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THE COMPRESSIBLE EULER SYSTEM WITH NONLOCAL PRESSURE: GLOBAL EXISTENCE AND RELAXATION

RAPHAEL DANCHIN & PIOTR BOGUSŁAW MUCHA

ABSTRACT. We here investigate a modification of the compressible barotropic Euler system with friction, involving a fuzzy nonlocal pressure term in place of the conventional one. This nonlocal term is parameterized by $\varepsilon > 0$ and formally tends to the classical pressure when ε approaches zero. The central challenge is to establish that this system is a reliable approximation of the classical compressible Euler system. We establish the global existence and uniqueness of regular solutions in the neighborhood of the static state with density 1 and null velocity. Our results are demonstrated independently of the parameter ε , which enable us to prove the convergence of solutions to those of the classical Euler system. Another consequence is the rigorous justification of the convergence of the mass equation to various versions of the porous media equation in the asymptotic limit where the friction tends to infinity. Note that our results are demonstrated in the whole space, which necessitates to use the $L^1(\mathbb{R}_+; \dot{B}_{2,1}^\sigma(\mathbb{R}^d))$ spaces framework.

MSC: 35Q35, 76N10

KEY WORDS: damped Euler system, nonlocal pressure, aggregation, almost hyperbolic systems, Besov spaces, time integrability, regular solutions, relaxation, porous media equation.

1. INTRODUCTION

The phenomena of collective behavior are at the crossroads of various scientific disciplines and are currently the subject of active research. They find their roots in diverse fields such as sociology, biology, and classical physics [8, 17, 18]. At the microscale level, these phenomena are often described by simple Ordinary Differential Equations, as in e.g. the N -body problem. However, when the number of agents or particles becomes prohibitively large, such naive descriptions prove to be ineffective. Consequently, at the macroscale, it becomes suitable to adopt a hydrodynamical approach to model and understand these complex systems [3, 11, 16].

This paper delves into the analysis of a modified version of the classical compressible Euler system, incorporating a nonlocal force designed to induce mass alignment among the constituent elements. This modification consists in replacing the classical pressure term by a non-local fuzzy approximation, which is designed to model the communication of each particle/agent with other particles located in a non-trivial neighborhood.

More precisely, we are concerned with the following class of systems in the whole space \mathbb{R}^d :

$$(1.1) \quad \begin{cases} \rho_t + \operatorname{div}(\rho u) = 0 \\ \rho u_t + \rho u \cdot \nabla u + \mathfrak{f} \rho u = -\rho \nabla K_\varepsilon * \rho. \end{cases}$$

Above, $\rho = \rho(t, x) \in \mathbb{R}_+$ and $u = u(t, x) \in \mathbb{R}^d$ denote the density and velocity functions of the studied “matter”, respectively. The positive real number \mathfrak{f} is the friction coefficient and the family of smooth potentials $(K_\varepsilon)_{\varepsilon>0}$ is assumed to tend to the Dirac measure at 0, when ε goes to 0. The convolution in the right-hand side of (1.1) is taken with respect to space

variables. Hence formally, in the limit, we obtain the following compressible Euler system with friction:

$$(1.2) \quad \begin{cases} \rho_t + \operatorname{div}(\rho u) = 0, \\ \rho u_t + \rho u \cdot \nabla u + \mathfrak{f}\rho u + \frac{1}{2}\nabla \rho^2 = 0. \end{cases}$$

Our primary goal is to establish the global well-posedness of System (1.1) supplemented with initial data (ρ_0, u_0) which are perturbations of the constant solution $(\rho, u) = (1, 0)$. Because the nonlocal term $\nabla K_\varepsilon * \rho$ is rather smooth, proving local well-posedness results in the case of sufficiently smooth data bounded away from zero presents no particular difficulty. Indeed, the velocity satisfies a damped Burgers equation with a smooth source term, that can be considered independently of the density equation. In this way however, it is difficult to prove the global existence since, typically, the source term $\nabla K_\varepsilon * \rho$, albeit smooth, causes a linear growth of L^1 -in-time norms of ∇u . Back to the transport equation, it is thus impossible to get uniform bounds in time for the density, and thus to close the estimates for all positive time. Likewise, getting a control independent of ε in this way is hopeless.

The main difficulty is that our system does not enter in the classical theory of hyperbolic equations. Even for fixed values of parameters \mathfrak{f} and ε , a nonstandard approach is necessitated. The main points of our analysis (that is also valid for more general pressure functions than $P(\rho) = \rho^2/2$) are the following:

- A fundamental challenge arises from the essential requirement of L^1 time integrability for ∇u , that is,

$$(1.3) \quad \int_0^\infty \|\nabla u\|_{L^\infty} dt < \infty.$$

This is the key to controlling for all time the transport terms of (1.1), namely $u \cdot \nabla \rho$ and $u \cdot \nabla u$. Given the hyperbolic nature of the system, (1.3) can only be achieved thanks to the dissipative term $\mathfrak{f}\rho u$. In the whole space context, there exists no inherent mechanism to induce rapid temporal decay (we shall showcase below that there is no ‘spectral’ gap for the linearized system). A way to overcome the difficulty is to use the framework of homogeneous Besov spaces of type $\dot{B}_{2,1}^s(\mathbb{R}^d)$. Here, the factor ‘1’ will enable us to attain the L^1 integrability over time, while the factor ‘2’ reflects the fact that our framework is related to the L^2 space, in keeping with the quasilinear hyperbolic nature of the system.

- In order to achieve global results with some uniformity with respect to ε , the mathematical analysis is subtle. In fact, instead of helping, the smoothing kernel K_ε destroys the nice partially dissipative symmetrizable structure of (1.2). The so-called Shizuta-Kawashima condition (first pointed out in [12]) is not satisfied, and the more modern approach of Beauchard-Zuazua [2] (revisited in [6, 9]) cannot be used as is. Compared to (1.2), the difficulty is that the operator ∇K_ε provides less dissipation than the full gradient, as may be already observed on the following linearization of (1.1):

$$(1.4) \quad \begin{cases} a_t + \operatorname{div} u = 0, \\ u_t + u + \nabla K_\varepsilon a = 0. \end{cases}$$

In Fourier variables, the matrix of the system reads reads

$$\begin{pmatrix} 0 & i\xi \\ i\xi^T \widehat{K}_\varepsilon(\xi) & 1 \end{pmatrix}, \quad \xi \in \mathbb{R}^d.$$

The eigenvalues are 1 with multiplicity $d - 1$ (incompressible part of u) and:

$$\lambda^\pm(\xi) = \frac{1}{2} \left(1 \pm \sqrt{1 - 4|\xi|^2 \widehat{K}_\varepsilon(\xi)} \right) \quad \text{if } 4|\xi|^2 \widehat{K}_\varepsilon(\xi) \leq 1;$$

$$\lambda^\pm(\xi) = \frac{1}{2} \left(1 \pm i \sqrt{4|\xi|^2 \widehat{K}_\varepsilon(\xi) - 1} \right) \quad \text{if } 4|\xi|^2 \widehat{K}_\varepsilon(\xi) \geq 1.$$

The Euler situation corresponds to $\varepsilon = 0$, that is $\widehat{K}_0 \equiv 1$. We then have two distinct regimes: low frequencies with one parabolic mode and d damped modes, and high frequencies with only damped modes.

If $\varepsilon > 0$, then the regime where $4|\xi|^2 \widehat{K}_\varepsilon(\xi) < 1$ is likely to include arbitrarily high frequencies, since the functions \widehat{K}_ε that we will consider here have algebraic decay at ∞ . Furthermore, for small values of $|\xi|^2 \widehat{K}_\varepsilon(\xi)$ we have $\lambda^-(\xi) \simeq |\xi|^2 \widehat{K}_\varepsilon(\xi)$ that is, a *degenerate* parabolic mode. A key observation is that in this regime the combination $w := u + \nabla K_\varepsilon a$ (often referred to in this article as the ‘damped mode’) tends to undergo an exponential dissipation.

- An essential requirement in our study is the establishment of uniform dependence on the parameter ε . This is clearly needed for justifying rigorously the convergence to the Euler system (1.2) in the asymptotics $\varepsilon \rightarrow 0$.

Leveraging energy-based techniques, we succeed in controlling the essential quantities required for our analysis, uniformly as $\varepsilon \rightarrow 0$. This enables us to precisely determine the diffusive limit of our system. It is worth noting that our approach and functional framework for solving (1.1) is inspired by the recent paper [6]. However, the loss of symmetry caused by the kernel K_ε will entail a number of difficulties that will be described in detail in the next section. For older global existence results concerning System (1.2) and the relaxation limit, the reader may consult [5, 13, 19, 20].

- To recover the optimal information coming from the basic spectral analysis that we performed above for (1.4), it is convenient to localize the system by means of a dyadic decomposition in the Fourier space (the so-called Littlewood-Paley decomposition) then to implement the method that was used in [6, 7] for (1.2). There is one more difficulty: in the process, in order to compensate the loss of symmetry with respect to (a, u) , one has somehow to look at $K_\varepsilon a$ as an ‘independent’ unknown. This leads us to consider commutators of nonlinear terms with K_ε . A central objective lies in the meticulous control of these commutators, *uniformly with respect to ε* . In this endeavor, we have to extend the techniques delineated in [1, Chap. 2] to accommodate more intricate scenarios, wherein paraproduct operator and expansion techniques become indispensable for addressing higher-order terms. Here, the key is to use a Taylor expansion at order two; which necessitates a control of $\|\nabla^2 u\|_{L^\infty}$. This leads us to use a dual level of regularity while, for the classical compressible Euler system, it is enough to control $\|\nabla u\|_{L^\infty}$, and thus to use only one level of regularity.

- The last part of our study concerns the relaxation limit $\mathfrak{f} \rightarrow \infty$. A distinctive feature of our functional setting is that it allows to deduce the general case $\mathfrak{f} > 0$ from the particular case $\mathfrak{f} = 1$ by mere rescaling, provided parameter ε has been suitably modified. Then, the key to proving the strong convergence is to look at w defined above as the beneficial and dissipative component of our system¹.

Depending on the type of asymptotics we are looking at, we will justify rigorously the transition to porous media type equations, namely:

$$\partial_t r - \operatorname{div}(r \nabla K_\varepsilon r) = 0 \quad \text{or} \quad \partial_t n - \operatorname{div}(n \nabla n) = 0.$$

It is noteworthy that when $\varepsilon > 0$, the resulting equation corresponds to some degenerate porous media equation, with no parabolic smoothing-out effect.

¹Here we can draw an analogy with our use in [10] of *the effective viscous flux* for viscous compressible flows, so as to justify the convergence to the inhomogeneous incompressible Navier-Stokes equations.

2. DERIVATION FROM THE PARTICLE SYSTEM

In order to have a better understanding of the model presented in the introduction, let us delve into the interactions occurring among particles at the microscopic level. We therefore look at second-order agent models in their general formulation: consider a set of N identical particles, each of which is identified by the index k , ranging from 1 to N . At any time t , particle k occupies the position $x_k(t)$ and moves with an instantaneous velocity $v_k(t)$.

In our analysis, we make the underlying assumption that communication between these particles solely depends on aggregation-repulsion effects, contingent upon the positions of the agents. Furthermore, we incorporate frictional effects into the model to govern and ensure the system's stability. Consequently, denoting by \mathfrak{f} the (nonnegative) friction coefficient, the temporal evolution of both position $\{x_k\}$ and velocity $\{v_k\}$ for each particle, where k spans the values from 1 to N , is governed by the following system of equations:

$$(2.1) \quad \begin{cases} \dot{x}_k = v_k \\ \dot{v}_k = -\mathfrak{f}v_k - \frac{1}{N} \sum_{l \in \{1, \dots, N\}} \nabla K_\varepsilon(x_k - x_l). \end{cases}$$

Changing the scale from micro to macro setting, jumping over the kinetic formulation, leads to System (1.1) (see details in Appendix). Then, assuming that $K_\varepsilon \rightarrow \delta$ as $\varepsilon \rightarrow 0$, we formally obtain $\rho \nabla K_\varepsilon * \rho \rightarrow \frac{1}{2} \nabla \rho^2$, and thus the Euler system (1.2).

A simple example of a family of potentials $(K_\varepsilon)_{\varepsilon > 0}$ can be built from the characteristic function of the ball, namely we set (for a suitable normalization constant c_d):

$$K_\varepsilon(x) := c_d \varepsilon^{-d} (1 - \varepsilon^{-1}|x|) \chi_{B(0, \varepsilon)}(x) \quad \text{so that} \quad \nabla K_\varepsilon = c_d \varepsilon^{-d-1} \chi_{B(0, \varepsilon)} \frac{x}{|x|}.$$

To better understand the effects modelled by this potential, let us concentrate on the monodimensional case. Then

$$K'_\varepsilon * \rho(x) = \varepsilon^{-2} \int_{|z| \leq \varepsilon} \frac{z}{|z|} \rho(x - z) dz = \varepsilon^{-2} \int_{|z| \leq \varepsilon} \text{sgn}(z) (\rho(x - z) - \rho(x)) dz.$$

We observe that the force term arising from the integral on the right-hand side of the equation stems from the necessity of maintaining mass balance over the intervals $(-\varepsilon, 0)$ and $(0, \varepsilon)$. For multidimensional systems, while the weightings may become somewhat more intricate, the underlying mechanism remains fundamentally unchanged. To gain a visual insight into the impact of this nonlocal term, the reader may pay attention to Fig. 2.1 below.

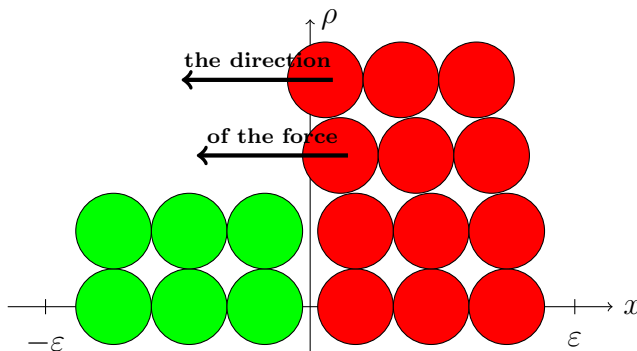


FIGURE 2.1. The mass contributed by the green balls on the segment $(-\varepsilon, 0)$ exerts a comparatively lesser influence compared to that of the red balls located in the segment $(0, \varepsilon)$. Consequently, the resultant force is oriented on the left.

The salient points of this analysis are valid in the specific case where the pressure is of the form $P(\rho) \sim \rho^2$. To achieve more general barotropic constitutive relations, one can introduce the density-induced communication protocol of [15, 14]. In that case we assume the communication between k -th and l -th agent to be of the form $\mathcal{N}(K_\varepsilon * \rho) \nabla K_\varepsilon(x_k - x_l)$, for some given function \mathcal{N} . At the level of the particle system, $\mathcal{N}(K_\varepsilon * \rho)$ measures the mass/number of particles in some vicinity of the agent x_k . At the hydrodynamical level, the convolution $K_\varepsilon * \rho$ describes an average value of the density function ρ in the vicinity of the examined point. Accordingly, the inclusion of the \mathcal{N} factor serves to augment or diminish the influence of communication relative to the average density in the given region. In this way, the effects showed at Fig. 2.1 are rescaled in terms of the magnitude of the mass in the considered neighborhood and our model System (1.1) has to be replaced with the following more general one:

$$(2.2) \quad \begin{cases} \rho_t + \operatorname{div}(\rho u) = 0 \\ u_t + u \cdot \nabla u + \mathfrak{f}u + \mathcal{N}(K_\varepsilon * \rho) \nabla K_\varepsilon * \rho = 0. \end{cases}$$

For ε tending to 0, we formally have

$$(2.3) \quad \rho \mathcal{N}(K_\varepsilon * \rho) \nabla K_\varepsilon * \rho \rightarrow \rho \mathcal{N}(\rho) \nabla \rho = \nabla P(\rho) \quad \text{with} \quad P'(\rho) = \rho \mathcal{N}(\rho).$$

Hence, we get the classical barotropic Euler system

$$(2.4) \quad \begin{cases} \rho_t + \operatorname{div}(\rho u) = 0, \\ u_t + u \cdot \nabla u + \mathfrak{f}u + \mathcal{N}(\rho) \nabla \rho = 0. \end{cases}$$

Note that the classical pressure law $P(\rho) = \rho^\gamma$ ($\gamma \geq 1$) (and thus the isentropic Euler system with friction) may be achieved if taking $\mathcal{N}(\rho) = \rho^{\gamma-2}$, up to a multiplicative constant.

3. RESULTS

Before presenting our main results, some definitions, assumptions and notation are in order. Let us first specify our assumptions on the family $(K_\varepsilon)_{\varepsilon>0}$. Since our approach bases essentially on the Fourier transform, the convergence of K_ε to the Dirac measure can be equivalently seen as $\widehat{K}_\varepsilon \rightarrow 1$ locally on \mathbb{R}^d . Our analysis requires \widehat{K}_ε to keep the same order of magnitude inside any annulus $\{2^{j-1} \leq |\xi| \leq 2^{j+1}\} \subset \mathbb{R}^d$ with $j \in \mathbb{Z}$. In fact, we shall assume throughout that

$$K_\varepsilon = L_\varepsilon * L_\varepsilon$$

with L_ε a real valued function such that \widehat{L}_ε is nonincreasing with range in $[0, 1]$, satisfies $\widehat{L}_\varepsilon(0) = 1$ and, for some $\kappa > 0$,

$$(3.1) \quad \begin{aligned} & \sup_{\varepsilon>0} (\|L_\varepsilon\|_{L^1} + \|z \nabla L_\varepsilon\|_{L^1} + \|(z \otimes z) \nabla^2 L_\varepsilon\|_{L^1}) < \infty, \\ & \kappa \widehat{L}_\varepsilon(\xi) \leq \widehat{L}_\varepsilon(2\xi) \leq \kappa^{-1} \widehat{L}_\varepsilon(\xi) \quad \text{and} \quad \xi_k \partial_\ell \widehat{L}_\varepsilon(\xi) \lesssim \widehat{L}_\varepsilon(\xi), \quad 1 \leq k, \ell \leq d, \quad \xi \in \mathbb{R}^d, \quad \varepsilon > 0. \end{aligned}$$

The above condition rules out sharp spectral cut-off or Gaussian functions. A simple example of a family $(K_\varepsilon)_{\varepsilon>0}$ satisfying (3.1) is to take $\widehat{K}_\varepsilon(\xi) = \widehat{K}(\varepsilon\xi)$ with

$$(3.2) \quad \widehat{K}(\xi) = (\widehat{L}(\xi))^2 \quad \text{and} \quad \widehat{L}(\xi) = \frac{1}{(1 + |\xi|^2)^{m/2}}.$$

For $m > d$, one can show from the standard properties of Fourier transform that (3.1) is indeed satisfied.

Let us next introduce the Littlewood-Paley decomposition on which an entire analysis is based. Fix a smooth function $\phi : \mathbb{R}_+ \rightarrow [0, 1]$ supported in $\{1/2 \leq r \leq 2\}$ such that

$$\sum_{k \in \mathbb{Z}} \phi(2^{-k}r) = 1 \quad \text{for all } r > 0.$$

Setting $\varphi(\xi) := \phi(|\xi|)$ for all $\xi \in \mathbb{R}^d$, one can define a homogeneous Littlewood-Paley decomposition $\{\dot{\Delta}_k\}_{k \in \mathbb{Z}}$ over the space \mathbb{R}^d in the following way:

$$\dot{\Delta}_k u := \varphi(2^{-k}D)u = \mathcal{F}^{-1}(\varphi 2^{-k \cdot})\mathcal{F}u \quad \text{with } iD := (\partial_{x_1}, \dots, \partial_{x_d}) \quad \text{for } u \in \mathcal{S}'(\mathbb{R}^d).$$

Homogeneous Besov ‘norms’ are defined as follows for all $s \in \mathbb{R}$ and $1 \leq p, q \leq \infty$:

$$\|u\|_{\dot{B}_{p,q}^s(\mathbb{R}^d)} := \left\| 2^{sk} \|\dot{\Delta}_k u\|_{L^p(\mathbb{R}^d)} \right\|_{\ell^q(\mathbb{Z})}.$$

Actually, as $\|P\|_{\dot{B}_{p,q}^s} = 0$ for any polynomial on \mathbb{R}^d , in the general tempered distribution setting $\|\cdot\|_{\dot{B}_{p,q}^s}$ is just a semi-norm. To get around the problem, we proceed as in [1], adopting the following definition:

$$\dot{B}_{p,q}^s(\mathbb{R}^d) := \left\{ u \in \mathcal{S}'_h(\mathbb{R}^d) : \|u\|_{\dot{B}_{p,q}^s} < \infty \right\},$$

where $\mathcal{S}'_h(\mathbb{R}^d)$ is the set of tempered distributions such that for all $\theta \in C_c^\infty(\mathbb{R}^d)$ we have

$$(3.3) \quad \lim_{\lambda \rightarrow \infty} \theta(\lambda D)u = 0 \quad \text{in } L^\infty(\mathbb{R}^d).$$

Next, in accordance with our preceding spectral analysis of the linear system (1.4), we introduce the following notation where the value of the small positive absolute constant ν_0 will be specified later in the paper:

$$(3.4) \quad \|z\|_{\dot{B}_{2,1}^\sigma}^\ell := \sum_{2^j \widehat{L}_\varepsilon(2^j) < \nu_0} 2^{j\sigma} \|\dot{\Delta}_j z\|_{L^2} \quad \text{and} \quad \|z\|_{\dot{B}_{2,1}^\sigma}^h := \sum_{2^j \widehat{L}_\varepsilon(2^j) \geq \nu_0} 2^{j\sigma} \|\dot{\Delta}_j z\|_{L^2},$$

$$(3.5) \quad z^\ell := \sum_{2^j \widehat{L}_\varepsilon(2^j) < \nu_0} \dot{\Delta}_j z \quad \text{and} \quad z^h := \sum_{2^j \widehat{L}_\varepsilon(2^j) \geq \nu_0} \dot{\Delta}_j z.$$

Note that this decomposition of frequencies does not quite correspond to what will be sometimes called, improperly, in the paper *low and high frequencies*. As said before, the fact that $2^j \widehat{L}_\varepsilon(2^j) < \nu_0$ does not exclude large values of 2^j .

Let us finally introduce the functional spaces that will be used for solving (1.1): for all $\sigma \in \mathbb{R}$ and kernel $K_\varepsilon = L_\varepsilon \star L_\varepsilon$ satisfying (3.1), the space $E_{K_\varepsilon}^\sigma$ stands for the set of all pairs $(a, u) \in \mathcal{C}_b(\mathbb{R}_+; \dot{B}_{2,1}^{\sigma-1} \cap \dot{B}_{2,1}^\sigma)$ such that, in addition:

$$(3.6) \quad (\nabla u, \nabla L_\varepsilon \star a) \in \mathcal{C}_b(\mathbb{R}_+; B_{2,1}^\sigma), \quad (u, \nabla u) \in L^1(\mathbb{R}_+; \dot{B}_{2,1}^\sigma)$$

$$\text{and} \quad \int_0^t (\|(K_\varepsilon a, \nabla K_\varepsilon \star a)\|_{\dot{B}_{2,1}^{\sigma+1}}^\ell + \|\nabla L_\varepsilon \star a\|_{\dot{B}_{2,1}^\sigma}^h) d\tau < \infty.$$

The version of $E_{K_\varepsilon}^\sigma$ corresponding to the case where $K_\varepsilon \star$ is the identity operator will be considered for solving the Euler system (1.2). We shall denote it by just E^σ .

We are now ready to state our main global existence result for System (1.1):

Theorem 3.1. *Assume that $d \geq 2$ and consider initial data $\rho_0 = 1 + a_0$ and u_0 such that*

$$u_0 \in \dot{B}_{2,1}^{\frac{d}{2}} \cap \dot{B}_{2,1}^{\frac{d}{2}+2}, \quad a_0 \in \dot{B}_{2,1}^{\frac{d}{2}} \cap \dot{B}_{2,1}^{\frac{d}{2}+1} \quad \text{and} \quad \nabla^2 L_\varepsilon a_0 \in \dot{B}_{2,1}^{\frac{d}{2}}.$$

There exists an absolute positive constant α_0 such that if

$$(3.7) \quad \|u_0\|_{\dot{B}_{2,1}^{\frac{d}{2}+1} \cap \dot{B}_{2,1}^{\frac{d}{2}+2}} + \|a_0\|_{\dot{B}_{2,1}^{\frac{d}{2}} \cap \dot{B}_{2,1}^{\frac{d}{2}+1}} + \|\nabla^2 L_\varepsilon a_0\|_{\dot{B}_{2,1}^{\frac{d}{2}}} \leq \alpha_0,$$

then System (1.1) with $\mathfrak{f} = 1$ supplemented with initial data (ρ_0, u_0) admits a unique global classical solution (ρ, u) such that (a, u) with $a := \rho - 1$ belongs to the space $E_{K_\varepsilon}^{\frac{d}{2}+1}$ defined in (3.6). Furthermore, there exists a constant C independent of ε such that for all $t \in \mathbb{R}_+$,

$$(3.8) \quad \|(a, \nabla a, \nabla^2 L_\varepsilon a)(t)\|_{\dot{B}_{2,1}^{\frac{d}{2}}} + \|(u, \nabla u)(t)\|_{\dot{B}_{2,1}^{\frac{d}{2}+1}} + \int_0^t (\|(u, \nabla u)\|_{\dot{B}_{2,1}^{\frac{d}{2}+1}} + \|(K_\varepsilon a, \nabla K_\varepsilon a)\|_{\dot{B}_{2,1}^{\frac{d}{2}+2}}^\ell + \|\nabla L_\varepsilon a\|_{\dot{B}_{2,1}^{\frac{d}{2}+1}}^h) d\tau \leq C (\|(a_0, \nabla a_0, \nabla^2 L_\varepsilon a_0)\|_{\dot{B}_{2,1}^{\frac{d}{2}}} + \|(u_0, \nabla u_0)\|_{\dot{B}_{2,1}^{\frac{d}{2}+1}}).$$

In addition, setting $w = u + \nabla K_\varepsilon a$, we have

$$(3.9) \quad \|(u, w)(t)\|_{\dot{B}_{2,1}^{\frac{d}{2}}} + \int_0^t \|w\|_{\dot{B}_{2,1}^{\frac{d}{2}}} d\tau + \left(\int_0^t \|u\|_{\dot{B}_{2,1}^{\frac{d}{2}}}^2 d\tau \right)^{1/2} \leq C (\|(a_0, \nabla a_0, \nabla^2 L_\varepsilon a_0)\|_{\dot{B}_{2,1}^{\frac{d}{2}}} + \|(u_0, \nabla u_0, \nabla^2 u_0)\|_{\dot{B}_{2,1}^{\frac{d}{2}}}).$$

Several important remarks are in order:

- The above statement is valid for any $\varepsilon > 0$, with constants α_0 and C independent of ε . We stated only the case $\mathfrak{f} = 1$ for simplicity. However, whenever the family $(L_\varepsilon)_{\varepsilon>0}$ is given by $L_\varepsilon = \varepsilon^{-d} L(\varepsilon^{-1} \cdot)$, then the rescaling

$$(3.10) \quad \rho(t, x) = \tilde{\rho}(\mathfrak{f}t, \mathfrak{f}x) \quad \text{and} \quad u(t, x) = \tilde{u}(\mathfrak{f}t, \mathfrak{f}x)$$

transforms the case $(\mathfrak{f}, \varepsilon)$ into the case $(1, \varepsilon\mathfrak{f})$. Hence, one may deduce from the above theorem a global well-posedness result that is valid for any $\varepsilon > 0$ and $\mathfrak{f} > 0$ (see the beginning of Section 6).

- The integrability property of the damped mode w is the key to proving strong convergence results in the asymptotics $\mathfrak{f} \rightarrow \infty$.
- Our approach is appropriate for dealing with the *multi-dimensional* case. In the one-dimensional case, the above result is still valid but the proof has to be slightly modified and it is very likely that stronger results may be obtained by different techniques (see a similar problem in [4]).
- A global existence result in the spirit of Theorem 3.1 can be established in the more general setting of System (2.2) (see Subsection 7.2).

Let us quickly outline the proof of Theorem 3.1. The core consists in establishing a priori estimates in the functional framework given above for the following linearization of (1.1):

$$(3.11) \quad \begin{cases} a_t + \operatorname{div} u + v \cdot \nabla a + b \operatorname{div} u = f, \\ u_t + u + v \cdot \nabla u + \nabla K_\varepsilon * a = g, \end{cases}$$

where the given pair (b, v) satisfies

$$(3.12) \quad b_t + \operatorname{div}((1+b)v) = 0.$$

We consider this linear system with *variable coefficients* since just looking at (1.4) with source terms cannot prevent a loss of derivatives. Here, we shall actually extend our analysis to the more general situation of $(1+c)\nabla K_\varepsilon * a$ in the second line of (3.11) and to a whole range of regularity exponents. The first extension is motivated by our wish to be able to consider more general pressure laws (see (2.3)) and the second one, to have a ready-to-use result for

proving stability estimates (and thus uniqueness) and the convergence from (1.1) to (1.2) by the same stroke.

Now, to get optimal a priori estimates for (3.11), we adapt the method of [6]. This consists in, first, localizing (3.11) by means of a Littlewood-Paley decomposition then:

- proving estimates for the (degenerate) parabolic mode a and the damped mode $w = u + \nabla K_\varepsilon a$ rather than for (a, u) , in the regime of frequencies ξ such that $|\xi| \widehat{L}_\varepsilon(\xi) \leq \nu_0$;
- considering a Lyapunov functional *depending on the coefficient b* that encodes the information on $a, u, \nabla u, \nabla L_\varepsilon a$ for frequencies such that $|\xi| \widehat{L}_\varepsilon(\xi) \geq \nu_0$. As for the Euler equation in [6], the dependence of this functional on b and c is designed to exactly compensate the loss of derivative coming from $b \operatorname{div} u$ and $c \nabla K_\varepsilon * a$. This could be seen as a symmetrization of (3.11) after spectral localization by means of a Littlewood-Paley decomposition.

Since, for proving global existence, we will have to take eventually $b = a$ and $v = u$, checking at every step of the proof that only norms of (b, v) that can be controlled in terms of the norms coming into play in Theorem 3.1 is fundamental.

The other steps of the proof are more standard: having at hand estimates for (3.11) in the spaces $E_{K_\varepsilon}^\sigma$, one can close the estimates globally for System (3.1) under Condition (3.7) in the space $E_{K_\varepsilon}^{\frac{d}{2}+1}$ and prove stability estimates in $E_{K_\varepsilon}^{\frac{d}{2}}$. These will enable us to prove the uniqueness part of the statement. As for the existence part, we first smooth out the data and prove the existence of a sequence of local-in-time solutions $(a^{(n)}, u^{(n)})_{n \in \mathbb{N}}$ with high Sobolev regularity. Combining our estimates in $E_{K_\varepsilon}^{\frac{d}{2}+1}$ with a continuation criterion, we then succeed in proving that these smooth solutions are actually global, and that $(a^{(n)}, u^{(n)})_{n \in \mathbb{N}}$ is bounded in $E_{K_\varepsilon}^{\frac{d}{2}}$. Combining with functional analysis arguments allow to conclude to convergence, up to subsequence, to a solution of (1.1) with the desired properties.

Our second aim is to justify that (1.1) is indeed an approximation of (1.2). More precisely, we show that the solution of (1.1) constructed above converges strongly and for all time for $\varepsilon \rightarrow 0$, to the unique solution of (1.2). This is achieved in the following theorem that essentially follows from a variation on stability estimates in $E_{K_\varepsilon}^{\frac{d}{2}}$.

Theorem 3.2. *Assume, in addition to (3.1), that $L_\varepsilon = \varepsilon^{-d} L(\varepsilon^{-1} \cdot)$ for a single function L . Consider initial data $(\rho_0 = 1 + a_0, u_0)$ with (a_0, u_0) in $\dot{B}_{2,1}^{\frac{d}{2}} \cap \dot{B}_{2,1}^{\frac{d}{2}+2}$. There exists a universal constant α_0 such that if*

$$(3.13) \quad \|(a_0, u_0)\|_{\dot{B}_{2,1}^{\frac{d}{2}} \cap \dot{B}_{2,1}^{\frac{d}{2}+2}} \leq \alpha_0,$$

then, for all $\varepsilon > 0$, System (1.1) has a unique global solution $(\rho^\varepsilon = 1 + a^\varepsilon, u^\varepsilon)$ with $(a^\varepsilon, u^\varepsilon)$ in $E_{K_\varepsilon}^{\frac{d}{2}+1}$ and System (1.2) has a unique global solution $(\rho = 1 + a, u)$ with (a, u) in $E^{\frac{d}{2}+1}$. Furthermore,

$$a^\varepsilon \rightarrow a \text{ in } L_{loc}^\infty(\mathbb{R}_+; \dot{B}_{2,1}^{\frac{d}{2}+\alpha}), \alpha \in [0, 1) \quad \text{and} \quad u^\varepsilon \rightarrow u \text{ in } L_{loc}^\infty(\mathbb{R}_+; \dot{B}_{2,1}^{\frac{d}{2}+\beta}), \beta \in [0, 2).$$

The above convergence holds uniformly on \mathbb{R}_+ if:

- either $\eta \mapsto |\eta|^{-1}(\widehat{L}(\eta) - 1)$ is bounded;
- or $a_0 \in \dot{B}_{2,1}^{\frac{d}{2}-1}$. In this case, we have a^ε and a in $\mathcal{C}_b(\mathbb{R}_+; \dot{B}_{2,1}^{\frac{d}{2}-1})$.

In the last part of the paper, we shall investigate the high friction limit $\mathfrak{f} \rightarrow \infty$ for (1.1). Our goal is to showcase the convergence of the (suitably rescaled) density toward a solution

of either the well-known porous media equation

$$(3.14) \quad \partial_t n - \operatorname{div}(n \nabla n) = 0$$

or of the following regularization of it:

$$(3.15) \quad \partial_t r - \operatorname{div}(r \nabla K_\varepsilon * r) = 0.$$

These two equations can be guessed after performing the following *diffusive* change of variables in (1.1):

$$(3.16) \quad \rho(t, x) = \check{\rho}(\mathfrak{f}^{-1}t, x) \quad \text{and} \quad u(t, x) = \mathfrak{f}^{-1}\check{u}(\mathfrak{f}^{-1}t, x).$$

We get

$$\begin{cases} \check{\rho}_t + \operatorname{div}(\check{\rho}\check{u}) = 0, \\ \mathfrak{f}^{-2}(\check{u}_t + \check{u} \cdot \nabla \check{u}) + \check{u} + \nabla K_\varepsilon * \check{\rho} = 0. \end{cases}$$

Hence, it can be expected that $\check{u} + \nabla K_\varepsilon * \check{\rho}$ goes to 0 when \mathfrak{f} tends to ∞ . Reverting to the mass equation and assuming that $\check{\rho} \rightarrow r$, one can conclude that r satisfies (3.15). In the same way, if both $\mathfrak{f} \rightarrow \infty$ and $\varepsilon \rightarrow 0$, then we will prove that $\check{\rho} \rightarrow n$ with n satisfying (3.14).

The rest of the paper is organized as follows. In Section 4, we establish a priori estimates for the linear System (3.11). To accommodate the general pressure case (2.2), we actually replace the term $\nabla K_\varepsilon * a$ with $(1+c)\nabla K_\varepsilon * a$ for some given function c . At first reading however, setting $c \equiv 0$ is advisable, as it corresponds to Theorems 3.1 and 3.2, (see Subsection 7.2 for the general case). The principal outcome, as presented in Theorem 4.1, furnishes a comprehensive estimate crucial for subsequent developments. Section 5 is dedicated to proving the existence of solutions. We outline the main steps of the construction procedure, followed by the proof of uniqueness, and ultimately, the convergence to the classical Euler system under the condition $\varepsilon \rightarrow 0$. All these aspects rely on the estimates established in Theorem 4.1. In Section 6, we delve into relaxation results, as presented in (3.14)–(3.16). We explore two types of relaxation, yielding modifications and classical versions of the porous equation. Subsection 7.1 is devoted to the study of various commutators, essential for our analysis. Subsequently, in Subsection 7.2, we examine the general case of pressure, emphasizing the distinctions from the original scenario. Lastly, we provide motivation for transitioning from the particle system (2.1) to the hydrodynamical equations under consideration (1.2) and (2.4).

4. STUDY OF A SUITABLE LINEARIZED SYSTEM

This part is devoted to proving a priori estimates for the following linear system:

$$(4.1) \quad \begin{cases} a_t + v \cdot \nabla a + (1+b) \operatorname{div} u = f, \\ u_t + u + v \cdot \nabla u + (1+c)\nabla K_\varepsilon a = g, \\ a|_{t=0} = a_0, \quad u|_{t=0} = u_0. \end{cases}$$

Note that the system (3.11) presented before corresponds to the special case $c = 0$. The reason for presenting here this more general class of systems is motivated by our desire to consider (1.1) with more general pressure laws (see Section 7.2). To short the notation from now we write $K_\varepsilon a$ instead of $K_\varepsilon * a$.

Throughout this section, we assume that the (given) triple (b, c, v) satisfies the relation (3.12) and the smallness condition²

$$(4.2) \quad \max(\|b\|_{L^\infty(0, T \times \mathbb{R}^d)}, \|c\|_{L^\infty(0, T \times \mathbb{R}^d)}) \leq 1/4.$$

²We assumed (4.2) for simplicity. A similar result holds if $0 < r \leq 1+b, 1+c < R$ for any real numbers r and R : it is just a matter of adapting the definition of the Lyapunov functional in (4.20) below, accordingly.

Theorem 4.1. *Let σ be in the range $(1 - d/2, 1 + d/2]$. Assume that a_0 and u_0 are such that*

$$(4.3) \quad u_0 \in \dot{B}_{2,1}^\sigma \cap \dot{B}_{2,1}^{\sigma+1}, \quad a_0 \in \dot{B}_{2,1}^{\sigma-1} \cap \dot{B}_{2,1}^\sigma \quad \text{and} \quad L_\varepsilon \nabla a_0 \in \dot{B}_{2,1}^\sigma,$$

and that the source terms f and g verify

$$(4.4) \quad g \in L^1(0, T; \dot{B}_{2,1}^\sigma \cap \dot{B}_{2,1}^{\sigma+1}), \quad f \in L^1(0, T; \dot{B}_{2,1}^{\sigma-1} \cap \dot{B}_{2,1}^\sigma) \quad \text{and} \quad L_\varepsilon \star \nabla f \in L^1(0, T; \dot{B}_{2,1}^\sigma).$$

Finally, let us assume that the triple (b, c, v) satisfies (3.12) and (4.2) and has enough regularity. Consider a smooth enough solution (a, u) of (3.11) on $[0, T] \times \mathbb{R}^d$, and set

$$X_{a,u}^\sigma(t) := \|(a, \nabla a, \nabla^2 L_\varepsilon a)(t)\|_{\dot{B}_{2,1}^{\sigma-1}} + \|(u, \nabla u)(t)\|_{\dot{B}_{2,1}^\sigma}$$

and $H_{a,u}^\sigma(t) := \|(u, \nabla u)\|_{\dot{B}_{2,1}^\sigma} + \|(\nabla K_\varepsilon a, \nabla^2 K_\varepsilon a)\|_{\dot{B}_{2,1}^\sigma}^\ell + \|\nabla L_\varepsilon a\|_{\dot{B}_{2,1}^\sigma}^h.$

There exists a constant C independent of ε and of T such that for all $t \in [0, T]$, there holds:

$$(4.5) \quad X_{a,u}^\sigma(t) + \int_0^t H_{a,u}^\sigma d\tau \leq C \left(X_{a,u}^\sigma(0) + \int_0^t X_{f,g}^\sigma d\tau + \int_0^t \|\nabla v\|_{\dot{B}_{2,1}^{\frac{d}{2}} \cap \dot{B}_{2,1}^{\frac{d}{2}+1}} X_{a,u}^\sigma d\tau \right. \\ \left. + \int_0^t \|b, \nabla b\|_{\dot{B}_{2,1}^{\frac{d}{2}}} \|u, \nabla u\|_{\dot{B}_{2,1}^\sigma} d\tau \right. \\ \left. + \int_0^t \left(\|b, \nabla v, \nabla L_\varepsilon b\|_{\dot{B}_{2,1}^\sigma} \|\nabla u\|_{L^\infty} + \|\nabla v\|_{\dot{B}_{2,1}^\sigma} \|\nabla L_\varepsilon a\|_{L^\infty} \right) d\tau \right. \\ \left. + \int_0^t \left(\|c\|_{\dot{B}_{2,1}^{\frac{d}{2}}} \left(\|\nabla L_\varepsilon a\|_{\dot{B}_{2,1}^\sigma}^h + \|\nabla K_\varepsilon a\|_{\dot{B}_{2,1}^\sigma}^\ell + \|\nabla^2 K_\varepsilon a\|_{\dot{B}_{2,1}^\sigma}^\ell \right) \right. \right. \\ \left. \left. + \left(\|\nabla c\|_{\dot{B}_{2,1}^{\frac{d}{2}}} + \|c_t + \operatorname{div}((1+c)v)\|_{L^\infty} \right) \|\nabla L_\varepsilon a\|_{\dot{B}_{2,1}^\sigma} + \|\nabla c\|_{\dot{B}_{2,1}^{\frac{d}{2}}} \|L_\varepsilon a\|_{\dot{B}_{2,1}^\sigma} \right. \right. \\ \left. \left. + \|c\|_{\dot{B}_{2,1}^\sigma} \left(\|\nabla L_\varepsilon a\|_{L^\infty}^h + \|\nabla K_\varepsilon a\|_{L^\infty}^\ell + \|\nabla^2 K_\varepsilon a\|_{L^\infty}^\ell \right) + \|\nabla c\|_{\dot{B}_{2,1}^\sigma} \|\nabla L_\varepsilon a\|_{L^\infty} \right) d\tau \right),$$

and the terms involving the L^∞ norm of ∇u or $\nabla L_\varepsilon a$ and so on, are not needed if $\sigma \leq d/2$.

If, in addition, u_0 belongs to $\dot{B}_{2,1}^{\sigma-1}$ and g , to $L^1(0, T; \dot{B}_{2,1}^{\sigma-1})$ then we also have

$$(4.6) \quad \|(u, w)(t)\|_{\dot{B}_{2,1}^{\sigma-1}} + \int_0^t \|w\|_{\dot{B}_{2,1}^{\sigma-1}} d\tau \lesssim X_{a,u}^\sigma(0) + \|u_0\|_{\dot{B}_{2,1}^{\sigma-1}}^\ell + \int_0^t (\|g\|_{\dot{B}_{2,1}^{\sigma-1}}^\ell + X_{f,g}^\sigma) d\tau \\ + \int_0^t \|\nabla v\|_{\dot{B}_{2,1}^{\frac{d}{2}} \cap \dot{B}_{2,1}^{\frac{d}{2}+1}} (\|u\|_{\dot{B}_{2,1}^{\sigma-1}}^\ell + X_{a,u}^\sigma) d\tau + \int_0^t \|c\|_{\dot{B}_{2,1}^{\frac{d}{2}}} \|K_\varepsilon a\|_{\dot{B}_{2,1}^\sigma} d\tau + \text{last 5 lines of (4.5)},$$

where the ‘damped mode’ w is defined by

$$(4.7) \quad w = u + \nabla K_\varepsilon a.$$

Proof. In all that follows, $\{c_j\}_{j \in \mathbb{Z}}$ will denote a nonnegative sequence with sum equal to 1, and we use the notation

$$z_j := \dot{\Delta}_j z, \quad j \in \mathbb{Z}, \quad z \in \mathcal{S}'(\mathbb{R}^d).$$

4.1. First step: Low frequencies estimates. The first step consists in estimating the ‘low frequencies’ of a at level of regularity $\sigma - 1$, then of w at level σ , where w has been defined in (4.7). Estimates for u and w at level $\sigma - 1$ (that is, Inequality (4.6)) are extra informations that can be obtained afterward.

As a start, we look at the evolution of (a, w) , namely,

$$(4.8) \quad \begin{cases} a_t - \Delta K_\varepsilon a + v \cdot \nabla a = -\operatorname{div} w - b \operatorname{div} u, \\ w_t + w + v \cdot \nabla w = [v, \nabla K_\varepsilon] \cdot \nabla a - \nabla K_\varepsilon \operatorname{div}(w - \nabla K_\varepsilon a) - \nabla K_\varepsilon (b \operatorname{div} u) - c \nabla K_\varepsilon a. \end{cases}$$

Up to lower order terms, this is a diagonalization of System (3.11): a may be seen as a (degenerate) parabolic mode, while w is a damped mode. Now, to prove estimates in Besov spaces for a and w , the unavoidable first step is to localize (4.8) by means of $\dot{\Delta}_j$. We have:

$$(4.9) \quad \begin{cases} a_{j,t} - \Delta K_\varepsilon a_j + v \cdot \nabla a_j = [v, \dot{\Delta}_j] \cdot \nabla a - \dot{\Delta}_j (\operatorname{div} w + b \operatorname{div} u), \\ w_{j,t} + w_j + v \cdot \nabla w_j = [v, \dot{\Delta}_j] \cdot \nabla w + \dot{\Delta}_j ([v, \nabla K_\varepsilon] \cdot \nabla a) \\ \quad - \dot{\Delta}_j \nabla K_\varepsilon \operatorname{div}(w - \nabla K_\varepsilon a) - \dot{\Delta}_j \nabla K_\varepsilon (b \operatorname{div} u) - \dot{\Delta}_j (c \nabla K_\varepsilon a). \end{cases}$$

Estimate of a . Taking the L^2 scalar product of the first equation of (4.9) with a_j and integrating by parts in the second and third terms on the left yields:

$$(4.10) \quad \begin{aligned} \frac{1}{2} \frac{d}{dt} \|a_j\|_{L^2}^2 + \|\nabla L_\varepsilon a_j\|_{L^2}^2 &= \frac{1}{2} \int_{\mathbb{R}^d} (\operatorname{div} v) a_j^2 dx \\ &\quad + \int_{\mathbb{R}^d} ([v, \dot{\Delta}_j] \cdot \nabla a) a_j dx - \int_{\mathbb{R}^d} \dot{\Delta}_j (\operatorname{div} w + b \operatorname{div} u) a_j dx. \end{aligned}$$

Provided $-d/2 < \sigma - 1 < d/2 + 1$, combining Hölder inequality, embedding (7.3) and the commutator estimate (7.4) ensures that

$$\frac{1}{2} \int_{\mathbb{R}^d} (\operatorname{div} v) a_j^2 dx + \int_{\mathbb{R}^d} ([v, \dot{\Delta}_j] \cdot \nabla a) a_j dx \leq C c_j 2^{-j(\sigma-1)} \|\nabla v\|_{\dot{B}_{2,1}^{\frac{d}{2}}} \|a\|_{\dot{B}_{2,1}^{\sigma-1}} \|a_j\|_{L^2}.$$

Furthermore, for $-d/2 < \sigma - 1 \leq d/2$, Cauchy-Schwarz inequality, the product law (7.1) and the definition of the space $\dot{B}_{2,1}^{\sigma-1}$ guarantee that

$$\int_{\mathbb{R}^d} \dot{\Delta}_j (b \operatorname{div} u) a_j dx \leq C c_j 2^{-j(\sigma-1)} \|b\|_{\dot{B}_{2,1}^{\frac{d}{2}}} \|\operatorname{div} u\|_{\dot{B}_{2,1}^{\sigma-1}} \|a_j\|_{L^2}.$$

Now, owing to the spectral localization given by $\dot{\Delta}_j$, Bernstein inequality and (3.1), we have

$$(4.11) \quad \|\nabla L_\varepsilon a_j\|_{L^2}^2 \approx 2^{2j} \widehat{L}_\varepsilon^2(2^j) \|a_j\|_{L^2}^2 \approx \|\Delta K_\varepsilon a_j\|_{L^2} \|a_j\|_{L^2}.$$

Hence after ‘simplification by $\|a_j\|_{L^2}$ ’ in (4.10), integration on $[0, t]$ and use of (4.11), we get for some absolute constant κ_0 ,

$$\begin{aligned} \|a_j(t)\|_{L^2} + \kappa \int_0^t \|K_\varepsilon \Delta a_j\|_{L^2} d\tau &\leq \|a_{0,j}\|_{L^2} + \int_0^t \|\operatorname{div} w_j\|_{L^2} d\tau \\ &\quad + C 2^{-j(\sigma-1)} \int_0^t c_j (\|\nabla v\|_{\dot{B}_{2,1}^{\frac{d}{2}}} \|a\|_{\dot{B}_{2,1}^{\sigma-1}} + \|\operatorname{div} u\|_{\dot{B}_{2,1}^{\sigma-1}} \|b\|_{\dot{B}_{2,1}^{\frac{d}{2}}}) d\tau. \end{aligned}$$

Then, multiplying by $2^{j(\sigma-1)}$ and summing up on all j ’s such that $2^j \widehat{L}_\varepsilon(2^j) < \nu_0$ gives

$$(4.12) \quad \begin{aligned} \|a(t)\|_{\dot{B}_{2,1}^{\sigma-1}}^\ell + \kappa_0 \int_0^t \|K_\varepsilon \Delta a\|_{\dot{B}_{2,1}^{\sigma-1}}^\ell d\tau &\leq \|a_0\|_{\dot{B}_{2,1}^{\sigma-1}}^\ell \\ &\quad + \int_0^t \|\operatorname{div} w\|_{\dot{B}_{2,1}^{\sigma-1}}^\ell d\tau + C \int_0^t (\|\nabla v\|_{\dot{B}_{2,1}^{\frac{d}{2}}} \|a\|_{\dot{B}_{2,1}^{\sigma-1}} + \|\operatorname{div} u\|_{\dot{B}_{2,1}^{\sigma-1}} \|b\|_{\dot{B}_{2,1}^{\frac{d}{2}}}) d\tau. \end{aligned}$$

Estimate of w at regularity level σ . Let us take the L^2 scalar product of the second equation of (4.9) with w_j . Handling the terms containing to v as previously, simplifying by $\|w_j\|_{L^2}$ and integrating, we get for $\sigma \in (-d/2, 1 + d/2]$,

$$\begin{aligned} \|w_j(t)\|_{L^2} + \int_0^t \|w_j\|_{L^2} d\tau &\leq \|w_{0,j}\|_{L^2} + C2^{-j\sigma} \int_0^t c_j \|\nabla v\|_{\dot{B}_{2,1}^{\frac{d}{2}}} \|w\|_{\dot{B}_{2,1}^{\sigma}} d\tau \\ &+ \int_0^t \left(\|\nabla K_\varepsilon \operatorname{div}(w_j - \nabla K_\varepsilon a_j)\|_{L^2} + \|\dot{\Delta}_j([v, \nabla K_\varepsilon] \cdot \nabla a)\|_{L^2} + \|\dot{\Delta}_j \nabla K_\varepsilon (b \operatorname{div} u)\|_{L^2} \right) d\tau \\ &+ \int_0^t \|\dot{\Delta}_j (c \nabla K_\varepsilon a)\|_{L^2} d\tau. \end{aligned}$$

Hence, multiplying by $2^{j\sigma}$ and summing on all j 's such that $2^j \widehat{L}_\varepsilon(2^j) < \nu_0$,

$$(4.13) \quad \begin{aligned} \|w(t)\|_{\dot{B}_{2,1}^\sigma}^\ell + \int_0^t \|w\|_{\dot{B}_{2,1}^\sigma}^\ell d\tau &\leq \|w_0\|_{\dot{B}_{2,1}^\sigma}^\ell + \int_0^t \|\nabla K_\varepsilon \operatorname{div}(w - \nabla K_\varepsilon a)\|_{\dot{B}_{2,1}^\sigma}^\ell d\tau \\ &+ C \int_0^t \|\nabla v\|_{\dot{B}_{2,1}^{\frac{d}{2}}} \|w\|_{\dot{B}_{2,1}^\sigma} d\tau + \int_0^t \left(\|[v, \nabla K_\varepsilon] \cdot \nabla a\|_{\dot{B}_{2,1}^\sigma}^\ell + \|\nabla K_\varepsilon (b \operatorname{div} u)\|_{\dot{B}_{2,1}^\sigma}^\ell + \|c \nabla K_\varepsilon a\|_{\dot{B}_{2,1}^\sigma}^\ell \right) d\tau. \end{aligned}$$

Looking at (3.4), we see that

$$(4.14) \quad \|\nabla K_\varepsilon \operatorname{div}(w - \nabla K_\varepsilon a)\|_{\dot{B}_{2,1}^\sigma}^\ell \leq C\nu_0^2 \|w - \nabla K_\varepsilon a\|_{\dot{B}_{2,1}^\sigma}^\ell.$$

For the last term of (4.13), using (7.1) and the low frequency cut-off yields

$$(4.15) \quad \|\nabla K_\varepsilon (b \operatorname{div} u)\|_{\dot{B}_{2,1}^\sigma}^\ell \leq C\nu_0^2 \|b \operatorname{div} u\|_{\dot{B}_{2,1}^{\sigma-1}}^\ell \leq C\nu_0^2 \|b\|_{\dot{B}_{2,1}^{\frac{d}{2}}} \|\operatorname{div} u\|_{\dot{B}_{2,1}^{\sigma-1}}.$$

To handle the commutator term, we use $K_\varepsilon = L_\varepsilon^2$. This enables us to write that:

$$[v, \nabla K_\varepsilon] \cdot \nabla a = [v, \nabla L_\varepsilon] \cdot \nabla L_\varepsilon a + \nabla L_\varepsilon [v, L_\varepsilon] \cdot \nabla a.$$

Therefore, thanks to Inequalities (7.5) and (7.6) with $c = v^k$ (for $k = 1, \dots, d$) and $h = \nabla L_\varepsilon a$ or ∇a , respectively, we have

$$\begin{aligned} \|[v, \nabla K_\varepsilon] \cdot \nabla a\|_{\dot{B}_{2,1}^\sigma}^\ell &\lesssim \|[v, \nabla L_\varepsilon] \cdot \nabla L_\varepsilon a\|_{\dot{B}_{2,1}^\sigma}^\ell + \|\nabla L_\varepsilon [v, L_\varepsilon] \cdot \nabla a\|_{\dot{B}_{2,1}^\sigma}^\ell \\ &\lesssim \|[v, \nabla L_\varepsilon] \cdot \nabla L_\varepsilon a\|_{\dot{B}_{2,1}^\sigma} + \nu_0 \|[v, L_\varepsilon] \cdot \nabla a\|_{\dot{B}_{2,1}^\sigma} \\ &\lesssim \|\nabla v\|_{\dot{B}_{2,1}^{\frac{d}{2}}} \|\nabla L_\varepsilon a\|_{\dot{B}_{2,1}^\sigma} + \|\nabla v\|_{\dot{B}_{2,1}^\sigma} \|\nabla L_\varepsilon a\|_{L^\infty} + \nu_0 \|\nabla v\|_{\dot{B}_{2,1}^{\frac{d}{2}}} \|a\|_{\dot{B}_{2,1}^\sigma}, \end{aligned}$$

and the second term in the right-hand side is not needed if $\sigma \leq d/2$.

Finally, we have

$$\|c \nabla K_\varepsilon a\|_{\dot{B}_{2,1}^\sigma}^\ell \lesssim \|c\|_{\dot{B}_{2,1}^{\frac{d}{2}}} \|\nabla K_\varepsilon a\|_{\dot{B}_{2,1}^\sigma} + \|c\|_{\dot{B}_{2,1}^\sigma} \|\nabla K_\varepsilon a\|_{L^\infty}$$

and the second term in the right-hand side is not needed if $\sigma \leq d/2$. In the end, reverting to (4.13) yields

$$(4.16) \quad \begin{aligned} \|w(t)\|_{\dot{B}_{2,1}^\sigma}^\ell + \int_0^t \|w\|_{\dot{B}_{2,1}^\sigma}^\ell d\tau &\leq \|w_0\|_{\dot{B}_{2,1}^\sigma}^\ell + \nu_0^2 \int_0^t \|w - \nabla K_\varepsilon a\|_{\dot{B}_{2,1}^\sigma}^\ell d\tau \\ &+ C \int_0^t \left(\nu_0^2 \|b\|_{\dot{B}_{2,1}^{\frac{d}{2}}} \|u\|_{\dot{B}_{2,1}^\sigma} + \|c\|_{\dot{B}_{2,1}^{\frac{d}{2}}} \|\nabla K_\varepsilon a\|_{\dot{B}_{2,1}^\sigma} + \|c\|_{\dot{B}_{2,1}^\sigma} \|\nabla K_\varepsilon a\|_{L^\infty} \right. \\ &\quad \left. + \|\nabla v\|_{\dot{B}_{2,1}^{\frac{d}{2}}} \|(w, a, \nabla L_\varepsilon a)\|_{\dot{B}_{2,1}^\sigma} + \|\nabla L_\varepsilon a\|_{L^\infty} \|v\|_{\dot{B}_{2,1}^{\sigma+1}} \right) d\tau, \end{aligned}$$

where the terms with L^∞ norms of $\nabla K_\varepsilon a$ or $\nabla L_\varepsilon a$ are absent if $\sigma \leq d/2$.

Putting this inequality together with (4.12) allows to absorb all the linear terms in the right-hand side provided that ν_0 is chosen small enough. We get

$$(4.17) \quad \|a(t)\|_{\dot{B}_{2,1}^{\sigma-1}}^\ell + \|w(t)\|_{\dot{B}_{2,1}^\sigma}^\ell + \frac{1}{2} \int_0^t \|(K_\varepsilon \nabla a, w)\|_{\dot{B}_{2,1}^\sigma}^\ell d\tau \lesssim \|a_0\|_{\dot{B}_{2,1}^{\sigma-1}}^\ell + \|w_0\|_{\dot{B}_{2,1}^\sigma}^\ell \\ + \int_0^t \left(\|b\|_{\dot{B}_{2,1}^{\frac{d}{2}}} \|u\|_{\dot{B}_{2,1}^\sigma} + \|c\|_{\dot{B}_{2,1}^{\frac{d}{2}}} \|\nabla K_\varepsilon a\|_{\dot{B}_{2,1}^\sigma} + \|c\|_{\dot{B}_{2,1}^\sigma} \|\nabla K_\varepsilon a\|_{L^\infty} \right. \\ \left. + \|\nabla v\|_{\dot{B}_{2,1}^{\frac{d}{2}}} (\|(a, \nabla a, \nabla^2 L_\varepsilon a)\|_{\dot{B}_{2,1}^{\sigma-1}} + \|w\|_{\dot{B}_{2,1}^\sigma}) + \|\nabla L_\varepsilon a\|_{L^\infty} \|v\|_{\dot{B}_{2,1}^{\sigma+1}} \right) d\tau.$$

Again, the terms with L^∞ norms of $\nabla K_\varepsilon a$ or $\nabla L_\varepsilon a$ are not needed if $\sigma \leq d/2$.

Estimate of w at regularity level $\sigma - 1$. Note that we also have

$$\|\nabla K_\varepsilon \operatorname{div}(w - \nabla K_\varepsilon a)\|_{\dot{B}_{2,1}^{\sigma-1}}^\ell \leq C(\nu_0^2 \|w\|_{\dot{B}_{2,1}^{\sigma-1}}^\ell + c \|\nabla K_\varepsilon a\|_{\dot{B}_{2,1}^\sigma}^\ell).$$

Hence, arguing as for proving (4.16) but using this time that

$$\|\nabla K_\varepsilon(b \operatorname{div} u)\|_{\dot{B}_{2,1}^{\sigma-1}}^\ell \leq \nu_0 \|b \operatorname{div} u\|_{\dot{B}_{2,1}^{\sigma-1}}^\ell \leq C\nu_0 \|b\|_{\dot{B}_{2,1}^{\frac{d}{2}}} \|u\|_{\dot{B}_{2,1}^\sigma},$$

we get if ν_0 is small enough:

$$(4.18) \quad \|w(t)\|_{\dot{B}_{2,1}^{\sigma-1}}^\ell + \frac{1}{2} \int_0^t \|w\|_{\dot{B}_{2,1}^{\sigma-1}}^\ell d\tau \leq \|w_0\|_{\dot{B}_{2,1}^{\sigma-1}}^\ell + \nu_0 \int_0^t \|\nabla K_\varepsilon a\|_{\dot{B}_{2,1}^\sigma}^\ell d\tau \\ + C \int_0^t (\nu_0^2 \|b\|_{\dot{B}_{2,1}^{\frac{d}{2}}} \|u\|_{\dot{B}_{2,1}^\sigma} + \|\nabla v\|_{\dot{B}_{2,1}^{\frac{d}{2}}} \|w, a, \nabla L_\varepsilon a\|_{\dot{B}_{2,1}^{\sigma-1}}) d\tau + C \int_0^t \|c\|_{\dot{B}_{2,1}^{\frac{d}{2}}} \|\nabla K_\varepsilon a\|_{\dot{B}_{2,1}^{\sigma-1}} d\tau.$$

Note that for all $\sigma' \in \mathbb{R}$, we may write

$$(4.19) \quad \|w - u\|_{\dot{B}_{2,1}^{\sigma'}}^\ell \leq C \|\nabla K_\varepsilon a\|_{\dot{B}_{2,1}^{\sigma'}}^\ell \leq C\nu_0^2 \|a\|_{\dot{B}_{2,1}^{\sigma'-1}}.$$

This allows to replace $\|w\|_{\dot{B}_{2,1}^\sigma}^\ell$ by $\|u\|_{\dot{B}_{2,1}^\sigma}^\ell$ in the left-hand side of (4.17) and $\|w\|_{\dot{B}_{2,1}^{\sigma-1}}^\ell$ by $\|u\|_{\dot{B}_{2,1}^{\sigma-1}}^\ell$ in the first term of the left-hand side of (4.18), and thus to complete the proof of the low frequency parts of (4.5) and (4.6).

Remark 1. *If $c = F(K_\varepsilon a)$ for some smooth function F vanishing at 0, the last term of (4.18) lacks time integrability. However, one can apply (4.17) with $\sigma - 1$ instead of (4.18) to bound u and $\nabla K_\varepsilon a$ in $L^1(\mathbb{R}_+; \dot{B}_{2,1}^{\sigma-1})$. We deduce a bound for $L_\varepsilon a$ in $L^2(\mathbb{R}_+; \dot{B}_{2,1}^{\sigma-1})$.*

4.2. Second step: High frequencies estimates. We adapt the approach of [6] for the dissipative Euler system, and introduce the following ‘‘Lyapunov’’ and ‘‘dissipation rate’’ functionals

$$(4.20) \quad \mathcal{L}_j^2 := \|(a_j, L_\varepsilon a_j, u_j)\|_{L^2}^2 - 2 \int_{\mathbb{R}^d} a_j \operatorname{div} u_j dx + 2 \int_{\mathbb{R}^d} (1+c) |\nabla L_\varepsilon a_j|^2 dx$$

$$(4.21) \quad + \int_{\mathbb{R}^d} |\nabla \mathcal{P} u_j|^2 dx + 2 \int_{\mathbb{R}^d} (1+b) (\operatorname{div} u_j)^2 dx,$$

$$\mathcal{H}_j^2 := \|u_j\|_{L^2}^2 + \|\nabla \mathcal{P} u_j\|_{L^2}^2 + \int_{\mathbb{R}^d} (1+c) |\nabla L_\varepsilon a_j|^2 dx + \int_{\mathbb{R}^d} (1+b) (\operatorname{div} u_j)^2 dx.$$

Note that, owing to (4.2) and Young inequality, we have

$$(4.22) \quad \begin{aligned} \mathcal{L}_j &\approx \|(\nabla L_\varepsilon a_j, \nabla u_j)\|_{L^2} \approx \mathcal{H}_j \quad \text{if } 2^j \widehat{L}_\varepsilon(2^j) \geq \nu_0, \quad \text{and} \\ \mathcal{L}_j &\approx \|(a_j, u_j, \nabla u_j)\|_{L^2}, \quad \mathcal{H}_j \gtrsim 2^j \widehat{L}_\varepsilon(2^j) \mathcal{L}_j \quad \text{if } 2^j \widehat{L}_\varepsilon(2^j) \leq \nu_0. \end{aligned}$$

Hence, if multiplying \mathcal{H}_j and \mathcal{L}_j by $2^{j\sigma}$ and summing up on those j 's such that $2^j \widehat{L}_\varepsilon(2^j) \geq \nu_0$, one gets the parts of $X_{a,u}^\sigma$ and $H_{a,u}^\sigma$ corresponding to the high frequencies.

We shall implement an energy method so has to compute the time derivatives of all the terms that constitute \mathcal{L}_j^2 . To proceed, the first step is of course to localize (3.11) by means of $\dot{\Delta}_j$. However, in order to avoid loss of derivatives, one needs to be very careful how one writes the terms with nonconstant coefficients after localization. The idea is to obtain for (a_j, u_j) a system with the same structure as (3.11), up to 'manageable' commutator terms. Having this in mind, a suitable way of writing the localized system is

$$(4.23) \quad \begin{cases} a_{j,t} + v \cdot \nabla a_j + \dot{\Delta}_j((1+b)\operatorname{div} u) = [v, \dot{\Delta}_j] \cdot \nabla a, \\ u_{j,t} + v \cdot \nabla u_j + u_j + \dot{\Delta}_j((1+c)\nabla K_\varepsilon a) = [v, \dot{\Delta}_j] \cdot \nabla u. \end{cases}$$

Let us first look at the time derivative of $\|(a_j, u_j)\|_{L^2}^2$. Taking the L^2 scalar product of (4.23) with (a_j, u_j) , and integrating by parts in the convection terms gives:

$$(4.24) \quad \begin{aligned} &\frac{1}{2} \frac{d}{dt} \|(a_j, u_j)\|_{L^2}^2 + \|u_j\|_{L^2}^2 + \int_{\mathbb{R}^d} (\operatorname{Id} - K_\varepsilon) a_j \operatorname{div} u_j \, dx + \int_{\mathbb{R}^d} u_j \cdot \dot{\Delta}_j(c \nabla K_\varepsilon a) \, dx \\ &+ \int_{\mathbb{R}^d} a_j \dot{\Delta}_j(b \operatorname{div} u) \, dx = \int_{\mathbb{R}^d} ([v, \dot{\Delta}_j] \cdot \nabla a a_j + ([v, \dot{\Delta}_j] \cdot \nabla u) \cdot u_j) \, dx + \frac{1}{2} \int_{\mathbb{R}^d} (a_j^2 + |u_j|^2) \operatorname{div} v \, dx. \end{aligned}$$

To eliminate the third term of the left-hand side, we need to look at $\|L_\varepsilon a_j\|_{L^2}^2$. We have

$$L_\varepsilon a_{j,t} + v \cdot \nabla L_\varepsilon a_j + L_\varepsilon \dot{\Delta}_j((1+b)\operatorname{div} u) = [v, L_\varepsilon \dot{\Delta}_j] \cdot \nabla a.$$

Hence,

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \|L_\varepsilon a_j\|_{L^2}^2 + \int_{\mathbb{R}^d} L_\varepsilon a_j L_\varepsilon \operatorname{div} u_j \, dx + \int_{\mathbb{R}^d} L_\varepsilon a_j L_\varepsilon \dot{\Delta}_j(b \operatorname{div} u) \, dx \\ = \frac{1}{2} \int_{\mathbb{R}^d} (L_\varepsilon a_j)^2 \operatorname{div} v \, dx + \int_{\mathbb{R}^d} [v, L_\varepsilon \dot{\Delta}_j] \cdot \nabla a L_\varepsilon a_j \, dx. \end{aligned}$$

Remembering $L_\varepsilon^2 = K_\varepsilon$ and ${}^t L_\varepsilon = L_\varepsilon$, and adding up this relation to (4.24) gives

$$(4.25) \quad \frac{1}{2} \frac{d}{dt} \|(a_j, L_\varepsilon a_j, u_j)\|_{L^2}^2 + \|u_j\|_{L^2}^2 + \int_{\mathbb{R}^d} a_j \operatorname{div} u_j \, dx = I_j^1,$$

where

$$\begin{aligned} I_j^1 &= - \int_{\mathbb{R}^d} a_j \dot{\Delta}_j(b \operatorname{div} u) \, dx - \int_{\mathbb{R}^d} L_\varepsilon a_j L_\varepsilon \dot{\Delta}_j(b \operatorname{div} u) \, dx + \int_{\mathbb{R}^d} [v, \dot{\Delta}_j] \cdot \nabla a a_j \\ &\quad + \int_{\mathbb{R}^d} ([v, \dot{\Delta}_j] \cdot \nabla u) \cdot u_j \, dx + \int_{\mathbb{R}^d} [v, L_\varepsilon \dot{\Delta}_j] \cdot \nabla a L_\varepsilon a_j \, dx \\ &\quad + \frac{1}{2} \int_{\mathbb{R}^d} (a_j^2 + (L_\varepsilon a_j)^2 + |u_j|^2) \operatorname{div} v \, dx + \int_{\mathbb{R}^d} \dot{\Delta}_j(c \nabla K_\varepsilon a) \cdot u_j \, dx. \end{aligned}$$

Next, to show the third term of \mathcal{H}_j^2 , one can compute the time derivative of $(a_j | \operatorname{div} u_j)_{L^2}$. To do this, it is better to rewrite the equation for a_j as follows:

$$a_{j,t} + v \cdot \nabla a_j + (1+b)\operatorname{div} u_j = [v, \dot{\Delta}_j] \cdot \nabla a + [b, \dot{\Delta}_j] \operatorname{div} u.$$

Then, using the fact that

$$\int_{\mathbb{R}^d} (a_j \operatorname{div}(v \cdot \nabla u_j) + \operatorname{div} u_j v \cdot \nabla a_j) dx = \int_{\mathbb{R}^d} a_j (\operatorname{Tr}(\nabla v \cdot \nabla u_j) - \operatorname{div} v \operatorname{div} u_j) dx,$$

and that

$$\begin{aligned} \int_{\mathbb{R}^d} a_j \operatorname{div} \dot{\Delta}_j (c \nabla K_\varepsilon a) dx &= \int_{\mathbb{R}^d} a_j \operatorname{div} \dot{\Delta}_j [c, L_\varepsilon] \nabla L_\varepsilon a dx \\ &\quad + \int_{\mathbb{R}^d} \nabla L_\varepsilon a_j \cdot [c, \dot{\Delta}_j] \nabla L_\varepsilon a dx - \int_{\mathbb{R}^d} c |\nabla L_\varepsilon a_j|^2 dx, \end{aligned}$$

we get

$$(4.26) \quad \frac{d}{dt} \int_{\mathbb{R}^d} a_j \operatorname{div} u_j dx + \int_{\mathbb{R}^d} a_j \operatorname{div} u_j dx + \int_{\mathbb{R}^d} (1+b)(\operatorname{div} u_j)^2 dx - \int_{\mathbb{R}^d} (1+c) |\nabla L_\varepsilon a_j|^2 dx = I_j^2,$$

where

$$\begin{aligned} I_j^2 &= \int_{\mathbb{R}^d} a_j (\operatorname{div} v \operatorname{div} u_j - \operatorname{Tr}(\nabla v \cdot \nabla u_j)) dx + \int_{\mathbb{R}^d} a_j \operatorname{div} \dot{\Delta}_j [L_\varepsilon, c] \nabla L_\varepsilon a dx \\ &\quad + \int_{\mathbb{R}^d} \nabla L_\varepsilon a_j \cdot [\dot{\Delta}_j, c] \nabla L_\varepsilon a dx + \int_{\mathbb{R}^d} (a_j \operatorname{div} [v, \dot{\Delta}_j] \cdot \nabla u + \operatorname{div} u_j ([v, \dot{\Delta}_j] \nabla a + [b, \dot{\Delta}_j] \operatorname{div} u)) dx. \end{aligned}$$

So, subtracting (4.26) from (4.25) eventually yields

$$(4.27) \quad \begin{aligned} \frac{1}{2} \frac{d}{dt} \left(\| (a_j, L_\varepsilon a_j, u_j) \|_{L^2}^2 - 2 \int_{\mathbb{R}^d} a_j \operatorname{div} u_j dx \right) &+ \| u_j \|_{L^2}^2 + \int_{\mathbb{R}^d} (1+c) |\nabla L_\varepsilon a_j|^2 dx \\ &- \int_{\mathbb{R}^d} (1+b)(\operatorname{div} u_j)^2 dx = I_j^1 - I_j^2. \end{aligned}$$

Next, to handle the term with $\| \nabla \mathcal{P} u_j \|_{L^2}^2$, we apply $\dot{\Delta}_j \mathcal{P}$ to the velocity equation and get

$$\mathcal{P} u_{j,t} + \mathcal{P} u_j + v \cdot \nabla \mathcal{P} u_j = [v, \dot{\Delta}_j \mathcal{P}] \cdot \nabla u - \mathcal{P} \dot{\Delta}_j (c \nabla K_\varepsilon a),$$

which immediately implies after taking the scalar product with $\mathcal{P} u_j$ and integrating by parts in the convection term:

$$(4.28) \quad \begin{aligned} \frac{1}{2} \frac{d}{dt} \| \nabla \mathcal{P} u_j \|_{L^2}^2 + \| \nabla \mathcal{P} u_j \|_{L^2}^2 &= I_j^3 := \frac{1}{2} \int_{\mathbb{R}^d} \operatorname{div} v |\nabla \mathcal{P} u_j|^2 dx \\ - \int_{\mathbb{R}^d} \operatorname{Tr}(\nabla \mathcal{P} u_j \cdot D \mathcal{P} u_j \cdot D v) dx &+ \int_{\mathbb{R}^d} \nabla \mathcal{P} u_j \cdot (\nabla [v, \dot{\Delta}_j \mathcal{P}] \cdot \nabla u) dx - \int_{\mathbb{R}^d} \nabla \mathcal{P} u_j \cdot \nabla \mathcal{P} \dot{\Delta}_j (c \nabla K_\varepsilon a) dx. \end{aligned}$$

Let us finally compute the time derivative of $\| \sqrt{1+c} \nabla L_\varepsilon a_j \|_{L^2}^2 + \| \sqrt{1+b} \operatorname{div} u_j \|_{L^2}^2$. First, taking the L^2 scalar product of the following relation

$$\nabla L_\varepsilon a_{j,t} + \nabla L_\varepsilon \dot{\Delta}_j ((1+b) \operatorname{div} u) + \nabla L_\varepsilon \dot{\Delta}_j (v \cdot \nabla a) = 0$$

with $(1+c) \nabla L_\varepsilon a_j$, and using the fact that

$$\begin{aligned} \int_{\mathbb{R}^d} (1+c) \nabla L_\varepsilon \dot{\Delta}_j (v \cdot \nabla a) \cdot \nabla L_\varepsilon a_j dx &= \int_{\mathbb{R}^d} (1+c) [\nabla L_\varepsilon \dot{\Delta}_j, v \cdot \nabla] a \cdot \nabla L_\varepsilon a_j dx \\ &\quad - \frac{1}{2} \int_{\mathbb{R}^d} |\nabla L_\varepsilon a_j|^2 \operatorname{div} ((1+c)v) dx, \end{aligned}$$

we find that

$$(4.29) \quad \frac{1}{2} \frac{d}{dt} \left(\int_{\mathbb{R}^d} (1+c) |\nabla L_\varepsilon a_j|^2 dx - \int_{\mathbb{R}^d} (c_t + \operatorname{div}((1+c)v)) |\nabla L_\varepsilon a_j|^2 dx \right) \\ + \int_{\mathbb{R}^d} (1+c) \nabla L_\varepsilon \dot{\Delta}_j ((1+b) \operatorname{div} u) \cdot \nabla L_\varepsilon a_j dx + \int_{\mathbb{R}^d} (1+c) [\nabla L_\varepsilon \dot{\Delta}_j, v \cdot \nabla] a \cdot \nabla L_\varepsilon a_j dx = 0.$$

Next, because

$$\operatorname{div} u_{j,t} + \operatorname{div} u_j + \operatorname{div}((1+c) \nabla K_\varepsilon a_j) + \operatorname{div}(v \cdot \nabla u_j) = \operatorname{div}[v, \dot{\Delta}_j] \cdot \nabla u + \operatorname{div}[c, \dot{\Delta}_j] \cdot \nabla K_\varepsilon a,$$

we discover that

$$(4.30) \quad \frac{1}{2} \left(\frac{d}{dt} \int_{\mathbb{R}^d} (1+b) (\operatorname{div} u_j)^2 dx - \int_{\mathbb{R}^d} b_t (\operatorname{div} u_j)^2 dx \right) + \int_{\mathbb{R}^d} (1+b) (\operatorname{div} u_j)^2 dx \\ + \int_{\mathbb{R}^d} ((1+b) \operatorname{div} u_j) \operatorname{div}((1+c) \nabla K_\varepsilon a_j) dx + \int_{\mathbb{R}^d} (1+b) \operatorname{div} u_j \operatorname{div}(v \cdot \nabla u_j) dx \\ = \int_{\mathbb{R}^d} (1+b) \operatorname{div} u_j \operatorname{div}[v, \dot{\Delta}_j] \cdot \nabla u dx.$$

Due to (3.12), we have

$$\int_{\mathbb{R}^d} (1+b) (\operatorname{div} u_j) \operatorname{div}(v \cdot \nabla u_j) dx - \frac{1}{2} \int_{\mathbb{R}^d} b_t (\operatorname{div} u_j)^2 dx = \int_{\mathbb{R}^d} (1+b) \operatorname{div} u_j \operatorname{Tr}(Dv \cdot Du_j) dx.$$

Hence adding up (4.29) and (4.30) leads to

$$(4.31) \quad \frac{1}{2} \frac{d}{dt} \int_{\mathbb{R}^d} (|\nabla L_\varepsilon a_j|^2 + (1+b) (\operatorname{div} u_j)^2) dx + \int_{\mathbb{R}^d} (1+b) (\operatorname{div} u_j)^2 dx = \sum_{i=0}^6 R_j^i$$

$$\text{with } R_j^0 := \int_{\mathbb{R}^d} (1+b) \operatorname{div} u_j \operatorname{div}[c, \dot{\Delta}_j] \nabla K_\varepsilon a dx, \\ R_j^1 := \int_{\mathbb{R}^d} (1+c) \nabla L_\varepsilon [b, \dot{\Delta}_j] \operatorname{div} u \cdot \nabla L_\varepsilon a_j dx, \\ R_j^2 := - \int_{\mathbb{R}^d} (1+c) [\nabla L_\varepsilon \dot{\Delta}_j, v \cdot \nabla] a \cdot \nabla L_\varepsilon a_j dx, \\ R_j^3 := - \int_{\mathbb{R}^d} (1+b) \operatorname{div} u_j \operatorname{Tr}(Dv \cdot Du_j) dx, \\ R_j^4 := \int_{\mathbb{R}^d} (1+b) \operatorname{div} u_j \operatorname{div}[v, \dot{\Delta}_j] \cdot \nabla u dx \\ R_j^5 := \int_{\mathbb{R}^d} (1+b) \operatorname{div} u_j \operatorname{div}[L_\varepsilon, c] \nabla L_\varepsilon a_j dx. \\ \text{and } R_j^6 := \frac{1}{2} \int_{\mathbb{R}^d} (c_t + \operatorname{div}((1+c)v)) |\nabla L_\varepsilon a_j|^2 dx.$$

To get (4.31), the key point is the following cancellation property between the third term of (4.29) and the fourth term of (4.30):

$$\int_{\mathbb{R}^d} ((1+b) \operatorname{div} u_j) \operatorname{div}((1+c) \nabla K_\varepsilon a_j) dx = -R_j^5 - \int_{\mathbb{R}^d} ((1+c) \nabla L_\varepsilon ((1+b) \operatorname{div} u_j) \cdot \nabla L_\varepsilon a_j) dx \\ = -R_j^5 - R_j^1 - \int_{\mathbb{R}^d} (1+c) \nabla L_\varepsilon \dot{\Delta}_j ((1+b) \operatorname{div} u) \cdot \nabla L_\varepsilon a_j dx.$$

So finally, adding (4.28) and twice (4.31) to (4.27), we discover that

$$(4.32) \quad \frac{1}{2} \frac{d}{dt} \mathcal{L}_j^2 + \mathcal{H}_j^2 = I_j^1 - I_j^2 + I_j^3 + 2 \sum_{i=0}^5 R_j^i.$$

Now, it is just a matter of bounding all the terms of the right-hand side. The most tricky part is to estimate the commutator terms in I_j^1 , R_j^1 and R_j^2 . For expository purpose, we admit these estimates, the reader being referred to Subsection 7.1 for the proof.

Estimating I_j^1 . From Hölder inequality, we have

$$\begin{aligned} I_j^1 &\leq \|a_j\|_{L^2} \|\dot{\Delta}_j(b \operatorname{div} u)\|_{L^2} + \|L_\varepsilon a_j\|_{L^2} \|L_\varepsilon \dot{\Delta}_j(b \operatorname{div} u)\|_{L^2} + \|[\dot{\Delta}_j, v] \cdot \nabla a\|_{L^2} \|a_j\|_{L^2} \\ &\quad + \|u_j\|_{L^2} \|\dot{\Delta}_j(c \nabla K_\varepsilon a)\|_{L^2} + \|[v, \dot{\Delta}_j] \cdot \nabla u\|_{L^2} \|u_j\|_{L^2} \\ &\quad + \|[L_\varepsilon \dot{\Delta}_j, v] \cdot \nabla a\|_{L^2} \|L_\varepsilon a_j\|_{L^2} + \frac{1}{2} (\|a_j\|_{L^2}^2 + \|L_\varepsilon a_j\|_{L^2}^2 + \|u_j\|_{L^2}^2) \|\operatorname{div} v\|_{L^\infty}. \end{aligned}$$

The terms with $b \operatorname{div} u$ may be bounded thanks to the product laws (7.1) and (7.2) with $f = b$ and $g = \operatorname{div} u$, and the commutators, by means of (7.4) and (7.15).

In the end, we get

$$(4.33) \quad \begin{aligned} I_j^1 &\leq C c_j 2^{-j\sigma} (\|\nabla v\|_{\dot{B}_{2,1}^{\frac{d}{2}}} \| (a, u) \|_{\dot{B}_{2,1}^\sigma} + \|b\|_{\dot{B}_{2,1}^{\frac{d}{2}}} \|\operatorname{div} u\|_{\dot{B}_{2,1}^\sigma} + \|c\|_{\dot{B}_{2,1}^{\frac{d}{2}}} \|\nabla K_\varepsilon a\|_{\dot{B}_{2,1}^\sigma} \\ &\quad + \|c\|_{\dot{B}_{2,1}^\sigma} \|\nabla K_\varepsilon a\|_{L^\infty} + \|\operatorname{div} u\|_{L^\infty} \|b\|_{\dot{B}_{2,1}^\sigma}) \| (a_j, u_j) \|_{L^2}, \end{aligned}$$

and the last two terms are not needed if $\sigma \leq d/2$.

Estimating I_j^2 . From Hölder inequality, we infer that

$$\begin{aligned} I_j^2 &\leq \|a_j\|_{L^2} (\|\nabla u_j\|_{L^2} \|\nabla v\|_{L^\infty} + \|\operatorname{div} [v, \dot{\Delta}_j] \cdot \nabla u\|_{L^2} + \|\operatorname{div} \dot{\Delta}_j [L_\varepsilon, c] \nabla L_\varepsilon a\|_{L^2}) \\ &\quad + \|\nabla L_\varepsilon a_j\|_{L^2} \|[\dot{\Delta}_j, c] \nabla L_\varepsilon a\|_{L^2} + \|\operatorname{div} u_j\|_{L^2} (\|[v, \dot{\Delta}_j] \nabla a\|_{L^2} + \|[b, \dot{\Delta}_j] \operatorname{div} u\|_{L^2}). \end{aligned}$$

The last two terms may be bounded by means of (7.4). For the one with $\operatorname{div} [v, \dot{\Delta}_j] \cdot \nabla u$, we use the decomposition

$$\partial_k [v, \dot{\Delta}_j] \cdot \nabla u = \partial_k v \cdot \nabla u_j - \dot{\Delta}_j (\partial_k v \cdot \nabla u) + [v, \dot{\Delta}_j] \nabla \partial_k u.$$

The L^2 norm of the last term of the right-hand side may be bounded according to (7.4). The L^2 norm of the first one is obviously bounded by $\|\nabla v\|_{L^\infty} \|\nabla u_j\|_{L^2}$. To bound the second term, one can take advantage of the product laws (7.1) and (7.2) with $f = \partial_k v$ and $g = \nabla u$. Finally, thanks to (7.6), we have

$$\begin{aligned} \|\operatorname{div} \dot{\Delta}_j [L_\varepsilon, c] \nabla L_\varepsilon a\|_{L^2} &\lesssim c_j 2^{-j\sigma} \|\operatorname{div} [L_\varepsilon, c] \nabla L_\varepsilon a\|_{\dot{B}_{2,1}^\sigma} \\ &\lesssim c_j 2^{-j\sigma} \|[L_\varepsilon, c] \nabla L_\varepsilon a\|_{\dot{B}_{2,1}^{\sigma+1}} \\ &\lesssim c_j 2^{-j\sigma} (\|\nabla c\|_{\dot{B}_{2,1}^{\frac{d}{2}}} \|L_\varepsilon a\|_{\dot{B}_{2,1}^{\sigma+1}} + \|\nabla c\|_{\dot{B}_{2,1}^\sigma} \|\nabla L_\varepsilon a\|_{L^\infty}) \end{aligned}$$

and, by virtue of (7.4),

$$\|[\dot{\Delta}_j, c] \nabla L_\varepsilon a\|_{L^2} \lesssim c_j 2^{-j\sigma} \|\nabla c\|_{\dot{B}_{2,1}^{\frac{d}{2}}} \|L_\varepsilon a\|_{\dot{B}_{2,1}^\sigma}.$$

This leads to

$$(4.34) \quad I_j^2 \lesssim c_j 2^{-j\sigma} \left(\|\operatorname{div} u_j\|_{L^2} \left(\|\nabla v\|_{\dot{B}_{2,1}^{\frac{d}{2}}} \|a\|_{\dot{B}_{2,1}^\sigma} + \|\nabla b\|_{\dot{B}_{2,1}^{\frac{d}{2}}} \|u\|_{\dot{B}_{2,1}^\sigma} \right) \right. \\ \left. + \|a_j\|_{L^2} \left(\|\nabla v\|_{\dot{B}_{2,1}^{\frac{d}{2}}} \|\nabla u\|_{\dot{B}_{2,1}^\sigma} + \|\nabla v\|_{\dot{B}_{2,1}^\sigma} \|\nabla u\|_{L^\infty} + \|\nabla c\|_{\dot{B}_{2,1}^{\frac{d}{2}}} \|L_\varepsilon a\|_{\dot{B}_{2,1}^{\sigma+1}} + \|\nabla c\|_{\dot{B}_{2,1}^\sigma} \|\nabla L_\varepsilon a\|_{L^\infty} \right) \right. \\ \left. + \|\nabla L_\varepsilon a_j\|_{L^2} \|\nabla c\|_{\dot{B}_{2,1}^{\frac{d}{2}}} \|L_\varepsilon a\|_{\dot{B}_{2,1}^\sigma} \right).$$

and the terms with L^∞ norms are not needed if $\sigma \leq d/2$.

Estimating I_j^3 . We start with the obvious inequality:

$$I_j^3 \leq C \|\nabla \mathcal{P} u_j\|_{L^2} \left(\|\nabla v\|_{L^\infty} \|\nabla \mathcal{P} u_j\|_{L^2} + \|\nabla[v, \dot{\Delta}_j \mathcal{P}] \cdot \nabla u\|_{L^2} + \|\nabla \mathcal{P} \dot{\Delta}_j (c \nabla K_\varepsilon a)\|_{L^2} \right)$$

and write that

$$\partial_k [v, \dot{\Delta}_j \mathcal{P}] \cdot \nabla u = \partial_k v \cdot \nabla \mathcal{P} u_j - \dot{\Delta}_j \mathcal{P} (\partial_k v \cdot \nabla u) + [v, \dot{\Delta}_j \mathcal{P}] \nabla \partial_k u.$$

The commutator with v may be bounded as the similar term in I_j^2 . As for the last term of I_j^3 , we observe that, since $\mathcal{P} \nabla = 0$, we have

$$\nabla \mathcal{P} (c \nabla K_\varepsilon a) = [\nabla \mathcal{P}, c] \nabla K_\varepsilon a.$$

Hence this term may be handled by (7.5) with the constant operator \mathcal{P} instead of L_ε . We end up with

$$(4.35) \quad I_j^3 \leq C c_j 2^{-j\sigma} \left(\|\nabla v\|_{\dot{B}_{2,1}^{\frac{d}{2}}} \|\nabla u\|_{\dot{B}_{2,1}^\sigma} + \|\nabla v\|_{\dot{B}_{2,1}^\sigma} \|\nabla u\|_{L^\infty} \right. \\ \left. + \|c\|_{\dot{B}_{2,1}^{\frac{d}{2}+1}} \|\nabla K_\varepsilon a\|_{\dot{B}_{2,1}^\sigma} + \|c\|_{\dot{B}_{2,1}^{\sigma+1}} \|\nabla K_\varepsilon a\|_{L^\infty} \right) \|\nabla \mathcal{P} u_j\|_{L^2},$$

and the terms with the L^∞ norm are not needed if $\sigma \leq d/2$.

Estimating R_j^0 . To bound this term, it suffices to apply Hölder inequality then to use (7.18) with L_0 (that is, the identity operator), $b = c$ and $z = \nabla K_\varepsilon a$. We get

$$(4.36) \quad R_j^0 \leq C c_j 2^{-j\sigma} \|\operatorname{div} u_j\|_{L^2} \left(\|\nabla c\|_{\dot{B}_{2,1}^{\frac{d}{2}}} \|\nabla K_\varepsilon a\|_{\dot{B}_{2,1}^\sigma} + \|\nabla c\|_{\dot{B}_{2,1}^\sigma} \|\nabla K_\varepsilon a\|_{L^\infty} \right)$$

and the last term is not needed if $\sigma \leq d/2$.

Estimating R_j^1 . Remembering that $\|c\|_{L^\infty}$ is small and applying Inequality (7.18) to $z = \operatorname{div} u$, we readily get

$$(4.37) \quad R_j^1 \leq C c_j 2^{-j\sigma} \|\nabla L_\varepsilon a_j\|_{L^2} \left(\|\nabla b\|_{\dot{B}_{2,1}^{\frac{d}{2}}} \|\operatorname{div} u\|_{\dot{B}_{2,1}^\sigma} + \|\operatorname{div} u\|_{L^\infty} \|\nabla L_\varepsilon b\|_{\dot{B}_{2,1}^\sigma} \right)$$

and the last term is not needed if $\sigma \leq d/2$.

Estimating R_j^2 . Because we strive for bounds that are independent of ε , we have to assume that $\nabla^2 v \in L^1(\mathbb{R}_+; L^\infty)$, that is one more space derivative than for the classical compressible Euler system. Now, leveraging Inequality (7.21), we readily get

$$(4.38) \quad R_j^2 \leq C c_j 2^{-j\sigma} \|\nabla L_\varepsilon a_j\|_{L^2} \left(\|a\|_{\dot{B}_{2,1}^\sigma} \|v\|_{\dot{B}_{2,1}^{\frac{d}{2}+2}} + \|v\|_{\dot{B}_{2,1}^{\frac{d}{2}+1}} \|\nabla L_\varepsilon a\|_{\dot{B}_{2,1}^\sigma} \right).$$

Estimating R_j^3 . Under assumption (4.2), it is obvious that

$$(4.39) \quad R_j^3 \lesssim \|\nabla v\|_{L^\infty} \|\nabla u_j\|_{L^2}^2 \lesssim c_j 2^{-j\sigma} \|\nabla u_j\|_{L^2} \|\nabla u\|_{\dot{B}_{2,1}^\sigma} \|\nabla v\|_{\dot{B}_{2,1}^{\frac{d}{2}}}.$$

Estimating R_j^4 . With the summation convention on repeated indices, we have

$$\operatorname{div}[v^m, \dot{\Delta}_j] \partial_m u = \partial_k v^m \partial_m u_j^k - \dot{\Delta}_j (\partial_k v^m \partial_m u^k) + [v^m, \dot{\Delta}_j] \partial_m \operatorname{div} u.$$

So, using (7.4) as well as product laws (7.1) and (7.2), we get

$$(4.40) \quad R_j^4 \leq C c_j 2^{-j\sigma} \|\operatorname{div} u_j\|_{L^2} \left(\|\nabla v\|_{\dot{B}_{2,1}^{\frac{d}{2}}} \|\nabla u\|_{\dot{B}_{2,1}^\sigma} + \|\nabla u\|_{L^\infty} \|\nabla v\|_{\dot{B}_{2,1}^\sigma} \right),$$

and the second term may be omitted if $\sigma \leq d/2$.

Estimating R_j^5 . By Hölder inequality and Condition (4.2), we have

$$R_j^5 \lesssim \|\operatorname{div} u_j\|_{L^2} \|\operatorname{div}[L_\varepsilon, c] \nabla L_\varepsilon a_j\|_{L^2}.$$

We observe that

$$\operatorname{div}[L_\varepsilon, c] \nabla L_\varepsilon a_j = L_\varepsilon (\nabla c \cdot \nabla L_\varepsilon a_j) - \nabla c \cdot L_\varepsilon \nabla L_\varepsilon a_j + [L_\varepsilon, c] \Delta L_\varepsilon a_j.$$

On the one hand, due to (3.1), it is obvious that

$$\|L_\varepsilon (\nabla c \cdot \nabla L_\varepsilon a_j)\|_{L^2} + \|\nabla c \cdot L_\varepsilon \nabla L_\varepsilon a_j\|_{L^2} \lesssim \|\nabla c\|_{L^\infty} \|\nabla L_\varepsilon a_j\|_{L^2}.$$

On the other hand, (7.19) and Bernstein inequality guarantee that

$$\|[L_\varepsilon, c] \Delta L_\varepsilon a_j\|_{L^2} \lesssim \|\nabla c\|_{L^\infty} \|\nabla L_\varepsilon a_j\|_{L^2}.$$

Hence

$$(4.41) \quad R_j^5 \leq C c_j 2^{-j\sigma} \|\operatorname{div} u_j\|_{L^2} \|\nabla c\|_{L^\infty} \|\nabla L_\varepsilon a_j\|_{\dot{B}_{2,1}^\sigma}.$$

Estimating R_j^6 . Finally, we have

$$(4.42) \quad \begin{aligned} R_j^6 &\leq \frac{1}{2} \|c_t + \operatorname{div}((1+c)v)\|_{L^\infty} \|\nabla L_\varepsilon a_j\|_{L^2}^2 \\ &\leq \frac{C_j}{2} 2^{-j\sigma} \|c_t + \operatorname{div}((1+c)v)\|_{L^\infty} \|\nabla L_\varepsilon a_j\|_{L^2} \|\nabla L_\varepsilon a_j\|_{\dot{B}_{2,1}^\sigma}. \end{aligned}$$

Third step: Putting everything together. Plugging (4.33), (4.34), (4.35), (4.36), (4.37), (4.38), (4.39), (4.40), (4.41) and (4.42) in (4.32) and remembering (4.22), we arrive (for some universal constant κ_0) at

$$\begin{aligned} &\frac{1}{2} \frac{d}{dt} \mathcal{L}_j^2 + \kappa_0 \min(1, 2^{2j} \widehat{K}_\varepsilon(2^j)) \mathcal{L}_j^2 \leq C c_j 2^{-j\sigma} \mathcal{L}_j \left(\|v\|_{\dot{B}_{2,1}^{\frac{d}{2}+1}} \|(a, \nabla L_\varepsilon a, u, \nabla u)\|_{\dot{B}_{2,1}^\sigma} \right. \\ &+ \|(b, \nabla b)\|_{\dot{B}_{2,1}^{\frac{d}{2}}} \|\operatorname{div} u\|_{\dot{B}_{2,1}^\sigma} + \|\nabla b\|_{\dot{B}_{2,1}^{\frac{d}{2}}} \|u\|_{\dot{B}_{2,1}^\sigma} + \|v\|_{\dot{B}_{2,1}^{\frac{d}{2}+2}} \|a\|_{\dot{B}_{2,1}^\sigma} + \|\nabla u\|_{L^\infty} \|(b, \nabla L_\varepsilon b, \nabla v)\|_{\dot{B}_{2,1}^\sigma} \\ &\quad \left(\|\nabla c\|_{\dot{B}_{2,1}^{\frac{d}{2}}} + \|c_t + \operatorname{div}((1+c)v)\|_{L^\infty} \right) \|\nabla L_\varepsilon a\|_{\dot{B}_{2,1}^\sigma} + \|\nabla c\|_{\dot{B}_{2,1}^{\frac{d}{2}}} \|L_\varepsilon a\|_{\dot{B}_{2,1}^\sigma} \\ &\quad \left. + \|c\|_{L^\infty} \|\nabla K_\varepsilon a\|_{\dot{B}_{2,1}^\sigma} + \|c\|_{\dot{B}_{2,1}^\sigma} \|\nabla K_\varepsilon a\|_{L^\infty} + \|\nabla c\|_{\dot{B}_{2,1}^\sigma} \|\nabla L_\varepsilon a\|_{L^\infty} \right), \end{aligned}$$

where κ_0 only depends on c , and the terms involving L^∞ norms of $L_\varepsilon a$ or u are not needed if $\sigma \leq d/2$.

Then, simplifying by \mathcal{L}_j and integrating on $[0, t]$ yields

$$\begin{aligned} \mathcal{L}_j(t) + \kappa_0 \min(1, 2^{2j} \widehat{K}_\varepsilon(2^j)) \int_0^t \mathcal{L}_j d\tau &\leq \mathcal{L}_j(0) + C2^{-j\sigma} \int_0^t c_j \left(\|v\|_{\dot{B}_{2,1}^{\frac{d}{2}+1}} \|(a, \nabla L_\varepsilon a, u, \nabla u)\|_{\dot{B}_{2,1}^\sigma} \right. \\ &+ \|b, \nabla b\|_{\dot{B}_{2,1}^{\frac{d}{2}}} \|\operatorname{div} u\|_{\dot{B}_{2,1}^\sigma} + \|\nabla b\|_{\dot{B}_{2,1}^{\frac{d}{2}}} \|u\|_{\dot{B}_{2,1}^\sigma} + \|v\|_{\dot{B}_{2,1}^{\frac{d}{2}+2}} \|a\|_{\dot{B}_{2,1}^\sigma} + \|\nabla u\|_{L^\infty} \|(b, \nabla L_\varepsilon b, \nabla v)\|_{\dot{B}_{2,1}^\sigma} \Big) d\tau \\ &+ C2^{-j\sigma} \int_0^t c_j \left((\|\nabla c\|_{\dot{B}_{2,1}^{\frac{d}{2}}} + \|c_t + \operatorname{div}((1+c)v)\|_{L^\infty}) \|\nabla L_\varepsilon a\|_{\dot{B}_{2,1}^\sigma} + \|\nabla c\|_{\dot{B}_{2,1}^{\frac{d}{2}}} \|L_\varepsilon a\|_{\dot{B}_{2,1}^\sigma} \right) d\tau \\ &+ C2^{-j\sigma} \int_0^t c_j \left(\|c\|_{L^\infty} \|\nabla K_\varepsilon a\|_{\dot{B}_{2,1}^\sigma} + \|c\|_{\dot{B}_{2,1}^\sigma} \|\nabla K_\varepsilon a\|_{L^\infty} + \|\nabla c\|_{\dot{B}_{2,1}^\sigma} \|\nabla L_\varepsilon a\|_{L^\infty} \right) d\tau. \end{aligned}$$

Multiplying by $2^{j\sigma}$, summing up on all $j \in \mathbb{Z}$ and using again (4.22), we conclude that

$$\begin{aligned} (4.43) \quad &\|(a, u, \nabla u)(t)\|_{\dot{B}_{2,1}^\sigma} + \|\nabla L_\varepsilon a(t)\|_{\dot{B}_{2,1}^h} + \int_0^t \left(\|(\nabla^2 K_\varepsilon u, \nabla^3 K_\varepsilon u)\|_{\dot{B}_{2,1}^\ell}^\ell + \|(u, \nabla u)\|_{\dot{B}_{2,1}^\sigma}^h \right) d\tau \\ &+ \int_0^t \left(\|\nabla^2 K_\varepsilon a\|_{\dot{B}_{2,1}^\ell}^\ell + \|\nabla L_\varepsilon a\|_{\dot{B}_{2,1}^h}^h \right) d\tau \lesssim \|(a_0, u_0, \nabla u_0)\|_{\dot{B}_{2,1}^\sigma} + \|\nabla L_\varepsilon a_0\|_{\dot{B}_{2,1}^h} \\ &+ \int_0^t \left(\|v\|_{\dot{B}_{2,1}^{\frac{d}{2}+1}} \|(a, \nabla L_\varepsilon a, u, \nabla u)\|_{\dot{B}_{2,1}^\sigma} + \|b, \nabla b\|_{\dot{B}_{2,1}^{\frac{d}{2}}} \|\operatorname{div} u\|_{\dot{B}_{2,1}^\sigma} \right. \\ &+ \|\nabla b\|_{\dot{B}_{2,1}^{\frac{d}{2}}} \|u\|_{\dot{B}_{2,1}^\sigma} + \|\nabla v\|_{\dot{B}_{2,1}^{\frac{d}{2}+1}} \|a\|_{\dot{B}_{2,1}^\sigma} + \|\nabla u\|_{L^\infty} \|(b, \nabla L_\varepsilon b, \nabla v)\|_{\dot{B}_{2,1}^\sigma} \Big) d\tau \\ &+ \int_0^t \left((\|\nabla c\|_{\dot{B}_{2,1}^{\frac{d}{2}}} + \|c_t + \operatorname{div}((1+c)v)\|_{L^\infty}) \|\nabla L_\varepsilon a\|_{\dot{B}_{2,1}^\sigma} + \|\nabla c\|_{\dot{B}_{2,1}^{\frac{d}{2}}} \|L_\varepsilon a\|_{\dot{B}_{2,1}^\sigma} \right) d\tau \\ &+ \int_0^t \left(\|c\|_{L^\infty} \|\nabla K_\varepsilon a\|_{\dot{B}_{2,1}^\sigma} + \|c\|_{\dot{B}_{2,1}^\sigma} \|\nabla K_\varepsilon a\|_{L^\infty} + \|\nabla c\|_{\dot{B}_{2,1}^\sigma} \|\nabla L_\varepsilon a\|_{L^\infty} \right) d\tau. \end{aligned}$$

where the terms with L^∞ norms of $\nabla L_\varepsilon a$ or u are not needed if $\sigma \leq d/2$.

Let us finally exhibit the L^1 -in-time control for $\|\nabla u\|_{\dot{B}_{2,1}^\ell}^\ell$ (note that it is not a consequence of the control of $\|u\|_{\dot{B}_{2,1}^\ell}^\ell$ since ‘low’ frequencies need not to be low!). We write that

$$\partial_k(\partial_k u) + \partial_k u + v \cdot \nabla \partial_k u = -\partial_k v \cdot \nabla u - \partial_k((1+c)\nabla K_\varepsilon a).$$

So, localizing this equation by means of $\dot{\Delta}_j$ then repeating essentially the same arguments as before (here we only need the most basic commutator estimate (7.4)), we arrive at

$$\begin{aligned} (4.44) \quad &\|\nabla u(t)\|_{\dot{B}_{2,1}^\ell}^\ell + \int_0^t \|\nabla u\|_{\dot{B}_{2,1}^\ell}^\ell d\tau \leq \|\nabla u_0\|_{\dot{B}_{2,1}^\ell}^\ell + \int_0^t \|\nabla((1+c)\nabla K_\varepsilon a)\|_{\dot{B}_{2,1}^\ell}^\ell d\tau \\ &+ C \int_0^t \|\nabla v\|_{\dot{B}_{2,1}^{\frac{d}{2}}} \|\nabla u\|_{\dot{B}_{2,1}^\sigma} d\tau + C \int_0^t \|\nabla u\|_{L^\infty} \|\nabla v\|_{\dot{B}_{2,1}^\sigma} d\tau, \end{aligned}$$

where, as usual, the last term is not needed if $\sigma \leq d/2$.

Note that $\|\nabla^2 K_\varepsilon a\|_{\dot{B}_{2,1}^\ell}^\ell$ can be controlled from (4.43). To handle the term $\nabla(c\nabla K_\varepsilon a)$, we use the decomposition (recall notation (3.5)):

$$\partial_k(c \partial_m K_\varepsilon a) = \partial_k c \partial_m K_\varepsilon a + c \partial_k \partial_m K_\varepsilon a^\ell + [c, \partial_k L_\varepsilon] \partial_m L_\varepsilon a^h + \partial_k L_\varepsilon (c \partial_m L_\varepsilon a^h).$$

Taking advantage of (7.1), (7.2) and (7.5), and using the low frequency cut-off in the last term, we get (with the usual convention if $\sigma \leq d/2$):

$$\begin{aligned} \|\partial_k c \partial_m K_\varepsilon a\|_{\dot{B}_{2,1}^\sigma} &\lesssim \|\partial_k c\|_{\dot{B}_{2,1}^{\frac{d}{2}}} \|\partial_m K_\varepsilon a\|_{\dot{B}_{2,1}^\sigma} + \|\partial_k c\|_{\dot{B}_{2,1}^\sigma} \|\partial_m K_\varepsilon a\|_{L^\infty}, \\ \|c \partial_k \partial_m K_\varepsilon a^\ell\|_{\dot{B}_{2,1}^\sigma} &\lesssim \|c\|_{\dot{B}_{2,1}^{\frac{d}{2}}} \|\partial_k \partial_m K_\varepsilon a^\ell\|_{\dot{B}_{2,1}^\sigma} + \|c\|_{\dot{B}_{2,1}^\sigma} \|\partial_k \partial_m K_\varepsilon a^\ell\|_{L^\infty}, \\ \|[c, \partial_k L_\varepsilon] \partial_m L_\varepsilon a^h\|_{\dot{B}_{2,1}^\sigma} &\lesssim \|\nabla c\|_{\dot{B}_{2,1}^{\frac{d}{2}}} \|\partial_m L_\varepsilon a^h\|_{\dot{B}_{2,1}^\sigma} + \|\nabla c\|_{\dot{B}_{2,1}^\sigma} \|\partial_m L_\varepsilon a^h\|_{L^\infty}, \\ \|\partial_k L_\varepsilon (c \partial_m L_\varepsilon a^h)\|_{\dot{B}_{2,1}^\sigma} &\lesssim \nu_0 (\|c\|_{\dot{B}_{2,1}^{\frac{d}{2}}} \|\partial_m L_\varepsilon a^h\|_{\dot{B}_{2,1}^\sigma} + \|c\|_{\dot{B}_{2,1}^\sigma} \|\partial_m L_\varepsilon a^h\|_{L^\infty}). \end{aligned}$$

Hence we have

$$\begin{aligned} \|\partial_k (c \partial_m K_\varepsilon a)\|_{\dot{B}_{2,1}^\sigma}^\ell &\lesssim \|\nabla c\|_{\dot{B}_{2,1}^{\frac{d}{2}}} \|\nabla L_\varepsilon a\|_{\dot{B}_{2,1}^\sigma} + \|\nabla c\|_{\dot{B}_{2,1}^\sigma} \|\nabla L_\varepsilon a\|_{L^\infty} \\ &+ \|c\|_{\dot{B}_{2,1}^\sigma} (\|\nabla^2 K_\varepsilon a^\ell\|_{L^\infty} + \|\nabla L_\varepsilon a^h\|_{L^\infty}) + \|c\|_{\dot{B}_{2,1}^{\frac{d}{2}}} (\|\nabla^2 K_\varepsilon a^\ell\|_{\dot{B}_{2,1}^\sigma} + \|\nabla L_\varepsilon a^h\|_{\dot{B}_{2,1}^\sigma}). \end{aligned}$$

Hence, putting (4.17), (4.43) and (4.44) together gives (4.5).

Let us finally consider the case where, in addition u_0 is in $\dot{B}_{2,1}^{\sigma-1}$. The starting point is (4.18). Since the term with $\nabla K_\varepsilon a$ in the right-hand side is controlled by (4.43) and because

$$\|w - u\|_{\dot{B}_{2,1}^{\sigma-1}}^\ell \leq C \|\nabla K_\varepsilon a\|_{\dot{B}_{2,1}^{\sigma-1}}^\ell \leq C \|a\|_{\dot{B}_{2,1}^{\sigma-1}}^\ell,$$

we have the low frequency part of (4.6). The high frequency part just stems from the fact that one can bound $\|u\|_{\dot{B}_{2,1}^{\sigma-1}}^h$ and $\|\nabla K_\varepsilon a\|_{\dot{B}_{2,1}^{\sigma-1}}^h$ by, say, $\|u\|_{\dot{B}_{2,1}^\sigma}^h$ and $\|\nabla a\|_{\dot{B}_{2,1}^\sigma}^h$, and thus by means of (4.5). In the end, we get (4.6). \square

5. PROVING WELL-POSEDNESS AND CONVERGENCE TO EULER

This section is devoted to proving Theorems 3.1 and 3.2. In the first subsection, we prove the existence part of Theorem 3.1 then, in second subsection, the uniqueness part. The end of the section is devoted to establishing the convergence of the solutions to (1.1) to those of (1.2) for ε tending to 0.

5.1. Existence. Before proving the existence part of Theorem 3.1, let us quickly explain why the results of the previous section allow to close the estimates for all time and uniformly with respect to ε in the desired functional space for any initial data satisfying (3.13).

To this end, we consider a smooth solution (ρ, u) of (1.1) on $[0, T) \times \mathbb{R}^d$ such that $a := \rho - 1$ satisfies $|a| \leq 1/4$. Then, applying Inequality (4.5) to the system satisfied by (a, u) (that is, to (3.11) with $b = a$, $c = 0$ and $v = u$) with $\sigma = d/2 + 1$ we get some absolute constant C such that for all $t \in [0, T)$,

$$X_{a,u}^{\frac{d}{2}+1}(t) + \int_0^t H_{a,u}^{\frac{d}{2}+1} d\tau \leq C \left(X_{a,u}^{\frac{d}{2}+1}(0) + \int_0^t \|\nabla u\|_{\dot{B}_{2,1}^{\frac{d}{2}} \cap \dot{B}_{2,1}^{\frac{d}{2}+1}} X_{a,u}^{\frac{d}{2}+1} d\tau \right).$$

Note indeed that the 2nd and 3rd lines of (4.5) then reduce to just $\|\nabla u\|_{\dot{B}_{2,1}^{\frac{d}{2}} \cap \dot{B}_{2,1}^{\frac{d}{2}+1}} X_{a,u}^{\frac{d}{2}+1}$. Now, it is obvious that

$$\|\nabla u\|_{\dot{B}_{2,1}^{\frac{d}{2}} \cap \dot{B}_{2,1}^{\frac{d}{2}+1}} \leq H_{a,u}^{\frac{d}{2}+1}.$$

Hence, we conclude by bootstrap that the smallness condition

$$2C^2 X_{a,u}^{\frac{d}{2}+1}(0) < 1$$

implies that

$$(5.1) \quad X_{a,u}^{\frac{d}{2}+1}(t) + \frac{1}{2} \int_0^t H_{a,u}^{\frac{d}{2}+1} d\tau \leq C X_{a,u}^{\frac{d}{2}+1}(0) \quad \text{for all } t \in [0, T].$$

Let us next move to the rigorous proof of existence of a solution for (1.1) under the assumptions of Theorem 3.1. For technical reasons, we will have assume that, in addition to (3.1), the kernel K_ε is such that $\nabla K_\varepsilon : L^2 \rightarrow H^1$. This is clearly achieved if $\nabla^2 K_\varepsilon$ is in $L^1_{loc}(\mathbb{R}^d)$ (since we already have $y^2 \nabla K_\varepsilon \in L^1(\mathbb{R}^d)$), and we note that there exists $M_\varepsilon \geq 0$ such that for all $s \in \mathbb{R}$,

$$(5.2) \quad \|\nabla K_\varepsilon\|_{\mathcal{L}(H^s; H^{s+1})} \leq M_\varepsilon.$$

Step 1. Solving Burgers equation with friction and smooth data. Here we consider

$$(5.3) \quad u_t + u \cdot \nabla u + u = f$$

supplemented with initial velocity $u_0 \in H^{s+1}$ and source term $f \in \mathcal{C}(\mathbb{R}_+; H^{s+1})$ with $s > d/2$.

The classical theory of symmetric hyperbolic systems (see e.g. [1, Chap. 4]) guarantees that (5.3) admits a unique maximal solution

$$u \in \mathcal{C}([0, T^*]; H^{s+1}) \cap \mathcal{C}^1([0, T^*]; H^s).$$

Furthermore, by combining an energy method and classical commutator estimates in Sobolev spaces, we have

$$(5.4) \quad \|u(t)\|_{H^{s+1}} \leq \|u_0\|_{H^{s+1}} + \int_0^t \|f\|_{H^{s+1}} + C \int_0^t \|\nabla u\|_{L^\infty} \|u\|_{H^{s+1}} d\tau \quad \text{for all } t \in [0, T^*],$$

whence, remembering the Sobolev embedding $H^s \hookrightarrow L^\infty$ (for $s > d/2$),

$$\sup_{\tau \in [0, t]} \|u(\tau)\|_{H^{s+1}} \leq \|u_0\|_{H^{s+1}} + \int_0^t \|f\|_{H^{s+1}} + Ct \sup_{\tau \in [0, t]} \|u(\tau)\|_{H^{s+1}}^2 \quad \text{for all } t \in [0, T^*].$$

This guarantees that there exists some constant c depending only on s and d , such that

$$T^* \geq \sup \left\{ t \geq 0, t \left(\|u_0\|_{H^{s+1}} + \int_0^t \|f\|_{H^{s+1}} d\tau \right) \leq c \right\}.$$

Step 2. Local existence for (1.1) supplemented with smooth data. Fix some $R_0 > 0$ and data $(a_0, u_0) \in H^s \times H^{s+1}$ with $s > d/2$, such that

$$\|a_0\|_{H^s} + \|u_0\|_{H^{s+1}} \leq R_0.$$

Our goal is to prove that there exists some $T > 0$ such that (1.1) has a solution

$$(5.5) \quad (a, u) \in \mathcal{C}([0, T]; H^s \times H^{s+1}) \cap \mathcal{C}^1([0, T]; H^{s-1} \times H^s).$$

To do so, we consider the map $\Psi : \tilde{a} \mapsto a$ where a is the solution to the transport equation

$$a_t + \operatorname{div}((1+a)u) = 0$$

and the transport field u is the solution in $\mathcal{C}([0, T]; H^{s+1}) \cap \mathcal{C}^1([0, T]; H^s)$ to the damped Burgers equation

$$u_t + u + u \cdot \nabla u = -\nabla K_\varepsilon \tilde{a}.$$

Owing to (5.2), the existence of u with the required regularity on some maximal time interval $[0, T^*)$ is guaranteed by the previous step. Then, the existence of $a \in \mathcal{C}([0, T^*]; H^s) \cap \mathcal{C}([0, T^*]; H^{s-1})$ follows from the standard theory of transport equations in Sobolev spaces.

We claim that one can find some $T \in (0, T^*)$ such that Ψ maps the closed ball $\bar{B}(0, R)$ of $\mathcal{C}([0, T]; H^s)$ to itself, with $R = 2R_0 + 1$. Indeed, combining an energy method and Gronwall lemma, it is easy to show that

$$\|a(t)\|_{H^s} \leq e^{C \int_0^t \|u\|_{H^{s+1}} d\tau} \|a_0\|_{H^s} + e^{C \int_0^t \|u\|_{H^{s+1}} d\tau} - 1, \quad t \in [0, T^*)$$

and, owing to (5.4) and (5.2),

$$\|u(t)\|_{H^{s+1}} \leq e^{C \int_0^t \|u\|_{H^{s+1}} d\tau} \left(\|u_0\|_{H^{s+1}} + M_\varepsilon \int_0^t \|\tilde{a}\|_{H^s} d\tau \right), \quad t \in [0, T^*).$$

If T is taken small enough then we have

$$C \int_0^T \|u\|_{H^{s+1}} d\tau \leq \log 2,$$

and thus, if $\|\tilde{a}\|_{L^\infty(0, T; H^s)} \leq R$,

$$\sup_{t \in [0, T]} \|a(t)\|_{H^s} \leq 2R_0 + 1 = R \quad \text{and} \quad \sup_{t \in [0, T]} \|u(t)\|_{H^{s+1}} \leq 2R_0 + 2RM_\varepsilon T.$$

Hence, to ensure our claim, it suffices to choose T such that

$$2RM_\varepsilon T \leq 1 \quad \text{and} \quad CTR \leq \log 2.$$

Next, since $a_t = -\operatorname{div}((1+a)u)$, we readily have $a_t \in \mathcal{C}([0, T]; H^{s-1})$ and

$$\sup_{t \in [0, T]} \|a_t(t)\|_{H^s} \leq C(1 + \sup_{t \in [0, T]} \|a(t)\|_{H^s}) \sup_{t \in [0, T]} \|u(t)\|_{H^s} \leq CR(1 + R).$$

Hence $(\Psi(\tilde{a}))_t$ remains in a bounded set of $\mathcal{C}([0, T]; H^{s-1})$. Remembering that the embedding of $H^s(\mathbb{R}^d)$ in $H^{s-1}(\mathbb{R}^d)$ is locally compact, Schauder theorem guarantees that Ψ admits a fixed point a in $L^\infty(0, T; H^s)$. Back to the equation of u , we deduce that u is in $\mathcal{C}([0, T]; H^{s+1})$ then, using once more the equation of a , that a is in $\mathcal{C}([0, T]; H^s)$. Finally, computing the time derivative of a and u from the equation, we conclude to (5.5).

Step 3. A continuation criterion. Let T^* be the lifespan of the solution (a, u) constructed in the previous step. On the one hand, applying (5.4) with $f = -\nabla K_\varepsilon a$, we see that

$$(5.6) \quad \|u(t)\|_{H^{s+1}} \leq \|u_0\|_{H^{s+1}} + M_\varepsilon \int_0^t \|a\|_{H^s} + C \int_0^t \|\nabla u\|_{L^\infty} \|u\|_{H^{s+1}} d\tau \quad \text{for all } t \in [0, T^*).$$

On the other hand, using the standard estimates in Sobolev spaces for the transport equation and the product law

$$\|a \operatorname{div} u\|_{H^s} \lesssim \|a\|_{L^\infty} \|\operatorname{div} u\|_{H^s} + \|a\|_{H^s} \|\operatorname{div} u\|_{L^\infty},$$

we get for all $t \in [0, T^*)$,

$$(5.7) \quad \|a(t)\|_{H^s} \leq \|a_0\|_{H^s} + C \int_0^t \|u\|_{H^{s+1}} (1 + \|a\|_{L^\infty}) d\tau + C \int_0^t \|\nabla u\|_{L^\infty} \|a\|_{H^s} d\tau.$$

Putting (5.6) and (5.7) together, then using Gronwall lemma yields for all $t \in [0, T^*)$,

$$\|a(t)\|_{H^s} + \|u(t)\|_{H^{s+1}} \leq (\|a_0\|_{H^s} + \|u_0\|_{H^{s+1}}) e^{(C+M_\varepsilon) \int_0^t (1 + \|a\|_{L^\infty} + \|\nabla u\|_{L^\infty}) d\tau},$$

whence the following blow-up criterion:

$$(5.8) \quad T^* < \infty \implies \int_0^{T^*} (\|a\|_{L^\infty} + \|\nabla u\|_{L^\infty}) dt = \infty.$$

Step 4. Global existence for System (1.1) with data in Sobolev spaces. Fix a pair (a_0, u_0) satisfying the smallness assumption of Theorem 3.1, and consider a sequence $(a_0^{(n)}, u_0^{(n)})_{n \in \mathbb{N}}$ of smooth initial data such that

$$(5.9) \quad \begin{aligned} (a_0^{(n)}, u_0^{(n)}) &\rightarrow (a_0, u_0) \text{ in } \dot{B}_{2,1}^{d/2} \cap \dot{B}_{2,1}^{d/2+1}(\mathbb{R}^d) \\ \text{and } (\nabla^2 K_\varepsilon a_0^{(n)}, \nabla^2 u_0^{(n)}) &\rightarrow (\nabla^2 K_\varepsilon a_0, \nabla^2 u_0) \text{ in } \dot{B}_{2,1}^{d/2}(\mathbb{R}^d). \end{aligned}$$

One can for instance set $a_0^{(n)} := (\dot{S}_n - \dot{S}_{-n})a_0$ and $u_0^{(n)} := (\dot{S}_n - \dot{S}_{-n})u_0$ so that (3.7) is satisfied by $(a_0^{(n)}, u_0^{(n)})$ for all $n \in \mathbb{N}$.

The previous steps guarantee that (1.1) supplemented with initial data $(a_0^{(n)}, u_0^{(n)})$ has a unique maximal solution $(a^{(n)}, u^{(n)})$ in, say,

$$\mathcal{C}([0, T^{(n)}); H^{\frac{d}{2}+3} \times H^{\frac{d}{2}+4}) \cap \mathcal{C}^1([0, T^{(n)}); H^{\frac{d}{2}+2} \times H^{\frac{d}{2}+3}).$$

We thus have enough regularity to apply the estimates of Theorem 4.1 with $\sigma = d/2 + 1$, $a = b = a^{(n)}$, $u = v = u^{(n)}$ and $c = 0$. Following the proof of (5.1), we conclude that if α_0 in (3.7) is small enough, then we have

$$\sup_{t \in [0, T^{(n)})} X_{a^{(n)}, u^{(n)}}^{\frac{d}{2}+1}(t) + \frac{1}{2} \int_0^{T^{(n)}} H_{a^{(n)}, u^{(n)}}^{\frac{d}{2}+1}(\tau) d\tau \leq C\alpha_0 \quad \text{for all } t \in [0, T^{(n)}).$$

This implies that both $a^{(n)}(t)$ and $\nabla u^{(n)}(t)$ remain in a bounded set of L^∞ . Hence the blow-up criterion (5.8) ensures that $T^{(n)} = \infty$.

Note also that the second estimate of Theorem 4.1 provides a uniform control on $u^{(n)}$ and $w^{(n)}$ in $L^\infty(\mathbb{R}_+; \dot{B}_{2,1}^{\frac{d}{2}})$.

Step 5. Passing to the limit. In the previous step, we constructed a sequence of smooth global solutions of (1.1) pertaining to smooth data, that is bounded in the space $E_{K_\varepsilon}^{\frac{d}{2}+1}$, and thus in

$$F_{K_\varepsilon}^{\frac{d}{2}+1} := \{(b, v) \in L^\infty(\mathbb{R}_+; \dot{B}_{2,1}^{\frac{d}{2}} \cap \dot{B}_{2,1}^{\frac{d}{2}+1}) \times L^\infty(\mathbb{R}_+; \dot{B}_{2,1}^{\frac{d}{2}} \cap \dot{B}_{2,1}^{\frac{d}{2}+2}), \nabla^2 K_\varepsilon \in L^\infty(\mathbb{R}_+; \dot{B}_{2,1}^{\frac{d}{2}})\}.$$

This latter space being the dual of some separable Banach space, one may deduce that there exists (a, u) in $F_{K_\varepsilon}^{\frac{d}{2}+1}$ such that, up to an omitted extraction, we have

$$(a^{(n)}, u^{(n)}) \rightharpoonup (a, u) \text{ weak } * \text{ in } F_{K_\varepsilon}^{\frac{d}{2}+1}.$$

Furthermore, as

$$u_t^{(n)} = -u^{(n)} - u^{(n)} \cdot \nabla u^{(n)} - \nabla K_\varepsilon a^{(n)},$$

the sequence $(u^{(n)})_{n \in \mathbb{N}}$ is bounded in $L^\infty(\mathbb{R}_+; \dot{B}_{2,1}^{\frac{d}{2}})$ (use product laws (7.1) and (7.2)) and, similarly, $(a^{(n)})_{n \in \mathbb{N}}$ is bounded in $L^\infty(\mathbb{R}_+; \dot{B}_{2,1}^{\frac{d}{2}})$. Using the fact that both sequences are, in particular, bounded in $L^\infty(\mathbb{R}_+; \dot{B}_{2,1}^{\frac{d}{2}} \cap \dot{B}_{2,1}^{\frac{d}{2}+1})$ and that the embedding from $\dot{B}_{2,1}^{\frac{d}{2}} \cap \dot{B}_{2,1}^{\frac{d}{2}+1}$ to $\dot{B}_{2,1}^{\frac{d}{2}}$ is locally compact, one discovers that for all $\theta \in \mathcal{C}_c^\infty(\mathbb{R}^d)$, we have (again, up to extraction),

$$(\theta a^{(n)}, \theta u^{(n)}) \rightarrow (\theta a, \theta u) \text{ strongly in } \mathcal{C}_b(\mathbb{R}_+; \dot{B}_{2,1}^{\frac{d}{2}}).$$

This allows to pass to the limit in (1.1) in the sense of distributions, fingers in the nose.

Furthermore, for all fixed $t \in \mathbb{R}_+$, the sequence $(\nabla u^{(n)}(t))_{n \in \mathbb{N}}$ is bounded in $\dot{B}_{2,1}^{\frac{d}{2}} \cap \dot{B}_{2,1}^{\frac{d}{2}+1}$ hence must converge to some function $z(t)$ weakly $*$ in $\dot{B}_{2,1}^{\frac{d}{2}} \cap \dot{B}_{2,1}^{\frac{d}{2}+2}$. Combining with the above property of strong convergence, we deduce that we must have $z(t) = \nabla u(t)$. Then,

using the properties of lower semi-continuity of the weak limit and Fatou lemma, one can write

$$\begin{aligned} \int_{\mathbb{R}_+} \|\nabla u\|_{\dot{B}_{2,1}^{\frac{d}{2}} \cap \dot{B}_{2,1}^{\frac{d}{2}+1}} dt &\leq \int_{\mathbb{R}_+} \liminf \|\nabla u^{(n)}\|_{\dot{B}_{2,1}^{\frac{d}{2}} \cap \dot{B}_{2,1}^{\frac{d}{2}+1}} dt \\ &\leq \liminf \int_{\mathbb{R}_+} \|\nabla u^{(n)}\|_{\dot{B}_{2,1}^{\frac{d}{2}} \cap \dot{B}_{2,1}^{\frac{d}{2}+1}} dt \leq C\alpha_0. \end{aligned}$$

Similar arguments may be employed to show that all the other L^1 -in-time properties of the space $E_{K_\varepsilon}^{\frac{d}{2}+1}$ are satisfied by (a, u) . Finally, the time continuity for u with values in $\dot{B}_{2,1}^{\frac{d}{2}} \cap \dot{B}_{2,1}^{\frac{d}{2}+2}$ stems from the properties of the transport equation and of the fact that (remember (5.2)):

$$u_t + u + u \cdot \nabla u = -\nabla K_\varepsilon a \in L^\infty(\mathbb{R}_+; \dot{B}_{2,1}^{\frac{d}{2}} \cap \dot{B}_{2,1}^{\frac{d}{2}+2}).$$

Similarly,

$$a_t + u \cdot \nabla a = -(1+a)\operatorname{div} u \in L^1(\mathbb{R}_+; \dot{B}_{2,1}^{\frac{d}{2}} \cap \dot{B}_{2,1}^{\frac{d}{2}+1}),$$

and thus $a \in \mathcal{C}_b(\mathbb{R}_+; \dot{B}_{2,1}^{\frac{d}{2}} \cap \dot{B}_{2,1}^{\frac{d}{2}+1})$. This completes the proof of existence in Theorem 3.1.

5.2. Uniqueness. This part is devoted to proving the uniqueness part of Theorem 3.1. We consider two solutions $(\rho_1 = 1 + a_1, u_1)$ and $(\rho_2 = 1 + a_2, u_2)$ of (1.1) with (a_i, u_i) for $i = 1, 2$ in the space $E_{K_\varepsilon}^{\frac{d}{2}+1}$. Then, we observe that $\delta a := \rho_2 - \rho_1$ and $\delta u := u_2 - u_1$ satisfy

$$\begin{cases} \partial_t \delta a + u_1 \cdot \nabla \delta a + \operatorname{div} \delta u + a_1 \operatorname{div} \delta u = f := -\delta u \cdot \nabla a_2 - \delta a \operatorname{div} u_2, \\ \partial_t \delta u + u_1 \cdot \nabla \delta u + \delta u + \nabla K_\varepsilon \delta a = g := -\delta u \cdot \nabla u_2. \end{cases}$$

Hence $(\delta a, \delta u)$ satisfies a linear system of type (3.11) with $v = u_1$ and $b = a_1$ and source terms (f, g) . Now, uniqueness on a finite interval $[0, T]$ will stem from Inequality (4.5) provided we have proved beforehand that $(\delta a, \delta u)$ has the regularity required in Theorem 4.1 in the case $\sigma = d/2$. After careful inspection of what is already known on (a_1, u_1) and (a_2, u_2) , we see that it suffices to check that

$$\delta a \in L^\infty(0, T; \dot{B}_{2,1}^{\frac{d}{2}-1}) \quad \text{and} \quad \delta u \in L^\infty(0, T; \dot{B}_{2,1}^{\frac{d}{2}}).$$

These two properties may be justified from the density and velocity equations and product laws in Besov spaces (that is, (7.1)) which guarantee that

$$\partial_t a_i \in L^1(0, T; \dot{B}_{2,1}^{\frac{d}{2}-1}) \quad \text{and} \quad \partial_t u_i + u_i \in L^1(0, T; \dot{B}_{2,1}^{\frac{d}{2}}), \quad i = 1, 2.$$

Hence, using the short notation $\delta X := X_{\delta a, \delta u}^{\frac{d}{2}}$ and $\delta H := H_{\delta a, \delta u}^{\frac{d}{2}}$, we have:

$$\begin{aligned} (5.10) \quad \delta X(t) + \int_0^t \delta H \, d\tau &\lesssim \delta X(0) + \int_0^t \|a_1, \nabla a_1, \nabla u_1, \nabla^2 u_1\|_{\dot{B}_{2,1}^{\frac{d}{2}}} \delta X \, d\tau \\ &\quad + \int_0^t \|f, \nabla f, \nabla^2 L_\varepsilon f\|_{\dot{B}_{2,1}^{\frac{d}{2}-1}} \, d\tau + \int_0^t \|g, \nabla g\|_{\dot{B}_{2,1}^{\frac{d}{2}}} \, d\tau. \end{aligned}$$

Thanks to (7.1), we readily have

$$\begin{aligned} \|f\|_{\dot{B}_{2,1}^{\frac{d}{2}-1}} &\lesssim \|\delta u\|_{\dot{B}_{2,1}^{\frac{d}{2}}} \|a_2\|_{\dot{B}_{2,1}^{\frac{d}{2}}} + \|\delta a\|_{\dot{B}_{2,1}^{\frac{d}{2}-1}} \|\operatorname{div} u_2\|_{\dot{B}_{2,1}^{\frac{d}{2}}}, \\ \|f\|_{\dot{B}_{2,1}^{\frac{d}{2}}} &\lesssim \|\delta u\|_{\dot{B}_{2,1}^{\frac{d}{2}}} \|\nabla a_2\|_{\dot{B}_{2,1}^{\frac{d}{2}}} + \|\delta a\|_{\dot{B}_{2,1}^{\frac{d}{2}}} \|\operatorname{div} u_2\|_{\dot{B}_{2,1}^{\frac{d}{2}}}, \\ \|g\|_{\dot{B}_{2,1}^{\frac{d}{2}}} &\lesssim \|\delta u\|_{\dot{B}_{2,1}^{\frac{d}{2}}} \|\nabla u_2\|_{\dot{B}_{2,1}^{\frac{d}{2}}}, \\ \|\nabla g\|_{\dot{B}_{2,1}^{\frac{d}{2}}} &\lesssim \|\nabla \delta u\|_{\dot{B}_{2,1}^{\frac{d}{2}}} \|\nabla u_2\|_{\dot{B}_{2,1}^{\frac{d}{2}}} + \|\delta u\|_{\dot{B}_{2,1}^{\frac{d}{2}}} \|\nabla^2 u_2\|_{\dot{B}_{2,1}^{\frac{d}{2}}}. \end{aligned}$$

Bounding $\nabla L_\varepsilon f$ in $\dot{B}_{2,1}^{\frac{d}{2}}$ is a bit more tricky. To achieve it, we use the decompositions:

$$\begin{aligned} \nabla L_\varepsilon(\delta u \cdot \nabla a_2) &= [\nabla L_\varepsilon, \delta u] \cdot \nabla a_2 + \delta u \cdot \nabla^2 L_\varepsilon a_2, \\ \nabla L_\varepsilon(\delta a \operatorname{div} u_2) &= [\nabla L_\varepsilon, \operatorname{div} u_2] \delta a + \operatorname{div} u_2 \nabla L_\varepsilon \delta a. \end{aligned}$$

Hence, taking advantage of (7.1) and of (7.5), we have

$$\begin{aligned} \|\nabla L_\varepsilon(\delta u \cdot \nabla a_2)\|_{\dot{B}_{2,1}^{\frac{d}{2}}} &\lesssim \|\nabla \delta u\|_{\dot{B}_{2,1}^{\frac{d}{2}}} \|\nabla a_2\|_{\dot{B}_{2,1}^{\frac{d}{2}}} + \|\delta u\|_{\dot{B}_{2,1}^{\frac{d}{2}}} \|\nabla^2 L_\varepsilon a_2\|_{\dot{B}_{2,1}^{\frac{d}{2}}}, \\ \|\nabla L_\varepsilon(\delta a \operatorname{div} u_2)\|_{\dot{B}_{2,1}^{\frac{d}{2}}} &\lesssim \|\delta a\|_{\dot{B}_{2,1}^{\frac{d}{2}}} \|\nabla \operatorname{div} u_2\|_{\dot{B}_{2,1}^{\frac{d}{2}}} + \|\operatorname{div} u_2\|_{\dot{B}_{2,1}^{\frac{d}{2}}} \|\nabla L_\varepsilon \delta a\|_{\dot{B}_{2,1}^{\frac{d}{2}}}. \end{aligned}$$

Plugging all the above inequalities in (5.10) yields

$$\delta X(t) + \int_0^t \delta H \, d\tau \lesssim \delta X(0) + \int_0^t \|a_1, \nabla a_1, \nabla u_1, \nabla^2 u_1, a_2, \nabla a_2, \nabla^2 L_\varepsilon a_2, \nabla u_2, \nabla^2 u_2\|_{\dot{B}_{2,1}^{\frac{d}{2}}} \delta X \, d\tau.$$

Since the prefactor of δX in the right-hand side is indeed locally integrable on time, Gronwall lemma ensures $\delta X \equiv 0$ that is, uniqueness, if the initial data of the two solutions are the same ones, and, more generally, stability with respect to the data. \square

5.3. Convergence to Euler. Justifying it is an easy adaptation of the proof of uniqueness that has been presented just above.

Indeed, consider initial data $(\rho_0 = 1 + a_0, u_0)$ such that the smallness condition (3.13) is satisfied. Then, even if it means a slight change in α_0 , Condition (3.7) is satisfied for all small enough $\varepsilon > 0$. Consequently, on the one hand, Theorem 3.1 provides us with a unique global solution $(\rho_\varepsilon = 1 + a_\varepsilon, u_\varepsilon)$ satisfying the properties described therein. On the other hand, by following faithfully the proof of estimates for (3.11) (formally replacing the convolution by K_ε with the identity operator everywhere) then adapting the proof of existence accordingly, one gets a global solution $(\rho = 1 + a, u)$ for Euler system (1.2) such that³ (a, u) belongs to the space $E^{\frac{d}{2}+1}$ defined in (3.6). To prove the convergence of $(a_\varepsilon, u_\varepsilon)$ to (a, u) , let us look at the system satisfied by $\delta a := a - a_\varepsilon$ and $\delta u := u - u_\varepsilon$:

$$\begin{cases} \partial_t \delta a + u \cdot \nabla \delta a + \operatorname{div} \delta u + a \operatorname{div} \delta u = f := -\delta u \cdot \nabla a_\varepsilon - \delta a \operatorname{div} u_\varepsilon, \\ \partial_t \delta u + u \cdot \nabla \delta u + \delta u + \nabla K_\varepsilon \delta a = g := -\delta u \cdot \nabla u_\varepsilon + \nabla (K_\varepsilon - Id) a. \end{cases}$$

Compared to the proof of uniqueness, only the last term of g is new. All the other terms may be bounded as above after replacing a_1 and u_1 (resp. a_2 and u_2) by a and u (resp. a_ε and u_ε). However, as we strive for a global-in-time result of convergence, putting all the terms

³The global well-posedness result of [6] only deals with regularity $\dot{B}_{2,1}^{\frac{d}{2}} \cap \dot{B}_{2,1}^{\frac{d}{2}+1}$.

concerning a or a_ε as prefactor of δX is not suitable : we need to be a little more precise, so we write that

$$\begin{aligned} \delta X(t) + \int_0^t \delta H \, d\tau &\lesssim \delta X(0) + \int_0^t \|\nabla u, \nabla^2 u, \nabla^2 L_\varepsilon a_\varepsilon, \nabla u_\varepsilon, \nabla^2 u_\varepsilon\|_{\dot{B}_{2,1}^{\frac{d}{2}}} \delta X \, d\tau \\ &+ \int_0^t \|a, \nabla a, a_\varepsilon, \nabla a_\varepsilon\|_{\dot{B}_{2,1}^{\frac{d}{2}}} \|\delta u, \nabla \delta u\|_{\dot{B}_{2,1}^{\frac{d}{2}}} \, d\tau + \int_0^t \|\nabla(K_\varepsilon - Id)a\|_{\dot{B}_{2,1}^{\frac{d}{2}} \cap \dot{B}_{2,1}^{\frac{d}{2}+1}} \, d\tau. \end{aligned}$$

Since $\|\delta u, \nabla \delta u\|_{\dot{B}_{2,1}^{\frac{d}{2}}}$ is a part of δH and

$$\sup_{t \in \mathbb{R}_+} \|(a, \nabla a, a_\varepsilon, \nabla a_\varepsilon)(t)\|_{\dot{B}_{2,1}^{\frac{d}{2}}} \text{ is small,}$$

the last but one term in the right-hand side may be absorbed by the left-hand side. Furthermore, the map $t \mapsto \|(\nabla u, \nabla^2 u, \nabla^2 L_\varepsilon a_\varepsilon, \nabla u_\varepsilon, \nabla^2 u_\varepsilon)(t)\|_{\dot{B}_{2,1}^{\frac{d}{2}}}$ is integrable on \mathbb{R}_+ and is also small. Hence, applying Gronwall lemma yields for all $t \in \mathbb{R}_+$,

$$\delta X(t) + \int_0^t \delta H \, d\tau \lesssim \delta X(0) + \int_0^t \|\nabla(K_\varepsilon - Id)a\|_{\dot{B}_{2,1}^{\frac{d}{2}} \cap \dot{B}_{2,1}^{\frac{d}{2}+1}} \, d\tau.$$

Finally, the properties of the solution (a, u) ensure in particular that

$$\nabla a \in L^2(\mathbb{R}_+; \dot{B}_{2,1}^{\frac{d}{2}}) \quad \text{and} \quad \nabla^2 a \in L^1(\mathbb{R}_+; \dot{B}_{2,1}^{\frac{d}{2}}).$$

Hence, by virtue of Lebesgue's dominated convergence theorem, we have

$$\int_0^\infty \|\nabla(K_\varepsilon - Id)a\|_{\dot{B}_{2,1}^{\frac{d}{2}+1}} \, d\tau \rightarrow 0 \quad \text{and} \quad \int_0^t \|\nabla(K_\varepsilon - Id)a\|_{\dot{B}_{2,1}^{\frac{d}{2}}} \, d\tau \rightarrow 0, \quad \text{for all } t \in \mathbb{R}_+.$$

From this, we conclude, among other, that for $\varepsilon \rightarrow 0$

$$(a_\varepsilon - a) \rightarrow 0 \text{ in } L_{loc}^\infty(\mathbb{R}_+; \dot{B}_{2,1}^{\frac{d}{2}-1}) \quad \text{and} \quad (u_\varepsilon - u) \rightarrow 0 \text{ in } L_{loc}^\infty(\mathbb{R}_+; \dot{B}_{2,1}^{\frac{d}{2}}).$$

Interpolating with the uniform bounds that are satisfied by $(a_\varepsilon, u_\varepsilon)$ and (a, u) , one can upgrade the convergence to e.g.

$$a_\varepsilon \rightarrow a \text{ in } L_{loc}^\infty(\mathbb{R}_+; \dot{B}_{2,1}^{\frac{d}{2}+\alpha}), \quad \alpha \in [0, 1) \quad \text{and} \quad u_\varepsilon \rightarrow u \text{ in } L_{loc}^\infty(\mathbb{R}_+; \dot{B}_{2,1}^{\frac{d}{2}+\beta}), \quad \beta \in [0, 2).$$

In order to have uniform-in-time convergence, one can assume in addition that $a_0 \in \dot{B}_{2,1}^{\frac{d}{2}-1}$.

Then, we have $\nabla a \in L^1(\mathbb{R}_+; \dot{B}_{2,1}^{\frac{d}{2}})$ and thus $\nabla(K_\varepsilon - Id)a \rightarrow 0$ in $L^1(\mathbb{R}_+; \dot{B}_{2,1}^{\frac{d}{2}})$ as $\varepsilon \rightarrow 0$.

To have a global result, another possibility is to assume that in addition to (3.1), we have $K_\varepsilon = \varepsilon^{-d} K(\varepsilon^{-1})$ with K such that $\eta \mapsto |\eta|^{-1}(\widehat{K}(\eta) - 1)$ is bounded. Indeed, one can write

$$\mathcal{F}(\nabla(K_\varepsilon - Id))(\xi) = i \left(\frac{\widehat{K}(\varepsilon\xi) - 1}{\varepsilon|\xi|} \right) \varepsilon\xi|\xi|\widehat{a}(\xi),$$

and thus $\nabla(K_\varepsilon - Id) = \mathcal{O}(\varepsilon)$ in $L^1(\mathbb{R}_+; \dot{B}_{2,1}^{\frac{d}{2}})$.

This completes the proof of Theorem 3.2.

6. ON THE HIGH FRICTION LIMIT

In the introductory part of the paper, we pointed out that, after performing the diffusive rescaling (3.16), the density formally tends to the solution of different avatars of the porous media equation. In this section, we aim at justifying rigorously this heuristics, getting in the small data case, strong and global-in-time results of convergence.

As a preliminary step, let us present the results that can be deduced from Theorem 3.1. Fix some data $(\rho_0 = 1 + a_0, u_0)$ satisfying the regularity requirements therein, and denote by $(\tilde{\rho}_0 = 1 + \tilde{a}_0, \tilde{u}_0)$ the data corresponding to the rescaling (3.10). If $(\tilde{a}_0, \tilde{u}_0)$ fulfills (3.7) with $\varepsilon\mathfrak{f}$ instead of ε , then Theorem 3.1 gives us a unique global solution $(1 + \tilde{a}, \tilde{u})$ satisfying (3.8) and (3.9) (with $\varepsilon\mathfrak{f}$). We have the following scaling properties for $\tilde{z}(x) = z(\mathfrak{f}^{-1}x)$:

$$\|\tilde{z}\|_{\dot{B}_{2,1}^{\frac{d}{2}+\sigma}} = \mathfrak{f}^{-\sigma} \|z\|_{\dot{B}_{2,1}^{\frac{d}{2}+\sigma}} \quad \text{and} \quad L_{\mathfrak{f}\varepsilon}\tilde{z}(x) = (L_\varepsilon z)(\mathfrak{f}^{-1}x).$$

Hence, reverting to the original variables, we deduce that provided

$$(6.1) \quad \|(a_0, \mathfrak{f}^{-1}\nabla a_0, \mathfrak{f}^{-2}\nabla^2 L_\varepsilon a_0)\|_{\dot{B}_{2,1}^{\frac{d}{2}}} + \mathfrak{f}^{-1}\|(u_0, \mathfrak{f}^{-1}\nabla u_0)\|_{\dot{B}_{2,1}^{\frac{d}{2}+1}} \leq \alpha_0,$$

System (1.1) has a unique global solution $(\rho = 1 + a, u)$ such that

$$(6.2) \quad X(t) + \int_0^t H \, d\tau \leq CX(0)$$

with

$$X(t) := \|(a, \mathfrak{f}^{-1}\nabla a, \mathfrak{f}^{-2}\nabla^2 L_\varepsilon a)(t)\|_{\dot{B}_{2,1}^{\frac{d}{2}}} + \mathfrak{f}^{-1}\|(u, \mathfrak{f}^{-1}\nabla u)(t)\|_{\dot{B}_{2,1}^{\frac{d}{2}+1}} \quad \text{and}$$

$$H(t) := \|(u, \mathfrak{f}^{-1}\nabla u)\|_{\dot{B}_{2,1}^{\frac{d}{2}+1}} + \mathfrak{f}^{-1}\|\nabla^2 K_\varepsilon a\|_{\dot{B}_{2,1}^{\frac{d}{2}}}^\ell + \mathfrak{f}^{-2}\|\nabla^2 K_\varepsilon a\|_{\dot{B}_{2,1}^{\frac{d}{2}+1}}^\ell + \|(a, \mathfrak{f}^{-1}\nabla L_\varepsilon a)\|_{\dot{B}_{2,1}^{\frac{d}{2}+1}}^h.$$

Furthermore, if, in addition, u_0 belongs to $\dot{B}_{2,1}^{\frac{d}{2}}$, then the damped mode $w := u + \mathfrak{f}^{-1}\nabla K_\varepsilon a$ satisfies

$$(6.3) \quad \|u(t)\|_{\dot{B}_{2,1}^{\frac{d}{2}}} + \|w(t)\|_{\dot{B}_{2,1}^{\frac{d}{2}}} + \mathfrak{f} \int_0^t \|w\|_{\dot{B}_{2,1}^{\frac{d}{2}}} \, d\tau \leq C(\|u_0\|_{\dot{B}_{2,1}^{\frac{d}{2}}}^\ell + X(0)).$$

Based on these uniform estimates, it will be rather easy to justify the high friction asymptotics pointed out in the introduction, after performing the diffusive rescaling (3.16).

6.1. High relaxation limit for fixed ε . In this section, we justify the convergence of $(\check{\rho}, \check{u})$ (obtained from (ρ, u) and (3.16)) to $(r, -\nabla K_\varepsilon r)$, with r satisfying (3.15) supplemented with initial data ρ_0 . Our main result reads as follows:

Theorem 6.1. *Fix some $\varepsilon > 0$ and data $(\rho_0 = 1 + a_0, u_0)$ such that a_0 and u_0 are in $\dot{B}_{2,1}^{\frac{d}{2}} \cap \dot{B}_{2,1}^{\frac{d}{2}+2}$. There exists an absolute constant α_0 such that, if*

$$(6.4) \quad \|a_0\|_{\dot{B}_{2,1}^{\frac{d}{2}}} \leq \alpha_0$$

then, for all large enough \mathfrak{f} , System (1.1) admits a unique global solution $(\rho_\mathfrak{f} = 1 + a_\mathfrak{f}, u_\mathfrak{f})$ satisfying (6.2) and (6.3), and Equation (3.15) supplemented with initial data ρ_0 has a unique global solution $r = 1 + \tilde{r}$ with $\tilde{r} \in \mathcal{C}_b(\mathbb{R}_+; \dot{B}_{2,1}^{\frac{d}{2}})$ and $\nabla^2 K_\varepsilon \tilde{r} \in L^1(\mathbb{R}_+; \dot{B}_{2,1}^{\frac{d}{2}-1})$.

Furthermore, if $(\check{\rho}_\mathfrak{f}, \check{u}_\mathfrak{f})$ is defined from $(\rho_\mathfrak{f}, u_\mathfrak{f})$ by (3.16), then we have

$$\|\check{u}_\mathfrak{f} + \nabla K_\varepsilon r\|_{L^1(\mathbb{R}_+; \dot{B}_{2,1}^{\frac{d}{2}})} + \|\check{\rho}_\mathfrak{f} - r\|_{L^\infty(\mathbb{R}_+; \dot{B}_{2,1}^{\frac{d}{2}-1})} + \|\nabla^2 K_\varepsilon(\check{\rho}_\mathfrak{f} - r)\|_{L^1(\mathbb{R}_+; \dot{B}_{2,1}^{\frac{d}{2}-1})} \rightarrow 0 \quad \text{as } \mathfrak{f} \rightarrow \infty$$

with convergence rate \mathfrak{f}^{-1} .

Proof. Since a_0 and u_0 are in $\dot{B}_{2,1}^{\frac{d}{2}} \cap \dot{B}_{2,1}^{\frac{d}{2}+2}$ and Hypothesis (6.4) holds, we are guaranteed that the smallness condition (6.1) is satisfied for large enough \mathfrak{f} . Hence, as explained at the beginning of Section 6, there exists a unique global solution $(\rho_{\mathfrak{f}}, u_{\mathfrak{f}})$ of (1.1) with the desired properties.

Next, in terms of $\tilde{r} := r - 1$, Equation (3.15) reads

$$(6.5) \quad \partial_t \tilde{r} - \nabla K_\varepsilon \tilde{r} \cdot \nabla \tilde{r} - \Delta K_\varepsilon \tilde{r} = \tilde{r} \Delta K_\varepsilon \tilde{r}, \quad \tilde{r}|_{t=0} = a_0.$$

This may be seen as a degenerate convection diffusion equation. We claim that there exists an absolute constant c_0 such that for all $t \geq 0$, we have

$$(6.6) \quad \|\tilde{r}(t)\|_{\dot{B}_{2,1}^{\frac{d}{2}}} + c_0 \int_0^t \|\nabla^2 K_\varepsilon \tilde{r}\|_{\dot{B}_{2,1}^{\frac{d}{2}}} \leq \|\tilde{r}_0\|_{\dot{B}_{2,1}^{\frac{d}{2}}} + C \int_0^t \|\nabla^2 K_\varepsilon \tilde{r}\|_{\dot{B}_{2,1}^{\frac{d}{2}}} \|\tilde{r}\|_{\dot{B}_{2,1}^{\frac{d}{2}}} d\tau.$$

Indeed, localizing (6.5) by means of $\dot{\Delta}_j$ gives

$$\partial_t \tilde{r}_j - \nabla K_\varepsilon \tilde{r} \cdot \nabla \tilde{r}_j - \Delta K_\varepsilon \tilde{r}_j = \dot{\Delta}_j(\tilde{r} \Delta K_\varepsilon \tilde{r}) - [\nabla K_\varepsilon \tilde{r}, \dot{\Delta}_j] \cdot \nabla \tilde{r}.$$

Hence, taking the L^2 scalar product with \tilde{r}_j and integrating by parts in the second and third term of the left-hand side:

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \|\tilde{r}_j\|_{L^2}^2 + \frac{1}{2} \int_{\mathbb{R}^d} |\tilde{r}_j|^2 \Delta K_\varepsilon \tilde{r} dx + \int_{\mathbb{R}^d} \nabla \tilde{r}_j \cdot \nabla K_\varepsilon \tilde{r}_j dx \\ = \int_{\mathbb{R}^d} \tilde{r}_j \dot{\Delta}_j(\tilde{r} \Delta K_\varepsilon \tilde{r}) dx - \int_{\mathbb{R}^d} \tilde{r}_j [\nabla K_\varepsilon \tilde{r}, \dot{\Delta}_j] \cdot \nabla \tilde{r} dx. \end{aligned}$$

So, using the usual integration procedure and (4.11), we get a universal positive constant κ_0 such that

$$\begin{aligned} \|\tilde{r}_j(t)\|_{L^2} + \kappa_0 \int_0^t \|\nabla^2 K_\varepsilon \tilde{r}_j\|_{L^2} d\tau \leq \|\tilde{r}_{j,0}\|_{L^2} \\ + \int_0^t \|\dot{\Delta}_j(\tilde{r} \Delta K_\varepsilon \tilde{r})\|_{L^2} d\tau + \int_0^t \|[\nabla K_\varepsilon \tilde{r}, \dot{\Delta}_j] \cdot \nabla \tilde{r}\|_{L^2} d\tau + \frac{1}{2} \int_0^t \|\Delta K_\varepsilon \tilde{r}\|_{L^\infty} \|\tilde{r}_j\|_{L^2} d\tau. \end{aligned}$$

Taking advantage of (7.1), (7.3) and (7.4), we discover that

$$\|\dot{\Delta}_j(\tilde{r} \Delta K_\varepsilon \tilde{r})\|_{L^2} + \|[\nabla K_\varepsilon \tilde{r}, \dot{\Delta}_j] \cdot \nabla \tilde{r}\|_{L^2} + \|\Delta K_\varepsilon \tilde{r}\|_{L^\infty} \|\tilde{r}_j\|_{L^2} \leq C c_j 2^{-j\frac{d}{2}} \|\nabla^2 K_\varepsilon \tilde{r}\|_{\dot{B}_{2,1}^{\frac{d}{2}}} \|\tilde{r}\|_{\dot{B}_{2,1}^{\frac{d}{2}}},$$

which after multiplication by $2^{j\frac{d}{2}}$ and summation on $j \in \mathbb{Z}$ completes the proof of (6.6).

Having this inequality at our disposal and assuming that α_0 in (6.4) is small enough, one can use the fixed point theorem (e.g. adapting the proof for the incompressible Navier-Stokes equations given in [1, Chap. 5]) to solve (6.5) globally in time. We get a unique solution \tilde{r} in $\mathcal{C}_b(\mathbb{R}_+; \dot{B}_{2,1}^{\frac{d}{2}})$ such that

$$\|\tilde{r}(t)\|_{\dot{B}_{2,1}^{\frac{d}{2}}} + \kappa_0 \int_0^t \|\nabla^2 K_\varepsilon \tilde{r}\|_{\dot{B}_{2,1}^{\frac{d}{2}}} \leq 2 \|a_0\|_{\dot{B}_{2,1}^{\frac{d}{2}}}, \quad t \in \mathbb{R}_+.$$

Let us drop index \mathfrak{f} for better readability. In order to prove the last part of the theorem, we observe that

$$\partial_t \tilde{\rho} - \operatorname{div}(\tilde{\rho} \nabla K_\varepsilon \tilde{\rho}) = -\operatorname{div}(\tilde{\rho} \tilde{w}) \quad \text{with} \quad \tilde{w} := \tilde{u} + \nabla K_\varepsilon \tilde{\rho}.$$

The key to the proof is that (6.3) after rescaling implies that

$$(6.7) \quad \int_0^t \|\tilde{w}\|_{\dot{B}_{2,1}^{\frac{d}{2}}} d\tau \leq C \mathfrak{f}^{-1} (\|u_0\|_{\dot{B}_{2,1}^{\frac{d}{2}}}^\ell + \alpha_0), \quad t > 0.$$

The difference $\delta r := \check{\rho} - r$ satisfies:

$$\partial_t \delta r - \operatorname{div}(\check{\rho} \nabla K_\varepsilon \delta r) = -\operatorname{div}(\delta r \nabla K_\varepsilon r) - \operatorname{div}(\check{\rho} \check{w}).$$

Putting $\check{a} := \check{\rho} - 1$ and remembering that $r = 1 + \tilde{r}$, the above equation may be rewritten:

$$(6.8) \quad \partial_t \delta r + \nabla K_\varepsilon \tilde{r} \cdot \nabla \delta r - \Delta K_\varepsilon \delta r = \operatorname{div}(\check{a} \nabla K_\varepsilon \delta r) - \delta r \Delta K_\varepsilon \tilde{r} - \operatorname{div}((1 + \check{a}) \check{w}).$$

Localizing (6.8) by means of Littlewood-Paley decomposition, then arguing as for proving (6.6) (with regularity index $d/2 - 1$ instead of $d/2$), we get for all $t > 0$,

$$(6.9) \quad \|\delta r(t)\|_{\dot{B}_{2,1}^{\frac{d}{2}-1}} + \kappa_0 \int_0^t \|\nabla^2 K_\varepsilon \delta r\|_{\dot{B}_{2,1}^{\frac{d}{2}-1}} \lesssim \int_0^t \|\nabla^2 K_\varepsilon \tilde{r}\|_{\dot{B}_{2,1}^{\frac{d}{2}}} \|\delta r\|_{\dot{B}_{2,1}^{\frac{d}{2}-1}} d\tau \\ + \|\operatorname{div}(\check{a} \nabla K_\varepsilon \delta r)\|_{L^1(\mathbb{R}_+; \dot{B}_{2,1}^{\frac{d}{2}-1})} + \|\delta r \Delta K_\varepsilon \tilde{r}\|_{L^1(\mathbb{R}_+; \dot{B}_{2,1}^{\frac{d}{2}-1})} + \|\operatorname{div}((1 + \check{a}) \check{w})\|_{L^1(\mathbb{R}_+; \dot{B}_{2,1}^{\frac{d}{2}-1})}.$$

According to the product law (7.1), and to (6.2), we have:

$$\|\operatorname{div}(\check{a} \nabla K_\varepsilon \delta r)\|_{L^1(\mathbb{R}_+; \dot{B}_{2,1}^{\frac{d}{2}-1})} \lesssim \|\check{a}\|_{L^\infty(\mathbb{R}_+; \dot{B}_{2,1}^{\frac{d}{2}})} \|\nabla K_\varepsilon \delta r\|_{L^1(\mathbb{R}_+; \dot{B}_{2,1}^{\frac{d}{2}})} \lesssim \alpha_0 \|\nabla^2 K_\varepsilon \delta r\|_{L^1(\mathbb{R}_+; \dot{B}_{2,1}^{\frac{d}{2}-1})},$$

so this term may be absorbed by the left-hand side of (6.9).

For the next term, we have:

$$\|\delta r \Delta K_\varepsilon \tilde{r}\|_{L^1(\mathbb{R}_+; \dot{B}_{2,1}^{\frac{d}{2}-1})} \lesssim \|\delta r\|_{L^\infty(\mathbb{R}_+; \dot{B}_{2,1}^{\frac{d}{2}-1})} \|\Delta K_\varepsilon \tilde{r}\|_{L^1(\mathbb{R}_+; \dot{B}_{2,1}^{\frac{d}{2}})} \lesssim \alpha_0 \|\delta r\|_{L^\infty(\mathbb{R}_+; \dot{B}_{2,1}^{\frac{d}{2}-1})}.$$

Hence, remembering (6.7), Inequality (6.9) implies that

$$\|\delta r\|_{L^\infty(\mathbb{R}_+; \dot{B}_{2,1}^{\frac{d}{2}-1})} + \|\nabla^2 K_\varepsilon \delta r\|_{L^1(\mathbb{R}_+; \dot{B}_{2,1}^{\frac{d}{2}-1})} \leq C \mathfrak{f}^{-1} (\|u_0\|_{\dot{B}_{2,1}^{\frac{d}{2}}}^\ell + \alpha_0).$$

Since $\check{u} = \check{w} - \nabla K_\varepsilon \check{\rho}$, the above inequality and (6.7) imply that \check{u} tends to the limit ‘velocity’ $z := -\nabla K_\varepsilon r$ with convergence rate \mathfrak{f}^{-1} in $L^1(\mathbb{R}_+; \dot{B}_{2,1}^{\frac{d}{2}})$. \square

6.2. Convergence of the relaxed system to the porous media equation. In this part, we want to justify the limit of solutions of Equation (3.15) to those of the porous media equation (3.14), when ε goes to 0. Our main result is stated below:

Theorem 6.2. *Consider initial data $r_{\varepsilon,0}$ and n_0 such that $\tilde{r}_{\varepsilon,0} := r_{\varepsilon,0} - 1$ and $\tilde{n}_0 := n_0 - 1$ are in $\dot{B}_{2,1}^{d/2}(\mathbb{R}^d)$. There exists an absolute constant α_0 such that if*

$$\max(\|\tilde{r}_{\varepsilon,0}\|_{\dot{B}_{2,1}^{d/2}}, \|\tilde{n}_0\|_{\dot{B}_{2,1}^{d/2}}) \leq \alpha_0,$$

then Equations (3.15) and (3.14) have a unique global solution $r_\varepsilon = 1 + \tilde{r}_\varepsilon$ and $n = 1 + \tilde{n}$ with r_ε given by Theorem 6.1 and $\tilde{n} \in \mathcal{C}_b(\mathbb{R}_+; \dot{B}_{2,1}^{\frac{d}{2}}) \cap L^1(\mathbb{R}_+; \dot{B}_{2,1}^{\frac{d}{2}+2})$, and we have

$$(6.10) \quad \|\tilde{r}_\varepsilon, \tilde{n}\|_{L^\infty(\mathbb{R}_+; \dot{B}_{2,1}^{\frac{d}{2}})} + \|\nabla^2 K_\varepsilon r_\varepsilon, \nabla^2 n\|_{L^1(\mathbb{R}_+; \dot{B}_{2,1}^{\frac{d}{2}})} \leq C \|\tilde{r}_{\varepsilon,0}, \tilde{n}_0\|_{\dot{B}_{2,1}^{\frac{d}{2}}}.$$

If, furthermore, $\tilde{r}_{\varepsilon,0}$ tends to \tilde{n}_0 in $\dot{B}_{2,1}^{\frac{d}{2}}$, then we have⁴:

$$(6.11) \quad \tilde{r}_\varepsilon \rightarrow \tilde{n} \quad \text{in} \quad L^\infty(\mathbb{R}_+; \dot{B}_{2,1}^{\frac{d}{2}}).$$

⁴Unless stronger assumptions are made on K_ε , we do not have any rate of convergence.

Proof. The existence of r_ε with the desired properties follows from Theorem 6.1. Next, as for (3.15), since $\tilde{n}_0 := n_0 - 1$ is small in $\dot{B}_{2,1}^{\frac{d}{2}}$, it is easy to see by variations on the fixed point theorem that there exists a unique solution n to (3.14) satisfying the properties mentioned in the above statement. Let us prove the convergence of r_ε to n . Set $\delta n := \tilde{n} - \tilde{r}_\varepsilon$. We have:

$$\partial_t \delta n - \operatorname{div}(n \nabla K_\varepsilon \delta n) = \operatorname{div}(\delta n \nabla K_\varepsilon r_\varepsilon) + \operatorname{div}(n(\operatorname{Id} - K_\varepsilon) \nabla n).$$

We rewrite this expression in the form of a degenerate convection diffusion equation as follows:

$$\partial_t \delta n - \nabla \delta n \cdot \nabla K_\varepsilon r_\varepsilon - \Delta K_\varepsilon \delta n = \nabla \tilde{n} \cdot \nabla K_\varepsilon \delta n + \tilde{n} \Delta K_\varepsilon \delta n + \delta n \Delta K_\varepsilon r_\varepsilon + \operatorname{div}((1 + \tilde{n})(\operatorname{Id} - K_\varepsilon) \nabla n).$$

Hence, arguing as in the proof of Theorem 6.1, we get

$$(6.12) \quad \begin{aligned} \|\delta n(t)\|_{\dot{B}_{2,1}^{\frac{d}{2}}} + \kappa_0 \int_0^t \|\nabla^2 K_\varepsilon \delta n\|_{\dot{B}_{2,1}^{\frac{d}{2}}} d\tau &\leq \|\delta n_0\|_{\dot{B}_{2,1}^{\frac{d}{2}}} + \int_0^t \|\nabla^2 K_\varepsilon \tilde{r}_\varepsilon\|_{\dot{B}_{2,1}^{\frac{d}{2}}} \|\delta n\|_{\dot{B}_{2,1}^{\frac{d}{2}}} d\tau \\ &+ \int_0^t \|\nabla \tilde{n} \cdot \nabla K_\varepsilon \delta n\|_{\dot{B}_{2,1}^{\frac{d}{2}}} d\tau + \int_0^t \|\tilde{n} \Delta K_\varepsilon \delta n\|_{\dot{B}_{2,1}^{\frac{d}{2}}} d\tau + \int_0^t \|\delta n \Delta K_\varepsilon \tilde{r}_\varepsilon\|_{\dot{B}_{2,1}^{\frac{d}{2}}} d\tau \\ &+ \int_0^t \|(1 + \tilde{n})(\operatorname{Id} - K_\varepsilon) \Delta \tilde{n}\|_{\dot{B}_{2,1}^{\frac{d}{2}}} d\tau + \int_0^t \|\nabla \tilde{n} \cdot (\operatorname{Id} - K_\varepsilon) \nabla \tilde{n}\|_{\dot{B}_{2,1}^{\frac{d}{2}}} d\tau. \end{aligned}$$

From product law (7.1), we have:

$$\begin{aligned} \|\nabla \tilde{n} \cdot \nabla K_\varepsilon \delta n\|_{\dot{B}_{2,1}^{\frac{d}{2}}} &\lesssim \|\nabla \tilde{n}\|_{\dot{B}_{2,1}^{\frac{d}{2}}} \|\nabla K_\varepsilon \delta n\|_{\dot{B}_{2,1}^{\frac{d}{2}}}, \\ \|\tilde{n} \Delta K_\varepsilon \delta n\|_{\dot{B}_{2,1}^{\frac{d}{2}}} &\lesssim \|\tilde{n}\|_{\dot{B}_{2,1}^{\frac{d}{2}}} \|\Delta K_\varepsilon \delta n\|_{\dot{B}_{2,1}^{\frac{d}{2}}}, \\ \|\delta n \Delta K_\varepsilon \tilde{r}_\varepsilon\|_{\dot{B}_{2,1}^{\frac{d}{2}}} &\lesssim \|\delta n\|_{\dot{B}_{2,1}^{\frac{d}{2}}} \|\Delta K_\varepsilon \tilde{r}_\varepsilon\|_{\dot{B}_{2,1}^{\frac{d}{2}}}, \\ \|(1 + \tilde{n})(\operatorname{Id} - K_\varepsilon) \Delta \tilde{n}\|_{\dot{B}_{2,1}^{\frac{d}{2}}} &\lesssim (1 + \|\tilde{n}\|_{\dot{B}_{2,1}^{\frac{d}{2}}}) \|(\operatorname{Id} - K_\varepsilon) \Delta \tilde{n}\|_{\dot{B}_{2,1}^{\frac{d}{2}}}, \\ \|\nabla \tilde{n} \cdot (\operatorname{Id} - K_\varepsilon) \nabla \tilde{n}\|_{\dot{B}_{2,1}^{\frac{d}{2}}} &\lesssim \|\nabla \tilde{n}\|_{\dot{B}_{2,1}^{\frac{d}{2}}} \|(\operatorname{Id} - K_\varepsilon) \nabla \tilde{n}\|_{\dot{B}_{2,1}^{\frac{d}{2}}}. \end{aligned}$$

As we work with small solutions, the second, fourth and fifth terms of the right-hand side above may be absorbed by the left-hand side of (6.12). Next, by interpolation, we have

$$\begin{aligned} \|\nabla \tilde{n}\|_{\dot{B}_{2,1}^{\frac{d}{2}}} \|\nabla K_\varepsilon \delta n\|_{\dot{B}_{2,1}^{\frac{d}{2}}} &\lesssim \|\tilde{n}\|_{\dot{B}_{2,1}^{\frac{d}{2}}}^{1/2} \|K_\varepsilon \delta n\|_{\dot{B}_{2,1}^{\frac{d}{2}}}^{1/2} \|\Delta \tilde{n}\|_{\dot{B}_{2,1}^{\frac{d}{2}}}^{1/2} \|\Delta K_\varepsilon \delta n\|_{\dot{B}_{2,1}^{\frac{d}{2}}}^{1/2} \\ &\lesssim \|\tilde{n}\|_{\dot{B}_{2,1}^{\frac{d}{2}}} \|\Delta K_\varepsilon \delta n\|_{\dot{B}_{2,1}^{\frac{d}{2}}} + \|\Delta \tilde{n}\|_{\dot{B}_{2,1}^{\frac{d}{2}}} \|\delta n\|_{\dot{B}_{2,1}^{\frac{d}{2}}}. \end{aligned}$$

Hence the corresponding term may also be absorbed with the left-hand side of (6.12).

Finally, in light of Lebesgue's dominated convergence theorem, since $\Delta \tilde{n} \in L^1(\mathbb{R}_+; \dot{B}_{2,1}^{\frac{d}{2}})$ and $\nabla \tilde{n} \in \tilde{L}^2(\mathbb{R}_+; \dot{B}_{2,1}^{\frac{d}{2}})$, we have

$$\lim_{\varepsilon \rightarrow 0} \|(\operatorname{Id} - K_\varepsilon) \Delta \tilde{n}\|_{L^1(\mathbb{R}_+; \dot{B}_{2,1}^{\frac{d}{2}})} = 0 \quad \text{and} \quad \lim_{\varepsilon \rightarrow 0} \|(\operatorname{Id} - K_\varepsilon) \nabla \tilde{n}\|_{L^2(\mathbb{R}_+; \dot{B}_{2,1}^{\frac{d}{2}})} = 0.$$

Plugging all this information in (6.12) completes the proof of (6.11). \square

6.3. Convergence of System (1.1) to the porous media equation. The convergence of the density, solution of System (1.1) to the solution of the porous media equation (3.14) when both $\mathfrak{f} \rightarrow \infty$ and $\varepsilon \rightarrow 0$ may be deduced from Theorems 6.1 and 6.2. Let $(\rho_{\mathfrak{f},\varepsilon}, u_{\mathfrak{f},\varepsilon})$ be the solution of (1.1) and $(\check{\rho}_{\mathfrak{f},\varepsilon}, \check{u}_{\mathfrak{f},\varepsilon})$ be the corresponding rescaled solution (see (3.16)). Let

r_ε be the solution of (6.5) with data $\rho_0 = 1 + a_0$ and, finally, n the solution to (3.14) with the same data (for simplicity). We have

$$\check{\rho}_{\mathfrak{f},\varepsilon} - n = (\check{\rho}_{\mathfrak{f},\varepsilon} - r_\varepsilon) + (r_\varepsilon - n).$$

Hence, in light of Theorems 6.1 and 6.2, one may conclude to the following result:

Theorem 6.3. *Take a_0 and u_0 as in Theorem 6.1. Let n be the solution to (3.14) with data $1 + a_0$. Let $(\check{\rho}_{\mathfrak{f},\varepsilon}, \check{u}_{\mathfrak{f},\varepsilon})$ be the solution of (1.1) after rescaling (3.16). Then*

$$\check{\rho}_{\mathfrak{f},\varepsilon} - n \rightarrow 0 \quad \text{in } L^\infty(\mathbb{R}_+; \dot{B}_{2,1}^{\frac{d}{2}-1} + \dot{B}_{2,1}^{\frac{d}{2}}) \quad \text{as } \mathfrak{f} \rightarrow \infty \quad \text{and } \varepsilon \rightarrow 0.$$

7. APPENDIX

7.1. Commutator estimates. As a first, we recall two product laws in Besov spaces that we used repeatedly in the paper (the reader may refer to [1, Chap. 2] for more details):

$$(7.1) \quad \|fg\|_{\dot{B}_{2,1}^\sigma} \leq C \|f\|_{\dot{B}_{2,1}^{\frac{d}{2}}} \|g\|_{\dot{B}_{2,1}^\sigma}, \quad -d/2 < \sigma \leq d/2,$$

$$(7.2) \quad \|fg\|_{\dot{B}_{2,1}^\sigma} \leq C \left(\|f\|_{L^\infty} \|g\|_{\dot{B}_{2,1}^\sigma} + \|g\|_{L^\infty} \|f\|_{\dot{B}_{2,1}^\sigma} \right), \quad \sigma > 0.$$

The latter inequality is often combined with the embedding

$$(7.3) \quad \dot{B}_{2,1}^{\frac{d}{2}} \hookrightarrow L^\infty.$$

In the rest of this part, we focus on the commutators estimates that we used for handling the terms $v \cdot \nabla a$, $v \cdot \nabla u$, $b \operatorname{div} u$ and $c \nabla K_\varepsilon a$ in System (4.1).

The following commutator estimate belongs to the mathematical folklore (see [1, Chap. 2]):

Lemma 7.1. *Let $-d/2 < \sigma \leq 1 + d/2$. Then, for $v \in \dot{B}_{2,1}^{d/2+1}$ and $z \in \dot{B}_{2,1}^\sigma$, it holds that*

$$(7.4) \quad \|[v, \dot{\Delta}_j] \nabla z\|_{L^2} \leq C c_j 2^{-j\sigma} \|\nabla v\|_{\dot{B}_{2,1}^{\frac{d}{2}}} \|z\|_{\dot{B}_{2,1}^\sigma},$$

where $(c_j)_{j \in \mathbb{Z}}$ denotes a nonnegative sequence with sum equal to 1.

The next commutator estimate is connected to the operator L_ε satisfying (3.1).

Lemma 7.2. *Let $\sigma > -d/2$. There exists a constant C independent of ε such that we have:*

$$(7.5) \quad \|[c, \partial_k L_\varepsilon] h\|_{\dot{B}_{2,1}^\sigma} \leq C \left(\|c\|_{\dot{B}_{2,1}^{\frac{d}{2}+1}} \|h\|_{\dot{B}_{2,1}^\sigma} + \|h\|_{L^\infty} \|c\|_{\dot{B}_{2,1}^{\sigma+1}} \right), \quad k \in \{1, \dots, d\}.$$

Moreover, if $-d/2 < \sigma \leq d/2$, then the second term is not needed.

We also have

$$(7.6) \quad \|[c, L_\varepsilon] h\|_{\dot{B}_{2,1}^\sigma} \leq C \left(\|c\|_{\dot{B}_{2,1}^{\frac{d}{2}+1}} \|h\|_{\dot{B}_{2,1}^{\sigma-1}} + \|h\|_{L^\infty} \|c\|_{\dot{B}_{2,1}^\sigma} \right),$$

and the second term is not needed for $-d/2 < \sigma \leq d/2 + 1$.

Proof. One can take advantage of the following (simplified) Bony decomposition:

$$(7.7) \quad fg = T_f g + T'_g f \quad \text{with} \quad T_f g := \sum_j \dot{S}_{j-1} f \dot{\Delta}_j g \quad \text{and} \quad T'_g f := \sum_j \dot{S}_{j+2} g \dot{\Delta}_j f.$$

Now, using the paraproduct operators T and T' , we have the decomposition:

$$(7.8) \quad [c, \partial_k L_\varepsilon] h = [T_c, \partial_k L_\varepsilon] h + T'_{\partial_k L_\varepsilon h} c - \partial_k L_\varepsilon T'_h c.$$

The last two terms may be bounded according to continuity results for the paraproduct (see [1, Chap. 2]): if $\sigma > -d/2$, then we have (using (3.1))

$$(7.9) \quad \|T'_{\partial_k L_\varepsilon h} c\|_{\dot{B}_{2,1}^\sigma} \lesssim \|\partial_k L_\varepsilon h\|_{\dot{B}_{2,1}^{\sigma-1}} \|c\|_{\dot{B}_{2,1}^{d/2+1}} \lesssim \|h\|_{\dot{B}_{2,1}^\sigma} \|c\|_{\dot{B}_{2,1}^{d/2+1}}$$

and, if $\sigma > 0$

$$(7.10) \quad \|\partial_k L_\varepsilon T'_h c\|_{\dot{B}_{2,1}^\sigma} \lesssim \|T'_h c\|_{\dot{B}_{2,1}^{\sigma+1}} \lesssim \|h\|_{L^\infty} \|c\|_{\dot{B}_{2,1}^{\sigma+1}}.$$

Note that for $-d/2 < \sigma \leq d/2$, then we have

$$(7.11) \quad \|\partial_k L_\varepsilon T'_h c\|_{\dot{B}_{2,1}^\sigma} \lesssim \|T'_h c\|_{\dot{B}_{2,1}^{\sigma+1}} \lesssim \|h\|_{\dot{B}_{2,1}^\sigma} \|c\|_{\dot{B}_{2,1}^{d/2+1}}.$$

For the first term in the right-hand side of (7.8) we write that, by definition of paraproduct,

$$[T_c, \partial_k L_\varepsilon] h = \sum_j [\dot{S}_{j-1} c, \partial_k L_\varepsilon] \dot{\Delta}_j h.$$

Now, from the mean value formula, we gather

$$[\dot{S}_{j-1} c, \partial_k L_\varepsilon] \dot{\Delta}_j h(x) = \int_0^1 \int_{\mathbb{R}^d} \nabla \dot{S}_{j-1} c(y + \tau(x-y)) \cdot (x-y) \partial_k L_\varepsilon(x-y) \dot{\Delta}_j h(y) dy.$$

Hence, for all $j \in \mathbb{Z}$,

$$\|[\dot{S}_{j-1} c, \partial_k L_\varepsilon] \dot{\Delta}_j h\|_{L^2} \leq \|z \partial_k L_\varepsilon\|_{L^1} \|\nabla \dot{S}_{j-1} c\|_{L^\infty} \|\dot{\Delta}_j h\|_{L^2}.$$

From this, Condition (3.1) and Lemma 2.23 in [1], we deduce that

$$(7.12) \quad \|[T_c, \partial_k L_\varepsilon] h\|_{\dot{B}_{2,1}^\sigma} \lesssim \|\nabla c\|_{L^\infty} \|h\|_{\dot{B}_{2,1}^\sigma}$$

and, in light of (7.3), we conclude to (7.5)

Proving (7.6) is similar : we start from the decomposition

$$[c, L_\varepsilon] h = [T_c, L_\varepsilon] h + T'_{L_\varepsilon h} c - L_\varepsilon T'_h c.$$

The last two terms may be bounded by means of standard continuity results for the paraproduct operator. To handle the first one, we introduce the function $L_{\varepsilon, j'} := \mathcal{F}^{-1}(\widehat{L}_\varepsilon \varphi(2^{-j'} \cdot))$ and write that

$$\begin{aligned} [\dot{S}_{j-1} c, L_\varepsilon] \dot{\Delta}_j h(x) &= \sum_{j' \sim j} [\dot{S}_{j-1} c, \dot{\Delta}_{j'} L_\varepsilon] \dot{\Delta}_j h(x) \\ &= \sum_{j' \sim j} \int_0^1 \int_{\mathbb{R}^d} \nabla \dot{S}_{j-1} c((y + \tau(x-y))) \cdot (x-y) L_{\varepsilon, j'}(x-y) \dot{\Delta}_j h(y) dy. \end{aligned}$$

Hence, for all $j \in \mathbb{Z}$,

$$(7.13) \quad \|[\dot{S}_{j-1} c, L_\varepsilon] \dot{\Delta}_j h\|_{L^2} \leq \sum_{j' \sim j} \|z L_{\varepsilon, j'}\|_{L^1} \|\nabla \dot{S}_{j-1} c\|_{L^\infty} \|\dot{\Delta}_j h\|_{L^2}.$$

Since we have

$$\mathcal{F}(z L_{\varepsilon, j'}) (\xi) = i 2^{-j'} \left(\widehat{L}_\varepsilon(\xi) \nabla \varphi(2^{-j'} \xi) + \psi(2^{-j'} \xi) \xi \cdot \nabla \widehat{L}_\varepsilon(\xi) \right) \quad \text{with} \quad \psi(\eta) := |\eta|^{-2} \eta \varphi(\eta),$$

we get from convolution inequalities and (3.1) that

$$(7.14) \quad \sup_{\varepsilon, j'} 2^{j'} \|z L_{\varepsilon, j'}\|_{L^1} < \infty.$$

So we have

$$2^{j\sigma} \|[\dot{S}_{j-1} c, L_\varepsilon] \dot{\Delta}_j h\|_{L^2} \leq C \|\nabla c\|_{L^\infty} 2^{j(\sigma-1)} \|\dot{\Delta}_j h\|_{L^2},$$

and it is now easy to complete the proof of (7.6). \square

Lemma 7.3. *Assume that $\sigma > -d/2$. Then we have*

$$(7.15) \quad \|[L_\varepsilon \dot{\Delta}_j, c]h\|_{L^2} \leq C c_j 2^{-j\sigma} (\|c\|_{\dot{B}_{2,1}^{\frac{d}{2}+1}} \|h\|_{\dot{B}_{2,1}^{\sigma-1}} + \|c\|_{\dot{B}_{2,1}^\sigma} \|h\|_{L^\infty})$$

and the second term is not needed if $\sigma \leq d/2 + 1$.

Proof. For conciseness, we only treat the case $\sigma \leq d/2 + 1$ (the easy adaptations for $\sigma > d/2 + 1$ are left to the reader). One can mimic the proof of (7.4) proposed in [1]: using Bony's decomposition (7.7), we write that

$$(7.16) \quad [L_\varepsilon \dot{\Delta}_j, c]h = [L_\varepsilon \dot{\Delta}_j, T_c]h + L_\varepsilon \dot{\Delta}_j T'_h c - T'_h L_\varepsilon \dot{\Delta}_j c.$$

For the last term, we have

$$T'_h L_\varepsilon \dot{\Delta}_j c = \sum_{|j'-j| \leq 1} \dot{S}_{j+2} h \dot{\Delta}_{j'} L_\varepsilon \dot{\Delta}_j c.$$

Hence, since $\sigma - 1 - d/2 \leq 0$, we have, thanks to Bernstein inequality, the definition of Besov space $\dot{B}_{\infty,1}^{\sigma-1-\frac{d}{2}}$ and embedding $\dot{B}_{2,1}^{\sigma-1} \hookrightarrow \dot{B}_{\infty,\infty}^{\sigma-1-\frac{d}{2}}$,

$$\begin{aligned} \|T'_h L_\varepsilon \dot{\Delta}_j c\|_{L^2} &\lesssim \|\dot{S}_{j+2} h\|_{L^\infty} \|\dot{\Delta}_j c\|_{L^2} \\ &\lesssim 2^{-j(\sigma-1-\frac{d}{2})} \|h\|_{\dot{B}_{\infty,1}^{\sigma-1-\frac{d}{2}}} \|\dot{\Delta}_j c\|_{L^2} \\ &\lesssim 2^{-j\sigma} \|h\|_{\dot{B}_{2,1}^{\sigma-1}} 2^{j(1+d/2)} \|\dot{\Delta}_j c\|_{L^2}. \end{aligned}$$

The last but one term of (7.16) may be bounded thanks to the fact that

$$T' : \dot{B}_{2,1}^{\sigma-1} \times \dot{B}_{2,1}^{\frac{d}{2}+1} \rightarrow \dot{B}_{2,1}^\sigma, \quad -d/2 < \sigma \leq d/2 + 1.$$

For bounding the first term, one can write that both $\dot{S}_{j'-1} c h_{j'}$ and $h_{j'}$ are localized in an annulus of size $2^{j'}$. Hence, by definition of the paraproduct, we have

$$[L_\varepsilon \dot{\Delta}_j, T_c]h = \sum_{j' \sim j} [L_\varepsilon \dot{\Delta}_j, \dot{S}_{j'-1} c] h_{j'}.$$

The mean value formula ensures that for all $x \in \mathbb{R}^d$, we have

$$[L_\varepsilon \dot{\Delta}_j, \dot{S}_{j'-1} c] h_{j'}(x) = \int_{\mathbb{R}^d} \int_0^1 L_{\varepsilon,j}(x-y) \nabla \dot{S}_{j'-1} c(x + \tau(y-x)) \cdot (y-x) h_{j'}(y) d\tau dy,$$

whence

$$\|[L_\varepsilon \dot{\Delta}_j, T_c]h\|_{L^2} \leq \sum_{j' \sim j} \|z L_{\varepsilon,j}\|_{L^1} \|\nabla \dot{S}_{j'-1} c\|_{L^\infty} \|h_{j'}\|_{L^2}.$$

Hence, owing to (7.14)

$$(7.17) \quad \|[L_\varepsilon \dot{\Delta}_j, T_c]h\|_{L^2} \leq C c_j 2^{-j\sigma} \|\nabla c\|_{L^\infty} \|h\|_{\dot{B}_{2,1}^\sigma},$$

which, combined with (7.3), completes the proof of the lemma. \square

Lemma 7.4. *Let $\sigma \in (-d/2, d/2 + 1]$. Let z be in $\dot{B}_{2,1}^{d/2} \cap \dot{B}_{2,1}^\sigma$ and b be a scalar function such that $\nabla b \in \dot{B}_{2,1}^{d/2}$ and $\nabla L_\varepsilon b \in \dot{B}_{2,1}^\sigma$. Then,*

$$(7.18) \quad \|\nabla L_\varepsilon [b, \dot{\Delta}_j] z\|_{L^2} \leq C c_j 2^{-j\sigma} (\|\nabla b\|_{\dot{B}_{2,1}^{\frac{d}{2}}} \|z\|_{\dot{B}_{2,1}^\sigma} + \|z\|_{L^\infty} \|\nabla L_\varepsilon b\|_{\dot{B}_{2,1}^\sigma}), \quad j \in \mathbb{Z}.$$

Furthermore, the second term is not needed if $-d/2 < \sigma \leq d/2$.

Proof. We start with the decomposition

$$\begin{aligned}\nabla L_\varepsilon[b, \dot{\Delta}_j]z &= L_\varepsilon[b, \dot{\Delta}_j]\nabla z + L_\varepsilon[\nabla b, \dot{\Delta}_j]z \\ &= L_\varepsilon[b, \dot{\Delta}_j]\nabla z + L_\varepsilon(\nabla b z_j) - \dot{\Delta}_j L_\varepsilon(\nabla b z) \\ &= R_j^{11} + R_j^{12} + R_j^{13} + R_j^{14} + R_j^{15}\end{aligned}$$

with $R_j^{11} := L_\varepsilon[b, \dot{\Delta}_j]\nabla z$, $R_j^{12} := L_\varepsilon(\nabla b z_j)$, $R_j^{13} := -\dot{\Delta}_j L_\varepsilon T'_{\nabla b} z$, $R_j^{14} := -T_z L_\varepsilon \dot{\Delta}_j \nabla b$ and $R_j^{15} := [T_z, L_\varepsilon \dot{\Delta}_j] \nabla b$.

Since $0 \leq \widehat{L}_\varepsilon \leq 1$, the commutator estimate (7.4) ensures that

$$\|R_j^{11}\|_{L^2} \leq \|[b, \dot{\Delta}_j]\nabla z\|_{L^2} \leq C c_j 2^{-j\sigma} \|\nabla b\|_{\dot{B}_{2,1}^{\frac{d}{2}}} \|z\|_{\dot{B}_{2,1}^\sigma}.$$

Next, owing to (7.3),

$$\|R_j^{12}\|_{L^2} \leq \|\nabla b z_j\|_{L^2} \leq \|\nabla b\|_{L^\infty} \|z_j\|_{L^2} \leq C c_j 2^{-j\sigma} \|\nabla b\|_{\dot{B}_{2,1}^{\frac{d}{2}}} \|z\|_{\dot{B}_{2,1}^\sigma},$$

and, because $T' : \dot{B}_{2,1}^{\frac{d}{2}} \times \dot{B}_{2,1}^\sigma \rightarrow \dot{B}_{2,1}^\sigma$ for $\sigma > -d/2$, we have

$$\|R_j^{13}\|_{L^2} \leq c_j 2^{-j\sigma} \|T'_{\nabla b} z\|_{\dot{B}_{2,1}^\sigma} \leq C c_j 2^{-j\sigma} \|\nabla b\|_{\dot{B}_{2,1}^{\frac{d}{2}}} \|z\|_{\dot{B}_{2,1}^\sigma}.$$

Next, since

$$R_j^{14} = \sum_{|j'-j| \leq 1} \dot{S}_{j'-1} z L_\varepsilon \dot{\Delta}_{j'} \nabla b_j,$$

we have

$$\|R_j^{14}\|_{L^2} \leq C \|z\|_{L^\infty} \|L_\varepsilon \nabla b_j\|_{L^2} \leq C c_j 2^{-j\sigma} \|z\|_{L^\infty} \|L_\varepsilon \nabla b\|_{\dot{B}_{2,1}^\sigma}.$$

Note that if $\sigma \leq d/2$, then we also have for $|j' - j| \leq 1$,

$$\|\dot{S}_{j'-1} z\|_{L^\infty} \lesssim 2^{-j(\sigma-d/2)} \|z\|_{\dot{B}_{\infty,1}^{\sigma-d/2}} \lesssim 2^{-j\sigma} \|z\|_{\dot{B}_{2,1}^\sigma}$$

so that

$$\|R_j^{14}\|_{L^2} \leq C c_j 2^{-j\sigma} \|\nabla b\|_{\dot{B}_{2,1}^{\frac{d}{2}}} \|z\|_{\dot{B}_{2,1}^\sigma}.$$

The term R_j^{15} may be treated by a small variation of (7.17). We get

$$\|R_j^{15}\|_{L^2} \leq C c_j 2^{-j\sigma} \|\nabla b\|_{\dot{B}_{2,1}^{\frac{d}{2}}} \|\nabla z\|_{B_{\infty,1}^{\sigma-1-\frac{d}{2}}}.$$

In the end, remembering the embedding $B_{2,1}^{\sigma-1} \hookrightarrow B_{\infty,1}^{\sigma-1-\frac{d}{2}}$, we obtain (7.18). \square

An alternative proof of (7.18). Inequality (7.18) can be alternatively demonstrated by means of an integral representation. In contrast to employing the para-decomposition, our approach necessitates the explicit elucidation of the paramount terms that engender limitations on the regularity. This particular scenario demands a more intricate analysis, wherein we focus on a three-dimensional space and impose constraints on the regularity with $\sigma = 5/2$.

Note that $\nabla L_\varepsilon[b, \dot{\Delta}_j] \operatorname{div} u = L_\varepsilon[\nabla b, \dot{\Delta}_j] \operatorname{div} u + L_\varepsilon[b, \dot{\Delta}_j] \nabla \operatorname{div} u$. The second part can be easily treated by the commutator rule (7.4)

$$2^{(d/2+1)j} \|L_\varepsilon[b, \dot{\Delta}_j] \nabla \operatorname{div} u\|_{L^2} \lesssim c_j \|\nabla b\|_{L^\infty} \|\operatorname{div} u\|_{\dot{B}_{2,1}^{d/2+1}} \quad \text{with} \quad \sum_j c_j = 1.$$

The first part can be seen as follows

$$L_\varepsilon[\nabla b, \dot{\Delta}_j] \operatorname{div} u = L_\varepsilon \nabla b \dot{\Delta}_j \operatorname{div} u - L_\varepsilon \dot{\Delta}_j (\nabla b \operatorname{div} u).$$

Above, the first term is bounded by

$$2^{(d/2+1)j} \|L_\varepsilon \nabla b \dot{\Delta}_j \operatorname{div} u\|_{L^2} \lesssim \|\nabla b\|_{L^\infty} 2^{(d/2+1)j} \|\nabla u_j\|_{L^2}.$$

Since $(\dot{\Delta}_j \nabla b) \operatorname{div} u$ is of a good form and

$$2^{(d/2+1)j} \|L_\varepsilon (\dot{\Delta}_j \nabla b) \operatorname{div} u\|_{L^2} \lesssim 2^{(d/2+1)j} \|\dot{\Delta}_j \nabla b\|_{L^2} \|\operatorname{div} u\|_{L^\infty},$$

the most difficult term we consider in the following form

$$L_\varepsilon \dot{\Delta}_j (\nabla b \operatorname{div} u) - L_\varepsilon (\dot{\Delta}_j \nabla b) \operatorname{div} u.$$

Let $L_\varepsilon^j = \dot{\Delta}_j L_\varepsilon$, then we restate the above term

$$\begin{aligned} & \int_{\mathbb{R}^d} L_\varepsilon^j(z) \nabla b(x-z) (\operatorname{div} u(x-z) - \operatorname{div} u(x)) dz \\ &= \int_{\mathbb{R}^d} L_\varepsilon^j(z) \nabla b(x-z) \left[\nabla^2 u(x) z + \int_0^1 (\nabla^2 u(x-tz) - \nabla^2 u(x)) z dt \right] dz = K_1 + K_2. \end{aligned}$$

Let fix $d = 3$. In order to get the general case, it is required to apply the induction method to get the bound for arbitrary dimension. First, we find the bound for

$$2^{5/2j} \|K_1\|_{L^2} \leq 2^{5/2j} \left\| \int_{\mathbb{R}^3} z L_\varepsilon^j(z) \nabla b(x-z) dz \right\|_{L^2} \|\nabla^2 u\|_{L^\infty} \leq C 2^{5/2j} \|b_j\|_{L^2} \|\nabla^2 u\|_{\dot{B}_{2,1}^{3/2}}.$$

Note that by definition

$$\int_{\mathbb{R}^3} z L_\varepsilon^j(z) \nabla b(x-z) dz = \int_{\mathbb{R}^3} z L_\varepsilon(z) \nabla \dot{\Delta}_j b(x-z) dz,$$

hence by (3.1)

$$\left\| \int_{\mathbb{R}^3} z L_\varepsilon^j(z) \nabla b(x-z) dz \right\|_{L^2} \lesssim \|\xi \partial_\xi \hat{L}(\xi) \hat{b}_j\|_{L^2} \lesssim \|b_j\|_{L^2}.$$

The term K_2 still needs to be restated. So (we use the transform $(t, s) \rightarrow (ts, s)$)

$$\begin{aligned} K_2 &= \int_{\mathbb{R}^3} z L_\varepsilon^j(z) \nabla b(x-z) \int_0^1 \int_0^1 \nabla^3 u(x-tsz) tz ds dt dz \\ &= \int_{\mathbb{R}^3} z^2 L_\varepsilon^j(z) \nabla b(x-z) [\nabla^3 u(x) + \int_0^1 (\nabla^3 u(x-sz) - \nabla^3 u(x)) ds] dz = K_{21} + K_{22}. \end{aligned}$$

And then

$$\begin{aligned} 2^{5/2j} \left\| \int_{\mathbb{R}^3} z^2 L_\varepsilon^j(z) \nabla b(x-z) dz \nabla^3 u(x) \right\|_{L^2} &\leq 2^{5/2j} \left\| \int_{\mathbb{R}^3} z^2 L_\varepsilon^j(z) \nabla b(x-z) dz \right\|_{L^6} \|\nabla^3 u\|_{L^3} \\ &\leq C 2^{5/2j} 2^{-j} \|L_\varepsilon b_j\|_{L^6} \|\nabla^3 u\|_{L^3} \leq C 2^{5/2j} \|L_\varepsilon b_j\|_{L^2} \|\nabla^3 u\|_{B_{2,1}^{1/2}}. \end{aligned}$$

Note the above case requires the highest regularity in both terms.

Term K_{22} requires some more care. In an direct way we get

$$\begin{aligned}
& 2^{5/2j} \left\| \int_{\mathbb{R}^3} z^2 L_\varepsilon^j(z) \nabla b(x-z) \int_0^1 (\nabla^3 u(x-sz) - \nabla^3 u(x)) ds dz \right\|_{L^2} \\
& \leq 2^{5/2j} \left\| \int_{\mathbb{R}^3} z^2 L_\varepsilon^j(z) \nabla b(x-z) \int_0^1 \frac{(\nabla^3 u(x-sz) - \nabla^3 u(x))}{(s|z|)^{1/2}} s^{1+1/2} |z|^{1/2} \frac{|z| ds}{s|z|} dz \right\|_{L^2} \\
& \leq C 2^{5/2j} \|z^{2+1/2} L_\varepsilon^j\|_{L^1} \|\nabla b\|_{L^\infty} \sup_{z \in \mathbb{R}^3} \int_0^\infty \frac{\|\nabla^3 u(x-h\hat{e}_z) - \nabla^3 u(x)\|_2 dh}{h^{1/2}} \frac{dh}{h} \\
& \leq C \|\nabla b\|_{L^\infty} \|\nabla^3 u\|_{B_{2,1}^{1/2}}.
\end{aligned}$$

By the assumption (3.1) we easily deduce that

$$2^{5/2j} \|z^{5/2} L_\varepsilon^j\|_{L^1} \leq \text{uniformly bounded in } j \quad \text{and} \quad \varepsilon.$$

The right-hand side is independent of j . The ℓ^1 summability is required. So we proved the existence of a map from $B_{2,1}^{1/2}(\mathbb{R}^3) \rightarrow B_{2,\infty}^{5/2}(\mathbb{R}^3)$, but it is not enough. Fortunately we can use interpolation. Note that $1/2$ can be replaced by any α close to $1/2$, a bit bigger and a bit smaller, then we get the map $T : \dot{B}_{2,1}^\alpha(\mathbb{R}^3) \rightarrow \dot{B}_{2,\infty}^\alpha(\mathbb{R}^3)$, so then

$$T : \dot{B}_{2,1}^{1/2}(\mathbb{R}^3) = (\dot{B}_{2,1}^{1/2-\sigma}(\mathbb{R}^3); \dot{B}_{2,1}^{1/2+\sigma}(\mathbb{R}^3))_{1/2,1} \rightarrow (\dot{B}_{2,\infty}^{1/2-\sigma}(\mathbb{R}^3); \dot{B}_{2,\infty}^{1/2+\sigma}(\mathbb{R}^3))_{1/2,1} = \dot{B}_{2,1}^{1/2}(\mathbb{R}^3)$$

with the suitable desired estimates. We are done. This approach takes more space, but in some special cases can deliver faster answers concerning the limitation on the required regularity. \square

Lemma 7.5. *There exists a constant C independent of $j \in \mathbb{Z}$ and $\varepsilon > 0$ such that the following inequality holds:*

$$(7.19) \quad \|[L_\varepsilon, c] \dot{\Delta}_j z\|_{L^2} \leq C 2^{-j} \|\nabla c\|_{L^\infty} \|\dot{\Delta}_j z\|_{L^2}.$$

Proof. It is based on the decomposition

$$(7.20) \quad [L_\varepsilon, c] \dot{\Delta}_j z = [L_\varepsilon, \dot{S}_{j-1} c] \dot{\Delta}_j z + L_\varepsilon ((\text{Id} - \dot{S}_{j-1}) c \dot{\Delta}_j z) - (\text{Id} - \dot{S}_{j-1}) c L_\varepsilon \dot{\Delta}_j z,$$

and on the fact that, in light of the properties of localization of $\dot{\Delta}_j$ and \dot{S}_{j-1} , we have

$$[L_\varepsilon, \dot{S}_{j-1} c] \dot{\Delta}_j z = \sum_{j' \sim j} [L_\varepsilon \dot{\Delta}_{j'}, \dot{S}_{j-1} c] \dot{\Delta}_j z.$$

Hence, (7.13) and (7.14) guarantee that

$$\|[L_\varepsilon, \dot{S}_{j-1} c] \dot{\Delta}_j z\|_{L^2} \leq 2^{-j} \|\nabla \dot{S}_{j-1} c\|_{L^\infty} \|\dot{\Delta}_j z\|_{L^2}$$

and, since

$$\|(\text{Id} - \dot{S}_{j-1}) c\|_{L^\infty} \leq C 2^{-j} \|\nabla c\|_{L^\infty},$$

the other two terms of (7.20) also satisfy the desired inequality. \square

Lemma 7.6. *We have for $\sigma \in (-d/2 - 1, d/2 + 1]$,*

$$\begin{aligned}
(7.21) \quad & \|\nabla L_\varepsilon \dot{\Delta}_j (v \cdot \nabla a) - v \cdot \nabla (\nabla L_\varepsilon \dot{\Delta}_j a)\|_{L^2} \\
& \leq C c_j 2^{-j\sigma} \left(\|a\|_{\dot{B}_{2,1}^\sigma} \|v\|_{\dot{B}_{2,1}^{\frac{d}{2}+2}} + \|v\|_{\dot{B}_{2,1}^{\frac{d}{2}+1}} \|\nabla L_\varepsilon a\|_{\dot{B}_{2,1}^\sigma} \right).
\end{aligned}$$

Proof. Using again Bony's decomposition (7.7) and the fact that $\dot{S}_{j'-1}v \cdot \nabla a_{j'}$ is localized in an annulus of size $2^{j'}$, we may write

$$\begin{aligned} \nabla L_\varepsilon \dot{\Delta}_j(v \cdot \nabla a) &= \nabla L_\varepsilon \dot{\Delta}_j T'_{\nabla a} \cdot v + \nabla L_\varepsilon \dot{\Delta}_j T_v \cdot \nabla a \\ &= \nabla L_\varepsilon \dot{\Delta}_j T'_{\nabla a} \cdot v + \nabla L_\varepsilon \dot{\Delta}_j \sum_{j' \sim j} \dot{S}_{j'-1}v \cdot \nabla a_{j'} \\ &= R_j^{21} + R_j^{22} + R_j^{23} + R_j^{24} + v \cdot \nabla(\nabla L_\varepsilon a_j) \end{aligned}$$

$$\text{with } R_j^{21} := \nabla L_\varepsilon T'_{\nabla a} \cdot v, \quad R_j^{22} := \nabla L_\varepsilon \dot{\Delta}_j \sum_{j' \sim j} (\dot{S}_{j'-1} - \dot{S}_{j-1})v \cdot \nabla a_{j'},$$

$$R_j^{23} := \sum_{j' \sim j} [\nabla L_\varepsilon \dot{\Delta}_j, \dot{S}_{j-1}v] \cdot \nabla a_{j'} \quad \text{and} \quad R_j^{24} := (\dot{S}_{j-1} - \text{Id})v \cdot (\nabla(\nabla L_\varepsilon a_j)).$$

For R_j^{21} , we use that $T' : \dot{B}_{2,1}^{\sigma-1} \times \dot{B}_{2,1}^{\frac{d}{2}+2} \rightarrow \dot{B}_{2,1}^{\sigma+1}$ for $-d/2 - 1 < \sigma \leq d/2 + 1$ and that $\nabla L_\varepsilon : \dot{B}_{2,1}^{\sigma+1} \rightarrow \dot{B}_{2,1}^\sigma$ uniformly with respect to ε to get

$$(7.22) \quad \|R_j^{21}\|_{L^2} \leq C c_j 2^{-j\sigma} \|\nabla a\|_{\dot{B}_{2,1}^{\sigma-1}} \|v\|_{\dot{B}_{2,1}^{\frac{d}{2}+2}}.$$

Next, by Bernstein inequality and the fact that $0 \leq \widehat{L}_\varepsilon \leq 1$, we have

$$\begin{aligned} \|R_j^{22}\|_{L^2} &\lesssim 2^j \sum_{j'' \sim j' \sim j} 2^{-2j''-j'(\sigma-1)} (2^{2j''} \|v_{j''}\|_{L^\infty}) (2^{j'(\sigma-1)} \|\nabla a_{j'}\|_{L^2}) \\ &\lesssim 2^{-j\sigma} \|v\|_{\dot{B}_{\infty,\infty}^2} \sum_{j' \sim j} (2^{j'(\sigma-1)} \|\nabla a_{j'}\|_{L^2}). \end{aligned}$$

Hence we have, owing to embedding $\dot{B}_{2,1}^{\frac{d}{2}+2} \hookrightarrow \dot{B}_{\infty,\infty}^2$,

$$(7.23) \quad \|R_j^{22}\|_{L^2} \leq C c_j 2^{-j\sigma} \|v\|_{\dot{B}_{2,1}^{\frac{d}{2}+2}} \|\nabla a\|_{\dot{B}_{2,1}^{\sigma-1}}.$$

To bound R_j^{24} , we use the fact that

$$\|(\dot{S}_{j-1} - \text{Id})v\|_{L^\infty} \lesssim 2^{-j} \|v\|_{\dot{B}_{\infty,\infty}^1}.$$

Hence, combining with the embedding $\dot{B}_{2,1}^{\frac{d}{2}+1} \hookrightarrow \dot{B}_{\infty,\infty}^1$ and Bernstein inequality,

$$(7.24) \quad \|R_j^{24}\|_{L^2} \leq C \|v\|_{\dot{B}_{2,1}^{\frac{d}{2}+1}} \|\nabla L_\varepsilon a_j\|_{L^2}.$$

To handle R_j^{23} , we have to go to the second order in the Taylor expansion. Using again the notation $L_{\varepsilon,j} = \mathcal{F}^{-1}(\widehat{L}_\varepsilon \varphi(2^{-j}\cdot))$, we write that for all $k \in \{1, \dots, d\}$, we have, with the summation convention,

$$[\partial_k L_\varepsilon \dot{\Delta}_j, \dot{S}_{j-1}v^\ell] \partial_\ell a_{j'} = R_{jj'k}^{231} + R_{jj'k}^{232}$$

$$\text{with } R_{jj'k}^{231}(x) := \int_{\mathbb{R}^d} \partial_k L_{\varepsilon,j}(y-x) (y-x) \cdot \nabla \dot{S}_{j-1}v^\ell(x) \partial_\ell a_{j'}(y) dy \quad \text{and}$$

$$R_{jj'k}^{232}(x) := \int_{\mathbb{R}^d} \left(\int_0^1 (1-\tau) D^2 \dot{S}_{j-1}v^\ell(x+\tau(y-x))(y-x, y-x) d\tau \right) \partial_k L_{\varepsilon,j}(x-y) \partial_\ell a_{j'}(y) dy.$$

First, by using Hölder inequality, we have

$$\|R_{jj'k}^{231}\|_{L^2} \leq \|\nabla \dot{S}_{j-1}v^\ell\|_{L^\infty} \|z \partial_k L_{\varepsilon,j} \star \partial_\ell a_{j'}\|_{L^2}.$$

Denoting $h_j := \mathcal{F}^{-1}(\varphi(2^{-j}\cdot))$, we have the identity

$$z\partial_{z_k}(L_\varepsilon \star h_j) = \partial_{z_k}L_\varepsilon \star (zh_j) - L_\varepsilon \star \partial_{z_k}(zh_j)$$

and thus

$$z\partial_k L_{\varepsilon,j} \star \partial_\ell a_{j'} = (zh_j) \star L_\varepsilon \partial_k \partial_\ell a_{j'} - \partial_k(zh_j) \star \partial_\ell L_\varepsilon a_{j'}.$$

Since

$$\|zh_j\|_{L^1} = 2^{-j}\|zh_0\|_{L^1} \quad \text{and} \quad \|\partial_k(zh_j)\|_{L^1} = \|\partial_k(zh_0)\|_{L^1},$$

we deduce (using once Bernstein inequality) that

$$(7.25) \quad \|R_{jj'k}^{231}\|_{L^2} \lesssim \|\nabla v\|_{L^\infty} \|L_\varepsilon \nabla a_{j'}\|_{L^2}.$$

For the other term, we have

$$\|R_{jj'k}^{232}\|_{L^2} \lesssim \|\nabla^2 v\|_{L^\infty} \|(z \otimes z) \nabla L_{\varepsilon,j}\|_{L^1} \|\nabla a_j\|_{L^2},$$

and one can show that

$$\|(z \otimes z) \nabla L_{\varepsilon,j}\|_{L^1} \lesssim 2^{-j} \|(z \otimes z) \nabla^2 L_\varepsilon\|_{L^1}.$$

Indeed, if we set $\tilde{\varphi}(\xi) = -i\xi|\xi|^{-2}\varphi(\xi)$ and $\tilde{h}_0 := \mathcal{F}^{-1}\tilde{\varphi}$, then $h_0 = \operatorname{div} \tilde{h}_0$, and thus $\tilde{h}_j = 2^{-j} \operatorname{div} \tilde{h}_j$ for all $j \in \mathbb{Z}$. Consequently, we have

$$(z \otimes z) \nabla L_{\varepsilon,j} = 2^{-j} (z \otimes z) (\Delta L_\varepsilon \star \tilde{h}_j).$$

Hence

$$(7.26) \quad \|R_{jj'k}^{232}\|_{L^2} \lesssim \|\nabla^2 v\|_{L^\infty} \|a_{j'}\|_{L^2}.$$

Putting (7.25) and (7.26) together yields

$$(7.27) \quad \|R_j^{23}\|_{L^2} \leq C \sum_{j' \sim j} (\|\nabla v\|_{L^\infty} \|L_\varepsilon \nabla a_{j'}\|_{L^2} + \|\nabla^2 v\|_{L^\infty} \|a_{j'}\|_{L^2}).$$

Hence, one can conclude from (7.22), (7.23), (7.24) and (7.27) that (7.21) holds true. \square

7.2. The general pressure case. Here we explain how to close the estimates for all time in the general pressure case, that is for System (2.2). Denoting $a := \rho - 1$, this corresponds to System (4.1) with $v = u$, $b = a$ and $c = F(K_\varepsilon a)$ with $F(z) := \mathcal{N}(1+z) - 1$. We assume that⁵

$$(7.28) \quad F(0) = 0 \quad \text{and} \quad F'(0) = 1.$$

We plan to use Inequality (4.5) with $\sigma = d/2 + 1$. Note that, at some point, we will have to bound the L^∞ norm of $c_t + \operatorname{div}((1+c)v)$ with $c = F(K_\varepsilon a)$. To do this, we observe from the first equation of (2.2) that

$$\partial_t(K_\varepsilon(a)) + \operatorname{div}((1+K_\varepsilon a)u) = \mathcal{R}_\varepsilon := K_\varepsilon a \operatorname{div} u + \sum_j [u^j, \partial_j K_\varepsilon] a + (\operatorname{Id} - K_\varepsilon) \operatorname{div} u,$$

whence

$$\partial_t(F(K_\varepsilon a)) + \operatorname{div}((1+F(K_\varepsilon a))u) = (1+F(K_\varepsilon a) - (1+K_\varepsilon a)F'(K_\varepsilon a)) \operatorname{div} u + F'(K_\varepsilon a) \mathcal{R}_\varepsilon.$$

Under condition (4.2) with $b = a$ and thanks to hypothesis (7.28), it is obvious that

$$\|(1+F(K_\varepsilon a) - (1+K_\varepsilon a)F'(K_\varepsilon a)) \operatorname{div} u\|_{L^\infty} \lesssim \|\operatorname{div} u\|_{L^\infty} \|a\|_{L^\infty}.$$

⁵The second assumption can be achieved after renormalization.

Next, using first order Taylor formula, we readily get

$$\left\| \sum_j [u^j, \partial_j K_\varepsilon] a \right\|_{L^\infty} \leq \|z \nabla K_\varepsilon\|_{L^1} \|\nabla u\|_{L^\infty} \|a\|_{L^\infty}.$$

Hence, keeping Assumption (3.1) in mind and assuming e.g. that $|a| \leq 1/4$, we conclude that

$$(7.29) \quad \left\| \partial_t (K_\varepsilon(a)) + \operatorname{div}((1 + K_\varepsilon a)u) \right\|_{L^\infty} \lesssim \|\nabla u\|_{L^\infty}.$$

Now, denoting $X := X_{a,u}^{\frac{d}{2}+1}$ and $H := H_{a,u}^{\frac{d}{2}+1}$, using (7.29) and observing that in the case $v = u$ and $b = a$ all the terms in lines two and three are of type $\|\nabla u\|_{\dot{B}_{2,1}^{\frac{d}{2}} \cap \dot{B}_{2,1}^{\frac{d}{2}+1}} X$, Inequality (4.5)

with $\sigma = d/2 + 1$ reduces to

$$\begin{aligned} X(t) + \int_0^t H \, d\tau &\lesssim X(0) + \int_0^t \|\nabla u\|_{\dot{B}_{2,1}^{\frac{d}{2}} \cap \dot{B}_{2,1}^{\frac{d}{2}+1}} X \, d\tau \\ &+ \int_0^t \left(\|F(K_\varepsilon a)\|_{\dot{B}_{2,1}^{\frac{d}{2}}} \left(\|\nabla L_\varepsilon a^h\|_{\dot{B}_{2,1}^{\frac{d}{2}+1}} + \|\nabla K_\varepsilon a^\ell\|_{\dot{B}_{2,1}^{\frac{d}{2}+1}} + \|\nabla^2 K_\varepsilon a^\ell\|_{\dot{B}_{2,1}^{\frac{d}{2}+1}} \right) \right. \\ &+ \|F(K_\varepsilon a)\|_{\dot{B}_{2,1}^{\frac{d}{2}+1}} \left(\|\nabla L_\varepsilon a^h\|_{\dot{B}_{2,1}^{\frac{d}{2}}} + \|\nabla K_\varepsilon a^\ell\|_{\dot{B}_{2,1}^{\frac{d}{2}}} + \|\nabla^2 K_\varepsilon a^\ell\|_{\dot{B}_{2,1}^{\frac{d}{2}}} \right) \\ &\left. + \|F(K_\varepsilon a)\|_{\dot{B}_{2,1}^{\frac{d}{2}+1}} \|(L_\varepsilon a, \nabla L_\varepsilon a)\|_{\dot{B}_{2,1}^{\frac{d}{2}+1}} + \|F(K_\varepsilon a)\|_{\dot{B}_{2,1}^{\frac{d}{2}+2}} \|\nabla L_\varepsilon a\|_{\dot{B}_{2,1}^{\frac{d}{2}}} \right) d\tau. \end{aligned}$$

Since $F(0) = 0$, the right-hand side may be simplified thanks to the following composition inequality that is valid whenever $\|z\|_{L^\infty}$ is small enough and $s > 0$:

$$\|F(z)\|_{\dot{B}_{2,1}^s} \lesssim \|z\|_{\dot{B}_{2,1}^s}.$$

In the end, after a few simplifications, we discover that

$$\begin{aligned} X(t) + \int_0^t H \, d\tau &\lesssim X(0) + \int_0^t \|\nabla u\|_{\dot{B}_{2,1}^{\frac{d}{2}} \cap \dot{B}_{2,1}^{\frac{d}{2}+1}} X \, d\tau \\ &+ \int_0^t \left(\|K_\varepsilon a\|_{\dot{B}_{2,1}^{\frac{d}{2}}} \left(\|\nabla L_\varepsilon a^h\|_{\dot{B}_{2,1}^{\frac{d}{2}+1}} + \|\nabla K_\varepsilon a^\ell\|_{\dot{B}_{2,1}^{\frac{d}{2}+1}} + \|\nabla^2 K_\varepsilon a^\ell\|_{\dot{B}_{2,1}^{\frac{d}{2}+1}} \right) \right. \\ &+ \|K_\varepsilon a\|_{\dot{B}_{2,1}^{\frac{d}{2}+1}} \left(\|L_\varepsilon a\|_{\dot{B}_{2,1}^{\frac{d}{2}+1}} + \|\nabla K_\varepsilon a^\ell\|_{\dot{B}_{2,1}^{\frac{d}{2}+1}} \right) \\ &\left. + \|K_\varepsilon a\|_{\dot{B}_{2,1}^{\frac{d}{2}+1}} \|\nabla L_\varepsilon a\|_{\dot{B}_{2,1}^{\frac{d}{2}+1}} + \|K_\varepsilon a\|_{\dot{B}_{2,1}^{\frac{d}{2}+2}} \|\nabla L_\varepsilon a\|_{\dot{B}_{2,1}^{\frac{d}{2}}} \right) d\tau. \end{aligned}$$

We observe that all the products in the integrals of the right-hand side may be bounded by HX except, maybe,

$$\|K_\varepsilon a\|_{\dot{B}_{2,1}^{\frac{d}{2}+1}}^\ell \|L_\varepsilon a\|_{\dot{B}_{2,1}^{\frac{d}{2}+1}}^\ell \quad \text{and} \quad \|K_\varepsilon a\|_{\dot{B}_{2,1}^{\frac{d}{2}+1}}^\ell \|\nabla L_\varepsilon a\|_{\dot{B}_{2,1}^{\frac{d}{2}+1}}^\ell.$$

However, by Cauchy-Schwarz inequality in the Fourier space and the fact that $K_\varepsilon = L_\varepsilon * L_\varepsilon$, we notice that

$$\begin{aligned} \|K_\varepsilon a\|_{\dot{B}_{2,1}^{\frac{d}{2}+1}}^\ell &\leq \|L_\varepsilon a\|_{\dot{B}_{2,1}^{\frac{d}{2}+1}}^\ell \leq \sqrt{\|a\|_{\dot{B}_{2,1}^{\frac{d}{2}}}^\ell \|\nabla^2 K_\varepsilon a\|_{\dot{B}_{2,1}^{\frac{d}{2}}}^\ell} \leq \sqrt{XH} \\ \|\nabla L_\varepsilon a\|_{\dot{B}_{2,1}^{\frac{d}{2}+1}}^\ell &\leq \sqrt{\|a\|_{\dot{B}_{2,1}^{\frac{d}{2}+1}}^\ell \|\nabla^2 K_\varepsilon a\|_{\dot{B}_{2,1}^{\frac{d}{2}+1}}^\ell} \leq \sqrt{XH}. \end{aligned}$$

Hence, we conclude that for some universal constant $C \geq 1$, we have for all $t \geq 0$

$$X(t) + \int_0^t H d\tau \leq C \left(X(0) + \int_0^t H X d\tau \right).$$

Now, provided $2C^2 X(0) < 1$, we get the following global-in-time and uniform in ε control:

$$X(t) + \frac{1}{2} \int_0^t H d\tau \leq C X(0).$$

Granted with the above a priori estimate, remembering Remark 1 and mimicking the proof of Theorem (3.1), one ends up with the following global uniform well-posedness result for System (2.2).

Theorem 7.1. *Let \mathcal{N} be any smooth function defined on some neighborhood of 1 and such that $\mathcal{N}(1) = \mathcal{N}'(1) = 1$. Assume that $d \geq 2$ and take initial data $\rho_0 = 1 + a_0$ and u_0 such that*

$$u_0 \in \dot{B}_{2,1}^{\frac{d}{2}} \cap \dot{B}_{2,1}^{\frac{d}{2}+2}, \quad a_0 \in \dot{B}_{2,1}^{\frac{d}{2}-1} \cap \dot{B}_{2,1}^{\frac{d}{2}+1} \quad \text{and} \quad \nabla^2 L_\varepsilon a_0 \in \dot{B}_{2,1}^{\frac{d}{2}}.$$

There exists an absolute positive constant α_0 such that if

$$\|u_0\|_{\dot{B}_{2,1}^{\frac{d}{2}+1} \cap \dot{B}_{2,1}^{\frac{d}{2}+2}} + \|a_0\|_{\dot{B}_{2,1}^{\frac{d}{2}} \cap \dot{B}_{2,1}^{\frac{d}{2}+1}} + \|\nabla^2 L_\varepsilon a_0\|_{\dot{B}_{2,1}^{\frac{d}{2}}} \leq \alpha_0,$$

then System (2.2) with $\mathfrak{f} = 1$ supplemented with initial data (ρ_0, u_0) admits a unique global classical solution (ρ, u) such that (a, u) with $a := \rho - 1$ belongs to the space $E_{K_\varepsilon}^{\frac{d}{2}+1}$ defined in (3.6). Furthermore, $a \in \mathcal{C}(\mathbb{R}_+; \dot{B}_{2,1}^{\frac{d}{2}-1})$ and Inequality (3.8) is satisfied.

In this general pressure setting, it is also possible to prove convergence to the compressible Euler System (2.4) and asymptotic results when the friction coefficient \mathfrak{f} tends to ∞ , in the spirit of Theorems 3.2, 6.1, 6.2 and 6.3. The details are left to the reader.

7.3. From the micro to the macro scale. In this part we aim at sketching the connection between (2.1) and (1.1). In case the number N of particles is large in (2.1), it is customary to treat the distribution of particles in terms of measures. By performing the so-called mean field limit, we are led to the following kinetic equation:

$$(7.30) \quad f_t + v \cdot \nabla_x f + \operatorname{div}_v (F(f)f) = 0$$

where, for some suitable kernel K_ε ,

$$F(f)(t, x, v) = \mathfrak{f}v + \int_{\mathbb{R}^d} \nabla K_\varepsilon(x - y) f(t, y, w) dy.$$

Note that the solution to (2.1) may be seen as a *measure solution* to (7.30). Indeed, the weak formulation of (7.30) reads, for all test function $\phi \in \mathcal{D}(\mathbb{R}^d \times [0, T))$,

$$\int_0^T \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} f(\partial_t + v \cdot \nabla_x + F(f)\nabla_v)\phi dx dv dt = - \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} f_0(x, v)\phi(0, x, v) dx dv.$$

Hence, if we set

$$f := \sum_k \delta_{x_k(t)} \otimes \delta_{v_k(t)},$$

then we have

$$\begin{aligned} \sum_k \int_0^T \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \left(\partial_t + v_k(t) \cdot \nabla_x + F \left(\sum_l \delta_{x_l(t)} \otimes \delta_{v_l(t)} \right) \nabla_v \right) \phi(x_k(t), v_k(t)) dx dv \\ = - \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} f_0(x, v) \phi(0, x, v) dx dv, \end{aligned}$$

since

$$\frac{d}{dt} \phi(t, x_k(t), v_k(t)) = \partial_t \phi + \frac{dx_k}{dt} \cdot \nabla_x \phi + \frac{dv_k}{dt} \cdot \nabla_v \phi.$$

Next, let us explain how (1.1) can be obtained from (7.30). The idea is to assume that we are in the *mono-kinetic regime*, namely

$$(7.31) \quad f(t, x, v) = \rho(t, x) \otimes \delta_{v=u(t,x)}$$

for some nonnegative function $\rho(t, x)$ and vector-field $u(t, x)$. In other words, all the particles at point x at time t have the same velocity $u(t, x)$, and their density is $\rho(t, x)$. Then, first integrating over the v -coordinate the equation (7.30) we obtain the simple continuity law

$$(7.32) \quad \rho_t + \operatorname{div}(\rho u) = 0.$$

Second, multiplying (7.30) by v and integrating over v gives:

$$\begin{aligned} 0 &= \int_{\mathbb{R}^d} v f_t dv + \int_{\mathbb{R}^d} v \otimes v \cdot \nabla_x f dv - \int_{\mathbb{R}^d} v \operatorname{div}_v((fv + \nabla K_\varepsilon *_x \rho) f) dv \\ &= \frac{d}{dt} \int_{\mathbb{R}^d} v f dv + \operatorname{div}_x \int_{\mathbb{R}^d} v \otimes v f dv + d \int_{\mathbb{R}^d} (fv + \nabla K_\varepsilon *_x \rho f) dv. \end{aligned}$$

Taking into account the ansatz (7.31) we obtain then

$$(7.33) \quad \partial_t(\rho u) + \operatorname{div}(\rho u \otimes u) + d f \rho u + d \rho \nabla K_\varepsilon * \rho = 0.$$

After a suitable rescaling of the constant parameters, using (7.32) to (7.33) we obtain the original system (1.1).

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