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## Schéma MPFA symétrique pour le problème de Stokes

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## NAVIER-STOKES EQUATIONS [1]

Find  $(\mathbf{u}(t), p(t)) \in \mathbf{H}_0^1(\Omega) \times L_0^2(\Omega)$  such that:

$$\begin{cases} \partial_t \mathbf{u} - \nu \Delta \mathbf{u} + (\mathbf{u} \cdot \nabla) \mathbf{u} + \nabla p = \mathbf{f}(t) \\ \operatorname{div} \mathbf{u} = 0 \\ \mathbf{u}(0) = \mathbf{u}_0 \end{cases}$$

Difficulties: unsteady, 3D, non linear, constrained.

- $\mathbf{u}$ : velocity of the fluid
- $\mathbf{f}$ : volumic external sources
- $\nu$ : kinematic viscosity
- $p$ : pressure / fluid density

Stokes problem:

Find  $(\mathbf{u}, p) \in \mathbf{H}_0^1(\Omega) \times L_0^2(\Omega)$  such that:

$$\begin{cases} -\nu \Delta \mathbf{u} + \nabla p = \mathbf{f} \\ \operatorname{div} \mathbf{u} = 0 \end{cases}$$

## TIME STEPPING SCHEME (SEMI-IMPLICIT CASE) [2]

Find  $(\underline{U}^{m+1}, \underline{P}^{m+1})$  such that:

$$\delta t^{-1} M(\underline{U}^{m+1} - \underline{U}^m) + K \underline{U}^{m+1} + L(\underline{U}^m) \underline{U}^{m+1} - B^T \underline{P}^{m+1} = \underline{F}^{m+1} \text{ and } B \underline{U}^{m+1} = 0.$$

1 **Prediction step:** Find  $\underline{U}^*$  such that

$$(M + K + L(\underline{U}^m)) \underline{U}^* = \underline{F}^{m+1} + \delta t^{-1} M \underline{U}^m + B^T \underline{P}^m \quad \text{GMRES}$$

2 **Pressure solver step:** Find  $\delta P$  such that

$$\begin{aligned} B M^{-1} B^T \delta P &= -\delta t^{-1} B \underline{U}^*, \quad \text{Cholesky or PCG} \\ \underline{P}^{m+1} &= \underline{P}^m + \delta P \end{aligned}$$

3 **Correction step:** Find  $\underline{U}^{m+1}$  such that

$$\delta t^{-1} M \underline{U}^{m+1} = \delta t^{-1} M \underline{U}^* + B^T \delta P \quad \text{direct computation}$$

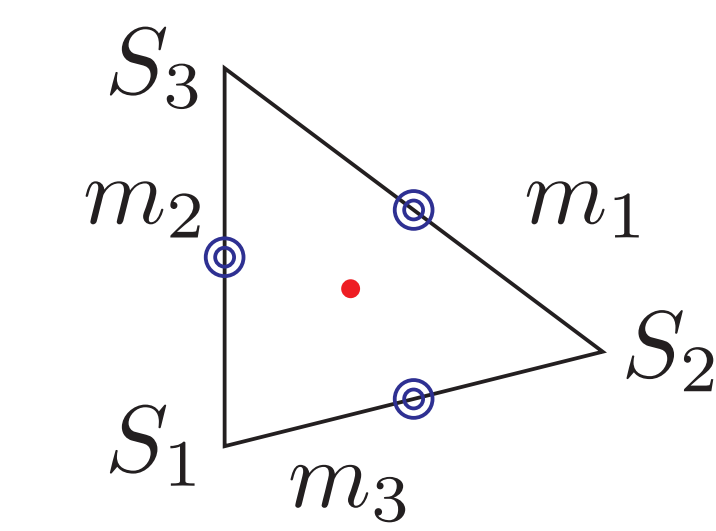
The mass matrix  $M$  is diagonal.

Relaxed Courant-Friedrichs-Levy condition vs explicit case.

## SPURIOUS VELOCITIES [3]

Non-conforming discretization  $P_{nc}^1/P^0$ :

- Velocity dof ( $d$  per facet) and **pressure dof**.
- Non conforming FEM:  $\int_F [\mathbf{v}_h] = 0$ .

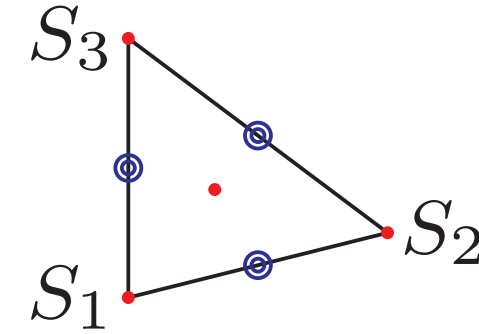
Let  $\mathbf{f} = \nabla \varphi$ ,  $\varphi \in L_0^2(\Omega) \cap H^1(\Omega)$ .Solution to Stokes pb:  $(\mathbf{u}, p) = (0, \varphi)$ .

$$\int_{\Omega} \mathbf{f} \cdot \mathbf{v}_h = - \sum_{T \in \mathcal{T}_h} \int_T \varphi \operatorname{div} \mathbf{v}_h + \sum_{F \in \mathcal{F}_h} \int_F \varphi [\mathbf{v}_h] \cdot \mathbf{n}_F$$

Second term  $\rightarrow$  numerical source.

## IMPROVEMENTS [3, 4, 5, 6]

- Increase the ndof of the pressure (TrioCFD):

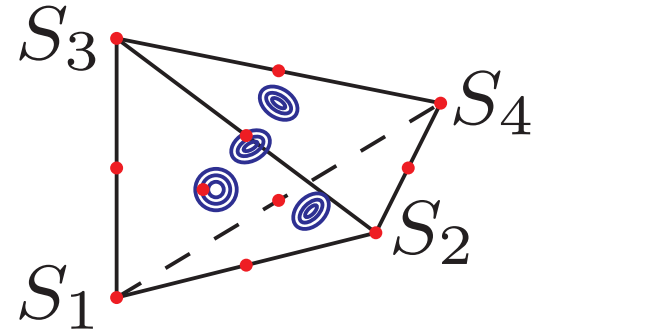
 $P_{nc}^1 - P^0 P^1$ 

Triangle

If  $\mathbf{f} = \nabla \varphi$ ,  $(\mathbf{u}, p) = (0, \varphi)$ ,  $\varphi \in P^k$ :

$$\|\mathbf{u}_h\|_h \lesssim h^{2d-1}/\nu, \quad \|p - p_h\|_0 \lesssim h^2.$$

- $P_{nc}^1 - P^0 P^1 P^a$ : too many dof.
- **H(div)-conforming projection**
- Modify  $F$ ,  $M$  and  $L(\underline{U}^m)$ .

 $P_{nc}^1 - P^0 P^1 P^a$ 

Tetrahedra

## MULTIPLE POINTS FLUX APPROX. [7]

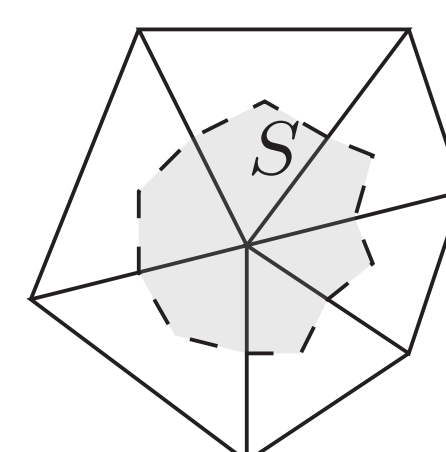
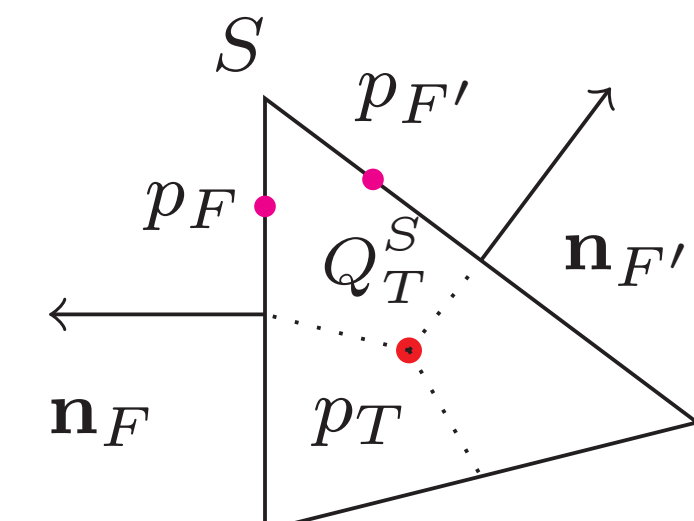
- Scheme for gradient: approached by  $G_h$
- Add auxiliary unknowns  $(p_F)_{F \in K_S}$  (2 per edges).
- Evaluate  $G_h(p_T)$ ,  $p_T = \mathbb{1}_T$ . **Modify B.**

$$G_h(p_T)|_{Q_T^S} = \frac{3}{2|T'|} \sum_{F|S \in F} (p_F - p_T) \mathbf{n}_F^{T'}.$$

- Continuity conditions to compute  $(p_F)_{F \in Q^S}$ :

 $Q^S = \cup_T Q_T^S$  (in gray, right picture):

$$[G_h(p_h)] \cdot \mathbf{n}_F = 0.$$



- Substitute the  $p_F$  in the system, with  $p_T$  and previous systems.
- **Boundary Condition:** To eliminate the  $(p_F)_{F \in K_S^b}$  on the boundary, we use Neumann/Dirichlet condition.
- **Properties:** Applied to the laplacian problem :
  - exact gradient for affine function
  - converge with similar assumptions of P1 F.E.M.
- **Stokes:** Applied MPFA on the pressure gradient with Crouzeix-Raviart element for the velocity :

$$\int_{\Omega} \mathbf{v} \cdot \nabla q \simeq \sum_{T \in \mathcal{T}_h} \sum_{Q \in \mathcal{Q}_h(T)} \int_Q \mathbf{v}_h \cdot G_h(q_h)$$

## MPFA DISCRETIZATION FOR THE GRADIENT OF PRESSURE ON THE NAVIER-STOKES PROBLEM

- System :

$$\begin{pmatrix} A & G_{int} & G_{bd} \\ D & 0 & 0 \end{pmatrix} \begin{pmatrix} U \\ P_{int} \\ P_{bd} \end{pmatrix} = \begin{pmatrix} RHS \\ 0 \end{pmatrix}$$

with:

$$(A_{\phi})_{FF'} := \sum_{K \in \mathcal{T}_h} \int_K \nabla \phi_F \cdot \nabla \phi_{F'},$$

$$(D^{\alpha})_{FT} := \sum_{K \in \mathcal{T}_h} \int_K \psi_T^{int} \operatorname{div}(\phi_F \mathbf{e}_{\alpha}),$$

$$(G_{int}^{\alpha})_{TF} := \sum_{K \in \mathcal{T}_h} \int_K \phi_F G_h(\psi_T^{int}) \cdot \mathbf{e}_{\alpha},$$

$$(G_{bd}^{\alpha})_{fF} := \sum_{K \in \mathcal{T}_h} \int_K \phi_F G_h(\psi_f^{bd}) \cdot \mathbf{e}_{\alpha}.$$

- $J_F = \{1, \dots, N_F\}$ ,  $F, F' \in J_F$
- $J_{fb} = \{1, \dots, 2N_{Fb}\}$ ,  $f \in J_{fb}$
- Velocity basis:  $(\phi_F \mathbf{e}_{\alpha})_{F \in J_F}$
- Pressure basis  $(\psi_T^{int})_{T \in J_T}$  and  $(\psi_f^{bd})_{f \in J_b}$

- To determine  $P_{bd}$ : we reinforce the zero divergence condition by :

1. Using velocity boundary conditions: Add velocity unknowns on the boundary and use them to reformulate the mass conservation on each border triangle.
2. Constraining the velocity space on the boundary : Let T on the boundary,

$$F \subset \Omega^{\circ}, q_1 \in P^1(T), \int_F [\phi_F \mathbf{e}_{\alpha}] q_1 = 0$$

3. Using approximate Neumann conditions for pressure :  $\int_f G_h(p_T) \cdot \mathbf{n}_{\partial\Omega} = \int_f \mathbf{f} \cdot \mathbf{n}_{\partial\Omega}$

**Rq:** Solutions 1 and 2 impose a mesh restriction. Two different triangles having a face on the boundary cannot share a face.

- Stokes,  $\nu = 1$ ,  $\mathbf{u} = 0$ ,  $p = \frac{(x^2 + y^2)}{2} - \frac{1}{3}$ .

	1/h	$P_{nc}^1 - P^0$	MPFA	TrioCFD
$\varepsilon_0(\mathbf{u}_h)$	40	$3.3 \times 10^{-5}$	$7.5 \times 10^{-8}$	$4.3 \times 10^{-10}$
	80	$8.4 \times 10^{-6}$	$1.1 \times 10^{-8}$	$5.1 \times 10^{-11}$
$\varepsilon_0(p_h)$	40	$1.9 \times 10^{-2}$	$4.8 \times 10^{-5}$	$1.0 \times 10^{-4}$
	80	$9.7 \times 10^{-3}$	$1.2 \times 10^{-5}$	$2.5 \times 10^{-5}$

- Stokes: 1/h = 20,  $\mathbf{u}$  sine function,  $p$  affine function

		$P_{nc}^1 - P^0$	MPFA	TrioCFD
Re 10	$\varepsilon_0(\mathbf{u}_h)$	$2.8 \times 10^{-2}$	$2.6 \times 10^{-2}$	$2.5 \times 10^{-2}$
	$\varepsilon_0(p_h)$	$1.3 \times 10^{-1}$	$2.6 \times 10^{-1}$	$1.4 \times 10^{-1}$
Re 100	$\varepsilon_0(\mathbf{u}_h)$	$8.7 \times 10^{-2}$	$2.6 \times 10^{-2}$	$2.5 \times 10^{-2}$
	$\varepsilon_0(p_h)$	$7.7 \times 10^{-2}$	$2.6 \times 10^{-2}$	$1.4 \times 10^{-2}$

\*we denote:  $\varepsilon_0(\mathbf{u}_h) := \|\mathbf{u} - \mathbf{u}_h\|_{L^2(\Omega)}$ ,  $\varepsilon_0(p_h) := \|p - p_h\|_{L^2(\Omega)}$ 

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