On the conjugacy classes of monomial birational maps

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ABSTRACT. This short note deals with the conjugacy classes of monomial birational maps in the n-dimensional Cremona group, $n \geq 2$.

1. Introduction

A map $f\colon \mathbb{P}^n_{\mathbb{C}} \dashrightarrow \mathbb{P}^n_{\mathbb{C}}$ given by

$$(x_0 : x_1 : \ldots : x_n) \dashrightarrow (f_0(x_0,x_1,\ldots,x_n) : f_1(x_0,x_1,\ldots,x_n) : \ldots : f_n(x_0,x_1,\ldots,x_n))$$

where the f_i's denote homogeneous polynomials of the same degree ν without common factor is a *rational self map* of $\mathbb{P}^n_{\mathbb{C}}$ of *degree* ν. A *birational self map* f of $\mathbb{P}^n_{\mathbb{C}}$ is a rational self map of $\mathbb{P}^n_{\mathbb{C}}$ such that there exists a rational self map g of $\mathbb{P}^n_{\mathbb{C}}$ with $f \circ g = g \circ f = \mathrm{id}$. The group of birational self maps of $\mathbb{P}^n_{\mathbb{C}}$ is the *Cremona group*; it is denoted by $\mathrm{Bir}(\mathbb{P}^n_{\mathbb{C}})$.

In [Bla06] the author describes the conjugacy classes of affine automorphisms in $\mathrm{Bir}(\mathbb{P}^n_{\mathbb{C}})$; from this he deduces the classification of conjugacy classes of automorphisms of $\mathbb{P}^n_{\mathbb{C}}$ in the Cremona group.

For a birational map $f \in \mathrm{Bir}(\mathbb{P}^n_{\mathbb{C}})$ there is a well-defined pull-back map $f^*\colon \mathrm{H}^{k,k}(\mathbb{P}^n_{\mathbb{C}},\mathbb{R}) \to \mathrm{H}^{k,k}(\mathbb{P}^n_{\mathbb{C}},\mathbb{R})$ for any $1 \leq k \leq n$. The k-th *dynamical degree* of f is then defined as $\lambda_k(f) = \lim_{\ell \to +\infty} ||(f^\ell)^*||^{1/\ell}$. The λ_k's are birational invariants, *i.e.* $\lambda_k(f) = \lambda_k(gfg^{-1})$ for any f, g in $\mathrm{Bir}(\mathbb{P}^n_{\mathbb{C}})$ and for any $1 \leq k \leq n$. In [BC16] Blanc and Cantat study the relationship between the value of $\lambda(f)$ and the structure of the conjugacy class of f. In particular, they get that $f, g \in \mathrm{Bir}(\mathbb{P}^2_{\mathbb{C}})$ of degree $\leq d$ and such that $\lambda_1(f) > 1$, $\lambda_1(g) > 1$ are conjugate if and only if they are conjugate by an element of degree $\leq (2d)^{57}$. Furthermore, given an element f of $\mathrm{Bir}(\mathbb{P}^2_{\mathbb{C}})$ they define the minimal degree in its conjugacy class as the positive integer $\mathrm{mcdeg}(f) = \min \deg(gfg^{-1})$ where g describes $\mathrm{Bir}(\mathbb{P}^2_{\mathbb{C}})$. The function mcdeg is constant on conjugacy classes, and $\lambda(f) \leq \mathrm{mcdeg}(f) \leq \deg f$ for any $f \in \mathrm{Bir}(\mathbb{P}^2_{\mathbb{C}})$. They provide the following reverse inequality:

- $\diamond$ if $\lambda(f) \geq 10$, then $\mathrm{mcdeg}(f) \leq 4700\lambda_1(f)^5$;
- $\diamond$ if $\lambda_1(f) > 1$, then $\mathrm{mcdeg}(f) \leq e^{18}\lambda_1(f)^{345}$.

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In this paper we are interested in the conjugacy classes of monomial birational maps of $\mathbb{P}^n_{\mathbb{C}}$. Let us specify what monomial birational maps of $\mathbb{P}^n_{\mathbb{C}}$ are. To any $n\times n$ integer matrix $M=(m_{ij})\in \mathrm{GL}(n,\mathbb{Z})$ we can associate the monomial map $f_M\colon (\mathbb{C}^*)^n\to(\mathbb{C}^*)^n$ defined by

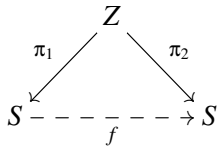
$$f_M(x_1,x_2,\dots,x_n)=\Big(\prod_{j=1}^n x_j^{m_{1j}},\prod_{j=1}^n x_j^{m_{2j}},\dots,\prod_{j=1}^n x_j^{m_{nj}}\Big)$$

Such a map can be viewed as a birational self map of $\mathbb{P}^n_{\mathbb{C}}$; these maps f_M are called *monomial birational maps* of $\mathbb{P}^n_{\mathbb{C}}$. Set $\mathbb{T}=(\mathbb{C}^*)^n$, and denote by $\mathrm{Bir}_{\mathbb{T}}(\mathbb{P}^n_{\mathbb{C}})$ the group of monomial birational maps of $\mathbb{P}^n_{\mathbb{C}}$. An element of $\mathrm{Bir}_{\mathbb{T}}(\mathbb{P}^n_{\mathbb{C}})$ is a birational map $f\colon \mathbb{P}^n_{\mathbb{C}}\dashrightarrow\mathbb{P}^n_{\mathbb{C}}$ defined on $\mathbb{T}$, and such that $f(\mathbb{T})\subset\mathbb{T}$. There is a natural isomorphism between $\mathrm{Bir}_{\mathbb{T}}(\mathbb{P}^n_{\mathbb{C}})$ and $\mathrm{Aut}(\mathbb{T})$. The monomial maps have been the subject of many studies, in particular:

Theorem 1.1 ([FW12, Lin12])**.** *Let f_M be a monomial birational map of $\mathbb{P}^n_{\mathbb{C}}$ induced by $M\in\mathrm{GL}(n,\mathbb{Z})$. Let $|\mu_1|\geq|\mu_2|\geq\ldots\geq|\mu_n|$ be the eigenvalues (counting multiplicities) of M. Then, the k-th dynamical degree of f_M is $\lambda_k(f_M)=|\mu_1\mu_2\ldots\mu_k|$.*

Remark 1.2. Consider the two elements $\left(\frac{1}{x},\frac{1}{y}\right)$ and (xy,y) of $\mathrm{Bir}_{\mathbb{T}}(\mathbb{P}^2_{\mathbb{C}})$. Note that $\lambda_1\left(\frac{1}{x},\frac{1}{y}\right)=\lambda_1(xy,y)$; nevertheless, $\left(\frac{1}{x},\frac{1}{y}\right)$ and (xy,y) are not birationally conjugate (the first one is an involution whereas the second one is of infinite order). In other words λ_1 is useless to determine whether f, $g\in\mathrm{Bir}_{\mathbb{T}}(\mathbb{P}^2_{\mathbb{C}})$ are birationally conjugate.

Remark 1.3. If S is a projective smooth surface, every birational map f from S into itself admits a resolution



where π_1, π_2 are sequences of blow-ups. Assume that the resolution is minimal, *i.e.* no (-1)-curve of Z are contracted by both π_1 and π_2; the *base-points* of f are the points blown-up by π_1, which can be points of S or infinitely near points. We denote by $\mathfrak{b}(f)$ the number of such points, which is also equal to the difference of the ranks of the Picard group of Z and the Picard group of S. Let us define the *dynamical number of base-points* of f by

$$\mu(f)=\lim_{k\to+\infty}\frac{\mathfrak{b}(f^k)}{k}$$

introduced in [BD15]. The integer μ is a birational invariant ([BD15]). Consider the two elements $\left(xy,\frac{1}{x}\right)$ and $\left(y,\frac{1}{x}\right)$ of $\mathrm{Bir}_{\mathbb{T}}(\mathbb{P}^2_{\mathbb{C}})$ Note that $\mu\left(xy,\frac{1}{x}\right)=\mu\left(y,\frac{1}{x}\right)=0$; nevertheless $\left(xy,\frac{1}{x}\right)$ and $\left(y,\frac{1}{x}\right)$ are not birationally conjugate (the first one is of order 6 whereas the second one is of order 4). In other words μ is useless to determine whether f, $g\in\mathrm{Bir}_{\mathbb{T}}(\mathbb{P}^2_{\mathbb{C}})$ are birationally conjugate.

Let G be a group, and let H be a subgroup of G. The elements f and g of G are H-*conjugate* if there exists $h \in \mathrm{H}$ such that $f = hgh^{-1}$.

Theorem A. *Two monomial birational maps of $\mathbb{P}^n_{\mathbb{C}}$ are $\mathrm{Bir}(\mathbb{P}^n_{\mathbb{C}})$-conjugate if and only if they are $\mathrm{Bir}_{\mathbb{T}}(\mathbb{P}^n_{\mathbb{C}})$-conjugate.*

To any $f_M \in \mathrm{Bir}_{\mathbb{T}}(\mathbb{P}^n_{\mathbb{C}})$, one can associate a finite sequence of an even length with positive integer elements called LLS period (*see* §3.2, Theorem-Definition 3.3) which is a complete invariant of $\mathrm{Bir}(\mathbb{P}^2_{\mathbb{C}})$-conjugacy classes of monomial birational maps of $\mathbb{P}^2_{\mathbb{C}}$:

Theorem B.
- ◇ *Let M be a $\mathrm{SL}(2,\mathbb{Z})$-matrix. If the characteristic polynomial of M has a pair of complex conjugate roots, then the associated monomial map f_M of $\mathbb{P}^2_{\mathbb{C}}$ is conjugate to one of the following periodic maps*

$$\left(xy, \frac{1}{x}\right), \qquad \left(y, \frac{1}{x}\right), \qquad \left(y, \frac{1}{xy}\right).$$

- ◇ *Let M be a $\mathrm{SL}(2,\mathbb{Z})$-matrix. If the characteristic polynomial of M has a double root, then the associated monomial map f_M of $\mathbb{P}^2_{\mathbb{C}}$ is conjugate to the Jonquières twist (xy^n, y) for some $n \geq 0$.*
- ◇ *Every LLS period is a complete invariant of conjugacy class for monomial birational maps f_M such that M is an element of $\mathrm{SL}(2,\mathbb{Z})$ with two distinct positive real eigenvalues.*

What happens in higher dimensions ? Let us first mention that:

Theorem C. *Let f_M and f_N be two monomial birational maps of $\mathbb{P}^n_{\mathbb{C}}$.*
There exists an algorithm for deciding if f_M and f_N are $\mathrm{Bir}(\mathbb{P}^n_{\mathbb{C}})$-conjugate.

To any $f_M \in \mathrm{Bir}_{\mathbb{T}}(\mathbb{P}^n_{\mathbb{C}})$ is associated a pair consisting of the Klein-Voronoi continued fraction $\mathrm{KVCF}(M)$ and the action P_M of M on $\mathrm{KVCF}(M)$ (*see* §4.2) which is a complete invariant of $\mathrm{Bir}(\mathbb{P}^n_{\mathbb{C}})$-conjugacy classes:

Theorem D. *Let f_M and f_N be two monomial birational maps of $\mathbb{P}^n_{\mathbb{C}}$ such that the characteristic polynomials of M and N are irreducible over $\mathbb{Q}$.*
The maps f_M and f_N are $\mathrm{Bir}(\mathbb{P}^n_{\mathbb{C}})$-conjugate if, and only if, the pairs $(\mathrm{KVCF}(M), P_M)$ et $(\mathrm{KVCF}(N), P_N)$ are in the same $\mathrm{GL}(n,\mathbb{Z})$-orbit.

The paper is organized as follows: in §2 we prove Theorem A; in §3 we deal wtih Theorem B, and finally, in §4 we focus on Theorems C and D.

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2. Proof of Theorem A

Proof of Theorem A. It is clear that if two birational maps are $\mathrm{Bir}_{\mathbb{T}}(\mathbb{P}^n_{\mathbb{C}})$-conjugate, then they are $\mathrm{Bir}(\mathbb{P}^n_{\mathbb{C}})$-conjugate.

Let us prove the converse. Let f and h be two elements of $\mathrm{Bir}_{\mathbb{T}}(\mathbb{P}^n_{\mathbb{C}})$ that are birationally conjugate, *i.e.* $h = gfg^{-1}$ with $g \in \mathrm{Bir}(\mathbb{P}^n_{\mathbb{C}})$. Note that $h = gfg^{-1}$ preserves $\mathbb{T}$, *i.e.* $gfg^{-1}\mathbb{T}gf^{-1}g^{-1} = \mathbb{T}$. As a consequence, $fg^{-1}\mathbb{T}gf^{-1} = g^{-1}\mathbb{T}g$. But f is a monomial birational map of $\mathbb{P}^n_{\mathbb{C}}$, hence f preserves $\mathbb{T}$, and only $\mathbb{T}$; therefore $g^{-1}\mathbb{T}g = \mathbb{T}$: the map g preserves $\mathbb{T}$, and g belongs to $\mathrm{Bir}_{\mathbb{T}}(\mathbb{P}^n_{\mathbb{C}})$. □

Corollary 2.1. *Let f_M and f_N be two monomial birational maps of $\mathbb{P}^n_{\mathbb{C}}$. The maps f_M and f_N are $\mathrm{Bir}(\mathbb{P}^n_{\mathbb{C}})$-conjugate if and only if M and N are $\mathrm{GL}(n,\mathbb{Z})$-conjugate.*

Corollary 2.2. *Any similarity invariant of $\mathrm{GL}(n,\mathbb{Z})$-matrices (trace, determinant, characteristic polynomial, minimal polynomial...) is a birational invariant of monomial birational maps of $\mathbb{P}^n_{\mathbb{C}}$.*

3. The 2-dimensional case

3.1. Reduced matrices. The matrices of $\mathrm{SL}(2,\mathbb{Z})$ are distributed as follows:

- ◇ case of complex spectra: consider $\mathrm{SL}(2,\mathbb{Z})$ matrices whose characteristic polynomials have a pair of complex conjugate roots, we call such a matrix a *complex spectrum matrix*. There are three integer conjugacy classes of such matrices represented by $\begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix}$, $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ and $\begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix}$.
- ◇ degenerate case: let us now study matrices whose characteristic polynomial has a double root (which actually equal to 1). Such matrices are $\mathrm{SL}(2,\mathbb{Z})$-conjugate to exactly one matrix of the following family
$$\begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix},\ n \geq 0$$
- ◇ case of real spectra: a matrix with two real distinct eigenvalues is called a *real spectrum matrix*.

This yields to:

Lemma 3.1. *Let f_M be a monomial birational map of $\mathbb{P}^2_{\mathbb{C}}$ with $M \in \mathrm{SL}(2,\mathbb{Z})$.*

- ◇ *If M is a complex spectrum matrix, then f_M is conjugate to one of the following monomial maps*
$$\left(xy, \frac{1}{x}\right), \qquad \left(y, \frac{1}{x}\right), \qquad \left(y, \frac{1}{xy}\right).$$
- ◇ *If the characteristic polynomial of M has a double root, then f_M is conjugate to (xy^n, y) for some $n \geq 0$.*

We will now focus on real spectrum matrices. There are two similar subfamilies of real spectrum $\mathrm{SL}(2,\mathbb{Z})$ matrices: those with positive eigenvalues and those with negative eigenvalues. The one-to-one correspondence between them is given by associating to a matrix M the matrix $-M$. For simplicity we consider only matrices with positive eigenvalues.

A matrix $M = \begin{pmatrix} p & r \\ q & s \end{pmatrix}$ in $\mathrm{SL}(2,\mathbb{Z})$ is *reduced* if $s > q > p \geq 0$. In [Kar22] Karpenkov gives an algorithm to construct reduced matrices:

Theorem 3.2 ([Kar22]). *For every real spectrum matrix A in $\mathrm{SL}(2,\mathbb{Z})$ either A, or $-A$ is $\mathrm{SL}(2,\mathbb{Z})$-conjugate to a reduced matrix.*

Algorithm to construct reduced matrices ([Kar22]). Let $M = \begin{pmatrix} p & r \\ q & s \end{pmatrix}$ be a $\mathrm{SL}(2,\mathbb{Z})$ matrix. Assume that the characteristic polynomial of M is irreducible over $\mathbb{Q}$ (or equivalently, that it does not have ± 1 as roots) and has two positive real roots.

- ◇ `Step 1`. If $q < 0$, then multiply $\begin{pmatrix} p & r \\ q & s \end{pmatrix}$ by $-\mathrm{Id}$. Go to `Step 2`.
- ◇ `Step 2`. We have $q \geq 0$. After conjugation of the matrix $\begin{pmatrix} p & r \\ q & s \end{pmatrix}$ by the matrix $\begin{pmatrix} 1 & \left\lfloor -\frac{p}{q} \right\rfloor \\ 0 & 1 \end{pmatrix}$ we get the matrix $\begin{pmatrix} p' & r' \\ q' & s' \end{pmatrix}$ where $0 \leq p' \leq q'$. Go to `Step 3.1`.
- ◇ `Step 3.1`. Suppose $q' = 1$. Then $p' = 0$ and $r' = -1$ (indeed, $0 \leq p' \leq q' = 1$ so $p' = 0$ or $p' = 1$, but if $p' = 1$, then $r' = 0$ and $s' = 1$, *i.e.* $\begin{pmatrix} p' & r' \\ q' & s' \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$: contradiction with the characteristic polynomial of $\begin{pmatrix} p & r \\ s & q \end{pmatrix}$ is irreducible over $\mathbb{Q}$). In addition $|s'| > 2$ otherwise the matrix has either complex roots, or rational roots. The algorithm stops, and the output of the algorithm is the matrix $\begin{pmatrix} 0 & -1 \\ 1 & s' \end{pmatrix}$.
- ◇ `Step 3.2.1`. Assume that $q' > 1$ and $s' > q'$. Then the algorithm stops and the output of the algorithm is the matrix $\begin{pmatrix} p' & r' \\ q' & s' \end{pmatrix}$.
- ◇ `Step 3.2.2`. Suppose that $q' > 1$ and $s' < -q'$. Conjugate the matrix $\begin{pmatrix} p' & r' \\ q' & s' \end{pmatrix}$ by the matrix $\begin{pmatrix} -1 & 1 \\ 0 & 1 \end{pmatrix}$ and multiply by $-\mathrm{Id}$, then you get $\begin{pmatrix} p'' & r'' \\ q'' & s'' \end{pmatrix}$ with $p'' = q' - p'$, $q'' = q'$, $s'' = -q' - s'$. In particular, $q'' > 1$ and $s'' > 0$. Go to `Step 3.2.1` or to `Step 3.2.3`. depending on s'' and q''.
- ◇ `Step 3.2.3`. Assume that $q' > 1$, and $|s'| \leq |q'|$. Notice that the absolute values of q' and s' coincide since the matrix has unit determinant. Hence $|s'| < |q'|$ and we have

$$|r'| = \left| \frac{p's' - 1}{q'} \right| \leq \frac{(q'-1)^2 + 1}{q'} \leq q' - 1.$$

Go to `Step 1`. with the matrix

$$\begin{pmatrix} s' & q' \\ r' & p' \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} p' & r' \\ q' & s' \end{pmatrix} \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$$

with $|r'| < |q'|$.

◇ `Output`: the reduced integer matrix that is conjugate to $\pm M$. The sign is defined by the sign of the trace of the original matrix.

3.2. A complete invariant. Let α, a_0, a_1, …, a_n be real numbers satisfying

$$\alpha = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{\ddots_{+\frac{1}{a_n}}}}}$$

The expression in the right-hand side of the equality is called a *continued fraction* of a given number α, and denoted by $[a_0; a_1 : a_2 : \ldots : a_n]$.

Theorem-Definition 3.3 ([Kar22])**.** *Consider a reduced matrix* $M = \begin{pmatrix} p & r \\ q & s \end{pmatrix}$ *of* $\mathrm{SL}(2,\mathbb{Z})$.

If $p = 0$, *then* $M = \begin{pmatrix} 0 & -1 \\ 1 & \lambda+2 \end{pmatrix}$ *for* $\lambda \geq 2$, *and the LLS*[1] *period of* M *is* $(1,\lambda)$.

If $p \neq 0$, *then suppose* $\frac{q}{p} = [a_1 : a_2 : \ldots : a_{2n-1}]$ *and* $\lambda = \left\lfloor \frac{s-1}{q} \right\rfloor$. *The LLS period of* M *is* $(a_1, a_2, \ldots, a_{2n-1}, \lambda)$.

Remark 3.4. The LLS period is a cyclically ordered sequence of an even number of integer elements. The LLS periods $(1,2,1,2)$ and $(2,1,2,1)$ are the same, whereas the LLS periods $(1,2,1,2)$ and $(1,2)$ are distinct.

Example 3.5. Let us consider the matrix $\begin{pmatrix} 7 & 18 \\ 5 & 13 \end{pmatrix}$. According to the `Step 2`. of the algorithm we get

$$\begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 7 & 18 \\ 5 & 13 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 2 & 7 \\ 5 & 18 \end{pmatrix}.$$

But $18 > 5 > 2 > 0$, so the matrix $\begin{pmatrix} 2 & 7 \\ 5 & 18 \end{pmatrix}$ is reduced. Since $\frac{5}{2} = [2;1:1]$ we get that the LLS of $\begin{pmatrix} 7 & 18 \\ 5 & 13 \end{pmatrix}$ is $(2,1,1,3)$.

Corollary 3.6 ([Kar22])**.** *The set of real spectrum reduced matrices is in one-to-one correspondence* (*defined in Remark 3.4*) *with the set of finite sequences consisting of an even number of positive integer elements.*

Remark 3.7. If we consider a matrix constructed from a sequence with an odd number of elements, it is a real spectrum $\mathrm{GL}(2,\mathbb{Z})$-matrix with a negative determinant.

[1] LLS for lattice length sine sequence, *see* [Kar22] for more details.

The reduced matrices are "almost" normal forms, since each matrix could have more than one normal form:

Corollary 3.8 ([Kar22]). *The number of reduced matrices in a* $\mathrm{SL}(2,\mathbb{Z})$*-conjugacy class coincides with the number of elements in the minimal period of the corresponding LLS sequence.*

Proposition 3.9 ([Kar22]). *Two real spectrum* $\mathrm{SL}(2,\mathbb{Z})$ *matrices with positive eigenvalues are* $\mathrm{SL}(2,\mathbb{Z})$*-conjugate if and only if their LLS periods coincide.*

Finally,

Theorem 3.10 ([Kar22]).
(i) *Every LLS period is a complete invariant of a* $\mathrm{SL}(2,\mathbb{Z})$*-conjugacy class for* $\mathrm{SL}(2,\mathbb{Z})$*-matrices with distinct positive real eigenvalues.*
(ii) *An arbitrary finite sequence of an even length with positive integer elements is realisable as a period of some LLS sequence.*

Example 3.11. Consider $M = \begin{pmatrix} 1519 & 1164 \\ -1964 & -1505 \end{pmatrix}$. The LLS period of M is $(1,2,1,2)$. Hence there are exactly two reduced matrices $\begin{pmatrix} p & r \\ q & s \end{pmatrix}$ represented by the sequences $(1,2,1,2)$ and $(2,1,2,1)$. The coefficients p and q for the corresponding reduced matrices are

$$\frac{q}{p} = [1;2:1] = \frac{4}{3}, \qquad \frac{q}{p} = [2;1:2] = \frac{8}{3}.$$

We find the elements r and s of the reduced matrices from conditions $\lambda = \left\lfloor \frac{s-1}{q} \right\rfloor$ and $ps - qr = 1$; finally we get that

$$\begin{pmatrix} 3 & 8 \\ 4 & 11 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 3 & 4 \\ 8 & 11 \end{pmatrix}$$

are both reduced matrices $\mathrm{SL}(2,\mathbb{Z})$-conjugate to M.

Example 3.12. Let us consider again the matrix $\begin{pmatrix} 7 & 18 \\ 5 & 13 \end{pmatrix}$. Using the algorithm we get that the reduced associated matrix is $\begin{pmatrix} 2 & 7 \\ 5 & 18 \end{pmatrix}$. Its LLS period is $(2,1,1,3)$. There are three reduced matrices in the conjugacy classes of $\begin{pmatrix} 7 & 18 \\ 5 & 13 \end{pmatrix}$ corresponding to $(1,1,3,2)$, $(1,3,2,1)$ and $(3,2,1,1)$.

3.3. Proof of Theorem B. Proposition 3.9 and Theorem A yield to:

Proposition 3.13. *Let* M *and* N *be two real spectrum* $\mathrm{SL}(2,\mathbb{Z})$ *matrices with positive eigenvalues. The monomial birational maps* f_M *and* f_N *of* $\mathbb{P}^2_{\mathbb{C}}$ *are* $\mathrm{Bir}(\mathbb{P}^2_{\mathbb{C}})$*-conjugate if and only if the LLS periods of* M *and* N *coincide.*

Proposition 3.13 directly implies

Theorem 3.14. *Every LLS period is a complete invariant of conjugacy class for the monomial birational map f_M of $\mathbb{P}^2_{\mathbb{C}}$ where M is a real spectrum $\mathrm{SL}(2,\mathbb{Z})$ matrix with positive eigenvalues.*

From Lemma 3.1 and Theorem 3.14 we get Theorem B.

4. The n-dimensional case

4.1. An algorithm. Let M and N be elements of $\mathrm{GL}(n,\mathbb{Q})$. The rational conjugacy problem asks if there exists $P \in \mathrm{GL}(n,\mathbb{Q})$ such that $PMP^{-1} = N$. This can be decided effectively by computing and comparing the canonical forms of M and N. More difficult is the integral conjugacy problem: decide whether or not there exists $P \in \mathrm{GL}(n,\mathbb{Z})$ such that $PMP^{-1} = N$. Clearly, if M and N are not $\mathrm{GL}(n,\mathbb{Q})$-conjugate, then M and N are not $\mathrm{GL}(n,\mathbb{Z})$-conjugate, but the converse does not hold. Grunewald proves that the integral conjugacy is decidable ([Gru80]) but he does not provide practical algorithm.

In [EHO19] the authors provide practical algorithms. They begin by following the approach of Grunewald. Let $M = S+U$ be the Jordan-Chevalley form for M (that is, S is semisimple, U is nilpotent and $SU = US$), and let $N = S'+U'$ be the Jordan-Chevalley form for N. The problem of determining $\mathrm{GL}(n,\mathbb{Z})$-conjugacy can be reduced to the case where M and N are $\mathrm{GL}(n,\mathbb{Q})$-conjugate and S, S', U, U' are in $\mathrm{GL}(n,\mathbb{Z})$. Let $P(x)$ be the characteristic polynomial of S (and S') and let ℓ be the nilpotence class of U (and U'). Set $R = \mathbb{Z}[x]/(P(x))$ and $P_\ell(R) = R[y]/(y^\ell)$. Then $\mathbb{Z}^n$ has the structure of a $P_\ell(R)$-module in two different ways depending on the M-action: either $vx := vS$ and $vy := vU$ for all $v \in \mathbb{Z}^n$, or we use the corresponding N-action. Let us denote by κ_M and κ_N the two $P_\ell(R)$-modules; $\kappa_M \simeq \kappa_N$ if and only if M and N are $\mathrm{GL}(n,\mathbb{Z})$-conjugate. This allows various known techniques of algebraic number theory to be applied. The authors then introduce a new tool (the so-called standard submodule) and use it to develop a detailed description of such an algorithm. They have implemented their algorithm in Magma and they report that their approach appears to be much faster and more general than earlier attempts to determine $\mathrm{GL}(n,\mathbb{Z})$-conjugacy. However, although their program is often fast, there are examples of quite small degree ($n = 10$) where it runs very slowly and indeed these examples may be intrinsically hard. Incidentally, they point out that solving the decision problem over $\mathrm{GL}(n,\mathbb{Z})$ leads directly to the solution of the conjugacy problem for $\mathrm{SL}(n,\mathbb{Z})$ and $\mathrm{PSL}(n,\mathbb{Z})$.

Hence both Theorem A and [Gru80, EHO19] yield to Theorems C and D.

4.2. Complete geometric invariants. Let us now introduce a complete geometric invariant of $\mathrm{SL}(2,\mathbb{Z})$-conjugacy classes proposed by Klein in [Kle95], further extended by both Voronoi in [Vor52] and Karpenkov in [Kar13].

Assume that $M \in \mathrm{GL}(n,\mathbb{R})$ has n distinct eigenvalues. Suppose that the real eigenvalues of M are r_1, r_2, …, r_k and the complex conjugate eigenvalues are c_1, $\overline{c_1}$, c_2, $\overline{c_2}$, …, c_ℓ, $\overline{c_\ell}$ where $k+2\ell = n$. Denote by $L_{\mathbb{R}}(M)$ the space spanned by real eigenvectors, and by $\mathrm{Spec}(N)$ the spectrum of N. Let us define the set T_M

as:

$$T_M = \{N \in \mathrm{GL}(n,\mathbb{R}) \,|\, MN = NM,\, \mathrm{Spec}(N) \subset \mathbb{S}^1,\, N_{|L_{\mathbb{R}}(M)} = \mathrm{id}_{|L_{\mathbb{R}}(M)}\}.$$

Actually, T_M is an abelian subgroup of $\mathrm{GL}(n,\mathbb{R})$. The orbit

$$T_M(v) = \{N(v) \,|\, N \in T_M\};$$

of $v \in \mathbb{R}^n$ under the action of T_M is homeomorphic to a torus of dimension $\leq \ell$; furthermore, in case v does not lie in any invariant plane of M, the equality $\dim T_M(v) = \ell$ holds.

For $1 \leq i \leq k$ denote by g_i a real eigenvector with eigenvalue r_i. For $1 \leq j \leq \ell$ denote by g_{k+2j-1} and g_{k+2j} vectors of the real and imaginary parts of some complex eigenvector with eigenvalue c_j. Set

$$\pi = \langle g_1, g_2, \ldots, g_k, g_{k+1}, g_{k+3}, \ldots, g_{k+2\ell-1}\rangle_{\mathbb{R}},$$

and let π_+ be the cone of π whose points have nonnegative last ℓ coordinates in the basis

$$(g_1, g_2, \ldots, g_k, g_{k+1}, g_{k+3}, \ldots, g_{k+2\ell-1}).$$

Remark that for any v the orbit $T_M(v)$ intersects the cone π_+ in a unique point.

Let us consider the arrangement of all k real invariant hyperplanes of M which are of the form

$$\langle g_1, g_2, \ldots, g_{i-1}, g_{i+1}, g_{i+2} \ldots, g_n\rangle_{\mathbb{R}};$$

we denote by $C_1(M)$, $C_2(M)$, …, $C_{2^k}(M)$ the connected components of their complement in $\mathbb{R}^n$. Let us set

$$\Omega = \mathrm{conv}\big(\{q \in \pi_+ \,|\, T_M(q) \cap C_i(M) \cap \mathbb{Z}^n \neq \emptyset,\, q \neq 0\}\big).$$

The set $S_i(M) = \bigcup_{p \in \partial\Omega} T_M(p)$ is called the *sail* of a cone C_i. The *Klein-Voronoi continued fraction* of M is the union of all sails:

$$\mathrm{KVCF}(M) = \bigcup_{i=1}^{2^k} S_i(M).$$

For an arbitrary m-dimensional plane $\mathcal{P} \subset \pi$ the set

$$\bigcup_{p \in S_i(M) \cap \mathcal{P}} T_M(p)$$

is called an *m-dimensional orbit-face* if the set $S_i(M) \cap \mathcal{P}$ is homeomorphic to the m-dimensional ball. The group of all elements of $\mathrm{GL}(n,\mathbb{Z})$ commuting with M is called the *Dirichlet group* of M and is denoted by $\Xi(M)$:

$$\Xi(M) = \{N \in \mathrm{GL}(n,\mathbb{Z}) \,|\, NM = MN\}.$$

The group $\Xi(M)$ takes $\mathrm{KVCF}(M)$ to itself and permutes the sails. By the Dirichlet unit theorem, $\Xi(M)$ is isomorphic to $\mathbb{Z}^{k+\ell-1} \oplus \mathrm{G}$ where G is a finite abelian group. It follows from the definition that $\mathrm{KVCF}(M)/\Xi(M)$ is homeomorphic to one or several copies of the $(n-1)$-dimensional torus. A fundamental domain of $\mathrm{KVCF}(M)$ is a collection of its orbit-faces, one from each equivalence class of $\mathrm{KVCF}(M)/\Xi(M)$.

Example 4.1. Let us consider the $\mathrm{SL}(3,\mathbb{Z})$-matrix $M = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 3 \end{pmatrix}$. It has one real and two complex conjugate eigenvalues. Hence, π_+ is a 2-dimensional half-plane. The invariant plane corresponding to the pair of complex eigenvectors separate π_+ onto two parts. The boundaries of convex hulls in each of the two parts are switched by $-\mathrm{id}$. The group $\Xi(M)$ is homeomorphic to $\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$ with generators M and $-\mathrm{id}$. A fundamental domain of $\mathrm{KVCF}(M)$ contains one orbit-vertex and one vertex edge. For instance, we choose the orbit-vertex $T_M(1,0,0)$ and the orbit-edge corresponding to the "tube" connecting orbit-vectors $T_M(1,0,0)$ and $T_M(M(1,0,0))$.

Theorem 4.2 ([Kar13])**.** *Assume that M, $N \in \mathrm{GL}(n,\mathbb{Z})$ have characteristic polynomials which are irreducible over $\mathbb{Q}$. Then $\Xi(M) = \Xi(N)$ if, and only if, $\mathrm{KVCF}(M) = \mathrm{KVCF}(N)$.*

Note that

- $\diamond$ $\mathrm{KVCF}(M)$ uniquely identifies $\Xi(M)$ (Theorem 4.2);
- $\diamond$ the matrix M acts on $\mathrm{KVCF}(M)$.

Let us denote by P_M the transformation induced by the action of M on $\mathrm{KVCF}(M)$. Distinct matrices of $\Xi(M)$ define equivalent shifts. As a result, the matrix $M \in \mathrm{GL}(n,\mathbb{Z})$ is uniquely identified with a pair $(\mathrm{KVCF}(M), P_M)$. The group $\mathrm{GL}(n,\mathbb{Z})$ acts on pairs $(\mathrm{KVCF}(M), P_M)$ by left multiplication on the first factor and by conjugation on the second factor. Hence Theorem 4.2 can be rephrased as follows:

Theorem 4.3 ([Kar13])**.** *Two matrices M, $N \in \mathrm{GL}(n,\mathbb{Z})$ whose characteristic polynomial are irreducible over $\mathbb{Q}$ are $\mathrm{GL}(n,\mathbb{Z})$-conjugate if, and only if, the pairs $(\mathrm{KVCF}(M), P_M)$ et $(\mathrm{KVCF}(N), P_N)$ are in the same $\mathrm{GL}(n,\mathbb{Z})$-orbit.*

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