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Schéma MPFA symétrique pour le problème de Stokes

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NAVIER-STOKES EQUATIONS [1]

Find $(\mathbf{u}(t), p(t)) \in \mathbf{H}_0^1(\Omega) \times L_0^2(\Omega)$ such that:

$$\begin{cases} \partial_t \mathbf{u} - \nu \Delta \mathbf{u} + (\mathbf{u} \cdot \nabla) \mathbf{u} + \nabla p = \mathbf{f}(t) \\ \operatorname{div} \mathbf{u} = 0 \\ \mathbf{u}(0) = \mathbf{u}_0 \end{cases}$$

Difficulties: unsteady, 3D, non linear, constrained.

- \mathbf{u} : velocity of the fluid
- \mathbf{f} : volumic external sources
- ν : kinematic viscosity
- p : pressure / fluid density

Stokes problem:

Find $(\mathbf{u}, p) \in \mathbf{H}_0^1(\Omega) \times L_0^2(\Omega)$ such that:

$$\begin{cases} -\nu \Delta \mathbf{u} + \nabla p = \mathbf{f} \\ \operatorname{div} \mathbf{u} = 0 \end{cases}$$

TIME STEPPING SCHEME (SEMI-IMPLICIT CASE) [2]

Find $(\underline{U}^{m+1}, \underline{P}^{m+1})$ such that:

$$\delta t^{-1} M(\underline{U}^{m+1} - \underline{U}^m) + K \underline{U}^{m+1} + L(\underline{U}^m) \underline{U}^{m+1} - B^T \underline{P}^{m+1} = \underline{F}^{m+1} \text{ and } B \underline{U}^{m+1} = 0.$$

1 **Prediction step:** Find \underline{U}^* such that

$$(M + K + L(\underline{U}^m)) \underline{U}^* = \underline{F}^{m+1} + \delta t^{-1} M \underline{U}^m + B^T \underline{P}^m \quad \text{GMRES}$$

2 **Pressure solver step:** Find δP such that

$$\begin{aligned} B M^{-1} B^T \delta P &= -\delta t^{-1} B \underline{U}^*, & \text{Cholesky or PCG} \\ \underline{P}^{m+1} &= \underline{P}^m + \delta P \end{aligned}$$

3 **Correction step:** Find \underline{U}^{m+1} such that

$$\delta t^{-1} M \underline{U}^{m+1} = \delta t^{-1} M \underline{U}^* + B^T \delta P \quad \text{direct computation}$$

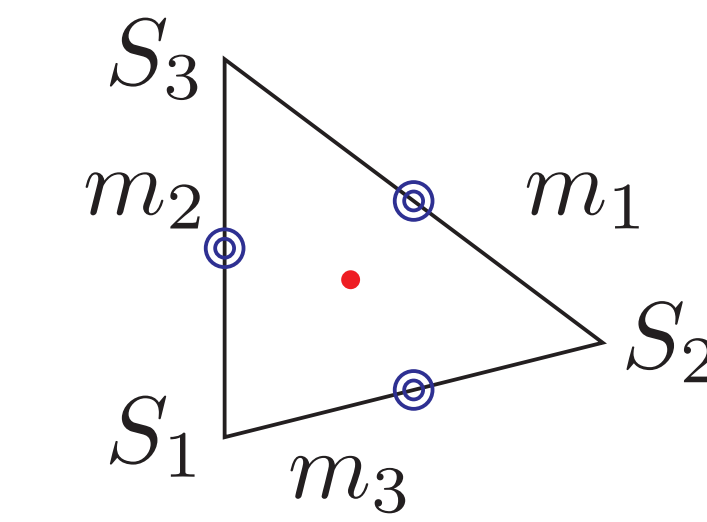
The mass matrix M is diagonal.

Relaxed Courant-Friedrichs-Levy condition vs explicit case.

SPURIOUS VELOCITIES [3]

Non-conforming discretization P_{nc}^1/P^0 :

- Velocity dof (d per facet) and **pressure dof**.
- Non conforming FEM: $\int_F [\mathbf{v}_h] = 0$.

Let $\mathbf{f} = \nabla \varphi$, $\varphi \in L_0^2(\Omega) \cap H^1(\Omega)$.Solution to Stokes pb: $(\mathbf{u}, p) = (0, \varphi)$.

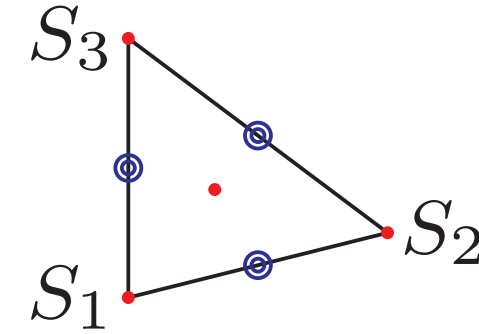
$$\int_{\Omega} \mathbf{f} \cdot \mathbf{v}_h = - \sum_{T \in \mathcal{T}_h} \int_T \varphi \operatorname{div} \mathbf{v}_h + \sum_{F \in \mathcal{F}_h} \int_F \varphi [\mathbf{v}_h] \cdot \mathbf{n}_F$$

Second term \rightarrow numerical source.

IMPROVEMENTS [3, 4, 5, 6]

- Increase the ndof of the pressure (TrioCFD):

$$P_{nc}^1 - P^0 P^1$$



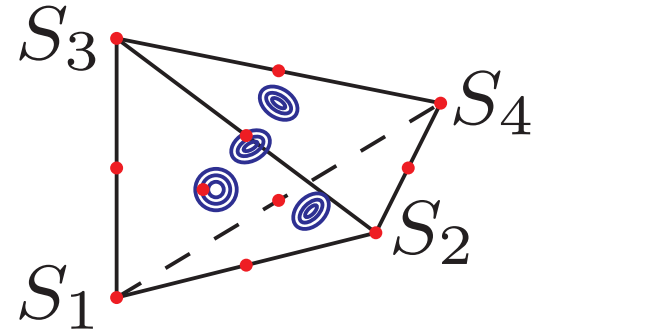
Triangle

If $\mathbf{f} = \nabla \varphi$, $(\mathbf{u}, p) = (0, \varphi)$, $\varphi \in P^k$:

$$\|\mathbf{u}_h\|_h \lesssim h^{2d-1}/\nu, \quad \|p - p_h\|_0 \lesssim h^2.$$

- $P_{nc}^1 - P^0 P^1 P^a$: too many dof.
- **H(div)-conforming projection**
- Modify F , M and $L(\underline{U}^m)$.

$$P_{nc}^1 - P^0 P^1 P^a$$



Tetrahedra

MULTIPLE POINTS FLUX APPROX. [7]

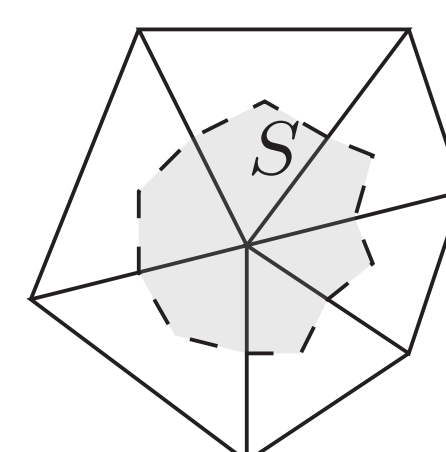
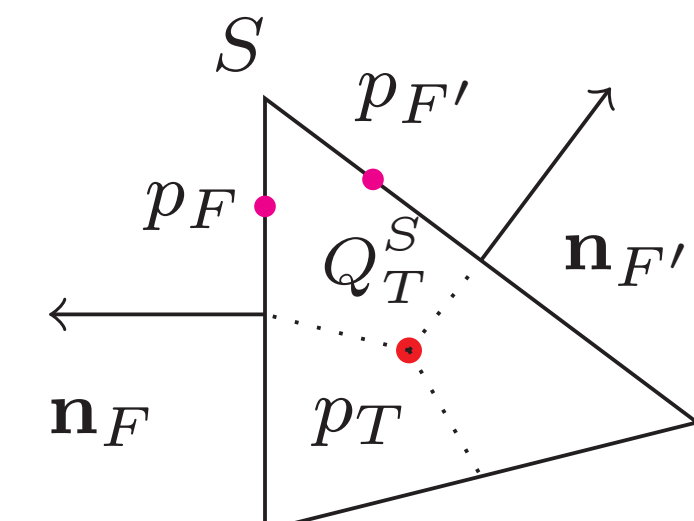
- Scheme for gradient: approached by G_h
- Add auxiliary unknowns $(p_F)_{F \in K_S}$ (2 per edges).
- Evaluate $G_h(p_T)$, $p_T = \mathbb{1}_T$. **Modify B.**

$$G_h(p_T)|_{Q_T^S} = \frac{3}{2|T'|} \sum_{F|S \in F} (p_F - p_T) \mathbf{n}_F^{T'}.$$

- Continuity conditions to compute $(p_F)_{F \in Q^S}$:

$$Q^S = \cup_T Q_T^S \text{ (in gray, right picture):}$$

$$[G_h(p_h)] \cdot \mathbf{n}_F = 0.$$



- Substitute the p_F in the system, with p_T and previous systems.

- **Boundary Condition:** To eliminate the $(p_F)_{F \in K_S^b}$ on the boundary, we use Neumann/Dirichlet condition.
- **Properties:** Applied to the laplacian problem :

- exact gradient for affine function
- converge with similar assumptions of P1 F.E.M.

- **Stokes:** Applied MPFA on the pressure gradient with Crouzeix-Raviart element for the velocity :

$$\int_{\Omega} \mathbf{v} \cdot \nabla q \simeq \sum_{T \in \mathcal{T}_h} \sum_{Q \in \mathcal{Q}_h(T)} \int_Q \mathbf{v}_h \cdot G_h(q_h)$$

MPFA DISCRETIZATION FOR THE GRADIENT OF PRESSURE ON THE NAVIER-STOKES PROBLEM

- System :

$$\begin{pmatrix} A & G_{int} & G_{bd} \\ D & 0 & 0 \end{pmatrix} \begin{pmatrix} U \\ P_{int} \\ P_{bd} \end{pmatrix} = \begin{pmatrix} RHS \\ 0 \end{pmatrix}$$

with:

$$(A_{\phi})_{FF'} := \sum_{K \in \mathcal{T}_h} \int_K \nabla \phi_F \cdot \nabla \phi_{F'},$$

$$(D^{\alpha})_{FT} := \sum_{K \in \mathcal{T}_h} \int_K \psi_T^{int} \operatorname{div}(\phi_F \mathbf{e}_{\alpha}),$$

$$(G_{int}^{\alpha})_{TF} := \sum_{K \in \mathcal{T}_h} \int_K \phi_F G_h(\psi_T^{int}) \cdot \mathbf{e}_{\alpha},$$

$$(G_{bd}^{\alpha})_{fF} := \sum_{K \in \mathcal{T}_h} \int_K \phi_F G_h(\psi_f^{bd}) \cdot \mathbf{e}_{\alpha}.$$

- $J_F = \{1, \dots, N_F\}$, $F, F' \in J_F$
- $J_T = \{1, \dots, N_T\}$, $T \in J_T$
- $J_{fb} = \{1, \dots, 2N_{Fb}\}$, $f \in J_{fb}$
- $(\mathbf{e}_{\alpha})^{\alpha \in \{x, y\}}$ Orthonormal basis.
- Velocity basis: $(\phi_F \mathbf{e}_{\alpha})_{F \in J_F}$
- Pressure basis $(\psi_T^{int})_{T \in J_T}$ and $(\psi_f^{bd})_{f \in J_b}$

- To determine P_{bd} : we reinforce the zero divergence condition by :

1. Using velocity boundary conditions: Add velocity unknowns on the boundary and use them to reformulate the mass conservation on each border triangle.

2. Constraining the velocity space on the boundary : Let T on the boundary,

$$F \subset \Omega^{\circ}, q_1 \in P^1(T), \int_F [\phi_F \mathbf{e}_{\alpha}] q_1 = 0$$

3. Using approximate Neumann conditions for pressure : $\int_f G_h(p_T) \cdot \mathbf{n}_{\partial \Omega} = \int_f \mathbf{f} \cdot \mathbf{n}_{\partial \Omega}$

Rq: Solutions 1 and 2 impose a mesh restriction. Two different triangles having a face on the boundary cannot share a face.

- Stokes, $\nu = 1$, $\mathbf{u} = 0$, $p = \frac{(x^2 + y^2)}{2} - \frac{1}{3}$.

	1/h	$P_{nc}^1 - P^0$	MPFA	TrioCFD
$\varepsilon_0(\mathbf{u}_h)$	40	3.3×10^{-5}	7.5×10^{-8}	4.3×10^{-10}
	80	8.4×10^{-6}	1.1×10^{-8}	5.1×10^{-11}
$\varepsilon_0(p_h)$	40	1.9×10^{-2}	4.8×10^{-5}	1.0×10^{-4}
	80	9.7×10^{-3}	1.2×10^{-5}	2.5×10^{-5}

- Stokes: 1/h = 20, \mathbf{u} sine function, p affine function

		$P_{nc}^1 - P^0$	MPFA	TrioCFD
Re 10	$\varepsilon_0(\mathbf{u}_h)$	2.8×10^{-2}	2.6×10^{-2}	2.5×10^{-2}
	$\varepsilon_0(p_h)$	1.3×10^{-1}	2.6×10^{-1}	1.4×10^{-1}
Re 100	$\varepsilon_0(\mathbf{u}_h)$	8.7×10^{-2}	2.6×10^{-2}	2.5×10^{-2}
	$\varepsilon_0(p_h)$	7.7×10^{-2}	2.6×10^{-2}	1.4×10^{-2}

*we denote: $\varepsilon_0(\mathbf{u}_h) := \|\mathbf{u} - \mathbf{u}_h\|_{L^2(\Omega)}$, $\varepsilon_0(p_h) := \|p - p_h\|_{L^2(\Omega)}$

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