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Lax Eigenvalues in the Zero-Dispersion Limit for the Benjamin–Ono Equation on the Torus

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Lax eigenvalues in the zero-dispersion limit for the Benjamin-Ono equation on the torus.

Louise Gassot

Abstract

We consider the zero-dispersion limit for the Benjamin-Ono equation on the torus for bell shaped initial data. Using the approximation by truncated Fourier series, we transform the eigenvalue equation for the Lax operator into a problem in the complex plane. Then, we use the steepest descent method to get asymptotic expansions of the Lax eigenvalues. As a consequence, we determine the weak limit of solutions as the dispersion parameter goes to zero, as long as the initial data is an even bell shaped potential.

Contents

1	Introduction	1
1.1	Zero-dispersion limit for bell shaped initial data	2
1.2	Asymptotic expansion of the Lax eigenvalues	4
1.3	Strategy of proof	6
2	Eigenvalue equation for trigonometric polynomials	7
2.1	Evans function and eigenvalue equation	8
2.2	Proof of the eigenvalue equation	9
2.3	Stationary points of the phase	13
3	Deformation of contours	16
3.1	Eigenvalues inside the bulk	16
3.2	Eigenvalues outside the bulk	22
3.3	Asymptotic expansion of small eigenvalues	24
3.4	Asymptotic expansion of large eigenvalues	27
4	Zero-dispersion limit for even trigonometric polynomials	28
4.1	Phase constants	28
4.2	Time evolution	29
5	Extension to bell shaped initial data	31
A	Appendices	32
A.1	Phase constants of even initial data	32
A.2	Approximation by truncated Fourier series	35

1 Introduction

In this paper, we focus on the zero-dispersion limit $\varepsilon \rightarrow 0$ for the Benjamin-Ono equation on the torus. The parameter ε balances the dispersive and the nonlinear effects in the Benjamin-Ono equation

$$\partial_t u = \partial_x(\varepsilon|\partial_x|u - u^2). \quad (\text{BO-}\varepsilon)$$

We recall that $|\partial_x|$ is the Fourier multiplier $\widehat{|\partial_x|u}(n) = |n|\widehat{u}(n)$.

The Benjamin-Ono equation [1, 27] describes a certain regime of long internal waves in a two-layer fluid of great depth. For $\varepsilon = 0$, this equation becomes the inviscid Burgers equation, for which shocks appear in finite time. When $\varepsilon > 0$, there is global well-posedness in H^2 [28] (see [11] for lower regularity results), hence the dispersive term prevents the shock formation. The shock is replaced by a dispersive shock, which manifests as a train of waves, we refer to [21] for numerical simulations on the real line.

1.1 Zero-dispersion limit for bell shaped initial data

The goal of this paper is to describe the weak limit of solutions as $\varepsilon \rightarrow 0$. We consider only *bell shaped* initial data $u_0(x) \in \mathcal{C}^3(\mathbb{T})$ as defined below.

Definition 1.1 (Bell shaped initial data). *We say that $u_0 \in \mathcal{C}^3(\mathbb{T})$ is a bell shaped initial data if the following holds:*

1. u_0 is real valued with zero mean;
2. there exist $x_{\min}, x_{\max} \in (0, 2\pi)$ such that $u_0' > 0$ on (x_{\min}, x_{\max}) and $u_0' < 0$ on $(x_{\max}, x_{\min} + 2\pi)$;
3. $x_{\min} = 0$;
4. there are exactly two inflection points $\xi_- \in (0, x_{\max})$ and $\xi_+ \in (x_{\max}, 2\pi)$ such that $u_0''(\xi_{\pm}) = 0$, and the inflection points are simple $u_0'''(\xi_{\pm}) \neq 0$.

When $u_0 \in \mathcal{C}^0(\mathbb{T})$ only satisfies 1. and 2., we say that u_0 is *weakly bell shaped*.

Note that conditions 1. and 3. are not restrictive because the real-valued property and the mean of the solution $\int_{\mathbb{T}} u(t, x) dx$ are preserved by the flow, and because the Benjamin-Ono equation is invariant by spatial translation. The other two conditions are more technical and aim at simplifying the calculation.

Given a bell shaped initial data, let u^ε be the solution to (BO- ε) with parameter ε . We prove that as $\varepsilon \rightarrow 0$, the weak limit of solutions has an explicit form in terms of the multivalued solution to Burgers' equation obtained by the method of characteristics. More precisely, we say that every point $u^B(t, x)$ is an image of the multivalued solution to Burgers' equation at (t, x) as soon as it solves the implicit equation

$$u^B(t, x) = u_0(x - 2u^B(t, x)t).$$

Given t and x , there may be several branches of solutions that are denoted $u_0^B(t, x) < \dots < u_{2P(t,x)}^B(t, x)$, see Figure 1 in [8]. We define the signed sum of branches as

$$u_{alt}^B(t, x) := \sum_{n=0}^{2P(t,x)} (-1)^n u_n^B(t, x).$$

In [8], we established that given a bell shaped initial data u_0 , there exists a family of approximate initial data u_0^ε such that the solution to (BO- ε) with initial data u_0^ε is weakly convergent to u_{alt}^B in $L^2(\mathbb{T})$, uniformly on compact time intervals, and such that $u_0^\varepsilon \rightarrow u_0$ in $L^2(\mathbb{T})$. In this paper, we prove that we can replace the approximate initial data u_0^ε by u_0 itself when u_0 is bell shaped. Our main result is the following.

Theorem 1.2 (Zero-dispersion limit). *Let $u_0 \in \mathcal{C}^3(\mathbb{T})$ be an even bell shaped initial data. Then uniformly on compact time intervals, the solution u^ε to the Benjamin-Ono equation (BO- ε) with parameter ε and initial data u_0 converges weakly to u_{alt}^B in $L^2(\mathbb{T})$ as $\varepsilon \rightarrow 0$: for every $T > 0$, there holds*

$$u^\varepsilon \rightharpoonup u_{alt}^B \quad \text{in} \quad \mathcal{C}([0, T], L_{weak}^2(\mathbb{T})).$$

In the paper, we mostly focus on the proof of the theorem when u_0 is an even bell shaped trigonometric polynomial. Then we rely on an explicit formula of the solution in terms of the Lax operator from Gérard [9] to extend this result to more general initial data.

Zero-dispersion limit for the Benjamin-Ono equation on the line Regarding the Benjamin-Ono equation on the line, the first formal approaches to the zero-dispersion limit problem for the Benjamin-Ono equation have been employed to justify the Whitham modulation theory [33] in [16,17] and in [12], and the case of more general nonlocal Benjamin-Ono equations which are not necessarily integrable is tackled in the paper [7]. A similar result as Theorem 1.2 from [8] (using an approximate initial data u_0^ε) was established by Miller and Xu in [21]. An extension to the Benjamin-Ono hierarchy has been investigated in [22].

In order to remove the approximate initial data u_0^ε , a better understanding of the direct scattering transform in the zero-dispersion limit is needed. The scattering data was studied by Miller and Wetzel in [19] on the line when the initial data is a rational potential of the form

$$u_0(x) = \sum_{n=1}^N \frac{c_n}{x - z_n} + c.c.$$

where $c_n \in \mathbb{C}^*$ and $\text{Im}(z_n) > 0$ for $1 \leq n \leq N$, moreover, the poles z_n have distinct real parts, and $\sum_{n=1}^N c_n + \bar{c}_n = 0$. In particular, every N -soliton (see Definition 1.1 in [29]) is rational. In the zero-dispersion limit, an asymptotic expansion of the scattering data for Klaus-Shaw initial data (a variation of the bell-shaped assumption adapted to the line) has been established in [20]. As a consequence, the choice of the scattering data of the approximate initial data u_0^ε in [21] is justified and it is likely that one could replace the approximate initial data u_0^ε by the actual initial data u_0 itself when u_0 is a rational Klaus-Shaw initial data.

Integrability for the Benjamin-Ono equation To generalize the class of initial data u_0 for which we do not need to rely on approximate initial data u_0^ε , we make use of the integrability properties of the Benjamin-Ono equation.

On the torus, the complete integrability for the Benjamin-Ono equation has been established by Gérard, Kappeler and Topalov [10,11] for general initial data in $H^s(\mathbb{T})$ as soon as $s > -\frac{1}{2}$, leading to global well-posedness in this range of Sobolev exponents. This integrability property could enable us to generalize the study of the zero-dispersion limit from trigonometric polynomials to bell-shaped initial data in our main theorem, only by estimating the error terms in the trigonometric approximation. However, in order to be successful, such an approach would require a huge decay of the Fourier coefficients of the initial data u_0 of the form

$$|\widehat{u_0}(n)| \leq Cn^{-Cn^2}.$$

This is why we rather rely on the explicit formula in [9] which requires less regularity of the initial data.

On the line, complete integrability of the Benjamin-Ono equation is known when restricted to the N -soliton manifold, see the recent breakthrough from Sun in [29]. However, its extension to more general potentials is an interesting open problem. As a consequence, if we try to estimate the error term in the N -soliton approximation of a general initial data, there is not much hope in removing the approximate initial data u_0^ε in the study of the zero-dispersion limit. With these challenges in mind, we believe that progress on the class of initial data for which we do not need approximate initial data will come from the fact that an explicit formula of solutions using the Lax operator also holds for the Benjamin-Ono equation on the line in [9], and from similar techniques as Section 5 from our paper.

Comparison with the KdV equation Historically, the first rigorous approach for zero-dispersion limit problem dates back to Lax and Levermore [14] for the Korteweg-de Vries (KdV) equation on the line

$$\partial_t u - 3\partial_x(u^2) + \varepsilon^2 \partial_{xxx} u = 0,$$

with initial data $u_0 \leq 0$ decaying at spatial infinity. The authors establish the existence of a weak limit up to approximate initial data, where the weak limit is different from u_{alt}^B as in the Benjamin-Ono equation. The case of positive initial data was investigated in [30].

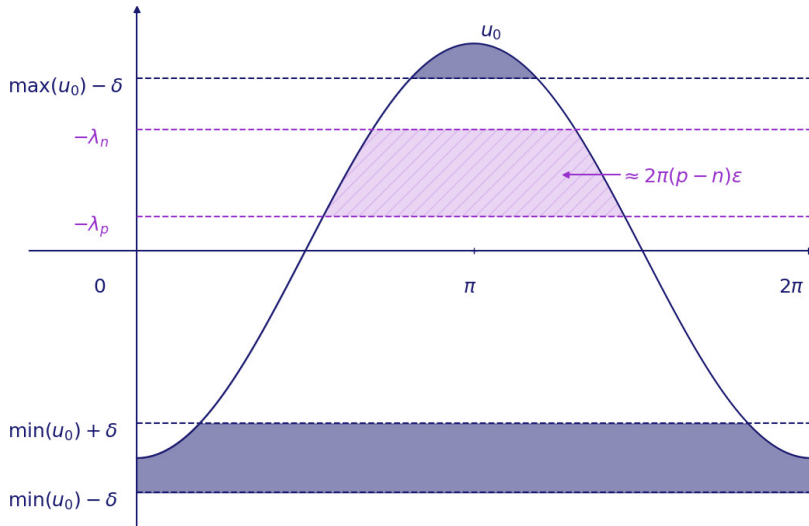


Figure 1: Distribution of Lax eigenvalues in the zero-dispersion limit

Many refinements of this result were then established for the KdV equation on the line, starting from a description of the oscillations in the dispersive shock [32], then strong asymptotics for the modulation equations using the steepest descent method [5]. For the KdV equation on the torus, one can mention a description of the dispersive shock in [31] and a justification of the Zabusky-Kruskal experiment for the cosine initial data in [6]. Finally, on the line, Claeys and Grava studied various asymptotic approximations in a neighborhood of the Whitham zone, in particular the gradient catastrophe in [2], the solitonic asymptotics in [4] and the Painlevé II asymptotics in [3]. Concerning the Benjamin-Ono equation, it is expected that the gradient catastrophe can be described via a universal profile at the breaking time [15]. We refer to the survey [18] and to [13] for more bibliographical information on zero-dispersion limit problems regarding the KdV equation and in other contexts.

1.2 Asymptotic expansion of the Lax eigenvalues

On the torus, the scattering data is replaced by the Birkhoff coordinates introduced by Gérard and Kappeler [10]. The goal of this paper is to better understand the Birkhoff coordinates of the initial data in the zero-dispersion limit. More precisely, in [8], Definition 1.5, we showed that a certain distribution of *Lax eigenvalues* and *phase constants* of the initial data u_0^ε as $\varepsilon \rightarrow 0$ would imply that the corresponding solution u^ε is weakly convergent to u_{alt}^B . Our goal in this paper is to show that the initial data u_0 itself satisfies those conditions, so that we do not need to rely on approximate initial data.

Let us first recall the definition of Lax eigenvalues and phase constants. Fix $\varepsilon > 0$. Denote by $L_+^2(\mathbb{T})$ the Hardy space of complex-valued functions in $L^2(\mathbb{T})$ with only nonnegative Fourier modes. The Lax operator $L_u(\varepsilon)$ is defined on $L_+^2(\mathbb{T})$ as:

$$L_u(\varepsilon)h = -i\varepsilon\partial_x h - \Pi(uh), \quad h \in L_+^2(\mathbb{T}).$$

The operator Π is the Szegő projector from $L^2(\mathbb{T})$ onto $L_+^2(\mathbb{T})$. We denote by $(\lambda_n(u; \varepsilon))_{n \geq 0}$ the eigenvalues of the Lax operator, and by $(f_n(u; \varepsilon))_{n \geq 0}$ its eigenfunctions. When $\langle f_n(u; \varepsilon) | \mathbf{1} \rangle \neq 0$, we define the phase constants as

$$\theta_n(u; \varepsilon) = \arg(\langle \mathbf{1} | f_n(u; \varepsilon) \rangle).$$

When $\langle f_n(u; \varepsilon) | \mathbf{1} \rangle = 0$, one can for instance use the convention $\theta_n(u; \varepsilon) = 0$.

The key result of this paper is the following. We prove that the Lax eigenvalues $(\lambda_n(u))_{n \geq 0}$ satisfy a Bohr-Sommerfeld quantization rule, see Figure 1. Roughly speaking, the area below

the curve of u but above the horizontal line $y = -\lambda_n$ should be approximately be equal to $n\varepsilon$. These conditions are similar to the assumptions of Definition 1.5 in [8], see also the statement of Corollary 2.2 in [8]. They are a refinement of the small-dispersion asymptotics for the Lax eigenvalues already investigated in [26] in the classical case, see also [25] in the quantum case.

We define the distribution function as follows: for $\eta \in \mathbb{R}$,

$$F(\eta) := \frac{1}{2\pi} \text{Leb}(x \in [0, 2\pi] \mid u_0(x) \geq \eta). \quad (1)$$

We also introduce a function ψ defined later in (21) for general trigonometric polynomials. In the case of an even trigonometric polynomial of degree N , ψ takes the simplified form (22)

$$\psi(\eta) = -\frac{1}{4} - \eta + (N + 1)F(\eta).$$

In this setting we get an approximation of Lax eigenvalues when the initial data is a trigonometric polynomial according to Definition 1.5 in [8]. We distinguish two areas:

- the small eigenvalues located in

$$\Lambda_-(\delta) = [-\max(u_0) + \delta, -\min(u_0) - \delta]$$

- the large eigenvalues located in

$$\Lambda_+(\delta) = [-\min(u_0) + \delta, +\infty).$$

For technical reasons we actually restrict our study so slightly smaller sets

$$\tilde{\Lambda}_\pm(\delta) = \Lambda_\pm(\delta) \setminus \bigcup_{k=1}^M (y_k - \delta, y_k + \delta).$$

Theorem 1.3 (Lax eigenvalues for trigonometric polynomials). *Let u be a bell-shaped trigonometric polynomial. There exists $C > 0$ such that for every $\delta > 0$ and $\varepsilon > 0$, there exists $C_u(\delta) > 0$ such that the following holds.*

1. (Small eigenvalues) If $\lambda_n + \varepsilon, \lambda_p + \varepsilon \in \tilde{\Lambda}_-(\delta)$, we have

$$\left| \int_{-\lambda_n(u)}^{-\lambda_p(u)} F(\eta) d\eta - (n - p + \psi(-\lambda_n) - \psi(-\lambda_p))\varepsilon \right| \leq C_u(\delta)\varepsilon\sqrt{\varepsilon}.$$

2. (Large eigenvalues) If $\lambda_n + \varepsilon \in \tilde{\Lambda}_+(\delta)$, then there holds if $n, p \leq K(\delta)/\varepsilon$

$$|\lambda_n - \lambda_p - (n - p)\varepsilon| \leq C_u(\delta)\varepsilon\sqrt{\varepsilon}$$

and if $n, p \geq K(\delta)/\varepsilon$,

$$|\lambda_n - \lambda_p - (n - p)\varepsilon| \leq \delta.$$

3. (Two-region eigenvalues) If $-\lambda_n - \varepsilon \in \Lambda_-(\delta)$ and $-\lambda_p - \varepsilon \in \Lambda_+(\delta)$, then $|p - n| \geq \frac{\delta}{C\varepsilon}$.

Compared to Definition 1.5 in [8], there is an extra term $(\psi(-\lambda_n) - \psi(-\lambda_p))\varepsilon$, but we will see later in Section 4 that as ψ is Lipschitz and n, p can be restricted to the condition $|n - p| \leq \varepsilon^{-r}$, this extra term can be treated as a remainder term. Note that the constant $C_u(\delta)$ actually depends both on the choice of bell shaped trigonometric polynomial u and on δ . The parameter δ is introduced for technical purposes. Indeed, we need to remove some bands $[y_k - \delta, y_k + \delta]$ which correspond to approaching a hidden complex stationary point of the phase with double multiplicity in order to get uniform bounds on the remainder terms. Similarly, we need to remove the bands $[-\min(u) - \delta, -\min(u) + \delta]$ and $[-\max(u) - \delta, -\max(u)]$ in order to have uniform bounds on the remainder terms when applying the stationary phase lemma when the stationary points are the antecedents of u by λ .

As a corollary, we get the following approximation for general bell-shaped functions according to Corollary 2.2 in [8].

Corollary 1.4 (Lax eigenvalues in the zero-dispersion limit). *Let $u \in \mathcal{C}^3(\mathbb{T})$ be bell-shaped. There exists $C > 0$ such that for every $N \geq 0$ and $\delta > 0$, the following holds. There exists $C_N(\delta) > 0$ and $\varepsilon_0(\delta) > 0$ such that if and $0 < \varepsilon < \varepsilon_0(\delta)$, then the following holds.*

1. For every $n \geq 0$, we have

$$\left| \int_{-\lambda_n(u)}^{\max(u)} F(\eta) d\eta - n\varepsilon \right| \leq \frac{C}{N\sqrt{N}} + C_N(\delta)\varepsilon\sqrt{\varepsilon} + C\delta.$$

2. (Large eigenvalues) If $\lambda_n + \varepsilon \in \Lambda_+(\delta) = [-\min(u_0) + \delta, +\infty)$, then there holds

$$\sum_{k=n+1}^{\infty} \gamma_k \leq \frac{C}{N\sqrt{N}} + C_N(\delta)\varepsilon\sqrt{\varepsilon} + C\delta.$$

3. (Two-region eigenvalues) If $\lambda_n + \varepsilon \in \Lambda_-(\delta) = [-\max(u_0) + \delta, -\min(u_0) - \delta]$ and $\lambda_p + \varepsilon \in \Lambda_+(\delta) = [-\min(u) + \delta, +\infty)$, then $|p - n| \geq \frac{\delta}{C\varepsilon}$.

In the statement, the constant $C_N(\delta)$ depends on the degree N of the trigonometric approximation and on δ , but also on the choice of the bell shaped initial data itself. Hence this statement is not uniform with respect to the choice of initial data. Moreover, the upper bound is not sufficient in general to prove Theorem 1.2. This is why we rather rely on the inversion formula from [9].

We also get some information of the phase constants of even trigonometric polynomials.

Theorem 1.5 (Phase constants in the zero-dispersion limit). *Assume that u_0 is a trigonometric polynomial, bell shaped, and even. Then the differences of two consecutive phase constants are multiples of π : for every n ,*

$$e^{i(\theta_{n+1}(u_0;\varepsilon) - \theta_n(u_0;\varepsilon))} = \pm 1.$$

Moreover, let

$$J(u_0; \varepsilon) = \left\{ n \geq 1 \mid \lambda_n + \varepsilon \in \Lambda_-(\delta), \quad e^{i(\theta_{n+1} - \theta_n)(u_0; \varepsilon)} = 1 \right\}.$$

Then for some $R(\varepsilon) \rightarrow 0$ as $\varepsilon \rightarrow 0$,

$$\varepsilon \sum_{n \in J(u_0; \varepsilon)} \operatorname{sinc}(\pi F(-\lambda_n)) \leq C(\delta)R(\varepsilon) + C\delta.$$

Theorems 1.3 and 1.5 present some differences with Definition 1.5 in [8]. Hence, once we prove this asymptotic distribution of Lax eigenvalues and phase constants, we will make precise how one can adapt the arguments in [8] in the study of the zero-dispersion limit and get Theorem 1.2.

1.3 Strategy of proof

The strategy for establishing Theorem 1.3 is inspired from the study of spectral data for the Benjamin-Ono equation on the line in [19, 20]. We transfer the problem into a problem of complex analysis. We use the identification between the spaces $L_+^2(\mathbb{T})$ and $L_+^2(\mathbb{D})$, where

$$L_+^2(\mathbb{T}) = \left\{ u : x \in \mathbb{T} \mapsto \sum_{k=0}^{+\infty} \hat{u}(k) e^{ikx} \mid \sum_{k=0}^{+\infty} |\hat{u}(k)|^2 < +\infty \right\},$$

$$L_+^2(\mathbb{D}) = \left\{ u : z \in \mathbb{D} \mapsto \sum_{k=0}^{+\infty} \hat{u}(k) z^k \mid \sup_{r < 1} \int_0^{2\pi} |u(re^{ix})|^2 dx < +\infty \right\}.$$

As a consequence, the eigenvectors $f_n(u; \varepsilon)$ of the Lax operator $L_u(\varepsilon)$ can be interpreted as holomorphic functions on \mathbb{D} .

A general bell shaped potential u , however, does not identify to a holomorphic function on the complex plane. For this reason, we approximate u by a trigonometric polynomial u_N of order N , obtained by truncation of its Fourier series. Then the trigonometric polynomial u_N extends to a meromorphic function on \mathbb{C} with a multiple (but finite) pole at zero:

$$u_N(z) = \sum_{k=1}^N \widehat{u}(k) z^k + \overline{\widehat{u}(k)} z^{-k}.$$

Hence, we transfer the eigenvalue problem

$$L_u(\varepsilon)f = \lambda f \tag{2}$$

to the complex plane, where the eigenvalue equation (2) becomes an ODE. We solve explicitly this equation in order to get an expression for f . However, one needs to check that the obtained formula for f is holomorphic on \mathbb{D} . This leads to the vanishing of an Evans function

$$\det(A_N(\lambda; \varepsilon)) = 0, \tag{3}$$

where every coefficient of the $N \times N$ matrix A_N is an oscillatory integral on a prescribed contour, with a fast oscillating phase of the form $S_\lambda(z)/\varepsilon$. The leading order of the integral is prescribed by the stationary points of the phase. Using the steepest descent method, we deduce an equation for λ to be an actual eigenvalue. It turns out that the stationary points of the phase on $\partial\mathbb{D}$ are the antecedents of u by λ , so that one can make a link with the distribution function F defined in (1).

To deduce Theorem 1.2, we rely on the complete integrability for the Benjamin-Ono equation on the torus established in [10,11], and generalizes the study of the cosine initial data from part 4 in [8].

Plan of the paper In Section 2 we establish an equation of the form (3) characterizing the eigenvalues of the Lax operator associated to a trigonometric polynomial. In Section 3, we deform the contours involved in this equation in order to apply the method of steepest descent, then we determine an asymptotic expansion of the Lax eigenvalues and establish Theorem 1.3. Then, in Section 4, we determine the weak limit of solutions of (BO- ε) as $\varepsilon \rightarrow 0$ using the asymptotic expansion of the Lax eigenvalues. Finally, we generalize this result from trigonometric polynomials to general bell shaped initial data in Section 5 to get Theorem 1.2.

In Appendix A.1 we show that the phase constants of an even initial data are multiples of π . Appendix A.2 is devoted to properties of the trigonometric approximation of a bell shaped initial data, from which we deduce Corollary 1.4.

Acknowledgments The author would like to warmly thank Patrick Gérard for providing a proof of Theorem 2.3 and for useful discussions about this problem.

2 Eigenvalue equation for trigonometric polynomials

In this part, we consider the eigenvalue equation (2) when u is a real-valued trigonometric polynomial of order N with zero mean

$$u(x) = \sum_{k=-N}^N c_k e^{ikx}.$$

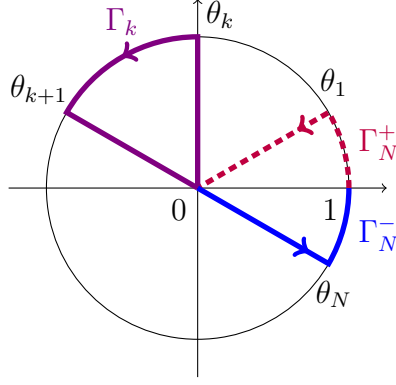


Figure 2: Contours of the eigenvalue equation

Since u has zero mean, then $c_0 = 0$, moreover, since u is real-valued, we have $c_{-k} = \overline{c_k}$, finally, one can assume that $c_N \neq 0$. Using the spatial translation invariance of equation (BO- ε), one can moreover assume that $c_N > 0$ (we do not assume that u is bell shaped here).

We first establish the eigenvalue equation, then we study the stationary points of the phase when $-\lambda$ has either zero or two antecedents by u on \mathbb{T} .

2.1 Evans function and eigenvalue equation

In this part, we state the eigenvalue equation in term of the vanishing of the Evans function $\det(A(\lambda; \varepsilon))$. The coefficients of the matrix $A(\lambda; \varepsilon)$ are oscillatory integrals on the following contours.

Definition 2.1 (Contours). For $1 \leq k \leq N$, we denote $\theta_k := \frac{(2k-1)\pi}{N}$.

- For $1 \leq k \leq N$, Γ_k is the closed contour made of the juxtaposition of the segment $[0, e^{i\theta_k}]$, the arc of circle of radius 1 where $\arg(\zeta)$ varies from θ_k to θ_{k+1} , and the segment $[e^{i\theta_{k+1}}, 0]$.
- The contour Γ_N^- is the juxtaposition of the segment $[0, e^{i\theta_N}]$ and the arc of circle of radius 1 where $\arg(\zeta)$ varies from θ_N to 2π .
- The contour Γ_N^+ is the juxtaposition the arc of circle of radius 1 where $\arg(\zeta)$ varies from 0 to θ_1 and of the segment $[e^{i\theta_1}, 0]$.

We orient these contours counterclockwise, see Figure 2.

We extend the function u as a meromorphic function on the complex plane with only one multiple pole at 0: for $z \in \mathbb{C} \setminus \{0\}$,

$$u(z) := \sum_{k=-N}^N c_k z^k.$$

Let us define Q in $\mathbb{C} \setminus \{0\}$ by the formula

$$Q(z) := \sum_{k=1}^N \left(-c_k \frac{z^k}{k} + \overline{c_k} \frac{z^{-k}}{k} \right), \quad (4)$$

so that $-zQ'(z) = u(z)$. The coefficients of the matrix $A(\lambda; \varepsilon)$ are the following oscillatory integrals.

Definition 2.2 (Oscillatory integrals). For $1 \leq k \leq N-1$ and $1 \leq \ell \leq N$, we define

$$A_{k,\ell} := \int_{\Gamma_k} e^{Q(\zeta)/\varepsilon} \zeta^{-\ell-\lambda/\varepsilon} \frac{d\zeta}{\zeta},$$

$$A_{N,\ell}^{\pm} := \int_{\Gamma_N^{\pm}} e^{Q(\zeta)/\varepsilon} \zeta^{-\ell-\lambda/\varepsilon} \frac{d\zeta}{\zeta},$$

$$A_{N,\ell} := A_{N,\ell}^+ + e^{-2i\pi\lambda/\varepsilon} A_{N,\ell}^-.$$

Let $A(\lambda; \varepsilon) = (A_{k,\ell})_{1 \leq k, \ell \leq N}$ be the $N \times N$ matrix with coefficients $A_{k,\ell}$.

We show that λ is an eigenvalue of the Lax operator $L_u(\varepsilon)$ if and only if the Evans function $\det(A(\lambda; \varepsilon))$ vanishes.

Theorem 2.3 (Eigenvalue equation). *Let u be a real-valued trigonometric polynomial of order N with zero mean. A real number λ is an eigenvalue of $L_u(\varepsilon)$ if and only if*

$$\det(A(\lambda; \varepsilon)) = 0. \quad (5)$$

2.2 Proof of the eigenvalue equation

Let us consider an eigenfunction f of $L_u(\varepsilon)$ with eigenvalue λ : $L_u(\varepsilon)f = \lambda f$, where we recall that

$$L_u(\varepsilon)f = -i\varepsilon\partial_x f - \Pi(uf) = -i\varepsilon\partial_x f - uf + (\text{Id} - \Pi)(uf).$$

We note that $L_u(\varepsilon)f = L_{u/\varepsilon}(1)(\varepsilon f)$. Hence, if $L_u(\varepsilon)f = \lambda f$, then $L_{u/\varepsilon}(1)(\varepsilon f) = \frac{\lambda}{\varepsilon}(\varepsilon f)$. Up to replacing f by εf , u by $\frac{u}{\varepsilon}$ and λ by $\frac{\lambda}{\varepsilon}$ at the end of the argument, we therefore assume that $\varepsilon = 1$.

Since $f \in L_+^2(\mathbb{T})$, we express f as a holomorphic function on the unit disk \mathbb{D} . The eigenvalue equation becomes: for every $0 < |z| < 1$,

$$\varepsilon z f'(z) - \left(\sum_{k=1}^N c_k z^k + \overline{c_k} z^{-k} \right) f(z) + \sum_{k=1}^N \overline{c_k} z^{-k} \left(\sum_{j=0}^{k-1} \frac{f^{(j)}(0)}{j!} z^j \right) = \lambda f(z).$$

This implies the existence of complex numbers v_0, \dots, v_N such that

$$z f'(z) - \left(\sum_{k=1}^N c_k z^k + \overline{c_k} z^{-k} \right) f(z) - \lambda f(z) = \sum_{j=1}^N v_j z^{-j}. \quad (6)$$

Conversely, if f is a holomorphic function near the origin solution to (6), then the constants v_1, \dots, v_N are uniquely expressed in terms of $f(0), \dots, f^{(N-1)}(0)$ by the triangular system

$$v_j = \sum_{k=j+1}^N \overline{c_k} \frac{f^{(k-j)}(0)}{(k-j)!}.$$

Moreover, if f is a solution to (6) on \mathbb{D} , and if f is holomorphic on \mathbb{D} and bounded on $\partial\mathbb{D}$, then f has a representative in $L_+^2(\mathbb{T})$ so that λ is an eigenvalue of $L_u(1)$ associated to the eigenfunction f . We deduce that the eigenvalues of L_u are the real numbers λ such that equation (6) admits a nonzero solution $f \in L_+^2(\mathbb{D})$, for some complex numbers v_1, \dots, v_N .

We first observe that equation (6) is a linear differential equation of order 1. On the simply connected open subset

$$\Omega = \mathbb{C} \setminus [0, +\infty),$$

this equation can be solved explicitly and the solutions are completely characterized by their value at one given point of Ω . We choose the determination of $\arg(z) \in (0, 2\pi)$ for $z \in \Omega$. Note that in the rest of the argument, one could choose the determination of the argument to be defined on $\mathbb{C} \setminus e^{i\theta}\mathbb{R}_+$ for any $\theta \in \mathbb{T}$ up to rotating everything by the angle θ in the proof, including the contours Γ_k . We get that (6) is equivalent to the equation

$$\frac{d}{dz} \left[z^{-\lambda} e^{Q(z)} f(z) \right] = z^{-\lambda} e^{Q(z)} \sum_{j=1}^N v_j z^{-j-1}, \quad (7)$$

where we recall (4)

$$Q(z) = \sum_{k=1}^N \left(-c_k \frac{z^k}{k} + \overline{c_k} \frac{z^{-k}}{k} \right).$$

Since we have assumed that $c_N > 0$, we know that for every $1 \leq k \leq N$, the real part of Q around the angle $\theta_k = \frac{(2k-1)\pi}{N}$ has the following asymptotic expansion as $r \rightarrow 0^+$:

$$\operatorname{Re}[Q(re^{i\theta_k})] \sim -\frac{c_N}{Nr^N}. \quad (8)$$

Moreover, for $z \rightarrow 0$ in the angular sector

$$|\arg(z) - \theta_k| \leq \frac{\pi}{2N} - \delta, \quad \delta \in \left(0, \frac{\pi}{2N}\right),$$

there exists $a_N(\delta) > 0$ such that

$$\operatorname{Re}[Q(z)] \leq -\frac{a_N(\delta)}{|z|^N}. \quad (9)$$

Fix $1 \leq k \leq N$. For every $z \in \Omega$, we choose a path $\gamma_{k,z}$ joining 0 to z in Ω such that for $t \geq 0$ small enough, we have

$$\gamma_{k,z}(t) = te^{i\theta_k}.$$

Thanks to (9), we get a well-defined and holomorphic solution f_k of (6) on Ω as follows:

$$f_k(z) = z^\lambda e^{-Q(z)} \int_{\gamma_{k,z}} e^{Q(\zeta)} \sum_{j=1}^N v_j \zeta^{-j-\lambda} \frac{d\zeta}{\zeta}. \quad (10)$$

Proposition 2.4 (Boundedness of f_k). *The function f_k is bounded on the angular sector*

$$A_k = \left\{ z \in \Omega \mid |z| \leq 1, \quad \arg(z) \in \left[\theta_k - \frac{\pi}{N}, \theta_k + \frac{\pi}{N} \right] \right\}.$$

Proof. In the proof, we first Taylor expand f_k around zero. Then we show that f_k is bounded in the angular sector $|\arg(z) - \theta_k| \leq \frac{\pi}{2N} - \delta$ for $\delta > 0$. Finally we show that f_k is also bounded in the angular sector $\frac{\pi}{2N} + \delta \leq |\arg(z) - \theta_k| \leq \frac{\pi}{N}$. We conclude to the boundedness for the missing angles by using the Phragmen-Lindelöf principle.

• We first observe that for every $M > N$, there exist coefficients b_0, \dots, b_M and a polynomial function P_M such that

$$e^{Q(z)} \sum_{j=1}^N v_j z^{-j-\lambda-1} = \frac{d}{dz} \left(\sum_{\ell=0}^M b_\ell z^{\ell-\lambda} e^{Q(z)} \right) + z^{M-N-\lambda} P_M(z) e^{Q(z)}.$$

Indeed, the coefficients b_0, \dots, b_M are the solutions of a triangular system when plugged into (6). Hence f_k has the following expression:

$$f_k(z) = \sum_{\ell=0}^M b_\ell z^\ell + z^\lambda e^{-Q(z)} \int_{\gamma_{k,z}} e^{Q(\zeta)} \zeta^{M-N-\lambda} P_M(\zeta) d\zeta. \quad (11)$$

• We first assume that there exists $\delta > 0$ such that

$$|\arg(z) - \theta_k| \leq \frac{\pi}{2N} - \delta.$$

Then we can choose the path $\gamma_{k,z}$ to stay inside the sector $|\arg(\zeta) - \theta_k| \leq \frac{\pi}{2N} - \delta$. In this case, estimate (9) combined with the holomorphy of f_k implies that one can transform the contour

$\gamma_{k,z}$ into the segment $[0, z]$ without changing the value of the integral in (11). Moreover, when $|z|$ is small enough in this angular sector, there exists $a_N(\delta) > 0$ such that

$$\operatorname{Re}[\zeta Q'(\zeta)] \geq \frac{a_N(\delta)}{|\zeta|^N},$$

therefore the function $t \in (0, 1] \mapsto \operatorname{Re}[Q(tz)]$ is increasing and $\operatorname{Re}[Q(\zeta) - Q(z)] \leq 0$ on $[0, z]$. We deduce that when M is large enough, the right hand side of (11) is bounded in the angular sector $|\arg(z) - \theta_k| \leq \frac{\pi}{2N} - \delta$.

- We now assume that there exists $\delta > 0$ such that

$$\frac{\pi}{2N} + \delta \leq |\arg(z) - \theta_k| \leq \frac{\pi}{N}.$$

Then one has that for some $a_N(\delta) > 0$, when z is in this angular sector,

$$\operatorname{Re}[Q(z)] \geq \frac{a_N(\delta)}{|z|^N}. \quad (12)$$

We choose $\gamma_{k,z}$ to be the juxtaposition of the following three paths:

1. the segment $[0, e^{i\theta_k}]$;
2. the arc of circle of radius 1 where $\arg(\zeta)$ goes from θ_k to $\arg(z)$;
3. the segment $[z/|z|, z]$.

Now, we show that the right hand side of equality (11) is bounded. Given the expression of $f_k(z)$ in (11), it is enough to prove that this expression is bounded when $z \rightarrow 0$.

1. Because of inequality (12), one can see that $z^\lambda e^{-Q(z)} \rightarrow 0$ as $z \rightarrow 0$. On $[0, e^{i\theta_k}]$, one has seen in (8) that $\operatorname{Re}[Q(re^{i\theta_k})] \sim -\frac{c_N}{Nr^N}$ as $r \rightarrow 0^+$. In particular the integral $\int_{[0, e^{i\theta_k}]} e^{Q(\zeta)} \zeta^{M-N-\lambda} P_M(\zeta) d\zeta$ stays bounded on $[0, e^{i\theta_k}]$.
2. In the arc of circle of radius 1 where $\arg(\zeta)$ goes from θ_k to $\arg(z)$, there is nothing to prove.
3. We now consider the segment $[z/|z|, z]$. Our goal is to show that when M is large enough, the following integral stays bounded as $z \rightarrow 0$:

$$z^\lambda e^{-Q(z)} \int_{[z/|z|, z]} e^{Q(\zeta)} \zeta^{M-N-\lambda} P_M(\zeta) d\zeta. \quad (13)$$

We use the fact that if $|\zeta| \leq 1$ and $\frac{\pi}{2N} + \delta \leq |\arg(\zeta) - \theta_k| \leq \frac{\pi}{N}$, then

$$\operatorname{Re}[\zeta Q'(\zeta)] \leq -\frac{a_N(\delta)}{|\zeta|^N}$$

to deduce that the function $t \in [1, |z|^{-1}] \mapsto \operatorname{Re}[Q(tz)]$ is decreasing. As a consequence, for every $\zeta \in [z/|z|, z]$, there holds $\operatorname{Re}[Q(\zeta) - Q(z)] \leq 0$. We conclude that when $M > N + \lambda$, the quantity (13) is bounded.

- We have shown in the latter steps that for every $\delta > 0$, f_k is bounded on the complement of $|\arg(z) - \theta_k - \frac{\pi}{2N}| \leq \delta$ in A_k . Furthermore, we see that there exists $C, A > 0$ such that for every $z \in A_k$,

$$|f_k(z)| \leq C e^{A|z|^{-N}}.$$

Making the change of variable $z \mapsto \frac{1}{z}$, Proposition 2.4 is now a consequence of the following lemma applied to $\theta = \theta_k + \frac{\pi}{2N}$, $\beta = \frac{\pi}{2N}$ and $\delta > 0$.

Lemma 2.5. *Let f be a holomorphic function in a neighborhood of the sector*

$$\Sigma_\beta := \{z \in \mathbb{C} \mid |z| > 1, \quad \arg(z) \in [\theta - \beta, \theta + \beta]\}$$

for some $\beta > 0$ and $\theta \in \mathbb{T}$. Assume that there exists $C, A, N > 0$ such that the following holds:

1. for every $z \in \Sigma_\beta$, $|f(z)| \leq Ce^{A|z|^N}$;
2. for every $\delta > 0$, f is bounded on $\Sigma_\beta \setminus \Sigma_\delta$.

Then f is bounded on Σ_β .

The proof of this lemma is as follows. Let $\delta > 0$ such that $2\delta N < \pi$. We apply the Phragmen-Lindelöf principle to f on Σ_δ . Therefore f is bounded on Σ_δ , hence on Σ_β . \square

As a consequence of Proposition 2.4, we have defined N solutions f_1, \dots, f_N of (6) on Ω with the same Taylor expansion around 0, and such that for every k , f_k is bounded on A_k . If there exists a solution to (6) which is holomorphic near the origin, then every function f_k should coincide with f on A_k . This implies the following system of $N - 1$ equations

$$f_k(e^{i(\theta_k + \frac{\pi}{N})}) = f_{k+1}(e^{i(\theta_k + \frac{\pi}{N})}), \quad 1 \leq k \leq N - 1. \quad (14)$$

For $1 \leq k \leq N - 1$, we note that from the integral expression (10), we have

$$f_k(e^{i(\theta_k + \frac{\pi}{N})}) - f_{k+1}(e^{i(\theta_k + \frac{\pi}{N})}) = e^{i\lambda(\theta_k + \frac{\pi}{N}) - Q(e^{i(\theta_k + \frac{\pi}{N})})} \int_{\Gamma_k} e^{Q(\zeta)} \sum_{j=1}^N v_j \zeta^{-j-\lambda} \frac{d\zeta}{\zeta},$$

where we recall that the contours Γ_k have been introduced in Definition 2.1. The equations (14) mean that all these quantities cancel, meaning that all the integrals on Γ_k cancel:

$$\int_{\Gamma_k} e^{Q(\zeta)} \sum_{j=1}^N v_j \zeta^{-j-\lambda} \frac{d\zeta}{\zeta} = 0. \quad (15)$$

Given expression (10), this is a system of $N - 1$ linear equation on v_1, \dots, v_N .

Moreover, integrating (7) on the circle of radius 1, we also get the equation

$$f(1)(e^{-2i\pi\lambda} - 1)e^{Q(1)} = \int_{\partial\mathbb{D}} z^{-\lambda} e^{Q(z)} \sum_{j=1}^N v_j z^{-j-1} dz. \quad (16)$$

Replacing $f(1)$ by $f_1(1 + i0)$, we get a new linear equation on v_1, \dots, v_N .

Proposition 2.6. *The linear system of N equations (15) and (16) admits a nontrivial solution (v_1, \dots, v_N) if and only if λ is an eigenvalue of $L_u(1)$.*

Proof. It only remains to prove the converse. Assume that the linear system of N equations (14) and (16) admits a nontrivial solution (v_1, \dots, v_N) , and let us prove that λ is an eigenvalue of $L_u(1)$.

Since the solutions to equation (6) in Ω are unique, we get from (14) that all the f_k coincide, so that Proposition 2.4 implies that f is bounded on the union of the angular sectors S_k , hence on $\Omega \cap \mathbb{D}$.

- We first assume that λ is not an integer. Define for $z \neq 0$

$$g(z) := \frac{ie^{-Q(z)}}{e^{-2i\pi\lambda} - 1} \int_0^{2\pi} e^{Q(ze^{i\theta}) - i\lambda\theta} \sum_{j=1}^N v_j z^{-j} e^{-ij\theta} d\theta.$$

Then g is holomorphic on $\mathbb{C} \setminus \{0\}$ and one can check that g satisfies equation (6).

Moreover, the choice of constant term implies that $g(1) = f(1)$, therefore $f = g$ on $\mathbb{D} \cap \Omega$. Moreover, f is bounded on $\mathbb{D} \cap \Omega$, therefore g is bounded on \mathbb{D} by continuity. Hence g is also holomorphic near the origin. We conclude that λ is an eigenvalue of $L_u(1)$.

• We now assume that λ is an integer. Withdrawing the sum of the integrals in (15) from the integral (16) on $\partial\mathbb{D}$, which cancels since equation (16) becomes

$$0 = \int_{\partial\mathbb{D}} z^{-\lambda} e^{Q(z)} \sum_{j=1}^N v_j z^{-j-1} dz,$$

we deduce that

$$\int_{\Gamma_N} e^{Q(\zeta)} \sum_{j=1}^N v_j \zeta^{-j-\lambda} \frac{d\zeta}{\zeta} = 0.$$

For $0 < r < 1$, we consider the contour $\Gamma_N(r)$ defined the closed contour made of the juxtaposition of the segment $[0, re^{i\theta_k}]$, the arc of circle of radius r where $\arg(\zeta)$ varies from θ_k to θ_{k+1} , and the segment $[re^{i\theta_{k+1}}, 0]$. Since the above integrand is holomorphic outside the origin, the integral on Γ_N is equal to the one on $\Gamma_N(r)$:

$$\int_{\Gamma_N(r)} e^{Q(\zeta)} \sum_{j=1}^N v_j \zeta^{-j-\lambda} \frac{d\zeta}{\zeta} = 0.$$

This implies that $f(r + i0) = f(r - i0)$ for $0 < r \leq 1$. Since f solves (6) on Ω , we deduce that f is holomorphic outside the origin. Moreover, since f is bounded on $\Omega \cap \mathbb{D}$, then f is also holomorphic near the origin. We conclude that λ is an eigenvalue of $L_u(1)$. \square

Hence, we have seen that λ is an eigenvalue of $L_u(1)$ if and only if there exists a nontrivial solution (v_1, \dots, v_N) such that (15) and (16) hold.

We now replace $f(1)$ by its integral expression (10) on $\gamma_{1,1} = \Gamma_N^+$ in equation (16). Then, removing (15) for $1 \leq k \leq N-1$ on the right hand side of equation (16), equation (16) becomes equivalent to

$$-(e^{-2i\pi\lambda} - 1) \sum_{\ell=1}^N A_{N,\ell}^+ v_\ell = \sum_{\ell=1}^N (A_{N,\ell}^+ + A_{N,\ell}^-) v_\ell.$$

We conclude that the linear system (15), (16) is equivalent to the existence of $V = (v_1, \dots, v_N)$ such that

$$A(u; 1)V = 0$$

where we recall that $A(u; \varepsilon)$ was introduced in Definition 2.2. Moreover, $f \not\equiv 0$ if and only if $V \not\equiv 0$, therefore this property is equivalent to

$$\det(A(u; 1)) = 0.$$

2.3 Stationary points of the phase

In order to find an asymptotic expansion of the Lax eigenvalues, the general strategy is to now apply the method of steepest descent to every coefficient of $A(u; \varepsilon)$ in the determinant formula $\det(A(u; \varepsilon)) = 0$. Intuitively, in the limit $\varepsilon \rightarrow 0$, the leading asymptotics are driven by the stationary points of the phase $e^{iS_\lambda(z)/\varepsilon}$, where

$$S_\lambda(z) = Q(z) - \lambda \log(z).$$

Since $-zQ'(z) = u(z)$, the stationary points satisfy $S'_\lambda(z) = 0$ whenever

$$u(z) + \lambda = 0.$$

In this part, we describe basic properties of the holomorphic extension of $u + \lambda$ in \mathbb{C}^* when $u + \lambda$ has zero or two antecedents on \mathbb{T} .

Lemma 2.7 (Critical points).

- (Small eigenvalues) Assume that $p_{\pm}(\lambda) = e^{ix_{\pm}(-\lambda)}$ are the only solutions to the equation $u(p_{\pm}(\lambda)) = -\lambda$ on $\partial\mathbb{D}$. Then there exist $p_1(\lambda), \dots, p_{N-1}(\lambda) \in \mathbb{D}$ such that for every $z \in \mathbb{C}^*$,

$$u(z) + \lambda = c_N z^{-N} (z - p_+) (z - p_-) \prod_{k=1}^{N-1} (z - p_k) (z - 1/\bar{p}_k). \quad (17)$$

- (Large eigenvalues) Similarly, assume that the equation $u + \lambda = 0$ has no solution on $\partial\mathbb{D}$. Then there exist $p_1(\lambda), \dots, p_N(\lambda) \in \mathbb{D}$ such that for every $z \in \mathbb{C}^*$,

$$u(z) + \lambda = c_N z^{-N} \prod_{k=1}^N (z - p_k) (z - 1/\bar{p}_k).$$

Lemma 2.8 (Spacing of roots). Let u be a bell-shaped trigonometric polynomial of order N . There exists $y_1 < \dots < y_M$, $M \leq 2N$, such that if

$$\lambda \notin \bigcup_{k=1}^M (y_k - \delta, y_k + \delta) \cup (K(\delta), +\infty),$$

then all the roots $p_1, \dots, p_{N-1}, p_+, p_-$ in the small eigenvalues case (or p_1, \dots, p_N in the large eigenvalues case) have single multiplicity, moreover they satisfy the following uniform bounds:

- if $\lambda \in \Lambda_-(\delta)$, then

$$|p_k(\lambda) - p_{\pm}(\lambda)| \geq \frac{1}{C_u(\delta)};$$

- if $k \neq l$, then

$$|p_k(\lambda) - p_l(\lambda)| \geq \frac{1}{C_u(\delta)}, \quad |p_k(\lambda)| \geq \frac{1}{C_u(\delta)}.$$

Proof. Since $P_{\lambda}(z) = z^N(u(z) + \lambda)$ is a polynomial of order $2N$, one can see that its derivative $P'_{\lambda}(z) = Nz^{N-1}(u(z) + \lambda) + z^N u'(z)$ is a polynomial of order $2N - 1$. We note that a root z in $\mathbb{C} \setminus \{0\}$ of P_{λ} is a multiple nonzero root if and only if $u(z) = -\lambda$ and $u'(z) = 0$. This happens at most $2N$ times since P_{λ} is a polynomial of order $2N$. Denote $z_1, \dots, z_{N'}$, $N' \leq 2N$, the nonzero roots of $z^{N+1}u'(z)$, where we remove the eventual multiplicity. If P_{λ} has a multiple nonzero root, then it must be one of the z_k , hence $u(z_k) + \lambda = 0$. Let us denote $y_1 < \dots < y_M$, $M \leq 2N$, the real numbers y such that there exists a multiple nonzero root z_k of P_{λ} , satisfying $u(z_k) = -y$.

Without loss of generality, we replace the set of small eigenvalues $\Lambda_-(\delta) = (-\max(u) + \delta, -\min(u) - \delta)$ by the compact set

$$\tilde{\Lambda}_-(\delta) := [-\max(u) + \delta, -\min(u) - \delta] \setminus \bigcup_{k=1}^M (y_k - \delta, y_k + \delta).$$

On $\tilde{\Lambda}_-(\delta)$, we know that $p_+(\lambda), p_-(\lambda), p_k(\lambda)$ are simple roots of the polynomial $z^N(u(z) + \lambda)$ with smooth coefficients with respect to the parameter λ , therefore, these roots are smooth with respect to the parameter λ . In particular, they lie in a compact set of $\mathbb{C} \setminus \{0\}$ and have a bounded derivative for $\lambda \in \Lambda_-(\delta)$.

The same applies by removing the problematic bands to the set $\Lambda_+(\delta) = [-\min(u) + \delta, K(\delta)]$ of large eigenvalues:

$$\tilde{\Lambda}_+(\delta) := [-\min(u) + \delta, K(\delta)] \setminus \bigcup_{k=1}^M (y_k - \delta, y_k + \delta). \quad \square$$

We recall that the oscillatory phase satisfies

$$S_\lambda(z) = Q(z) - \lambda \log(z), \quad S'_\lambda(z) = -\frac{u(z) + \lambda}{z}.$$

As a consequence of Lemma 2.8, when $\lambda \in \widetilde{\Lambda}_-(\delta) \cup \widetilde{\Lambda}_+(\delta)$, we get by compactness and continuity the following uniform bounds on the phase on a vicinity of the unit disk and far enough from $z = 0$.

Lemma 2.9 (Bounds for the phase). *Consider the constant $C_u(\delta) > 0$ from Lemma 2.8. Let $V(\delta) := B_{\mathbb{C}}(0, 2) \setminus B_{\mathbb{C}}(0, \frac{1}{2C_u(\delta)})$. There exists $C'_u(\delta) > 0$ such that for every $\lambda \in \widetilde{\Lambda}_-(\delta) \cup \widetilde{\Lambda}_+(\delta)$, for every $z \in V(\delta)$,*

- $\|S_\lambda - S_\lambda(p_k)\|_{C^4(V(\delta))} \leq C'_u(\delta)$;
- $\frac{|z - p_k|}{|S'_\lambda(z)|} \leq C'_u(\delta)$;
- $\|z^{-\ell-1}\|_{C^2(V(\delta))} \leq C'_u(\delta)$ for $1 \leq \ell \leq N$.

We also note that at the critical points p_k and p_\pm , the real part $\operatorname{Re}(S_\lambda)$ has at saddle point, whereas for z outside of these critical points, $\operatorname{Re}(S_\lambda)$ splits the vicinity of z into two parts $\operatorname{Re}(S_\lambda) > \operatorname{Re}(S_\lambda)(z)$ and $\operatorname{Re}(S_\lambda) < \operatorname{Re}(S_\lambda)(z)$.

Lemma 2.10 (Real part of the phase). *Assume that $\lambda \in \widetilde{\Lambda}_-(\delta) \cup \widetilde{\Lambda}_+(\delta)$.*

- *At the vicinity of the critical points $z = p_k$ and $z = p_\pm$, the level set $\operatorname{Re}(S_\lambda(\zeta)) = \operatorname{Re}(S_\lambda(z))$ is composed of two paths which intersect perpendicularly at z . Moreover, z is at saddle point.*
- *At the vicinity of z outside of these critical points, the level set $\operatorname{Re}(S_\lambda(\zeta)) = \operatorname{Re}(S_\lambda(z))$ is a path, which splits the vicinity of z into two parts $\operatorname{Re}(S_\lambda) > \operatorname{Re}(S_\lambda)(z)$ and $\operatorname{Re}(S_\lambda) < \operatorname{Re}(S_\lambda)(z)$.*

The proof of this lemma consists in a Taylor expansion of S_λ around z .

We now simplify the expressions of S_λ and S''_λ that we will obtain when applying the steepest descent method, as a direct consequence of the factorization of $u + \lambda$.

Lemma 2.11 (Second derivative of the phase). *Assume that $u + \lambda$ has two zeroes on \mathbb{T} and that $p_1(\lambda), \dots, p_{N-1}(\lambda), p_+(\lambda), p_-(\lambda)$ are distinct. Then at the stationary points $p_\pm(\lambda)$, we have*

$$|S''_\lambda(p_\pm(\lambda))| = c_N |p_+(\lambda) - p_-(\lambda)| \prod_{k=1}^{N-1} \frac{|p_k(\lambda) - p_\pm(\lambda)|^2}{|p_k(\lambda)|}.$$

Proof. We treat the case $p_\pm = p_+$ to simplify the notation. Since $u(p_+) + \lambda = 0$ and $|p_+| = 1$, we compute

$$|S''_\lambda(p_+)| = |u'(p_+)| = c_N |p_+ - p_-| \prod_{k=1}^{N-1} |p_+ - p_k| |p_+ - 1/\overline{p_k}|.$$

Finally, we note that

$$|p_+ - 1/\overline{p_k}| = |p_+| |p_k|^{-1} |\overline{p_k} - 1/p_+| = |p_k|^{-1} |\overline{p_k} - \overline{p_+}|. \quad \square$$

Finally, we make a link between the integral in the Bohr-Sommerfeld quantization formulation for the Lax eigenvalues in Theorem 1.4, and the value of the phase at the critical points $p_\pm(\lambda) = e^{ix_\pm(-\lambda)}$.

Lemma 2.12 (Phase and distribution function). *Assume that $-\lambda \in (\min(u), \max(u))$. Then there holds*

$$S_\lambda(p_+(\lambda)) - S_\lambda(p_-(\lambda)) = 2i\pi \int_{-\lambda}^{\max(u)} F(\nu) d\nu.$$

Proof. We write the difference of phase values as an integral from $x_- = x_-(-\lambda)$ to $x_+ = x_+(-\lambda)$. We get by using the original notation for the function u defined on \mathbb{T} that

$$\begin{aligned} S_\lambda(p_+(\lambda)) - S_\lambda(p_-(\lambda)) &= \int_{x_-(\lambda)}^{x_+(\lambda)} i e^{ix} S'_\lambda(e^{ix}) dx \\ &= -i \int_{x_-(\lambda)}^{x_+(\lambda)} (u(x) + \lambda) dx. \end{aligned}$$

Let $x_{\max} \in (x_-(\lambda), x_+(\lambda))$ be the point on \mathbb{T} such that $u(x_{\max}) = \max(u)$. Using an integration by parts then a change of variables $u(x) = \nu$, leading to $u'(x) dx = d\nu$ and $x_\pm(\nu) = x$, we get

$$\begin{aligned} \int_{x_-(\lambda)}^{x_+(\lambda)} u(x) dx &= \left[u(x) \right]_{x_-(\lambda)}^{x_+(\lambda)} - \int_{x_-(\lambda)}^{x_{\max}} x u'(x) dx - \int_{x_{\max}}^{x_+(\lambda)} x u'(x) dx \\ &= -\lambda(x_+(\lambda) - x_-(\lambda)) - \int_{-\lambda}^{\max(u)} (x_-(\nu) - x_+(\nu)) d\nu. \end{aligned}$$

This leads to the formula. \square

3 Deformation of contours

In order to apply the steepest descent method, we will choose contours such that the stationary points maximize the real part of the phase $\operatorname{Re}(S_\lambda(z)) = \operatorname{Re}(Q(z)) + \lambda \log(|z|)$ on the contour instead of Γ_k or Γ_N^\pm . We therefore study the level sets of $\operatorname{Re}(S_\lambda)$ in order to deform the contours Γ_k and Γ_N^\pm into more suitable ones in parts 3.1 and 3.2. Then we apply the method of steepest descent in parts 3.3 and 3.4.

3.1 Eigenvalues inside the bulk

In this part, we choose a small eigenvalue $\lambda \in \Lambda_-(\delta)$. We justify that we can deform the contours so that every coefficient in the matrix A is suitable for a steepest descent expansion. For the same equation on the line, this work was tackled in Proposition 5.1 in [20].

Definition 3.1 (Suitable contours). *The sequence of loops W_1, \dots, W_N is suitable if:*

1. for every $1 \leq k \leq N$, the loop W_k starts along the segment $[0, r_k e^{i\theta_k}]$ and finishes along the segment $[r_k e^{i\theta_{k+1}}, 0]$ for small enough $r_k > 0$;
2. when $k \leq N - 1$, each contour W_k passes through exactly one critical point $p_k(\lambda)$ of S_λ , and $\operatorname{Re}(S_\lambda)$ is maximal along W_k exactly at $p_k(\lambda)$;
3. when $k = N$, the contour passes only through the two critical points $p_+(\lambda)$ and $p_-(\lambda)$, and $\operatorname{Re}(S_\lambda)$ is maximal along W_N exactly at $p_+(\lambda)$ and $p_-(\lambda)$.

By construction of the suitable contours, we will get that for every critical point of S_λ in $\overline{\mathbb{D}}$, there is exactly one contour which passes through this point.

Proposition 3.2 (Existence of suitable contours). *There exists a family of suitable contours $(W_k)_{1 \leq k \leq N}$ according to Definition 3.1, with $W_N = W_N^+ \cup W_N^-$, and such that the following holds. Let $B(\lambda; \varepsilon)$ be the matrix with coefficients given for $1 \leq k \leq N - 1$ and $1 \leq \ell \leq N$ by*

$$\begin{aligned} B_{k,\ell} &= \int_{W_k} e^{Q(\zeta)/\varepsilon} \zeta^{-\ell-1-\lambda/\varepsilon} d\zeta, \\ B_{k,N}^\pm &= \int_{W_N^\pm} e^{Q(\zeta)/\varepsilon} \zeta^{-\ell-1-\lambda/\varepsilon} d\zeta, \\ B_{k,N} &= B_{k,N}^+ + e^{-2i\pi\lambda/\varepsilon} B_{k,N}^-. \end{aligned}$$

Then $\det(B(\lambda; \varepsilon)) = 0$ if and only if $\det(A(\lambda; \varepsilon)) = 0$.

The rest of this part is devoted to the proof of Proposition 3.2.

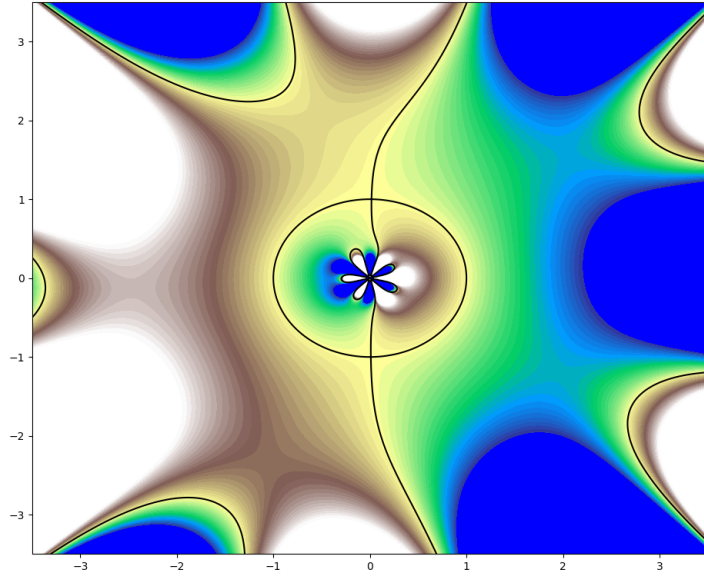


Figure 3: Map of the landscape when $u(x) = 8 \cos(x) + \sin(2x) + \cos(6x)/5$ and $\lambda = 0$. The green to dark blue color represents very low values of $\text{Re}(S)$ (lakes) whereas the brown to white color represents very high values of $\text{Re}(S)$ (mountains with snow). The black curve is the zero level set.

Description of the level sets We first describe the level sets of $\text{Re}(S_\lambda)$. This description is inspired from Appendix B of [20] for the same equation on the real line. One example of illustration of the level sets of $\text{Re}(S_\lambda)$ on the complex plane is given in Figure 3.

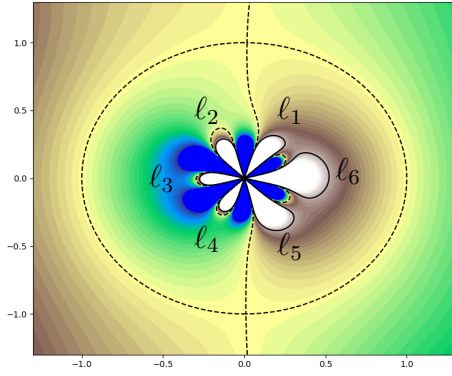
We consider the level curve $\text{Re}(S_\lambda) = L$ for L decreasing from $+\infty$ to $-\infty$. For fixed L , we consider that L is the sea level, therefore $\{z \in \mathbb{C}^* \mid \text{Re}(S_\lambda) < L\}$ will be the part of the landscape under the water and $\{z \in \mathbb{C}^* \mid \text{Re}(S_\lambda) > L\}$ will be the part above water.

We first note that for $z = e^{ix} \in \partial\mathbb{D}$, there holds $Q(z) \in i\mathbb{R}$, therefore $\text{Re}(S_\lambda) = 0$ on $\partial\mathbb{D}$. Moreover, in a neighborhood $B(0, r_0)$ of 0, write $z = re^{ix}$, $0 < r < r_0$. Then the main contribution to $\text{Re}(S_\lambda)$ is the term

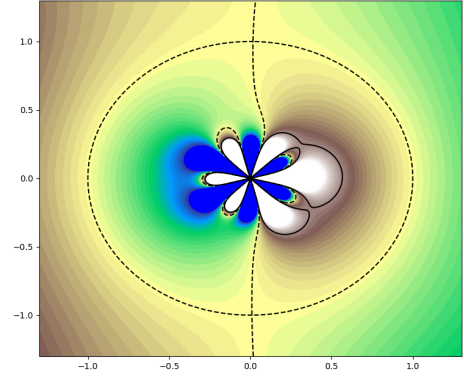
$$\text{Re} \left(c_N \frac{z^{-N}}{N} \right) = c_N \frac{r^N}{N} \cos(Nx).$$

Choosing r_0 small enough, one has that $\text{Re}(S_\lambda) \gg 0$ on the branches $(0, r_0 e^{2ik\pi/N})$ whereas $\text{Re}(S_\lambda) \ll 0$ on the branches $(0, r_0 e^{(2k+1)i\pi/N})$. As a consequence, inside of $B(0, r_0)$, there is a alternation of lakes and islands when turning around 0. Let $L \gg 0$ be large enough so that the compact set $\bar{\mathbb{D}} \setminus B(0, r_0)$ is covered by water. There are isolated infinite mountains which are localized around the directions $(0, r_0 e^{i\tilde{\theta}_k})$, $\tilde{\theta}_k = \frac{2k\pi}{N}$, see Figure 4a. The same happens when $L \ll 0$ is small enough, there are isolated infinite lakes localized around the directions $(0, r_0 e^{i\theta_k})$, $\theta_k = \frac{(2k+1)\pi}{N}$.

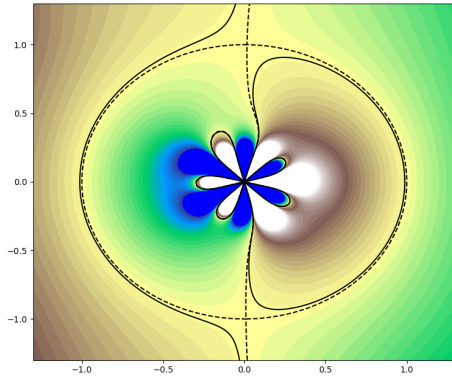
We start from a very high sea level $L \gg 0$, where there are isolated infinite mountains localized around $(0, r_0 e^{i\tilde{\theta}_k})$. Under a long drought the sea level will decrease, see Figure 4. Therefore at the critical values $L = L_k > 0$, some of the infinite mountains will become connected. According to Lemma 2.10, the fusion takes place at one of the points p_k , $2 \leq k \leq N$, and at most two connected groups of mountains fuse at this point. At the critical value $L = 0$, one of the groups of mountains becomes connected to a vicinity of the disk in the complex plane, totally encircling the remaining lakes in \mathbb{D} . Then, at critical values $L = L_k < 0$, groups of mountains



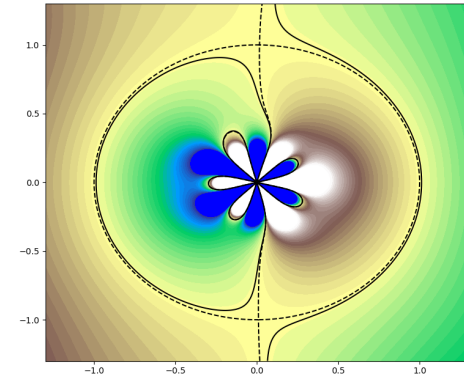
(a) $L > L_1 > 0$
Isolated mountains



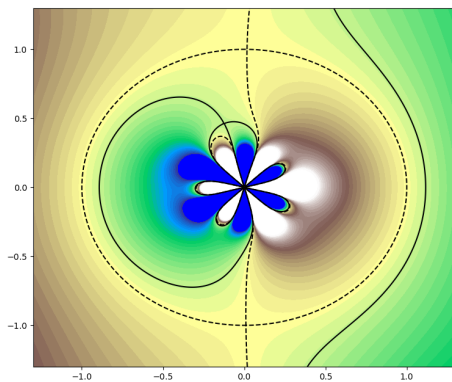
(b) $L_1 > L > L_2$
Two mountains merge



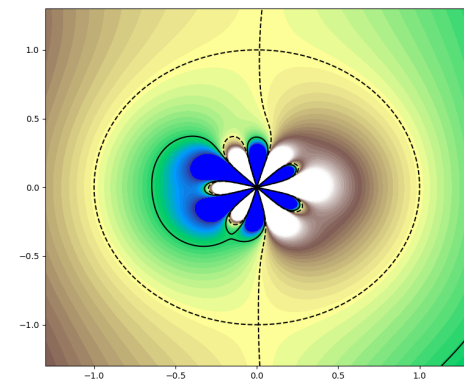
(c) $L_2 > L > 0$
Three mountains have merged



(d) $L_3 < L < 0$
A continent joins $\partial\mathbb{D}$



(e) $L = L_3 < 0$
Two groups of lakes separate



(f) $L_4 < L < 0$
Low sea level

Figure 4: Plots of several level sets of $\text{Re}(S_\lambda)$. The solid line is the level set L , the dotted line is the level set $L_4 = 0$.

fuse together again, and similarly, at every point p_k where this happens, at most two groups mountains become connected. Given that initially there are N infinite pits and N infinite peaks, a counting argument implies that a connected component of mountains does not fuse with itself again, and every infinite mountain still isolated fuses exactly once with a group of mountains in the process. Up to renumbering the critical points, the critical values L_k ordered such that $L_1 > \dots > L_{K-1} > L_K = 0 > L_{K+1} > \dots > L_N$ each correspond to the critical point p_k if $k \neq K$, or to the critical points p_{\pm} if $k = K$.

Tree pruning algorithm We use the “tree pruning algorithm” construction from Appendix B of [20] in order to properly describe how the mountains merge. The goal is to show that the steepest descent contours W_k will form a bijection with the original contours Γ_k .

We first construct a tree as follows. We label by ℓ_k , $1 \leq k \leq N$, the infinite mountains localized around the lines $(0, r_0 e^{2ik\pi/N})$ for small r_0 . The mountains ℓ_k will be leaves of the tree, and the critical points p_k will be internal modes. At the end of the construction, the descendants of each internal mode p_k are the mountains that get connected when the sea level goes from L_k^+ to L_k^- .

We proceed in order of decreasing sea level L_k . At the level set L_k for $k \neq K$, because the sea level decreases, two distinct connected groups of mountains fuse together through the critical point p_k . If the connected group is composed of several mountains, we draw an edge from p_k to the internal mode p_l at which this group became fully connected. If the connected group is a single leaf ℓ_m , we draw an edge from p_k to this leaf. As a consequence p_k has two direct children.

We note that when $L_k > 0$, then the mountains are subsets of $\{z \in \mathbb{D} \mid \text{Re}(S_\lambda(z)) \geq L_k\}$, therefore a connected group of mountains does not cross the zero level set. At the level set $L_K = 0$, no groups of mountains inside of \mathbb{D} fuse with each other, but one group of mountains becomes connected with a vicinity of $\partial\mathbb{D}$. Before this, this group of mountains either became fully connected at some node p_l , or is an isolated mountain ℓ_l . We decide that the node labeled p_{\pm} has exactly one child which is either the internal mode p_l or to the leaf ℓ_l . In particular, the leaves that descend from p_{\pm} form a connected component of $\{z \in \mathbb{D} \mid \text{Re}(S_\lambda(z)) > 0\}$ called *continent*. Its characteristic is that it becomes connected as $L \rightarrow 0^-$ to a vicinity of the level set $\{z \in \mathbb{C}^* \mid \text{Re}(S_\lambda(z)) > 0^-\}$, hence to a vicinity of $\partial\mathbb{D}$.

We get a tree with N leaves ℓ_1, \dots, ℓ_N and $(N - 1)$ internal nodes p_k for $1 \leq k \neq N$ and $k \neq K$ with two children each, plus one extra node p_{\pm} with only one child.

Now we chop the tree at the node p_{\pm} , which becomes both a leaf for one sub-tree and a root for the other. Therefore we have one sub-tree of $M \geq 1$ leaves and $(M - 1)$ internal modes p_k , rooted at p_{\pm} , and a sub-tree with $(N - M + 1)$ leaves including p_{\pm} and $(N - M)$ internal modes.

For the sub-tree of M leaves and then the sub-tree of $(N - M + 1)$ leaves, we apply the tree-pruning algorithm. At one step, we only keep a sub-tree of the tree such that each child of a node p_k is represents one of the two groups of mountains that become connected at p_k . We obtain this sub-tree by repeating the following operation:

1. We consider an internal node p_k connected to two leaves. By construction, the two leaves that come from the two edges in the original sub-tree are the two connected groups of mountains that fuse together when $L = L_k^-$.
2. We remove one of the two leaves ℓ_m , the internal node p_k and the edge between them. We can always choose this leaf to be different from p_{\pm} . Note that at the critical point p_k , one could construct a contour staying in the level set $\{z \in \mathbb{C} \mid \text{Re}(S_\lambda(z)) < L_k\}$ in two ways by encircling either one of the two groups of mountains that are merging. The choice of a leaf ℓ_m represents the choice of the group of mountains that we decide to encircle.
3. We define the set D_k as the set of leaves that have been pruned from the sub-tree up to now, and that are descendant of p_k in the original sub-tree. This represents the set of mountains that are allowed to be encircled in our choice of contour.

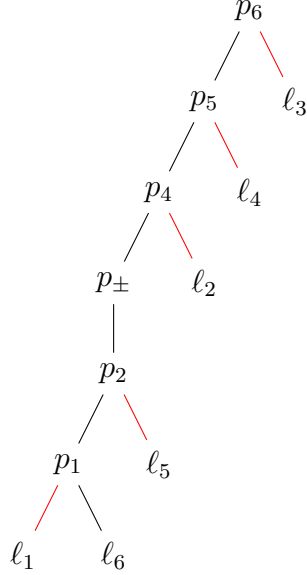


Figure 5: Construction of a tree from Figure 4. The colored edges represent the pairings (leaves and internal modes removed together) obtained during the tree pruning algorithm.

4. Let p_l be the parent of p_k , the other leaf which was connected to p_k becomes a leaf connected p_l . If p_l does not exist, then the process is over. Otherwise, we search for an internal mode which has p_l as an ancestor and which is connected to two leaves. We apply step 1 to this internal mode.

We refer to Figure 5 for an example of tree and tree pruning in the context of Figure 4.

At the end of the process, all the leaves have been pruned, except for one leaf $\ell'_N = \ell_{\sigma(N)}$ for the first sub-tree and the leaf p_{\pm} for the second sub-tree. Since ℓ'_N is a descendant of p_{\pm} in the big tree, it belongs to the “continent”, in particular, the complex points p_+ , $e^{i\theta_{\sigma(N)}}$ and p_- are placed in this order in the counterclockwise orientation. We choose the determination of the logarithm to have its branch cuts at the line $e^{i\theta_{\sigma(N)}}\mathbb{R}_+$. This implies that we rotate everything and instead of splitting $\Gamma_N = \Gamma_N^+ \cup \Gamma_N^-$, we split the contour encircling $e^{i\theta_{\sigma(N)}}$ instead.

Moreover, there is a permutation $\ell'_k = \ell_{\sigma(k)}$ of the leaves and a permutation $p'_k = p_{\tau(k)}$ of the critical points with $p'_K = p_N = p_{\pm}$, such that the leaves $\ell'_1, \dots, \ell'_{N-1}$ and internal modes p'_1, \dots, p'_N are pruned in this order. We denote $D'_k = D_{\tau(k)}$.

We will construct steepest descent contours W'_k such that the following properties hold.

- If $k \leq K$, the contour W'_k stays inside the level set $\{z \in \mathbb{C}^* \mid \text{Re}(S_{\lambda}(z)) \leq L'_k\}$, it encloses ℓ'_k and eventually some other leaves in $D'_k \subseteq \{\ell'_1, \dots, \ell'_{k-1}\}$, and eventually some leaves in $\{\ell'_K, \dots, \ell'_{N-1}\}$. This is possible by construction: at p'_k , the mountain at ℓ'_k merges with all the leaves in S'_k (and eventually ℓ'_N), so that they form a connected component of $\{z \in \mathbb{C}^* \mid \text{Re}(S_{\lambda}(z)) \leq L'_k\}$. There is always a way not to encircle ℓ'_N in our choice of contour.
- if $K + 1 \leq k \leq N$, the contour W'_k stays inside the level set $\{z \in \mathbb{D}^* \mid \text{Re}(S_{\lambda}(z)) \leq L'_k\}$, and encloses ℓ'_k , plus eventually some other leaves in $S'_k \subseteq \{\ell'_K, \dots, \ell'_{k-1}\}$. Note that when $k \geq K + 1$, then the land is very dry so every group of lakes is isolated inside one portion of \mathbb{D} delimited by $\{z \in \mathbb{C}^* \mid \text{Re}(S_{\lambda}(z)) = 0\}$. Hence we can assume that the corresponding contour stays inside this portion of \mathbb{D} and does not encircle any leaf in $\{\ell'_1, \dots, \ell'_{K-1}\}$.

Choice of steepest descent contours It now remains to define the steepest descent contours W_l . Let $l \neq N$, and $k \neq K$ such that $W_l = W'_k$.

We recall that if $k \leq K - 1$, by construction, the connected group of mountains containing ℓ'_k fuses with an other group of mountains when the sea level decreases from L_l^+ to L_l^- . Moreover, the connected group of mountains containing ℓ'_k after fusion (when $L = L_l^-$) is only comprised from mountains that belong to D_k . Similarly, if $K + 1 \leq k \leq N$, the connected group of mountains containing ℓ'_k fuses with an other group of mountains. The connected group of mountains containing ℓ'_k after fusion is comprised from mountains that belong to D_k , maybe some mountains in the first sub-tree $\{\ell'_1, \dots, \ell'_{M-1}\}$.

As a consequence, in any of these cases, it is possible to enclose the mountain ℓ'_k while enclosing only the other leaves mentioned here by some path. We modify this path a little so that this path is a steepest descent path by taking:

- a short arc of the steepest decent path at the vicinity of p'_k , descending to the level $L_k - \delta_k$, hence staying in $\{z \in \mathbb{D}^* \mid \operatorname{Re}(S_\lambda(z)) \leq L_k\}$;
- going from each of the two extremities of the steepest descent path, any path staying inside the set $\{z \in \mathbb{D}^* \mid \operatorname{Re}(S_\lambda(z)) \leq L_k - \delta_k\}$, and enclosing the mountain ℓ'_k , then going to a neighborhood of 0, then going to two lakes $r_k e^{i\theta_1}$ resp. $r_k e^{i\theta_2}$;
- the segments of the form $[0, r_k e^{i\theta_m}]$ and $[0, r_k e^{i\theta_n}]$.

We orient this loop counterclockwise. The contour does not enclose the leaf ℓ'_N by assumption.

If $k = K$ and $l = N$, then as L goes to 0^+ to 0^- , we know that one continent fuses with the vicinity of the unit disk, whereas one lake localized around $(0, r_0 e^{i\theta_m})$ becomes separated from it. We define two paths $(W'_K)^\pm = W_N^\pm$ so that $W'_K = (W'_K)^+ \cup (W'_K)^-$ encircles ℓ'_N and a subset of $\{\ell'_1, \dots, \ell'_{K-1}\}$, and:

- a short arc of the steepest descent path at the vicinity of p_\pm , descending to the level set $\{z \in \mathbb{C}^* \mid \operatorname{Re}(S_\lambda(z)) \leq -\delta_K\}$;
- any path staying inside the set $\{z \in \mathbb{C}^* \mid \operatorname{Re}(S_\lambda(z)) \leq -\delta_K\}$ and joining the outside extremity of the steepest descent path around p_\pm to $e^{i\theta_{\sigma(N)}}$;
- any path staying inside the set $\{z \in \mathbb{C}^* \mid \operatorname{Re}(S_\lambda(z)) \leq -\delta_K\}$ and joining the inside extremity of the steepest descent path around p_\pm to $r_\pm e^{i\theta_m}$;
- the segment $[0, r_\pm e^{i\theta_m}]$.

We also orient these paths counterclockwise.

Vanishing of the Evans function Let $V = (v_1, \dots, v_N)$. We have seen that $AV = 0$ if and only if (15) and (16) combined with (10) hold, i.e. for $l \leq N - 1$,

$$\int_{\Gamma_l} e^{Q(\zeta)/\varepsilon} \sum_{j=1}^N v_j \zeta^{-j-\lambda/\varepsilon} \frac{d\zeta}{\zeta} = 0,$$

$$(e^{-2i\pi\lambda/\varepsilon} - 1) \int_{\Gamma_{\sigma(N)}^+} e^{Q(\zeta)/\varepsilon} \sum_{j=1}^N v_j \zeta^{-j-\lambda/\varepsilon-1} d\zeta = \int_{\partial\mathbb{D}} z^{-\lambda} e^{Q(z)/\varepsilon} \sum_{j=1}^N v_j z^{-j-1} dz.$$

Since each contour Γ_l encloses exactly the mountain ℓ_l , the W'_k for $k \neq K$ can be written up to deformation (outside of 0) as a sum of the $\Gamma_{\sigma(l)}$ for $l \neq N$. This deformation being in the domain where the integrand of the oscillatory integrals is holomorphic (see Definition 2.2), the deformation does not change the total value of the integral. The coefficients for passing from the two sets of contours $(\Gamma_{\sigma(l)})_{l \neq N}$ to $(W'_k)_{k \neq K}$ will be of the form

$$P = \begin{pmatrix} L_{K-1} & * \\ 0 & L_{N-K} \end{pmatrix},$$

with L_{K-1} and L_{N-K} being two lower-triangular matrices with indicated dimension with ones on the diagonal. This implies that the matrix P is invertible. As a consequence, the first system (15) is equivalent to the fact that for every $k \neq K$,

$$\int_{W'_k} e^{Q(\zeta)/\varepsilon} \sum_{j=1}^N v_j \zeta^{-j-\lambda/\varepsilon} \frac{d\zeta}{\zeta} = 0.$$

Finally, the contour $(W'_K)^+ \cup \Gamma_{\sigma(N)}^+$ when following $\Gamma_{\sigma(N)}^+$ clockwise is a loop not containing ℓ'_N . Removing the first equation (15) with the encircled leaves to the left-hand side of the second equation (16) for each non-encircled leaves ℓ_k in W'_K gives a path that one can deform into $(W'_K)^+$ by staying in the holomorphic part of the integrand, whereas $\partial\mathbb{D}$ can be deformed into $(W'_K)^+ \cup (W'_K)^-$. We transform this equation into

$$-(e^{-2i\pi\lambda/\varepsilon} - 1) \int_{(W'_K)^+} e^{Q(\zeta)/\varepsilon} \sum_{j=1}^N v_j \zeta^{-j-\lambda/\varepsilon-1} d\zeta = \int_{(W'_K)^+ \cup (W'_K)^-} z^{-\lambda} e^{Q(z)/\varepsilon} \sum_{j=1}^N v_j z^{-j-1} dz.$$

This leads to the fact that $AV = 0$ if and only if $BV = 0$, hence to Proposition 3.2.

3.2 Eigenvalues outside the bulk

The eigenvalues outside follow the description from Appendix A in [20]. The strategy is similar as part 3.1, but the node p_{\pm} becomes replaced by one of the critical points p_K that plays a special role.

Definition 3.3 (Suitable contours). *The sequence of loops W_1, \dots, W_N is suitable if:*

1. for every $1 \leq k \leq N$, the loop W_k starts along the segment $[0, r_k e^{i\theta_k}]$ and finishes along the segment $[r_k e^{i\theta_{k+1}}, 0]$ for small enough $r_k > 0$;
2. each contour W_k for $1 \leq k \leq N$ passes through exactly one critical point $p_k(\lambda)$ of S_λ , and $\text{Re}(S_\lambda)$ is maximized along W_k at $p_k(\lambda)$.

As a consequence of this definition, a family of suitable contours satisfies that for every critical point of S_λ in $\overline{\mathbb{D}}$, there is exactly one contour which passes through this point.

Proposition 3.4 (Existence of suitable contours). *There exists a family of suitable contours $(W_k)_{1 \leq k \leq N}$ with $W_N = W_N^+ \cup W_N^-$ such that the following holds. Let $B(\lambda; \varepsilon)$ be the matrix with coefficients given for $1 \leq k \leq N-1$ and $1 \leq \ell \leq N$ by*

$$B_{k,\ell} = \int_{W_k} e^{Q(\zeta)/\varepsilon} \zeta^{-\ell-1-\lambda/\varepsilon} d\zeta,$$

$$B_{k,N}^\pm = \int_{W_N^\pm} e^{Q(\zeta)/\varepsilon} \zeta^{-\ell-1-\lambda/\varepsilon} d\zeta,$$

$$B_{k,N} = B_{k,N}^+ + e^{-2i\pi\lambda/\varepsilon} B_{k,N}^-.$$

Then $\det(B(\lambda; \varepsilon)) = 0$ if and only if $\det(A(\lambda; \varepsilon)) = 0$.

Description of the level sets Again, we start with a description of the level sets. An illustration of the level sets appears in figure 6.

First, at the sea level $L = 0$, we know that $\text{Re}(S_\lambda) = 0$ on the circle $\partial\mathbb{D}$, moreover, since $S'_\lambda(z) = -(u + \lambda)/z$, a Taylor expansion of $S_\lambda(rz)$ around $z \in \partial\mathbb{D}$ and $r \rightarrow 1$ implies that $\text{Re}(S_\lambda(z)) > 0$ for $|z| \rightarrow 1^-$ and $\text{Re}(S_\lambda(z)) < 0$ for $|z| \rightarrow 1^+$.

When $L \gg 0$, there are isolated infinite mountains localized around the lines $(0, r_0 e^{2ik\pi/N})$ for small enough δ . These mountains are separated by a big ocean connecting 0 to ∞ . As L

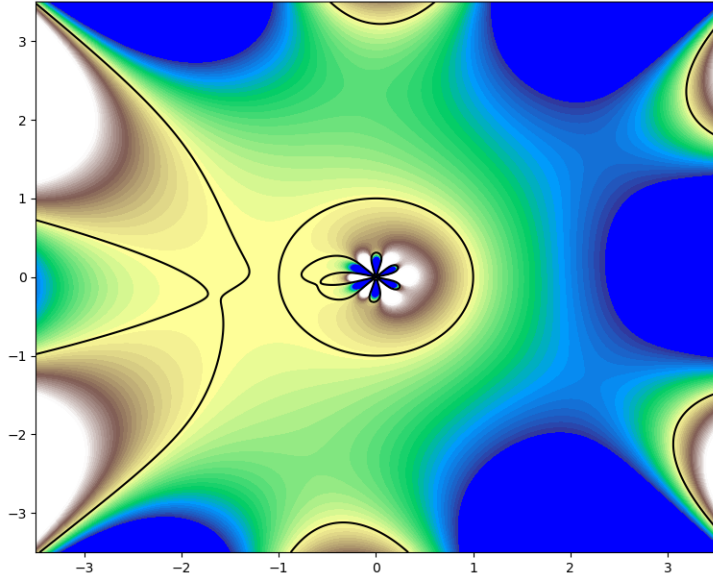


Figure 6: Map of the landscape when $u(x) = 8 \cos(x) - \sin(2x)/2 + \sin(4x)/4 + \cos(6x)/10$ and $\lambda = 7.95$. The green to dark blue color represents very low values of $\text{Re}(S)$ (lakes) whereas the brown to white color represents very high values of $\text{Re}(S)$ (mountains with snow). The black curve is the zero level set.

decreases, the isolated mountains become connected at critical sea levels $L = L_k$, forming lakes which are not connected to the big ocean. Moreover, there exists one critical sea level $L = L_K$ such that going from L_K^+ to L_K^- , the point 0 becomes disconnected from ∞ . Equivalently, the mountains that merge together at the critical point were already connected before, and at $L = L_K^-$ they encircle 0, so that they become a *continent*. We note that at $L = 0$, the circle $\partial\mathbb{D}$ is already part of the continent, so this particular sea level satisfies $L_K > 0$. When $L \ll 0$, all the mountains are connected, and there are isolated infinite lakes localized around the lines $(0, r_0 e^{i(2k+1)\pi/N})$ for small enough r_0 .

By a counting argument, at every level L_k , $1 \leq k \leq N$, $k \neq K$, two groups of mountains that were not connected before merge together at the node p_k , and at some level L_K , one group of mountains becomes a continent at the node p_K .

We denote the leaves ℓ_1, \dots, ℓ_N as a label for the mountains localized around the lines $(0, r_0 e^{2ik\pi/N})$ when $L \gg 0$.

Tree pruning algorithm We construct a tree as before, for which the node p_K at the level set L_K where one group of mountains becomes a continent replaces the role of the level p_{\pm} .

Choice of steepest descent contours The choice of contours is also similar except for the path $(W'_K)^{\pm} = W_N^{\pm}$, which both path through p_K but with different orientations. In this case, the ocean, which contains some lake around $e^{i\theta_m}$ when $L = L_K^+$, becomes separated from ∞ . We take:

- a short arc of the steepest descent path at the vicinity of p_K , descending to the level set $L_K - \delta_K$ for δ_K small enough so that $L_K - \delta_K > 0$;
- any path staying inside the set $\{z \in \mathbb{C} \mid \text{Re}(S_{\lambda}(z)) \leq L_K - \delta_K\}$ and joining the inside extremity of the steepest descent path to $r_K e^{i\theta_m}$ for some θ_m ;

- the segment $[0, r_K e^{i\theta_m}]$;
- any path staying inside the set $\{z \in \mathbb{C} \mid 0 \leq \operatorname{Re}(S_\lambda(z)) \leq L_K - \delta_K\}$ and joining the outside extremity of the steepest descent path to a point $e^{i\theta} \in \partial\mathbb{D}$;
- the arc of circle on $\partial\mathbb{D}$ from $e^{i\theta}$ to 1 in the clockwise direction if we consider $(W'_N)^+$, in the counterclockwise direction if we consider $(W'_N)^-$.

We choose the determination of the logarithm to be defined in $\mathbb{C} \setminus \mathbb{R}_+$ in this case.

Vanishing of the Evans function We recall that since each contour Γ_l encloses exactly the mountain ℓ_l , the steepest descent contours W'_k for $k \neq K$ defined as in part 3.1 can be written up to deformation (outside of 0) as a sum of the $\Gamma_{\sigma(l)}$ for $l \neq N$. This deformation being in the domain where the integrand of the oscillatory integrals is holomorphic, the deformation does not change the total value of the integral, so that the coefficients for passing from the two sets of contours $(\Gamma_{\sigma(l)})_{l \neq N}$ to $(W'_k)_{k \neq K}$ are of the form

$$P = \begin{pmatrix} L_{K-1} & * \\ 0 & L_{N-K} \end{pmatrix},$$

with L_{K-1} and L_{N-K} being two lower-triangular matrices with ones on the diagonal. This implies that the matrix P is invertible. As a consequence, the first system (15) is equivalent to the fact that for every $k \neq K$,

$$\int_{W'_k} e^{Q(\zeta)/\varepsilon} \sum_{j=1}^N v_j \zeta^{-j-\lambda/\varepsilon} \frac{d\zeta}{\zeta} = 0.$$

The contour $(W'_K)^+ \cup \Gamma_{\sigma(N)}^+$ when following $\Gamma_{\sigma(N)}^+$ clockwise is a loop encircling some leaves ℓ'_k , but not ℓ'_N . However adding the first equation (15) with index k for each leaf ℓ_k encircled to the left-hand side of the second equation (16), we get

$$-(e^{-2i\pi\lambda/\varepsilon} - 1) \int_{\Gamma_{\sigma(N)}^+} e^{Q(\zeta)/\varepsilon} \sum_{j=1}^N v_j \zeta^{-j-\lambda/\varepsilon-1} d\zeta = \int_{\partial\mathbb{D}} z^{-\lambda} e^{Q(z)/\varepsilon} \sum_{j=1}^N v_j z^{-j-1} dz$$

gives a path that one can deform into $(W'_K)^+$, whereas $\partial\mathbb{D}$ can be deformed into $(W'_K)^+ \cup (W'_K)^-$ without changing the value of the integral. We get

$$-(e^{-2i\pi\lambda/\varepsilon} - 1) \int_{(W'_K)^+} e^{Q(\zeta)/\varepsilon} \sum_{j=1}^N v_j \zeta^{-j-\lambda/\varepsilon-1} d\zeta = \int_{(W'_K)^+ \cup (W'_K)^-} z^{-\lambda} e^{Q(z)/\varepsilon} \sum_{j=1}^N v_j z^{-j-1} dz.$$

Hence one can replace the contours Γ_k by W'_k , which leads to the fact that $AV = 0$ if and only if $BV = 0$, hence to Proposition 3.2.

3.3 Asymptotic expansion of small eigenvalues

In this part, we assume that $\lambda \in \widetilde{\Lambda}_-(\delta)$. Thanks to Proposition 3.2 and Theorem 2.3, the eigenvalue equation (5) becomes

$$\det(B(\lambda; \varepsilon)) = 0.$$

We deduce the asymptotics of the eigenvalues by using the method of steepest descent on every coefficient of the matrix $B(\lambda; \varepsilon)$.

For $1 \leq k \leq N-1$ and $1 \leq \ell \leq N$, we recall that the coefficients of $B(\lambda; \varepsilon)$ are given in Proposition 3.2 by

$$B_{k,\ell} = \int_{W_k} e^{S_\lambda(\zeta)/\varepsilon} \zeta^{-\ell-1} d\zeta,$$

$$B_{N,\ell} := B_{N,\ell}^+ + e^{-2i\pi\lambda} B_{N,\ell}^-,$$

$$B_{N,\ell}^\pm = \int_{W_N^\pm} e^{S_\lambda(\zeta)/\varepsilon} \zeta^{-\ell-1} d\zeta,$$

with

$$S_\lambda(z) = Q(z) - \lambda \log(z).$$

Let us apply the method of steepest descent to these oscillatory integrals.

Method of steepest descent Let $1 \leq \ell \leq N$. For $1 \leq k \leq N-1$, we have

$$B_{k,\ell} = \sqrt{\frac{2\pi\varepsilon}{|S_\lambda''(p_k)|}} e^{i\theta_k - S_\lambda(p_k)/\varepsilon} p_k^{-\ell-1} + R_{k,\ell}(\varepsilon),$$

where θ_k is the steepest descent angle. Similarly,

$$B_{N,\ell}^\pm = \sqrt{\frac{2\pi\varepsilon}{|S_\lambda''(p_\pm)|}} e^{i\theta_N^\pm - S_\lambda(p_\pm)/\varepsilon} p_\pm^{-\ell-1} + R_{N,\ell}^\pm(\varepsilon),$$

and the steepest descent angles are $\theta_N^+ = -\frac{\pi}{4}$ and $\theta_N^- = \frac{\pi}{4}$ since this corresponds to the stationary phase method. The remainder terms are bounded by

$$|R_{k,l}(\varepsilon)| \leq C_u(\delta)\varepsilon.$$

Asymptotic expansion of the Evans function Expanding the determinant we deduce that $\det(B(\lambda; \varepsilon))$ has the form

$$\begin{aligned} \det(B(\lambda; \varepsilon)) &= (2\pi\varepsilon)^{N/2} \prod_{k=1}^{N-1} \frac{e^{i\theta_k - S_\lambda(p_k)/\varepsilon} e^{-i\pi/4 - S_\lambda(p_+)/\varepsilon}}{\sqrt{|S_\lambda''(p_k)|} \sqrt{|S_\lambda''(p_+)|}} \det(V_+) \\ &+ (2\pi\varepsilon)^{N/2} \prod_{k=1}^{N-1} \frac{e^{i\theta_k - S_\lambda(p_k)/\varepsilon} e^{-2i\pi\lambda + i\pi/4 - S_\lambda(p_-)/\varepsilon}}{\sqrt{|S_\lambda''(p_k)|} \sqrt{|S_\lambda''(p_-)|}} \det(V_-) + (2\pi\varepsilon)^{N/2} R(\sqrt{\varepsilon}), \end{aligned}$$

where V_\pm is the Vandermonde matrix with coefficients

$$(V_\pm)_{k,\ell} = p_k^{-\ell-1}, \quad p_N = p_\pm,$$

and the remainder term satisfies that if $|S_\lambda''(p_k)| \geq \frac{1}{C_u(\delta)}$ and $|p_k| \leq C_u(\delta)$ for every $1 \leq k \leq N-1$ or $k = \pm$, then

$$|R(\sqrt{\varepsilon})| \leq C2^N N! (C_u(\delta))^{CN^2} \sqrt{\varepsilon}.$$

We simplify the Vandermonde determinants

$$\det(V_\pm) = \prod_{1 \leq j < k \leq N} (p_j^{-1} - p_k^{-1}) \prod_{k=1}^N p_k^{-2} = \prod_{1 \leq j < k \leq N} (p_k - p_j) \prod_{k=1}^N p_k^{-(N+1)}.$$

We deduce that

$$\det(B(\lambda; \varepsilon)) = (2\pi\varepsilon)^{N/2} (P(\lambda; \varepsilon)\Delta(\lambda; \varepsilon) + R(\sqrt{\varepsilon})), \quad (18)$$

$$P(\lambda; \varepsilon) = \frac{e^{-i\pi/4 - S_\lambda(p_+)/\varepsilon}}{\sqrt{|S_\lambda''(p_+)|}} \det(V_+) \prod_{k=1}^{N-1} \frac{e^{i\theta_k - S_\lambda(p_k)/\varepsilon}}{\sqrt{|S_\lambda''(p_k)|}}, \quad (19)$$

$$\Delta(\lambda; \varepsilon) = 1 + e^{-2i\pi\lambda + i\pi/2 - (S_\lambda(p_-) - S_\lambda(p_+))/\varepsilon} \frac{\sqrt{|S_\lambda''(p_+)|} p_-^{-(N+1)}}{\sqrt{|S_\lambda''(p_-)|} p_+^{-(N+1)}} \prod_{k=1}^{N-1} \frac{p_k - p_-}{p_k - p_+}.$$

Using Lemma 2.11, we see that the quotient $\frac{\sqrt{|S''_\lambda(p_+)|} p_-^{-(N+1)}}{\sqrt{|S''_\lambda(p_-)|} p_+^{-(N+1)}} \prod_{k=1}^{N-1} \frac{p_k - p_-}{p_k - p_+}$ has modulus 1, hence there exists $\psi(\lambda) \in \mathbb{T}$ such that

$$\Delta(\lambda; \varepsilon) = 1 - e^{2i\pi\psi(\lambda) - (S_\lambda(p_-) - S_\lambda(p_+))/\varepsilon}. \quad (20)$$

Since $\arg(p_+) - \arg(p_-) = 2\pi F(\lambda)$, the angle $\psi(\lambda)$ is equal to

$$\psi(\lambda) = -\frac{1}{4} - \lambda + (N+1)F(\lambda) + \frac{1}{2\pi} \sum_{k=1}^{N-1} \arg(p_-(\lambda) - p_k(\lambda)) - \arg(p_+(\lambda) - p_k(\lambda)). \quad (21)$$

In this case, ψ is a $C_u(\delta)$ -Lipschitz function of λ when the critical points are on a compact set of values for which the p_k have single multiplicity, for instance when $\lambda \in \widetilde{\Lambda}_-(\delta) \cup \widetilde{\Lambda}_+(\delta)$. In this case, we get smoothness of the simple roots p_k, p_+, p_- along the parameter λ , so that ψ is a Lipschitz function of λ . Note that when u is even, then $p_+ = \overline{p_-}$, moreover p_k and $\overline{p_k}$ both appear in the sum for every k , so that

$$\psi(\lambda) = -\frac{1}{4} - \lambda + (N+1)F(\lambda). \quad (22)$$

We now estimate $P(\lambda; \varepsilon)$. We know that by definition, $S''_\lambda(p_k) = u'(p_k)$, so

$$|S''_\lambda(p_k)| = c_N |p_k|^{-N} |p_+ - p_k| |p_- - p_k| \prod_{\substack{j=1 \\ j \neq k}}^{N-1} |p_k - p_j| \prod_{j=1}^{N-1} |p_k - 1/\overline{p_j}|.$$

Using Lemma 2.11 for p_+ , we deduce that

$$\begin{aligned} |S''_\lambda(p_+)| \prod_{k=1}^{N-1} |S''_\lambda(p_k)| \\ = c_N^N |p_+ - p_-| \prod_{k=1}^{N-1} \frac{|p_+ - p_k|^3 |p_- - p_k|}{|p_k|^{N+1}} \prod_{1 \leq j \neq k \leq N-1} |p_k - p_j| \prod_{1 \leq j, k \leq N-1} |p_k - 1/\overline{p_j}| \end{aligned}$$

belongs to $\left[\frac{1}{C_u(\delta)}, C_u(\delta)\right]$. Similarly, $|\det(V_\pm)| \in \left[\frac{1}{C_u(\delta)}, C_u(\delta)\right]$. As a consequence,

$$|P(\lambda; \varepsilon)| \in \left[\frac{1}{C_u(\delta)}, C_u(\delta)\right].$$

The eigenvalue equation $\det(B(\lambda; \varepsilon)) = 0$ combined with expression (18) therefore becomes

$$\frac{S_\lambda(p_+) - S_\lambda(p_-)}{i\varepsilon} = 2\pi (\psi(\lambda) + n + R'_n(\sqrt{\varepsilon})),$$

$$|R'_n(\sqrt{\varepsilon})| \leq C_u(\delta) \sqrt{\varepsilon}, \quad n \in \mathbb{Z}.$$

Finally, Lemma 2.12 $S_\lambda(p_+) - S_\lambda(p_-) = 2i\pi \int_{-\lambda}^{\max(u)} F(\nu) d\nu$ implies that

$$\int_{-\lambda}^{\max(u)} F(\nu) d\nu = \varepsilon(n + \psi(\lambda)) + R'(\varepsilon\sqrt{\varepsilon}), \quad (23)$$

$$|R'(\varepsilon\sqrt{\varepsilon})| \leq C_u(\delta) \varepsilon \sqrt{\varepsilon}.$$

3.4 Asymptotic expansion of large eigenvalues

We now focus on the case when $\lambda \in \widetilde{\Lambda}_+(\delta) \subseteq [-\min(u) + \delta, K(\delta)]$. Thanks to Proposition 3.4 and Theorem 2.3, the eigenvalue equation (5) becomes

$$\det(B(\lambda; \varepsilon)) = 0.$$

We recall that using Proposition 3.4, the coefficients of B are given for $1 \leq k \leq N-1$ and $1 \leq \ell \leq N$ by

$$\begin{aligned} B_{k,\ell} &= \int_{W_k} e^{S_\lambda(\zeta)/\varepsilon} \zeta^{-\ell-1} d\zeta, \\ B_{N,\ell} &:= B_{N,\ell}^+ + e^{-2i\pi\lambda} B_{N,\ell}^-, \\ B_{N,\ell}^\pm &= \int_{W_N^\pm} e^{S_\lambda(\zeta)/\varepsilon} \zeta^{-\ell-1} d\zeta. \end{aligned}$$

We deduce the asymptotics of the eigenvalues by using the method of steepest descent on every coefficient of the matrix B : for $1 \leq k \leq N-1$ and $1 \leq \ell \leq N$,

$$B_{k,\ell} = \sqrt{\frac{2\pi\varepsilon}{|S_\lambda''(p_k)|}} e^{i\theta_k - S_\lambda(p_k)/\varepsilon} p_k^{-\ell-1} + R_{k,\ell}(\varepsilon),$$

where θ_k is the steepest descent angle, and for $k = N$ and $1 \leq \ell \leq N$, we have

$$B_{N,\ell} = -\left(1 - e^{-2i\pi\lambda}\right) \sqrt{\frac{2\pi\varepsilon}{|S_\lambda''(p_N)|}} e^{i\theta_N - S_\lambda(p_N)/\varepsilon} p_N^{-\ell-1} + R_{N,\ell}(\varepsilon),$$

moreover, the remainder terms are bounded by

$$|R_{k,\ell}(\varepsilon)| \leq C_u(\delta)\varepsilon.$$

The determinant of B therefore admits the asymptotic expansion

$$\det(B(\lambda; \varepsilon)) = (2\pi\varepsilon)^{N/2} \left(1 - e^{-2i\pi\lambda}\right) \left(\prod_{k=1}^N \frac{e^{i\theta_k - iS_\lambda(p_k)/\varepsilon}}{\sqrt{|S_\lambda''(p_k)|}}\right) \det(V) + (2\pi\varepsilon)^{N/2} R(\sqrt{\varepsilon}),$$

where V is the Vandermonde matrix with coefficients $V_{k,\ell} = p_k^{-\ell-1}$. The determinant of V is equal to

$$\det(V) = \prod_{1 \leq j < k \leq N} (p_j^{-1} - p_k^{-1}) \prod_{k=1}^N p_k^{-2} = \prod_{1 \leq j < k \leq N} (p_j - p_k) \prod_{k=1}^N p_k^{-(N+1)},$$

therefore it does not vanish. More precisely, if $|S_\lambda''(p_k)| \geq \frac{1}{C_u(\delta)}$, $|p_k| \leq C_u(\delta)$ and $|p_k - p_\ell| \geq \frac{1}{C_u(\delta)}$ for every $k, \ell \in \{1, \dots, N-1, +, -\}$ and $k \neq \ell$, then

$$\left(\prod_{k=1}^N \frac{1}{\sqrt{|S_\lambda''(p_k)|}}\right) |\det(V)| \geq \frac{1}{C_u(\delta) C_{N^2}}$$

whereas

$$|R(\sqrt{\varepsilon})| \leq C 2^N N! (C_u(\delta))^{CN^2} \sqrt{\varepsilon}.$$

We deduce that for ε small enough,

$$\left| \left(\prod_{k=1}^N \frac{e^{i\theta_k - S_\lambda(p_k)/\varepsilon}}{\sqrt{|S_\lambda''(p_k)|}}\right) \det(V) + R(\sqrt{\varepsilon}) \right| \geq \frac{1}{C_u(\delta)}.$$

As a consequence, $\det(B(\lambda; \varepsilon)) = 0$ implies that $2\pi\lambda/\varepsilon = 2\pi(R'(\sqrt{\varepsilon}) + n)$, or

$$\begin{aligned} \lambda &= n\varepsilon + \varepsilon R'(\sqrt{\varepsilon}), \\ |R'(\sqrt{\varepsilon})| &\leq C_u(\delta)\sqrt{\varepsilon}. \end{aligned} \tag{24}$$

4 Zero-dispersion limit for even trigonometric polynomials

We finally show that when u is an even weakly bell shaped trigonometric polynomial (see Definition 1.1), then the asymptotic expansion of the Lax eigenvalues is precise enough to determine the weak limit of solutions as $\varepsilon \rightarrow 0$. We rely on the fact that the difference of consecutive phase constants are multiple of π when u is even, see part A.1.

First, in part 4.1 we show that the phase constants satisfy a weak form of convergence

$$e^{i(\theta_{n+1}-\theta_n)(u;\varepsilon)} \approx e^{i\frac{x_+(-\lambda_n)+x_-(-\lambda_n)}{2}+i\pi} = -1,$$

As this form of convergence is weaker than the required assumptions in Definition 1.5 of [8], in part 4.2 we finally explain how to adapt the general strategy in order to prove our main Theorem 1.2.

4.1 Phase constants

Let us first estimate the phase constants in the zero-dispersion limit.

Proposition 4.1 (Phase constants in the zero-dispersion limit). *Let $0 < c < r < \frac{1}{3}$, and define the “upside down” terms as*

$$J := \left\{ n \geq 1 \mid \lambda_n + \varepsilon \in \Lambda_-(\delta), \quad e^{i(\theta_{n+1}-\theta_n)(u;\varepsilon)} = 1 \right\}.$$

Then for $\varepsilon < \varepsilon_0(\delta)$, there holds

$$\varepsilon \sum_{n \in J} \text{sinc}(\pi F(-\lambda_n)) \leq C(\delta)\varepsilon^{r-c} + C\delta.$$

Note that this series is only comprised of nonnegative terms since $F(-\lambda_n) \in [0, 1]$, so that the general term of the series satisfies $\text{sinc}(\pi F(-\lambda_n)) \in [0, 1]$.

Proof. In proof of Theorem 3.9 in [8], one can see that inequality (17) is still valid in the case $k = 1$. This implies that for some fixed choice of constants $0 < c < r < 1/3$,

$$\left| \widehat{u}_0(1) - \varepsilon \sum_{\substack{n_2=n_1 \\ \lambda_{n_1}+\varepsilon \in \Lambda_-(\delta)}} \text{sinc}(\pi F(-\lambda_{n_1})) \frac{F(-\lambda_{n_1})e^{i(\theta_{n_1+1}-\theta_{n_1})}}{n_2 - n_1 - F(-\lambda_{n_1})} \right| \leq C(\delta)\varepsilon^{r-c} + C\delta,$$

which simplifies as

$$\left| \widehat{u}_0(1) - \varepsilon \sum_{\substack{n \geq 1 \\ \lambda_n + \varepsilon \in \Lambda_-(\delta)}} \text{sinc}(\pi F(-\lambda_n)) (-1) e^{i(\theta_{n+1}-\theta_n)(u;\varepsilon)} \right| \leq C(\delta)\varepsilon^{r-c} + C\delta. \quad (25)$$

We now divide the sum between the indices for which $e^{i(\theta_{n+1}-\theta_n)} = -1$ and $e^{i(\theta_{n+1}-\theta_n)} = 1$.

According Theorem 3.9 in [8], we already know that the *admissible* the initial data u_0^ε chosen such that for every n , $\lambda_n(u_0^\varepsilon; \varepsilon) = \lambda_n(u_0; \varepsilon)$ and $e^{i(\theta_{n+1}-\theta_n)(u_0^\varepsilon; \varepsilon)} = -1$ satisfies

$$\left| \widehat{u}_0^\varepsilon(1) - \varepsilon \sum_{\substack{n \geq 1 \\ \lambda_n + \varepsilon \in \Lambda_-(\delta)}} \text{sinc}(\pi F(-\lambda_n)) \right| \leq C(\delta)\varepsilon^{r-c} + R(\varepsilon) + C\delta. \quad (26)$$

Moreover, Theorem 3.14 from [8] at $t = 0$ implies that there exists $\varepsilon_0(\delta) > 0$ such that if $0 < \varepsilon < \varepsilon_0(\delta)$, then

$$\left| \widehat{u_0^\varepsilon}(1) + \frac{i}{2\pi} \int_{\min(u_0)}^{\max(u_0)} e^{-ix_+(\eta)} - e^{-ix_-(\eta)} d\eta \right| \leq C\delta.$$

We recognize $\widehat{u_0^\varepsilon}(1)$ in the second term of the left-hand side, so that

$$\left| \widehat{u_0^\varepsilon}(1) - \widehat{u_0}(1) \right| \leq C\delta. \quad (27)$$

We conclude from (25), (26) and (27) that when $0 < \varepsilon < \varepsilon_0(\delta)$, then the indices $n \in J$ for which $(-1)e^{i(\theta_{n+1}-\theta_n)(u;\varepsilon)} \neq 1$ satisfy

$$2\varepsilon \sum_{n \in J} \text{sinc}(\pi F(-\lambda_n)) e^{i(\theta_{n+1}-\theta_n)(u;\varepsilon)} \leq C(\delta)\varepsilon^{r-c} + R(\varepsilon) + C\delta. \quad \square$$

4.2 Time evolution

We now proceed to the proof of Theorem 3.9 in [8], where we replace the solution to (BO- ε) with approximate initial data u_0^ε by the solution to (BO- ε) with the initial data u_0 itself. Since we did not prove that the approximation $e^{i(\theta_{n+1}-\theta_n)(u_0;\varepsilon)} \approx -1$ is valid in the sense of Definition 1.5 in [8], one should instead make use of Proposition 4.1 for the rest of this proof. Theorem 3.9 in [8], taking into account the time evolution, becomes as follows.

Proposition 4.2 (Theorem 3.9 in [8], with weaker phase constant assumption and time evolution). *Let $0 < r < 1$ and $T > 0$. There exists $C(\delta), \varepsilon_0(\delta, T) > 0$ such that for every $0 < \varepsilon < \varepsilon_0(\delta, T)$,*

$$\left| \widehat{u^\varepsilon}(t)(k) - \varepsilon \sum_{\substack{n \geq 1 \\ \lambda_n + \varepsilon \in \Lambda_-(\delta + \delta_N)}} \text{sinc}(k\pi F(-\lambda_n)) (-1)^k e^{2ikt\lambda_n} \right| \leq C(\delta)\varepsilon^{r-c} + C\delta.$$

In order to take into account the time evolution, we recall that the phase constants satisfy $\theta_n(u^\varepsilon(t); \varepsilon) = \theta_n(u_0; \varepsilon) + \omega_n(u_0; \varepsilon)t$ with

$$\omega_{n+1}(u_0; \varepsilon) - \omega_n(u_0; \varepsilon) = 2\lambda_n(u_0; \varepsilon) + \varepsilon, \quad (28)$$

see for instance the proof of Theorem 3.14 in [8].

Proof of Proposition 4.2. Thanks to Corollary 1.4, the proof of Theorem 3.9 in [8] is still valid as long as we do not use the approximation on the phase constants but only the approximation of the Lax eigenvalues. Since the Lax eigenvalues are constant over time, the estimates are also valid for any $t \in \mathbb{R}$. Hence inequality (17) in [8] still holds:

$$\left| \widehat{u^\varepsilon}(t)(k) - \varepsilon \sum_{\substack{n_{k+1}=n_1 \\ |n_{i+1}-n_i| \leq \varepsilon^{-r} \\ \lambda_{n_1} + \varepsilon \in \Lambda_-(\delta)}} \text{sinc}(\pi F(-\lambda_{n_1}))^k \prod_{i=1}^k \frac{F(-\lambda_{n_1}) e^{i(\theta_{n_{i+1}} - \theta_{n_i} + (\omega_{n_{i+1}} - \omega_{n_i})t)}}{n_{i+1} - n_i - F(-\lambda_{n_1})} \right| \leq C(\delta)\varepsilon^{r-c} + C\delta.$$

Thanks to (28) and the fact that $|\lambda_p - \lambda_n| \leq C(\delta)\varepsilon^{1-r}$ when $|n - p| \leq \varepsilon^{-r}$, we can replace $\omega_{n_{i+1}} - \omega_{n_i}$ by $\lambda_{n_1} + \varepsilon$. We get that for every $\varepsilon < \varepsilon_0(\delta, T)$, for every $t \in [0, T]$,

$$\left| \widehat{u^\varepsilon}(t)(k) - \varepsilon \sum_{\substack{n_{k+1}=n_1 \\ |n_{i+1}-n_i| \leq \varepsilon^{-r} \\ \lambda_{n_1} + \varepsilon \in \Lambda_-(\delta)}} \text{sinc}(\pi F(-\lambda_{n_1}))^k e^{2ik\lambda_{n_1}t} \prod_{i=1}^k \frac{F(-\lambda_{n_1}) e^{i(\theta_{n_{i+1}} - \theta_{n_i})}}{n_{i+1} - n_i - F(-\lambda_{n_1})} \right| \leq C(\delta)\varepsilon^{r-c} + C\delta.$$

We now want to use the fact that the approximation $e^{i(\theta_{n_i+1}-\theta_{n_i})} \approx -1$ is valid on a large set of indices. Thanks to the Toeplitz absolute convergence from Lemma 3.12 in [8], there holds the upper bound

$$\varepsilon \sum_{\substack{n_{k+1}=n_1 \\ |n_{i+1}-n_i| \leq \varepsilon^{-r} \\ n_1 \in J}} \text{sinc}(\pi F(-\lambda_{n_1}))^k \prod_{i=1}^k \frac{F(-\lambda_{n_1})}{|n_{i+1} - n_i - F(-\lambda_{n_1})|} \leq C\varepsilon \sum_{n_1 \in J} \text{sinc}(\pi F(-\lambda_{n_1}))^k.$$

Using that $\text{sinc}(\pi F(-\lambda_{n_1}))^k \leq \text{sinc}(\pi F(-\lambda_{n_1}))$ since $\text{sinc}(\pi F(-\lambda_{n_1})) \in [0, 1]$, Proposition 4.1 implies

$$\varepsilon \sum_{\substack{n_{k+1}=n_1 \\ |n_{i+1}-n_i| \leq \varepsilon^{-r} \\ n_1 \in J}} \text{sinc}(\pi F(-\lambda_{n_1}))^k \prod_{i=1}^k \frac{F(-\lambda_{n_1})}{|n_{i+1} - n_i - F(-\lambda_{n_1})|} \leq C(\delta)\varepsilon^{r-c} + C\delta.$$

As a consequence, the contribution of the ‘‘upside down’’ terms such that $n_i \in J$ for some index $1 \leq i \leq k$ is bounded by $C(\delta)\varepsilon^{r-c} + C\delta$, so that

$$\left| \widehat{u^\varepsilon}(t)(k) - \varepsilon \sum_{\substack{n_{k+1}=n_1 \\ |n_{i+1}-n_i| \leq \varepsilon^{-r} \\ \lambda_{n_1} + \varepsilon \in \Lambda_-(\delta)}} \text{sinc}(\pi F(-\lambda_{n_1}))^k e^{2ik\lambda_{n_1}t} (-1)^k \prod_{i=1}^k \frac{F(-\lambda_{n_1})}{n_{i+1} - n_i - F(-\lambda_{n_1})} \right| \leq C(\delta)\varepsilon^{r-c} + C\delta.$$

We then use the Toeplitz identity from Lemma 3.12 in [8] and get

$$\left| \widehat{u^\varepsilon}(t)(k) - \varepsilon \sum_{\substack{n_1 \geq 1 \\ n_1 \in \Lambda_-(\delta + \delta_N)}} \text{sinc}(k\pi F(-\lambda_{n_1}))^k e^{2ik\lambda_{n_1}t} (-1)^k \right| \leq C(\delta)\varepsilon^{r-c} + R(\varepsilon) + C\delta. \quad \square$$

We now proceed to the proof of our main theorem.

Proof of Theorem 1.2. We follow the proof of Theorem 3.14 in [8], where we transform the sum involved in the statement of Proposition 4.2 into a Riemann sum. We deduce that for every $\varepsilon < \varepsilon_0(\delta, T)$,

$$\left| \widehat{u^\varepsilon}(t)(k) + \frac{i}{2k\pi} \int_{\min(u_0)}^{\max(u_0)} e^{-ik(x_+(\eta)+2\eta t)} - e^{-ik(x_-(\eta)+2\eta t)} d\eta \right| \leq C\delta.$$

To conclude, using Proposition 3.15 in [8], we have proven that

$$\limsup_{\varepsilon \rightarrow 0} \left| \widehat{u^\varepsilon}(t)(k) + \frac{i}{2k\pi} \int_{\min(u_0)}^{\max(u_0)} e^{-ik(x_+(\eta)+2\eta t)} - e^{-ik(x_-(\eta)+2\eta t)} d\eta \right| \leq C\delta.$$

Since δ is arbitrary, we obtain

$$\lim_{\varepsilon \rightarrow 0} \widehat{u^\varepsilon}(t)(k) = -\frac{i}{2k\pi} \int_{\min(u_0)}^{\max(u_0)} e^{-ik(x_+(\eta)+2\eta t)} - e^{-ik(x_-(\eta)+2\eta t)} d\eta. \quad (29)$$

When u_0 is bell shaped, we conclude from Proposition 3.15 in [8] that $\widehat{u^\varepsilon}(t)(k) \rightarrow \widehat{u_{alt}^B}(t)(k)$ as $\varepsilon \rightarrow 0$. \square

5 Extension to bell shaped initial data

To extend the study of the zero-dispersion limit from bell shaped trigonometric polynomials to general bell shaped initial data, we rely on an inversion formula from [9] for which one can pass to the limit both in the trigonometric approximation and in the dispersion parameter $\varepsilon \rightarrow 0$.

Let $u_0 \in L^\infty(\mathbb{T}) \cap L^2(\mathbb{T})$ with zero mean. The solution u^ε to (BO- ε) with initial data u_0 is well-defined [23, 24]. We decompose $u^\varepsilon = \Pi(u^\varepsilon) + \overline{\Pi(u^\varepsilon)}$ where Π is the Szegő projector. Then according to Theorem 3 in [9], the holomorphic extension of $\Pi(u^\varepsilon)$ to \mathbb{D} satisfies: for every $z \in \mathbb{D}$,

$$\Pi(u^\varepsilon(t, z)) = \left\langle \left(\text{Id} - z e^{i\varepsilon t} e^{2itL_{u_0}(\varepsilon)} S^* \right)^{-1} \Pi(u_0) \mid 1 \right\rangle. \quad (30)$$

Using this formula, we first establish the existence of a weak limit for u^ε as $\varepsilon \rightarrow 0$. Then we show that this weak limit is u_{alt}^B when u_0 is bell shaped, using the fact that this property is true for trigonometric polynomials.

We recall that

$$L_{u_0}(\varepsilon) = \varepsilon D - T_{u_0},$$

where $D = -i\partial_x$ and $T_{u_0} : h \in L_+^2 \rightarrow \Pi(u_0 h)$ is the Toeplitz operator of symbol u_0 .

Note that when $u_0 \in L^\infty(\mathbb{T}) \cap L^2(\mathbb{T})$, we have that T_{u_0} is a bounded self-adjoint operator in $\mathcal{L}(L_+^2)$. We deduce that the operator $e^{it} e^{2itL_{u_0}(\varepsilon)} S^*$ has norm at most 1 for every $\varepsilon \geq 0$. Hence for every $z \in \mathbb{R}$, formula (30) is continuous from $L^\infty(\mathbb{T}) \cap L^2(\mathbb{T})$ to \mathbb{C} .

Proposition 5.1 (Existence of a weak limit). *Let $u_0 \in L^\infty(\mathbb{T}) \cap L^2(\mathbb{T})$. Define $\tilde{u} := \Pi(\tilde{u}) + \overline{\Pi(\tilde{u})}$ by the formula*

$$\Pi(\tilde{u}(t, z)) = \left\langle \left(\text{Id} - z e^{-2itT_{u_0}} S^* \right)^{-1} \Pi(u_0) \mid 1 \right\rangle.$$

The solution u^ε to (BO- ε) with initial data u_0 is weakly convergent to \tilde{u} as $\varepsilon \rightarrow 0$: uniformly on finite time intervals

$$u^\varepsilon \xrightarrow{\varepsilon \rightarrow 0} \tilde{u}.$$

Proof. But for every $\varepsilon > 0$, we know from the conservation of mass that $\|\Pi u^\varepsilon(t)\|_{L_+^2} = \|\Pi u_0\|_{L_+^2}$ is bounded independently of ε . Using the sequential version of Banach-Alaoglu's theorem on the Hilbert space $L_+^2(\mathbb{T})$, we deduce that there is a sequence ε_n and $f \in L_+^2$ such that $\Pi u^{\varepsilon_n}(t) \xrightarrow{\varepsilon \rightarrow 0} f$ in $L_+^2(\mathbb{T})$.

Using formula (30), we know that when $|z| < 1$ there holds as $\varepsilon \rightarrow 0$ the pointwise convergence $\Pi u^\varepsilon(t, z) \rightarrow \Pi \tilde{u}(t, z)$. As a consequence, the unique possible accumulation point is $f = \Pi \tilde{u}(t)$. This implies that $\Pi u^\varepsilon(t) \xrightarrow{\varepsilon \rightarrow 0} \Pi \tilde{u}(t)$ in L_+^2 .

Finally, since $\|\Pi u^\varepsilon(t)\|_{L_+^2}$ is bounded for $t \in \mathbb{R}$ and $\varepsilon > 0$ thanks to the conservation of mass, one can see that the weak convergence is actually uniform on finite time intervals. \square

Remark 5.2. *One may be tempted to use this formula directly for the approximate initial data u_0^ε from [8] without relying on the present work. However, one cannot easily ensure that the chosen approximate initial data $(u_0^\varepsilon)_\varepsilon$ are uniformly bounded in L^∞ , whereas the proof of Proposition 5.1 relies on the continuity of formula (30) on $L^\infty \cap L^2$.*

Proposition 5.3 (Formula for the weak limit). *Let u_0 be a bell shaped initial data. Then the function \tilde{u} from Proposition 5.1 satisfies*

$$\tilde{u} = u_{alt}^B.$$

Proof. Let u_0 be an even bell shaped initial data. Let $u_{0,N}$ be the truncated Fourier series of u_0 up to order N . Then according to Proposition A.4, $u_{0,N}$ is increasing on $(x_{\min}(N), x_{\max}(N))$ and decreasing on $(x_{\max}(N), x_{\min}(N) + 2\pi)$ where $x_{\min}(N) \rightarrow 0$ and $x_{\max}(N) \rightarrow \pi$. We denote

by u_N^ε the solution to (BO- ε) with initial data $u_{0,N}$. Using Proposition 5.1, we denote by \tilde{u}_N the weak limit of u_N^ε as $\varepsilon \rightarrow 0$, and by \tilde{u} the weak limit of u^ε .

We know from Section 4 that if $u_{0,N}$ is bell shaped, then \tilde{u}_N is the signed sum of Branches for the multivalued solution of Burgers' equation obtained using the method of characteristics: $\tilde{u}_N = (u_N)_{alt}^B$. Since this assumption is not entirely true here, we instead use the formula for the limit of Fourier coefficients given by equality (29): denoting $x_{N,\pm}(\eta)$ the antecedents of η by $u_{0,N}$, for every $k \in \mathbb{Z}$ there holds

$$\lim_{\varepsilon \rightarrow 0} \widehat{u_N^\varepsilon(t)}(k) = -\frac{i}{2k\pi} \int_{\min(u_{0,N})}^{\max(u_{0,N})} e^{-ik(x_{N,+}(\eta)+2\eta t)} - e^{-ik(x_{N,-}(\eta)+2\eta t)} d\eta.$$

This implies that

$$\widehat{\tilde{u}_N(t)}(k) = -\frac{i}{2k\pi} \int_{\min(u_{0,N})}^{\max(u_{0,N})} e^{-ik(x_+^N(\eta)+2\eta t)} - e^{-ik(x_-^N(\eta)+2\eta t)} d\eta.$$

Finally, according to Proposition A.5, one can pass to the limit $N \rightarrow +\infty$ in the right hand side to get that

$$\widehat{\tilde{u}_N(t)}(k) \xrightarrow{N \rightarrow +\infty} -\frac{i}{2k\pi} \int_{\min(u_0)}^{\max(u_0)} e^{-ik(x_+(\eta)+2\eta t)} - e^{-ik(x_-(\eta)+2\eta t)} d\eta.$$

Proposition 3.15 in [8], applied to the bell shaped initial data u_0 , implies that the right-hand side is actually equal to $u_{alt}^B(t)$.

We now make use of the continuity of formula (30) on $L^\infty(\mathbb{T}) \cap L^2(\mathbb{T})$ for every $\varepsilon \geq 0$ and $t \in \mathbb{R}$. As a consequence, as $N \rightarrow +\infty$, we have that

$$\tilde{u}_N(t) \xrightarrow{N \rightarrow +\infty} \tilde{u}(t).$$

This implies that for every $k \in \mathbb{Z}$,

$$\widehat{\tilde{u}(t)}(k) = u_{alt}^B(t).$$

We conclude that $\tilde{u}(t) = u_{alt}^B(t)$ for every $t \in \mathbb{R}$. □

A Appendices

A.1 Phase constants of even initial data

In this part, we prove that if u is even, then for every $n \in \mathbb{N}$ such that $\zeta_n(u) \neq 0$, there holds that $\overline{\zeta_n(u; \varepsilon)} \zeta_{n+1}(u; \varepsilon) \in \mathbb{R}$.

Theorem A.1 (Phase constants for an even function). *Let u be an even bell shaped potential. Then for every $n \in \mathbb{N}$, $\overline{\zeta_n(u; \varepsilon)} \zeta_{n+1}(u; \varepsilon) \in \mathbb{R}$.*

We show this result for even bell shaped trigonometric polynomials of degree N . Indeed, for general even bell shaped function u , one can use the truncated Fourier series u_N of degree N , that satisfies Proposition A.4. The theorem applied to u_N implies that $\overline{\zeta_n(u_N; \varepsilon)} \zeta_{n+1}(u_N; \varepsilon) \in \mathbb{R}$ for every n . Moreover, this term goes to $\overline{\zeta_n(u; \varepsilon)} \zeta_{n+1}(u; \varepsilon) \in \mathbb{R}$ as $N \rightarrow +\infty$ by continuity of the Birkhoff map.

We have seen that the trigonometric polynomial u_N extends to a meromorphic function on the unit disk \mathbb{D} as

$$u_N(z) = \sum_{k=-N}^N c_k z^k.$$

Since u is even, then for every $-N \leq k \leq N$, $c_k \in \mathbb{R}$. In particular, the function Q defined in (4) as $Q(z) = \sum_{k=1}^N \left(-c_k \frac{z^k}{k} + c_k \frac{z^{-k}}{k} \right)$ satisfies $Q(\bar{z}) = \overline{Q(z)}$ for every $z \in \mathbb{D}$.

We first study the matrix A involved in the eigenvalue equation and introduced in Definition 2.2: for $1 \leq k \leq N-1$ and $1 \leq \ell \leq N$, we have

$$\begin{aligned} A_{k,\ell} &:= \int_{\Gamma_k} e^{Q(\zeta)/\varepsilon} \zeta^{-\ell-\lambda/\varepsilon} \frac{d\zeta}{\zeta}, \\ A_{N,\ell}^\pm &:= \int_{\Gamma_N^\pm} e^{Q(\zeta)/\varepsilon} \zeta^{-\ell-\lambda/\varepsilon} \frac{d\zeta}{\zeta}, \\ A_{N,\ell} &:= A_{N,\ell}^+ + e^{-2i\pi\lambda/\varepsilon} A_{N,\ell}^-. \end{aligned}$$

Lemma A.2 (Matrix coefficients). *For $1 \leq k \leq N-1$ and $1 \leq \ell \leq N$, there holds*

$$A_{k,\ell} = -\overline{A_{N-k,\ell}}.$$

Moreover, for $1 \leq \ell \leq N$, we have

$$e^{i\pi\lambda/\varepsilon} A_{N,\ell} \in \mathbb{R}.$$

Proof. Let $1 \leq k \leq N-1$ and $1 \leq \ell \leq N$. We recall that Γ_k is the union of the segment $[0, e^{i\theta_k}]$, the arc of circle of radius 1 going from $e^{i\theta_k}$ to $e^{i\theta_{k+1}}$ and the segment $[e^{i\theta_{k+1}}, 0]$ for $\theta_k = \frac{2k-1}{N}\pi$ and $\theta_{k+1} = \frac{2k+1}{N}\pi$.

- First, we parameterize the segment $[0, e^{i\theta_k}]$ by $\zeta = te^{i\theta_k}$ for $t \in [0, 1]$, so that

$$\int_{[0, e^{i\theta_k}]} e^{Q(\zeta)/\varepsilon} \zeta^{-\ell-\lambda/\varepsilon} \frac{d\zeta}{\zeta} = \int_0^1 e^{Q(te^{i\theta_k})/\varepsilon} (te^{i\theta_k})^{-\ell-\lambda/\varepsilon} \frac{dt}{t}.$$

Taking the conjugate, we have

$$\overline{\int_{[0, e^{i\theta_k}]} e^{Q(\zeta)/\varepsilon} \zeta^{-\ell-\lambda/\varepsilon} \frac{d\zeta}{\zeta}} = \int_0^1 e^{Q(te^{-i\theta_k})/\varepsilon} (te^{-i\theta_k})^{-\ell-\lambda/\varepsilon} \frac{dt}{t}.$$

But $-\theta_k = \frac{2N-2k+1}{N}\pi = \theta_{N-k+1}$ modulo 2π , therefore

$$\overline{\int_{[0, e^{i\theta_k}]} e^{Q(\zeta)/\varepsilon} \zeta^{-\ell-\lambda/\varepsilon} \frac{d\zeta}{\zeta}} = \int_{[0, e^{i\theta_{N-k+1}}]} e^{Q(\zeta)/\varepsilon} \zeta^{-\ell-\lambda/\varepsilon} \frac{d\zeta}{\zeta}.$$

In particular

$$\overline{\int_{[0, e^{i\theta_k}]} e^{Q(\zeta)/\varepsilon} \zeta^{-\ell-\lambda/\varepsilon} \frac{d\zeta}{\zeta}} = - \int_{[e^{i\theta_{N-k+1}}, 0]} e^{Q(\zeta)/\varepsilon} \zeta^{-\ell-\lambda/\varepsilon} \frac{d\zeta}{\zeta}.$$

The same relationship holds by considering $[0, e^{i\theta_{k+1}}]$ and $[0, e^{i\theta_{N-k}}]$.

- Now, we consider the integral on the arc of circle $\gamma(\theta_k, \theta_{k+1})$ of radius 1 going from $e^{i\theta_k}$ to $e^{i\theta_{k+1}}$. We parameterize $\zeta = e^{i\theta}$ for $\theta \in [\theta_k, \theta_{k+1}]$, so that

$$\int_{\gamma(\theta_k, \theta_{k+1})} e^{Q(\zeta)/\varepsilon} \zeta^{-\ell-\lambda/\varepsilon} \frac{d\zeta}{\zeta} = i \int_{\theta_k}^{\theta_{k+1}} e^{Q(e^{i\theta})/\varepsilon} (e^{i\theta})^{-\ell-\lambda/\varepsilon} d\theta.$$

Taking the conjugate we get

$$\overline{\int_{\gamma(\theta_k, \theta_{k+1})} e^{Q(\zeta)/\varepsilon} \zeta^{-\ell-\lambda/\varepsilon} \frac{d\zeta}{\zeta}} = -i \int_{\theta_k}^{\theta_{k+1}} e^{Q(e^{-i\theta})/\varepsilon} (e^{-i\theta})^{-\ell-\lambda/\varepsilon} d\theta.$$

We make the change of variable $\alpha = -\theta$ and get

$$\overline{\int_{\gamma(\theta_k, \theta_{k+1})} e^{Q(\zeta)/\varepsilon} \zeta^{-\ell-\lambda/\varepsilon} \frac{d\zeta}{\zeta}} = -i \int_{\theta_{N-k}}^{\theta_{N-k+1}} e^{Q(e^{i\alpha})/\varepsilon} (e^{-i\alpha})^{-\ell-\lambda/\varepsilon} d\alpha.$$

As a consequence,

$$\overline{\int_{\gamma(\theta_k, \theta_{k+1})} e^{Q(\zeta)/\varepsilon} \zeta^{-\ell-\lambda/\varepsilon} \frac{d\zeta}{\zeta}} = - \int_{\gamma(\theta_{N-k}, \theta_{N-k+1})} e^{Q(\zeta)/\varepsilon} \zeta^{-\ell-\lambda/\varepsilon} \frac{d\zeta}{\zeta}.$$

Similarly in the case $k = N$, we obtain that

$$\overline{\int_{\gamma(\theta_N, 0)} e^{Q(\zeta)/\varepsilon} \zeta^{-\ell-\lambda/\varepsilon} \frac{d\zeta}{\zeta}} = - \int_{\gamma(0, \theta_1)} e^{Q(\zeta)/\varepsilon} \zeta^{-\ell-\lambda/\varepsilon} \frac{d\zeta}{\zeta}. \quad \square$$

Lemma A.3. *Let $f \in L_+^2$ be an eigenvector of $L_u(\varepsilon)$ associated to the eigenvalue λ . Then $\langle f|Sf \rangle \in \mathbb{R}$.*

Proof. We recall that the vector $V = (v_1, \dots, v_N)$ is solution to $AV = 0$.

Let C_1^T, \dots, C_N^T be the lines of A . Then $e^{i\pi\lambda/\varepsilon} C_N^T$ is real valued whereas $C_{N-1-k} = -\overline{C_k}$ for every $1 \leq k \leq N-1$. The fact that $AV = 0$ implies that for every $1 \leq k \leq N$,

$$C_k^T V = 0.$$

In particular, one can see that if $1 \leq k \leq N-1$,

$$C_k^T \overline{V} = -\overline{C_{N-k}^T} V = 0$$

and also

$$e^{i\pi\lambda/\varepsilon} C_N^T \overline{V} = \overline{e^{i\pi\lambda/\varepsilon} C_N^T} V = 0.$$

This implies that \overline{V} also satisfies $A\overline{V} = 0$. Since λ is a simple eigenvalue (see [10]) and the coefficients of the eigenvectors solve a triangular system depending on the Taylor expansion of the eigenfunction at 0, we deduce that there exists $\alpha \in \mathbb{C}$ such that $\overline{V} = \alpha V$. However this implies that $\overline{V} = |\alpha|^2 \overline{V}$ therefore $|\alpha| = 1$. Writing $\alpha = e^{2i\theta}$ for some $\theta \in \mathbb{T}$, we get that $\overline{e^{i\theta} V} = e^{i\theta} V$, so that $e^{i\theta} V$ is a real-valued vector.

We recall the equation (6) satisfied by f :

$$zf'(z) - \left(\sum_{k=1}^N c_k z^k + \overline{c_k} z^{-k} \right) f(z) - \lambda f(z) = \sum_{j=1}^N v_j z^{-j}.$$

We note that, up to replacing f by $e^{-i\theta} f$, which transforms V into $e^{-i\theta} V$, one can assume that V is real-valued.

Now, we check that $z \mapsto \overline{f(\overline{z})}$ is a holomorphic function on \mathbb{D} also satisfying (6). Since λ is a simple eigenvalue, this function is colinear to f , so that there exists $\alpha \in \mathbb{C}$ such that for every $z \in \mathbb{D}$, we have $\overline{f(\overline{z})} = \alpha f(z)$. We write the Taylor expansion of f around zero as $f(z) = \sum_{\ell \geq 0} b_\ell z^\ell$ for some $b_\ell \in \mathbb{C}$. Then $\overline{f(\overline{z})} = \sum_{\ell \geq 0} \overline{b_\ell} z^\ell$. Therefore the number $\alpha \in \mathbb{C}$ is so that for every $\ell \geq 0$, $\overline{b_\ell} = \alpha b_\ell$. Since f is nonzero, there exists ℓ such that $b_\ell \neq 0$. We deduce that $|\alpha| = 1$. Writing $\alpha = e^{2i\theta}$ for some $\theta \in \mathbb{T}$, we deduce that $\overline{e^{i\theta} b_\ell} = e^{i\theta} b_\ell$ and therefore $e^{i\theta} b_\ell \in \mathbb{R}$ for every $\ell \geq 0$. As a consequence, the parametrization $z = e^{ix}$ leads to

$$\langle f|Sf \rangle = \int_0^{2\pi} |f(e^{ix})|^2 e^{-ix} dx = -i \int_{\partial\mathbb{D}} \sum_{k, \ell \geq 0} b_k \overline{b_\ell} z^{k-\ell-2} dz.$$

The residue formula implies $k - \ell - 2 = -1$, hence

$$\langle f|Sf \rangle = 2\pi \sum_{\ell \geq 0} b_{\ell+1} \overline{b_\ell}.$$

We conclude that $\langle f|Sf \rangle \in \mathbb{R}$ by noting that $b_{\ell+1} \overline{b_\ell} = e^{i\theta} b_{\ell+1} \overline{e^{i\theta} b_\ell} \in \mathbb{R}$ for every $\ell \geq 0$. □

Proof of Theorem A.1. We use the fact that

$$\langle f_n(u; \varepsilon) | S f_n(u; \varepsilon) \rangle = -\frac{1}{\varepsilon} \sqrt{\mu_{n+1}(u; \varepsilon)} \frac{\sqrt{\kappa_n(u; \varepsilon)}}{\sqrt{\kappa_{n+1}(u; \varepsilon)}} \overline{\zeta_n(u; \varepsilon)} \zeta_{n+1}(u; \varepsilon),$$

see for instance equality (7.8) in [10]. As a consequence, since the term $\langle f_n(u; \varepsilon) | S f_n(u; \varepsilon) \rangle$ is real, so is $\overline{\zeta_n(u; \varepsilon)} \zeta_{n+1}(u; \varepsilon)$. □

A.2 Approximation by truncated Fourier series

In this part, we fix a bell shaped function u . Our goal is to show that the approximation of u by its truncated Fourier series u_N is comonotone so that it has weakened properties of a bell shaped function. We deduce in particular Corollary 1.4.

More precisely, we use the following result.

Proposition A.4 (Comonotone approximation). *Let u be a bell shaped function in $\mathcal{C}^3(\mathbb{T})$. We decompose u into a Fourier series*

$$u(x) = \sum_{k=-\infty}^{\infty} c_k e^{ikx}.$$

Then there exists $N_0 \geq 1$ such that for every $N \geq N_0$, the truncated Fourier series of order N

$$u_N(x) := \sum_{k=-N}^N c_k e^{ikx}$$

satisfies the following properties. There exists a sequence $x_{\min}(N) \rightarrow 0$ and $x_{\max}(N) \rightarrow x_{\max}$ as $N \rightarrow +\infty$ such that the 2π -periodic function u_N is increasing on $(x_{\min}(N), x_{\max}(N))$ and decreasing on $(x_{\max}(N), x_{\min}(N) + 2\pi)$, moreover,

$$\|u - u_N\|_{L^\infty(\mathbb{T})} \leq \frac{C}{N\sqrt{N}}. \quad (31)$$

Consequently, the trigonometric approximation is also bell shaped up to spatial translation and up to removing the mean value.

Proof. Let u be a bell shaped initial data. Then we estimate

$$\begin{aligned} \|u'' - u_N''\|_{L^\infty(\mathbb{T})} &\leq \sum_{|k| \geq N+1} k(k-1)|c_k| \\ &\leq \left(\sum_{|k| \geq N+1} k^6 |c_k|^2 \right)^{1/2} \left(\sum_{|k| \geq N+1} \frac{1}{k^2} \right)^{1/2} \\ &\leq C \frac{\|u\|_{H^3(\mathbb{T})}}{\sqrt{N}}. \end{aligned}$$

By assumption, there are only two inflection points ξ_\pm such that $u''(\xi_\pm) = 0$. Let $\alpha_1 > 0$ sufficiently small. We remove a box of size α_2 around ξ_\pm so that if $x \in [0, 2\pi] \setminus ([\xi_- - \alpha_2, \xi_- + \alpha_2] \cup [\xi_+ - \alpha_2, \xi_+ + \alpha_2])$, then $|u''(x)| \geq \alpha_1$. When $C \frac{\|u\|_{\mathcal{C}^3(\mathbb{T})}}{\sqrt{N}} \leq \alpha_1$, we deduce that u_N'' has the same sign as u'' on this set.

As a consequence, u_N' is strictly increasing on $[\xi_+ + \alpha_2 - 2\pi, \xi_- - \alpha_2]$ and strictly decreasing on $[\xi_- + \alpha_2, \xi_+ - \alpha_2]$. By choosing α_2 small enough, we also get that $u_N' > 0$ on $[\xi_- - \alpha_2, \xi_- + \alpha_2]$ and $u_N' < 0$ on $[\xi_+ - \alpha_2, \xi_+ + \alpha_2]$.

We deduce that there exist unique $x_{\min}(N), x_{\max}(N) \in [0, 2\pi]$ such that u_N is increasing on $(x_{\min}(N), x_{\max}(N))$ and decreasing on $(x_{\max}(N), x_{\min}(N) + 2\pi)$. Finally, an application of the Cauchy-Schwarz' inequality as in the beginning of this proof leads to (31). This implies that $x_{\min}(N) \rightarrow 0$ and $x_{\max}(N) \rightarrow x_{\max}$ as $N \rightarrow +\infty$. \square

Proof of Corollary 1.4 From the min-max formula

$$\lambda_n(u; \varepsilon) = \max_{\dim F=n} \min\{\langle L_u(\varepsilon)h|h \rangle \mid h \in H_+^1 \cap F^\perp, \|h\|_{L^2} = 1\},$$

one can see that for every $u, v \in \mathcal{C}^0(\mathbb{T})$, there holds

$$|\lambda_n(u; \varepsilon) - \lambda_n(v; \varepsilon)| \leq \|u - v\|_{L^\infty}.$$

Using truncated Fourier series, see Proposition A.4 below, we deduce that if $u \in \mathcal{C}^3(\mathbb{T})$, then for $N \geq N_0$, we have

$$|\lambda_n(u; \varepsilon) - \lambda_n(u_N; \varepsilon)| \leq C \frac{1}{N\sqrt{N}}$$

This approximation leads to the estimate

$$\left| \int_{-\lambda_n(u; \varepsilon)}^{-\lambda_p(u; \varepsilon)} F(\eta) \, d\eta - (n - p)\varepsilon \right| \leq \frac{C}{N\sqrt{N}} + C_N(\delta)\varepsilon\sqrt{\varepsilon}.$$

This also leads to point 1. in Corollary 1.4. However, this estimate is not sufficient compared to Definition 1.5 in [8] for an admissible approximate initial data, as the necessary condition is

$$\left| \int_{-\lambda_n(u; \varepsilon)}^{-\lambda_p(u; \varepsilon)} F(\eta) \, d\eta - (n - p)\varepsilon \right| \leq C_u(\delta)\varepsilon\sqrt{\varepsilon}. \quad (32)$$

This condition is especially important in the proof of Lemma 3.10 in [8]. One could study the growth of $C_N(\delta)$ with N to show that if u has very fast decaying Fourier coefficients as $|\hat{u}(n)| \leq Cn^{-Cn^2}$, then the growth of $C_N(\delta)$ is sufficiently slow so that for $N = N(\varepsilon)$ well chosen, for every $n, p \in \Lambda_-(\delta)$,

$$\frac{C}{N(\varepsilon)\sqrt{N(\varepsilon)}} + C_{N(\varepsilon)}(\delta)\varepsilon\sqrt{\varepsilon} \leq C_u(\delta)\varepsilon\sqrt{\varepsilon}.$$

As a consequence (32) still holds. However, we have seen that the approach in Section 5 does not require such a decay of the Fourier coefficients.

We can now turn to very large the Lax eigenvalues for bell shaped initial data to get point 2. of Corollary 1.4. Let u_0 be a bell shaped initial data, fix $N \geq N_0$ and let u_N be the truncated Fourier series of u up to order N .

Fix $\delta > 0$. Using the Parseval formula, for $K \in \mathbb{N}$, we have

$$K\varepsilon \sum_{k=K}^{\infty} \gamma_k(u; \varepsilon) \leq \varepsilon \sum_{k=1}^{\infty} k\gamma_k(u; \varepsilon) = \|u\|_{L^2}^2/2.$$

As a consequence, if $K(\delta) = \frac{C}{\delta}$, one has $\sum_{k=K(\delta)/\varepsilon}^{\infty} \gamma_k(u; \varepsilon) \leq \delta$, therefore for every $n \geq K(\delta)/\varepsilon$,

$$|\lambda_n - n\varepsilon| \leq \delta.$$

In particular one can show that for $n \geq \frac{K(\delta)}{\varepsilon}$ then $\lambda_n \geq K(\delta) - \delta$, whereas for $n = \frac{K(\delta)}{\varepsilon}$, then $\lambda_n \leq K(\delta) + \delta$ (and this property is also true for the smaller indices).

Estimate of x_{\pm} in the trigonometric approximation Consider $u \in \mathcal{C}^0(\mathbb{T})$ weakly bell shaped, strictly increasing from u_{\min} to u_{\max} on (x_{\min}, x_{\max}) and strictly decreasing on $(x_{\max}, x_{\min} + 2\pi)$. For $\eta \in (u_{\min}, u_{\max})$, we define $x_+(\eta)$ as the antecedent of η by u on (x_{\min}, x_{\max}) and by $x_-(\eta)$ as the antecedent of η by u on $(x_{\max}, x_{\min} + 2\pi)$. When considering a sequence $(u_n)_n$ we denote x_{\pm}^n the points x_{\pm} associated to u_n .

Proposition A.5. *Let $u \in \mathcal{C}^1(\mathbb{T})$ be a weakly bell shaped function and $(u_n)_n \in \mathcal{C}^0(\mathbb{T})$ a sequence of weakly bell shaped functions uniformly convergent to u . Then for every $\delta > 0$,*

$$\|x_+^n(\eta) - x_+(\eta)\|_{L^\infty([u_{\min} + \delta, u_{\max} - \delta])} + \|x_-^n(\eta) - x_-(\eta)\|_{L^\infty([u_{\min} + \delta, u_{\max} - \delta])} \xrightarrow{n \rightarrow +\infty} 0.$$

Proof. Let $\delta > 0$. Since $\|u_n - u\|_{L^\infty} \rightarrow 0$, we know that there exists n_0 such that for every $n \geq n_0$, $\|u_n - u\|_{L^\infty} \leq \delta/2$. In particular η has an antecedent by u_n for every $\eta \in [u_{\min} + \delta, u_{\max} - \delta]$. Moreover, by strict monotonicity, $|u'|$ is bounded from below by $\frac{1}{C(\delta)}$ when x is far from x_{\min} and x_{\max} in the sense that $u(x) \in [u_{\min} + \delta, u_{\max} - \delta]$. We consider for instance the values of x of the form $x = x_+(\eta)$ for some $\eta \in [u_{\min} + \delta, u_{\max} - \delta]$

Let x such that $u(x) \in [u_{\min} + \delta, u_{\max} - \delta]$, and $n \geq n_0$. We know that there exists $y_n := x_+^n(u_n)$ satisfying $u(x) = u_n(y_n)$. Since $|u'|$ is bounded from below, we have that

$$|u(x) - u(y_n)| \geq \frac{1}{C'(\delta)} |x - y_n|.$$

Moreover, $u(x) = u_n(y_n)$ so the left hand side is

$$|u(x) - u(y_n)| = |u_n(y_n) - u(y_n)| \leq \|u_n - u\|_{L^\infty}.$$

We deduce that

$$|x - y_n| \leq C(\delta) \|u_n - u\|_{L^\infty}$$

Let $\nu > 0$. Then for $n \geq n_1$ large enough, we have that for every x such that $u(x) \in [u_{\min} + \delta, u_{\max} - \delta]$, denoting $y_n \in \mathbb{T}$ the point at which $u(x) = u_n(y_n)$, there holds $|x - y_n| \leq \nu$. We deduce that for every $\eta \in [u_{\min} + \delta, u_{\max} - \delta]$,

$$|x_+(\eta) - x_+^n(\eta)| \leq \nu. \quad \square$$

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