1'(v)g_2(u) - g_2'(v)g_1(u)\right]\mathrm{d}v\mathrm{d}u}{\left(\int_0^z g_2(u)\mathrm{d}u\right)^2} \\
&= \frac{\int_0^z \int_0^u \left[g_1'(v)g_2(u) - g_2'(v)g_1(u)\right]\mathrm{d}v\mathrm{d}u + \int_0^z \int_u^z \left[g_1'(v)g_2(u) - g_2'(v)g_1(u)\right]\mathrm{d}v\mathrm{d}u}{\left(\int_0^z g_2(u)\mathrm{d}u\right)^2} \\
&= \frac{\int_0^z \int_0^u \left[g_1'(v)g_2(u) - g_2'(v)g_1(u)\right]\mathrm{d}v\mathrm{d}u + \int_0^z \int_0^v \left[g_1'(v)g_2(u) - g_2'(v)g_1(u)\right]\mathrm{d}u\mathrm{d}v}{\left(\int_0^z g_2(u)\mathrm{d}u\right)^2} \\
&= \frac{\int_0^z \int_0^v \left[g_1'(v)g_2(u) - g_2'(v)g_1(u)\right]\mathrm{d}u\mathrm{d}v}{\left(\int_0^z g_2(u)\mathrm{d}u\right)^2} > 0.
\end{aligned}$$

The last inequality holds since $\dfrac{g_1(x)}{g_2(x)}$ is an increasing function of x and so $g_1'(v)g_2(v) > g_2'(v)g_1(v)$ and $g_1(v)g_2(u) > g_1(u)g_2(v)$ for $u < v$. By multiplying these two inequalities we see that the integrand is a positive function which leads to a positive value for the integral and an increasing function $h(z)$. □

Lemma 2. *Let u be a positive and increasing function of x, then the following statements are true.*

(i) *For $\nu > 0$ and $x > 0$, the function $x \to \dfrac{u(x)I_\nu(u(x))}{I_{\nu-1}(u(x))}$ is an increasing function.*

(ii) *For $\nu_2 > \nu_1 > 1$ and $x > 0$, the function $x \to \dfrac{u^{\nu_2-1}(x)I_{\nu_2-1}(u(x))}{u^{\nu_1-1}(x)I_{\nu_1-1}(u(x))}$ is an increasing function.*

Proof. (i) By writing

$$\frac{u(x)I_\nu(u(x))}{I_{\nu-1}(u(x))} = \frac{u^\nu(x)I_\nu(u(x))}{u^{\nu-1}(x)I_{\nu-1}(u(x))},$$

and using the fact that for all real values of ν, we have:

$$\frac{\partial}{\partial u}\left(u^\nu I_\nu(u)\right) = u^\nu I_{\nu-1}(u),$$

the derivative of the function with respect to x is positive by setting $\kappa = \nu - 1$ in (A.1) which holds for $\nu > 0$.

(ii) Similarly, the derivative of the function with respect to x is positive due to (A.2) by setting $\mu = \nu_2 - 2$ and $\kappa = \nu_1 - 2$, i.e. for $\nu_2 > \nu_1 > 1$. □

Lemma 3. *Let* $G(y; z, \nu) = \int_0^z g(x, y; \nu)\mathrm{d}x$ *where*

$$g(x, y; \nu) = x^{\frac{\nu}{2}} \exp\left(-\frac{\lambda_1 x}{1-\rho}\right) I_\nu\left(\frac{2\sqrt{\rho\lambda_1\lambda_2 xy}}{1-\rho}\right),$$

then the following statements are true.

(i) *for* $\nu > 0$, *the function* $z \to \dfrac{G(y; z, \nu)}{G(y; z, \nu - 1)}$ *is an increasing function of* z.

(ii) *for* $\nu_2 > \nu_1 > 1$, *the function* $z \to \dfrac{G(y; z, \nu_2 - 1)}{G(y; z, \nu_1 - 1)}$ *is an increasing function of* z.

Proof. By defining $u(x) = \dfrac{2\sqrt{\rho\lambda_1\lambda_2 xy}}{1-\rho}$, we know that:
(i) The function

$$\frac{g(x, y; \nu)}{g(x, y; \nu - 1)} = \frac{1-\rho}{2\sqrt{\rho\lambda_1\lambda_2 y}} \frac{u(x) I_\nu\left(u(x)\right)}{I_{\nu-1}\left(u(x)\right)}$$

is an increasing function of x due to Lemma 2-(i). Therefore, the assumptions of Lemma 1 are satisfied, it can be deduced that $\dfrac{G(y; z, \nu)}{G(y; z, \nu - 1)}$ is an increasing function of z for $\nu > 0$.
(ii) Similarly, the function

$$\frac{g(x, y; \nu_2 - 1)}{g(x, y; \nu_1 - 1)} = \left(\frac{1-\rho}{2\sqrt{\rho\lambda_1\lambda_2 y}}\right)^{\nu_2 - \nu_1} \frac{u^{\nu_2 - 1}(x) I_{\nu_2 - 1}\left(u(x)\right)}{u^{\nu_1 - 1}(x) I_{\nu 1 - 1}\left(u(x)\right)}$$

is an increasing function of x due to Lemma 2-(ii). Hence, due to Lemma 1, $\dfrac{G(y; z, \nu_2 - 1)}{G(y; z, \nu_1 - 1)}$ is an increasing function of z for $\nu_2 > \nu_1 > 1$. □

Proof of Theorem 1:

Proof. As $M_X < L$, from equation (7)

$$P(X_t > M_X | Y_t = y, S > t) = \frac{\int_{M_X}^L f_1(y|x) f_2(x)\mathrm{d}x}{\int_0^L f_1(y|x) f_2(x)\mathrm{d}x}, \tag{A.3}$$

and then by substituting equations (4)-(6), we have:

$$\begin{aligned} P(X_t > M_X | Y_t = y, S > t) &= \frac{\int_{M_X}^L x^{\frac{\alpha t - 1}{2}} \exp\left(-\frac{\lambda_1 x}{1-\rho}\right) I_{\alpha t - 1}\left(\frac{2\sqrt{\rho\lambda_1\lambda_2 xy}}{1-\rho}\right)\mathrm{d}x}{\int_0^L x^{\frac{\alpha t - 1}{2}} \exp\left(-\frac{\lambda_1 x}{1-\rho}\right) I_{\alpha t - 1}\left(\frac{2\sqrt{\rho\lambda_1\lambda_2 xy}}{1-\rho}\right)\mathrm{d}x} \\ &= 1 - \frac{\int_0^{M_X} x^{\frac{\alpha t - 1}{2}} \exp\left(-\frac{\lambda_1 x}{1-\rho}\right) I_{\alpha t - 1}\left(\frac{2\sqrt{\rho\lambda_1\lambda_2 xy}}{1-\rho}\right)\mathrm{d}x}{\int_0^L x^{\frac{\alpha t - 1}{2}} \exp\left(-\frac{\lambda_1 x}{1-\rho}\right) I_{\alpha t - 1}\left(\frac{2\sqrt{\rho\lambda_1\lambda_2 xy}}{1-\rho}\right)\mathrm{d}x} \\ &= 1 - \frac{G(y; M_X, \alpha t - 1)}{G(y; L, \alpha t - 1)}. \end{aligned}$$

(i) By properties of the modified Bessel function of the first kind (see [46]), we have:

$$\frac{\partial}{\partial y} I_{\alpha t-1}\left(\frac{2\sqrt{\rho\lambda_1\lambda_2 xy}}{1-\rho}\right) = \frac{\alpha t-1}{2y} I_{\alpha t-1}\left(\frac{2\sqrt{\rho\lambda_1\lambda_2 xy}}{1-\rho}\right) + \frac{\sqrt{\rho\lambda_1\lambda_2 x}}{(1-\rho)\sqrt{y}} I_{\alpha t}\left(\frac{2\sqrt{\rho\lambda_1\lambda_2 xy}}{1-\rho}\right),$$

consequently

$$\frac{\partial}{\partial y} G(y; z, \alpha t-1) = \frac{\alpha t-1}{2y} G(y; z, \alpha t-1) + \frac{\sqrt{\rho\lambda_1\lambda_2}}{(1-\rho)\sqrt{y}} G(y; z, \alpha t).$$

Thus

$$\frac{\partial}{\partial y} P(X_t > M_X | Y_t = y, S > t) = -\frac{\frac{\sqrt{\rho\lambda_1\lambda_2}}{(1-\rho)\sqrt{y}}\Big[G(y; M_X, \alpha t) G(y; L, \alpha t-1) - G(y; L, \alpha t) G(y; M_X, \alpha t-1)\Big]}{G^2(y; L, \alpha t-1)}.$$

From Lemma 3-(i), $\dfrac{G(y; z, \alpha t)}{G(y; z, \alpha t-1)}$ is an increasing function of z for all $t > 0$ and as $M_X < L$, the expression in the brackets is negative. Thus, The derivative is positive which completes the proof.

(ii) For $1 < \alpha t_1 < \alpha t_2$, the assumptions of Lemma 3-(ii) are satisfied. Then, for $M_X < L$, we have:

$$\frac{G(y; M_X, \alpha t_2 - 1)}{G(y; M_X, \alpha t_1 - 1)} < \frac{G(y; L, \alpha t_2 - 1)}{G(y; L, \alpha t_1 - 1)},$$

in other words,

$$\frac{G(y; M_X, \alpha t_2 - 1)}{G(y; L, \alpha t_2 - 1)} < \frac{G(y; M_X, \alpha t_1 - 1)}{G(y; L, \alpha t_1 - 1)}.$$

Hence, for $t_2 > t_1 > \frac{1}{\alpha}$, $P(X_{t_1} > M_X | Y_{t_1} = y, S > t_1) < P(X_{t_2} > M_X | Y_{t_2} = y, S > t_2)$ which is the desired conclusion.

□

Appendix B: Proof of remarks in Subsection 4.1

Remark 1. *Given fixed values of M_X and T, increasing p results in greater values of ξ_k for each k.*

Proof. For a given k and M_X, from Theorem 1-(i) we know that $P(X_{T_k} > M_X | Y_{T_k} = y, S > T_k)$ is an increasing function of y. Hence, if $p^{(1)} < p^{(2)}$ then it is clear that

$$\xi_k^{(1)} = \inf\{y : P(X_{T_k} > M_X | Y_{T_k} = y, S > T_k) > p^{(1)}\} < \inf\{y : P(X_{T_k} > M_X | Y_{T_k} = y, S > T_k) > p^{(2)}\} = \xi_k^{(2)}$$

Figure B.5a shows this situation. □

Remark 2. *Given fixed values of p and M_X, the value of ξ_k decreases with increasing T for each k.*

Proof. For a given y, and M_X, from Theorem 1-(ii) the probability $P(X_t > M_X | Y_t = y, S > t)$ is an increasing function of t on $(\frac{1}{\alpha}, \infty)$. Hence, if $\frac{1}{\alpha} < T^{(1)} < T^{(2)}$ which is equivalent to $\frac{1}{\alpha} < T_k^{(1)} < T_k^{(2)}$ for a given k, then

$$P(X_{T_k^{(1)}} > M_X | Y_{T_k^{(1)}} = y, S > T_k^{(1)}) < P(X_{T_k^{(2)}} > M_X | Y_{T_k^{(2)}} = y, S > T_k^{(2)}), \text{ for all } y.$$

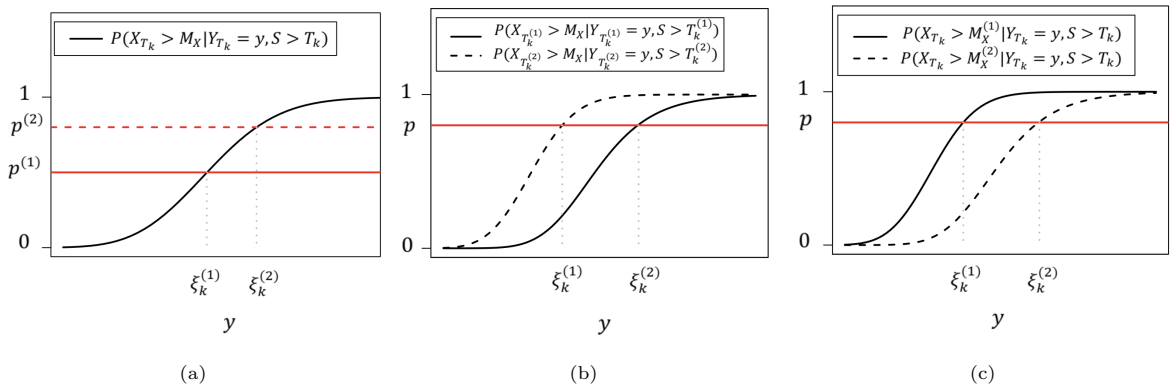

Figure B.5: Conditional probability and adaptive threshold for IMP as a function of decision variables

From Theorem 1-(i), we know that these functions are increasing in y. That means, for a given p, $\xi_k^{(1)} = \inf\{y : P(X_{T_k^{(1)}} > M_X|Y_{T_k^{(1)}} = y, S > T_k^{(1)}) > p\}$ is greater than $\xi_k^{(2)} = \inf\{y : P(X_{T_k^{(2)}} > M_X|Y_{T_k^{(2)}} = y, S > T_k^{(2)}) > p\}$. Figure B.5b depicts the situation. □

Remark 3. *Given fixed values of p and T, the value of ξ_k increases with increasing M_X for each k.*

Proof. For a given k and y, the probability $P(X_{T_k} > z|Y_{T_k} = y, S > T_k)$ is a decreasing function of z. Hence, if $M_X^{(1)} < M_X^{(2)}$ then

$$P(X_{T_k} > M_X^{(1)}|Y_{T_k} = y, S > T_k) > P(X_{T_k} > M_X^{(2)}|Y_{T_k} = y, S > T_k), \text{ for all } y.$$

From Theorem 1-(i), we know that these functions are increasing in y. That means, for a given p, $\xi_k^{(1)} = \inf\{y : P(X_{T_k} > M_X^{(1)}|Y_{T_k} = y, S > T_k) > p\}$ is less than $\xi_k^{(2)} = \inf\{y : P(X_{T_k} > M_X^{(2)}|Y_{T_k} = y, S > T_k) > p\}$. Figure B.5c depicts the situation. □

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