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DYNAMICAL NUMBER OF BASE-POINTS OF NON BASE-WANDERING JONQUIÈRES TWISTS

JULIE DÉSERTI

ABSTRACT. We give some properties of the dynamical number of base-points of birational self-maps of the complex projective plane.

In particular we give a formula to determine the dynamical number of base-points of non base-wandering Jonquières twists.

1. INTRODUCTION

The *plane Cremona group* $\text{Bir}(\mathbb{P}_{\mathbb{C}}^2)$ is the group of birational maps of the complex projective plane $\mathbb{P}_{\mathbb{C}}^2$. It is isomorphic to the group of \mathbb{C} -algebra automorphisms of $\mathbb{C}(X, Y)$, the function field of $\mathbb{P}_{\mathbb{C}}^2$. Using a system of homogeneous coordinates $(x : y : z)$ a birational map $f \in \text{Bir}(\mathbb{P}_{\mathbb{C}}^2)$ can be written as

$$(x : y : z) \dashrightarrow (P_0(x, y, z) : P_1(x, y, z) : P_2(x, y, z))$$

where P_0, P_1 and P_2 are homogeneous polynomials of the same degree without common factor. This degree does not depend on the system of homogeneous coordinates. We call it the *degree* of f and denote it by $\deg(f)$. Geometrically it is the degree of the pull-back by f of a general projective line. Birational maps of degree 1 are homographies and form the group $\text{Aut}(\mathbb{P}_{\mathbb{C}}^2) = \text{PGL}(3, \mathbb{C})$ of automorphisms of the projective plane.

◇ Four types of elements.

The elements $f \in \text{Bir}(\mathbb{P}_{\mathbb{C}}^2)$ can be classified into exactly one of the four following types according to the growth of the sequence $(\deg(f^k))_{k \in \mathbb{N}}$ (see [DF01, BD15]):

- (1) The sequence $(\deg(f^k))_{k \in \mathbb{N}}$ is bounded, f is either of finite order or conjugate to an automorphism of $\mathbb{P}_{\mathbb{C}}^2$; we say that f is an *elliptic element*.
- (2) The sequence $(\deg(f^k))_{k \in \mathbb{N}}$ grows linearly, f preserves a unique pencil of rational curves and f is not conjugate to an automorphism of any rational projective surface; we call f a *Jonquières twist*.
- (3) The sequence $(\deg(f^k))_{k \in \mathbb{N}}$ grows quadratically, f is conjugate to an automorphism of a rational projective surface preserving a unique elliptic fibration; we call f a *Halphen twist*.
- (4) The sequence $(\deg(f^k))_{k \in \mathbb{N}}$ grows exponentially and we say that f is *hyperbolic*.

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◇ **The Jonquière group.**

Let us fix an affine chart of $\mathbb{P}_{\mathbb{C}}^2$ with coordinates (x, y) . The *Jonquière group* J is the subgroup of the Cremona group of all maps of the form

$$(x, y) \dashrightarrow \left(\frac{A(y)x + B(y)}{C(y)x + D(y)}, \frac{ay + b}{cy + d} \right) \quad (1.1)$$

where $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{PGL}(2, \mathbb{C})$ and $\begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathrm{PGL}(2, \mathbb{C}(y))$. The group J is the group of all birational maps of $\mathbb{P}_{\mathbb{C}}^1 \times \mathbb{P}_{\mathbb{C}}^1$ permuting the fibers of the projection onto the second factor; it is isomorphic to the semi-direct product $\mathrm{PGL}(2, \mathbb{C}(y)) \rtimes \mathrm{PGL}(2, \mathbb{C})$.

We can check with (1.1) that if f belongs to J , then $(\deg(f^k))_{k \in \mathbb{N}}$ grows at most linearly; elements of J are either elliptic or Jonquière twists. Let us denote by \mathcal{J} the set of Jonquière twist:

$$\mathcal{J} = \{f \in \mathrm{Bir}(\mathbb{P}_{\mathbb{C}}^2) \mid f \text{ Jonquière twist}\}.$$

A Jonquière twist is called a *base-wandering Jonquière twist* if its action on the basis of the rational fibration has infinite order. Let us denote by J_0 the normal subgroup of J that preserves fiberwise the rational fibration, that is the subgroup of those maps of the form

$$(x, y) \dashrightarrow \left(\frac{A(y)x + B(y)}{C(y)x + D(y)}, y \right).$$

The group J_0 is isomorphic to $\mathrm{PGL}(2, \mathbb{C}(y))$. The group J_0 has three maximal (for the inclusion) uncountable abelian subgroups

$$J_a = \{(x + a(y), y) \mid a \in \mathbb{C}(y)\}, \quad J_m = \{(b(y)x, y) \mid b \in \mathbb{C}(y)^*\}$$

and

$$J_F = \left\{ (x, y), \left(\frac{c(y)x + F(y)}{x + c(y)}, y \right) \mid c \in \mathbb{C}[y] \right\}$$

where F denotes an element of $\mathbb{C}[y]$ that is not a square ([D06]).

Let us associate to $f = \left(\frac{A(y)x + B(y)}{C(y)x + D(y)}, y \right) \in J_0$ the matrix $M_f = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$. The *Baum Bott index* of f is $\mathrm{BB}(f) = \frac{(\mathrm{Tr}(M_f))^2}{\det(M_f)}$ (by analogy with the Baum Bott index of a foliation) which is well defined in PGL and is invariant by conjugation. This invariant BB indicates the degree growth:

Proposition 1.1 ([CD12]). *Let f be a Jonquière twist that preserves fiberwise the rational fibration. The rational function $\mathrm{BB}(f)$ is constant if and only if f is an elliptic element.*

A direct consequence is the following:

Corollary 1.2. *Let f be a non-base wandering Jonquière twist; the rational function $\mathrm{BB}(f)$ is constant if and only if f is an elliptic element.*

Every $f \in \text{Bir}(\mathbb{P}_{\mathbb{C}}^2)$ admits a resolution

$$\begin{array}{ccc} & S & \\ \pi_1 \swarrow & & \searrow \pi_2 \\ \mathbb{P}_{\mathbb{C}}^2 & \dashrightarrow f \dashrightarrow & \mathbb{P}_{\mathbb{C}}^2 \end{array}$$

where π_1, π_2 are sequences of blow-ups. The resolution is *minimal* if and only if no (-1) -curve of S are contracted by both π_1 and π_2 . Assume that the resolution is minimal; the *base-points* of f are the points blown-up by π_1 , which can be points of S or infinitely near points. If f belongs to \mathbf{J} , then f has one base-point p_0 of multiplicity $d - 1$ and $2d - 2$ base-points $p_1, p_2, \dots, p_{2d-2}$ of multiplicity 1. Similarly the map f^{-1} has one base-point q_0 of multiplicity $d - 1$ and $2d - 2$ base-points $q_1, q_2, \dots, q_{2d-2}$ of multiplicity 1. Let us denote by $f_{\#}$ the action of f on the Picard-Manin space of $\mathbb{P}_{\mathbb{C}}^2$, by $\mathbf{e}_m \in \text{NS}(S)$ the Néron-Severi class of the total transform of m under π_j (for $1 \leq j \leq 2$), and by ℓ the class of a line in $\mathbb{P}_{\mathbb{C}}^2$. The action of f on ℓ and the classes $(\mathbf{e}_{p_j})_{0 \leq j \leq 2d-2}$ is given by:

$$\begin{cases} f_{\#}(\ell) = d\ell - (d-1)\mathbf{e}_{q_0} - \sum_{i=1}^{2d-2} \mathbf{e}_{q_i} \\ f_{\#}(\mathbf{e}_{p_0}) = (d-1)\ell - (d-2)\mathbf{e}_{q_0} - \sum_{i=1}^{2d-2} \mathbf{e}_{q_i} \\ f_{\#}(\mathbf{e}_{p_i}) = \ell - \mathbf{e}_{q_0} - \mathbf{e}_{q_i} \quad \forall 1 \leq i \leq 2d-2 \end{cases}$$

◇ **Dynamical degree.**

Given a birational self-map $f: S \dashrightarrow S$ of a complex projective surface, its dynamical degree $\lambda(f)$ is a positive real number that measures the complexity of the dynamics of f . Indeed $\log(\lambda(f))$ provides an upper bound for the topological entropy of f and is equal to it under natural assumptions (*see* [BD05, DS05]). The dynamical degree is invariant under conjugacy; as shown in [BC16] precise knowledge on $\lambda(f)$ provides useful information on the conjugacy class of f . By definition a *Pisot number* is an algebraic integer $\lambda \in]1, +\infty[$ whose other Galois conjugates lie in the open unit disk; Pisot numbers include integers $d \geq 2$ as well as reciprocal quadratic integers $\lambda > 1$. A *Salem number* is an algebraic integer $\lambda \in]1, +\infty[$ whose other Galois conjugates are in the closed unit disk, with at least one on the boundary. Diller and Favre proved the following statement:

Theorem 1.3 ([DF01]). *Let f be a birational self-map of a complex projective surface.*

If $\lambda(f)$ is different from 1, then $\lambda(f)$ is a Pisot number or a Salem number.

One of the goal of [BC16] is the study of the structure of the set of all dynamical degrees $\lambda(f)$ where f runs over the group of birational maps $\text{Bir}(S)$ and S over the collection of all projective surfaces. In particular they get:

Theorem 1.4 ([BC16]). *Let Λ be the set of all dynamical degrees of birational maps of complex projective surfaces. Then*

- ◇ Λ is a well ordered subset of \mathbb{R}_+ ;
- ◇ if λ is an element of Λ , there is a real number $\varepsilon > 0$ such that $] \lambda, \lambda + \varepsilon]$ does not intersect Λ ;
- ◇ there is a non-empty interval $] \lambda_G, \lambda_G + \varepsilon]$, with $\varepsilon > 0$, on the right of the golden mean that contains infinitely many Pisot and Salem numbers, but does not contain any dynamical degree.

◇ **Dynamical number of base-points** ([BD15]).

If S is a projective smooth surface, every $f \in \text{Bir}(S)$ admits a resolution

$$\begin{array}{ccc} & Z & \\ \pi_1 \swarrow & & \searrow \pi_2 \\ S & \overset{f}{\dashrightarrow} & S \end{array}$$

where π_1, π_2 are sequences of blow-ups. The resolution is *minimal* if and only if no (-1) -curve of Z are contracted by both π_1 and π_2 . Assume that the resolution is minimal; the *base-points* of f are the points blown-up by π_1 , which can be points of S or infinitely near points. We denote by $\mathfrak{b}(f)$ the number of such points, which is also equal to the difference of the ranks of $\text{Pic}(Z)$ and $\text{Pic}(S)$, and thus equal to $\mathfrak{b}(f^{-1})$.

Let us define the *dynamical number of base-points* of f by

$$\mu(f) = \lim_{k \rightarrow +\infty} \frac{\mathfrak{b}(f^k)}{k}.$$

Since $\mathfrak{b}(f \circ \varphi) \leq \mathfrak{b}(f) + \mathfrak{b}(\varphi)$ for any $f, \varphi \in \text{Bir}(S)$ we see that $\mu(f)$ is a non-negative real number. Moreover, $\mathfrak{b}(f^{-1})$ and $\mathfrak{b}(f)$ being equal we get $\mu(f^k) = |k\mu(f)|$ for any $k \in \mathbb{Z}$. Furthermore, the dynamical number of base-points is an invariant of conjugation: if $\psi: S \dashrightarrow Z$ is a birational map between smooth projective surfaces and if f belongs to $\text{Bir}(S)$, then $\mu(f) = \mu(\psi \circ f \circ \psi^{-1})$. In particular if f is conjugate to an automorphism of a smooth projective surface, then $\mu(f) = 0$. The converse holds, *i.e.* $f \in \text{Bir}(S)$ is conjugate to an automorphism of a smooth projective surface if and only if $\mu(f) = 0$ ([BD15, Proposition 3.5]). This follows from the geometric interpretation of μ we will recall now. If $f \in \text{Bir}(S)$ is a birational map, a (possibly infinitely near) base-point p of f is a *persistent base-point* of f if there exists an integer N such that p is a base-point of f^k for any $k \geq N$ but is not a base-point of f^{-k} for any $k \geq N$. We put an equivalence relation on the set of points that belongs to S or are

infinitely near: take a minimal resolution of f

$$\begin{array}{ccc} & Z & \\ \pi_1 \swarrow & & \searrow \pi_2 \\ S & \dashrightarrow f \dashrightarrow & S \end{array}$$

where π_1, π_2 are sequences of blow-ups; the point p is *equivalent* to q if there exists an integer k such that $(\pi_2 \circ \pi_1^{-1})^k(p) = q$. Denote by \mathfrak{v} the number of equivalence classes of persistent base-points of f ; then the set

$$\{\mathfrak{b}(f^k) - \mathfrak{v}k \mid k \geq 0\} \subset \mathbb{Z}$$

is bounded. In particular, $\mu(f)$ is an integer, equal to \mathfrak{v} (see [BD15, Proposition 3.4]). This gives a bound for $\mu(f)$; indeed, if $f \in \text{Bir}(\mathbb{P}_{\mathbb{C}}^2)$ is a map whose base-points have multiplicities $m_1 \geq m_2 \geq \dots \geq m_r$ then (see for instance [ACn02, §2.5] and [ACn02, Corollary 2.6.7])

$$\begin{cases} \sum_{i=1}^r m_i = 3(\deg(f) - 1) \\ \sum_{i=1}^r m_i^2 = \deg(f)^2 - 1 \\ m_1 + m_2 + m_3 \geq \deg(f) + 1 \end{cases}$$

in particular, $r \leq 2 \deg(f) - 1$ so $\mathfrak{v} \leq 2 \deg(f) - 1$ and $\mu(f) \leq 2 \deg(f) - 1$.

If $f \in \text{Bir}(\mathbb{P}_{\mathbb{C}}^2)$ is a Jonquière twist, then there exists an integer $a \in \mathbb{N}$ such that

$$\lim_{k \rightarrow +\infty} \frac{\deg(f^k)}{k} = a^2 \frac{\mu(f)}{2};$$

moreover, a is the degree of the curves of the unique pencil of rational curves invariant by f (see [BD15, Proposition 4.5]). In particular, $a = 1$ if and only if f preserves a pencil of lines. On the one hand $\{\mu(f) \mid f \in \text{Bir}(\mathbb{P}_{\mathbb{C}}^2)\} \subseteq \mathbb{N}$ and on the other hand if f belongs to \mathcal{J} , then $\mu(f) > 0$; as a result

$$\{\mu(f) \mid f \in \mathcal{J}\} \subseteq \mathbb{N} \setminus \{0\}.$$

Let us recall that if $f_{\alpha,\beta} = \left(\frac{\alpha x + y}{x+1}, \beta y\right)$ then $\mu(f_{\alpha,\beta}) = 1$. Indeed, by induction one can prove that $f_{\alpha,\beta}^{2n} = \left(\frac{P_n(x,y)}{Q_n(x,y)}, \beta^{2n} y\right)$ with

$$P_n(x,y) = \sum_{0 \leq i+j \leq n+1} a_{ij} x^i y^j \quad Q_n(x,y) = \sum_{0 \leq i+j \leq n} b_{ij} x^i y^j$$

and $a_{ij} \geq 0, b_{ij} \geq 0$ for any $n \geq 0$, so that $\deg f_{\alpha,\beta}^{2n} = n+1$ for any $n \geq 0$; we conclude using

the fact that $\mu(f) = 2 \lim_{k \rightarrow +\infty} \frac{\deg(f^k)}{k}$. Furthermore, $\mu(f_{\alpha,\beta}^k) = |k\mu(f_{\alpha,\beta})| = |k|$ for any $k \in \mathbb{Z}$.

Hence

$$\{\mu(f) \mid f \in \mathcal{J}\} = \mathbb{N} \setminus \{0\}$$

and

$$\{\mu(f) \mid f \in \text{Bir}(\mathbb{P}_{\mathbb{C}}^2)\} = \mathbb{N}.$$

As we have seen if f belongs to \mathcal{J} , then $\mu(f) = 2 \lim_{k \rightarrow +\infty} \frac{\deg(f^k)}{k}$. Can we express $\mu(f)$ in a simpler way? We will see that if f is a non base-wandering Jonquières twist, the answer is yes.

◇ **Results.**

The dynamical number of base-points of birational self maps of the complex projective plane satisfies the following properties:

Theorem A. 1. *If f is a birational self-map from $\mathbb{P}_{\mathbb{C}}^2$ into itself, then its dynamical number of base-points is bounded: if $f \in \text{Bir}(\mathbb{P}_{\mathbb{C}}^2)$, then $\mu(f) \leq 2 \deg(f) - 1$.*

2. *We can precise the set of all dynamical numbers of base-points of birational maps of $\mathbb{P}_{\mathbb{C}}^2$ (resp. of Jonquières maps of $\mathbb{P}_{\mathbb{C}}^2$):*

$$\{\mu(f) \mid f \in \text{Bir}(\mathbb{P}_{\mathbb{C}}^2)\} = \mathbb{N} \quad \text{and} \quad \{\mu(f) \mid f \in \mathcal{J}\} = \mathbb{N} \setminus \{0\}.$$

3. *There exist sequences $(f_n)_n$ of birational self-maps of $\mathbb{P}_{\mathbb{C}}^2$ such that*

- ◇ $\mu(f_n) > 0$ for any $n \in \mathbb{N}$;
- ◇ $\mu\left(\lim_{n \rightarrow +\infty} f_n\right) = 0$.

4. *There exist sequences $(f_n)_n$ of birational self-maps of $\mathbb{P}_{\mathbb{C}}^2$ such that*

- ◇ $\mu(f_n) = 0$ for any $n \in \mathbb{N}$;
- ◇ $\mu\left(\lim_{n \rightarrow +\infty} f_n\right) > 0$.

Let us now give a formula to determine the dynamical number of base-points of Jonquières twists that preserves fiberwise the fibration.

Theorem B. *Let $f = \left(\frac{A(y)x+B(y)}{C(y)x+D(y)}, y\right)$ be a Jonquières twist that preserves fiberwise the fibration, and let M_f be its associated matrix. Denote by $\text{Tr}(M_f)$ the trace M_f , by χ_f the characteristic polynomial of M_f , and by Δ_f the discriminant of χ_f . Then exactly one of the following holds:*

1. *If χ_f has two distinct roots in $\mathbb{C}[y]$, then f is conjugate to $g = \left(\frac{\text{Tr}(M_f)+\delta_f}{\text{Tr}(M_f)-\delta_f}x, y\right)$, where $\delta_f^2 = \Delta_f$, and*

$$\mu(f) = \mu(g) = 2(\deg(g) - 1).$$

2. *If χ_f has no root in $\mathbb{C}[y]$, set*

$$\Omega_f = \gcd\left(\frac{\text{Tr}(M_f)}{2}, \left(\frac{\text{Tr}(M_f)}{2}\right)^2 - \det(M_f)\right)$$

and let us define P_f and S_f as

$$\frac{\text{Tr}(M_f)}{2} = P_f \Omega_f, \quad \left(\frac{\text{Tr}(M_f)}{2} \right)^2 - \det(M_f) = S_f \Omega_f.$$

2.a. If $\gcd(\Omega_f, S_f) = 1$, then

- ◊ if $\deg(S_f) \leq \deg(\Omega_f) + 2 \deg(P_f)$, then $\mu(f) = \deg(\Omega_f) + 2 \deg(P_f)$;
- ◊ otherwise $\mu(f) = \deg(S_f)$.

2.b. If $S_f = \Omega_f^p T_f$ with $p \geq 1$ and $\gcd(T_f, \Omega_f) = 1$, then

- ◊ if $\deg(S_f) \leq \deg(\Omega_f) + 2 \deg(P_f)$, then $\mu(f) = 2 \deg(P_f)$;
- ◊ otherwise $\mu(f) = \deg(S_f) - \deg(\Omega_f)$.

2.c. If $\Omega_f = S_f^p T_f$ with $p \geq 1$ and $\gcd(T_f, S_f) = 1$, then $\mu(f) = 2 \deg(P_f) + \deg(\Omega_f) - \deg(S_f)$.

As a consequence we are able to determine the dynamical number of base-points of non base-wandering Jonquières twists:

Corollary C. Let $f = (f_1, f_2)$ be a non base-wandering Jonquières twist.

If ℓ is the order of f_2 , then $\mu(f) = \frac{\mu(f^\ell)}{\ell}$ where $\mu(f^\ell)$ is given by Theorem B.

Combining the inequalities obtained in Theorem A and Theorem B we get the following statement (we use the notations introduced in Theorem B):

Corollary D. Let f be a Jonquières twist that preserves fiberwise the fibration. Assume that χ_f has two distinct roots in $\mathbb{C}[y]$.

Then there exists a conjugate g of f such that g belongs to J_m and $\deg(g) \leq \deg(f)$. For instance $g = \left(\frac{\text{Tr}(M_f) + \delta_f}{\text{Tr}(M_f) - \delta_f} x, y \right)$ suits.

2. DYNAMICAL NUMBER OF BASE-POINTS OF JONQUIÈRES TWISTS

In this section we will prove Theorem B.

Let f be an element of J_0 ; write f as $\left(\frac{A(y)x+B(y)}{C(y)x+D(y)}, y \right)$ with $A, B, C, D \in \mathbb{C}[y]$. The characteristic polynomial of $M_f = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$ is $\chi_f(X) = X^2 - \text{Tr}(M_f)X + \det(M_f)$. There are three possibilities:

- (1) χ_f has one root of multiplicity 2 in $\mathbb{C}[y]$;
- (2) χ_f has two distinct roots in $\mathbb{C}[y]$;
- (3) χ_f has no root in $\mathbb{C}[y]$.

Let us consider these three possibilities.

- (1) If χ_f has one root of multiplicity 2 in $\mathbb{C}[y]$, then f is conjugate to the elliptic birational map $(x + a(y), y)$ of J_a . In particular f does not belong to \mathcal{J} .

(2) Assume that χ_f has two distinct roots. The discriminant of χ_f is

$$\Delta_f = (\text{Tr}(M_f))^2 - 4\det(M_f) = \delta_f^2$$

and the roots of χ_f are

$$\frac{\text{Tr}(M_f) + \delta_f}{2} \quad \text{and} \quad \frac{\text{Tr}(M_f) - \delta_f}{2}.$$

Furthermore, M_f is conjugate to $\begin{pmatrix} \frac{\text{Tr}(M_f) + \delta_f}{2} & 0 \\ 0 & \frac{\text{Tr}(M_f) - \delta_f}{2} \end{pmatrix}$, i.e. f is conjugate to $g = (a(y)x, y) \in$

J_m with $a(y) = \frac{\text{Tr}(M_f) + \delta_f}{\text{Tr}(M_f) - \delta_f}$. Let us first express $\mu(g)$ thanks to $\deg(g)$. Remark that $g^k = (a(y)^k x, y)$. Write $a(y)^j$ as $\frac{P_j(y)}{Q_j(y)}$ where $P_j, Q_j \in \mathbb{C}[y]$, $\gcd(P_j, Q_j) = 1$, then $\deg(g^j) = \max(\deg(P_j), \deg(Q_j)) + 1$. But $\deg(P_j) = j \deg(P)$ and $\deg(Q_j) = j \deg(Q)$ so

$$\deg(g^k) = \max(k \deg(P_f), k \deg(Q_1)) + 1 = \underbrace{\max(\deg(P_f), \deg(Q_1))}_{\deg(g)-1} + 1.$$

As a consequence $\deg(g^k) = k \deg(g) - k + 1$. According to $\mu(g) = 2 \lim_{k \rightarrow +\infty} \frac{\deg(g^k)}{k}$ we get

$$\mu(g) = 2 \lim_{k \rightarrow +\infty} \left(\deg(g) - 1 + \frac{1}{k} \right) = 2(\deg(g) - 1).$$

Let us now express $\mu(f)$ thanks to f . Since f and g are conjugate $\mu(f) = \mu(g)$ hence $\mu(f) = 2(\deg(g) - 1)$. But $g = \left(\frac{\text{Tr}(M_f) + \delta_f}{\text{Tr}(M_f) - \delta_f} x, y \right)$; in particular

$$\deg(g) \leq 1 + \max(\deg(\text{Tr}(M_f) + \delta_f), \deg(\text{Tr}(M_f) - \delta_f))$$

and $\mu(f) \leq 2 \max(\deg(\text{Tr}(M_f) + \delta_f), \deg(\text{Tr}(M_f) - \delta_f))$.

(3) Suppose that χ_f has no root in $\mathbb{C}[y]$. This means that $(\text{Tr}(M_f))^2 - 4\det(M_f)$ is not a square in $\mathbb{C}[y]$ (hence $BC \neq 0$). Note that

$$\begin{pmatrix} C & \frac{D-A}{2} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} C & \frac{D-A}{2} \\ 0 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} \frac{\text{Tr}(M_f)}{2} & \left(\frac{\text{Tr}(M_f)}{2} \right)^2 - \det(M_f) \\ 1 & \frac{\text{Tr}(M_f)}{2} \end{pmatrix}.$$

In other words f is conjugate to

$$g = \left(\frac{\frac{\text{Tr}(M_f)}{2}x + \left(\frac{\text{Tr}(M_f)}{2} \right)^2 - \det(M_f)}{x + \frac{\text{Tr}(M_f)}{2}}, y \right) \in J_{\frac{\text{Tr}(M_f)}{2}}.$$

Set $P(y) = \frac{\text{Tr}(M_f)}{2} \in \mathbb{C}[y]$ and $F(y) = \left(\frac{\text{Tr}(M_f)}{2}\right)^2 - \det(M_f) \in \mathbb{C}[y]$, i.e. f is conjugate to $g = \left(\frac{P(y)x+F(y)}{x+P(y)}, y\right)$ with $P, F \in \mathbb{C}[y]$. Denote by d_P (resp. d_F) the degree of P (resp. F). Remark that $\deg(g) = \max(d_P + 1, d_F, 2)$.

Let us now express $\deg(g^k)$. Consider $M_g = \begin{pmatrix} P & F \\ 1 & P \end{pmatrix}$ and set

$$Q = \begin{pmatrix} \sqrt{F} & -\sqrt{F} \\ 1 & 1 \end{pmatrix} \quad \text{and} \quad D = \begin{pmatrix} P + \sqrt{F} & 0 \\ 0 & P - \sqrt{F} \end{pmatrix}$$

Then $M_g^k = QD^kQ^{-1}$ hence

$$M_g^k = \begin{pmatrix} \sqrt{F} \frac{(P+\sqrt{F})^k + (P-\sqrt{F})^k}{(P+\sqrt{F})^k - (P-\sqrt{F})^k} & F \\ 1 & \sqrt{F} \frac{(P+\sqrt{F})^k + (P-\sqrt{F})^k}{(P+\sqrt{F})^k - (P-\sqrt{F})^k} \end{pmatrix}$$

Let us set

$$\Upsilon_k = \sqrt{F} \frac{(P + \sqrt{F})^k + (P - \sqrt{F})^k}{(P + \sqrt{F})^k - (P - \sqrt{F})^k}$$

and let us denote by D_k (resp. N_k) the denominator (resp. numerator) of Υ_k .

Lemma 2.1. *Let $\Omega_f = \gcd(P, F)$ and write P (resp. F) as $\Omega_f P_f$ (resp. $\Omega_f S_f$). Assume $\gcd(S_f, \Omega_f) = 1$. Then*

- ◇ if $d_{S_f} \leq d_{\Omega_f} + 2d_{P_f}$, then $\mu(g) = d_{\Omega_f} + 2d_{P_f}$;
- ◇ otherwise $\mu(g) = d_{S_f}$.

Proof. (a) Assume k even, write k as 2ℓ . A straightforward computation yields to

$$\Upsilon_{2\ell} = \frac{\sum_{j=0}^{\ell} \binom{2\ell}{2j} \Omega_f^{\ell-j} P_f^{2(\ell-j)} S_f^j}{P_f \sum_{j=0}^{\ell-1} \binom{2\ell}{2j+1} \Omega_f^{\ell-1-j} P_f^{2(\ell-1-j)} S_f^j}$$

Recall that $\gcd(\Omega_f, S_f) = 1$ by assumption and $\gcd(\Omega_f, P_f) = 1$ by construction. On the one hand

$$\deg(N_{2\ell}) = \begin{cases} \ell(d_{\Omega_f} + 2d_{P_f}) & \text{if } d_{S_f} \leq d_{\Omega_f} + 2d_{P_f} \\ \ell d_{S_f} & \text{otherwise} \end{cases}$$

On the other hand

$$\deg(D_{2\ell}) = \begin{cases} d_{P_f} + (\ell - 1)(d_{\Omega_f} + 2d_{P_f}) & \text{if } d_{S_f} \leq d_{\Omega_f} + 2d_{P_f} \\ d_{P_f} + (\ell - 1)d_{S_f} & \text{otherwise} \end{cases}$$

Finally

$$\deg(g^{2\ell}) = \begin{cases} \max\left(\ell(d_{\Omega_f} + 2d_{P_f}) + 1, d_{S_f} + \ell d_{\Omega_f} + (2\ell - 1)d_{P_f}, (\ell - 1)d_{\Omega_f} + (2\ell - 1)d_{P_f} + 2\right) & \text{if } d_{S_f} \leq d_{\Omega_f} + 2d_{P_f} \\ \max\left(\ell d_{S_f} + 1, d_{\Omega_f} + d_{P_f} + \ell d_{S_f}, d_{P_f} + (\ell - 1)d_{S_f} + 2\right) & \text{otherwise} \end{cases}$$

(b) Suppose k odd, write k as $2\ell + 1$. A straightforward computation yields to

$$\Upsilon_{2\ell+1} = P_f \Omega_f \frac{\sum_{j=0}^{\ell} \binom{2\ell+1}{2j} \Omega_f^{\ell-j} P_f^{2(\ell-j)} S_f^j}{\sum_{j=0}^{\ell} \binom{2\ell+1}{2j+1} \Omega_f^{\ell-j} P_f^{2(\ell-j)} S_f^j}$$

Let us recall that $\gcd(\Omega_f, S_f) = 1$ by assumption and $\gcd(\Omega_f, P_f) = 1$ by construction.

On the one hand

$$\deg(N_{2\ell+1}) = \begin{cases} \ell(d_{\Omega_f} + 2d_{P_f}) + d_{\Omega_f} + d_{P_f} & \text{if } d_{S_f} \leq d_{\Omega_f} + 2d_{P_f} \\ \ell d_{S_f} + d_{P_f} + d_{\Omega_f} & \text{otherwise} \end{cases}$$

On the other hand

$$\deg(D_{2\ell+1}) = \begin{cases} \ell(d_{\Omega_f} + 2d_{P_f}) & \text{if } d_{S_f} \leq d_{\Omega_f} + 2d_{P_f} \\ \ell d_{S_f} & \text{otherwise} \end{cases}$$

Finally

$$\deg(g^{2\ell+1}) = \begin{cases} \max\left((\ell + 1)d_{\Omega_f} + (2\ell + 1)d_{P_f} + 1, (\ell + 1)d_{\Omega_f} + 2\ell d_{P_f} + d_{S_f}, \ell(d_{\Omega_f} + 2d_{P_f}) + 2\right) & \text{if } d_{S_f} \leq d_{\Omega_f} + 2d_{P_f} \\ \max\left(\ell d_{S_f} + d_{P_f} + d_{\Omega_f} + 1, (\ell + 1)d_{S_f} + d_{\Omega_f}, \ell d_{S_f} + 2\right) & \text{otherwise} \end{cases}$$

We conclude with the equality $\mu(g) = 2 \lim_{k \rightarrow +\infty} \frac{\deg(g^k)}{k}$. \square

Lemma 2.2. *Let $\Omega_f = \gcd(P, F)$ and write P (resp. F) as $\Omega_f P_f$ (resp. $\Omega_f S_f$). Suppose that $S_f = \Omega_f^p T_f$ with $p \geq 1$ and $\gcd(T_f, \Omega_f) = 1$. Then*

- \diamond if $d_{S_f} \leq d_{\Omega_f} + 2d_{P_f}$, then $\mu(g) = 2d_{P_f}$;
- \diamond otherwise $\mu(g) = d_{S_f} - d_{\Omega_f}$.

Proof. (a) Assume k even, write k as 2ℓ . We get

$$\Upsilon_{2\ell} = \frac{\sum_{j=0}^{\ell} \binom{2\ell}{2j} \Omega_f^{\ell-j} P_f^{2(\ell-j)} S_f^j}{P_f \sum_{j=0}^{\ell-1} \binom{2\ell}{2j+1} \Omega_f^{\ell-1-j} P_f^{2(\ell-1-j)} S_f^j} = \frac{\Omega_f \sum_{j=0}^{\ell} \binom{2\ell}{2j} \Omega_f^{(p-1)j} P_f^{2(\ell-j)} T_f^j}{P_f \sum_{j=0}^{\ell-1} \binom{2\ell}{2j+1} \Omega_f^{(p-1)j} P_f^{2(\ell-1-j)} T_f^j}$$

Recall that $\gcd(\Omega_f, T_f) = 1$ and that $d_{S_f} = pd_{\Omega_f} + d_{T_f}$, i.e. $d_{T_f} = d_{S_f} - pd_{\Omega_f}$. On the one hand

$$\deg(N_{2\ell}) = \begin{cases} 2\ell d_{P_f} + d_{\Omega_f} & \text{if } d_{S_f} \leq d_{\Omega_f} + 2d_{P_f} \\ \ell d_{S_f} + (1 - \ell)d_{\Omega_f} & \text{otherwise} \end{cases}$$

On the other hand

$$\deg(D_{2\ell}) = \begin{cases} (2\ell - 1)d_{P_f} & \text{if } d_{S_f} \leq d_{\Omega_f} + 2d_{P_f} \\ (\ell - 1)(d_{S_f} - d_{\Omega_f}) + d_{P_f} & \text{otherwise} \end{cases}$$

Finally

$$\deg(g^{2\ell}) = \begin{cases} \max\left(2\ell d_{P_f} + d_{\Omega_f} + 1, (2\ell - 1)d_{P_f} + d_{\Omega_f} + d_{S_f}, (2\ell - 1)d_{P_f} + 1\right) & \text{if } d_{S_f} \leq d_{\Omega_f} + 2d_{P_f} \\ \max\left(\ell d_{S_f} - (\ell - 1)d_{\Omega_f} + 1, \ell d_{S_f} + (2 - \ell)d_{\Omega_f} + d_{P_f}, (\ell - 1)(d_{S_f} - d_{\Omega_f}) + d_{P_f} + 1\right) & \text{otherwise} \end{cases}$$

(b) Suppose k odd, write k as $2\ell + 1$. We get

$$\Upsilon_{2\ell+1} = \frac{P_f \Omega_f \sum_{j=0}^{\ell} \binom{2\ell+1}{2j} \Omega_f^{\ell-j} P_f^{2(\ell-j)} S_f^j}{\sum_{j=0}^{\ell} \binom{2\ell+1}{2j+1} \Omega_f^{\ell-j} P_f^{2(\ell-j)} S_f^j} = \frac{P_f \Omega_f \sum_{j=0}^{\ell} \binom{2\ell+1}{2j} \Omega_f^{(p-1)j} P_f^{2(\ell-j)} T_f^j}{\sum_{j=0}^{\ell} \binom{2\ell+1}{2j+1} \Omega_f^{(p-1)j} P_f^{2(\ell-j)} T_f^j}$$

On the one hand

$$\deg(N_{2\ell+1}) = \begin{cases} (2\ell + 1)d_{P_f} + d_{\Omega_f} & \text{if } d_{S_f} \leq d_{\Omega_f} + 2d_{P_f} \\ \ell d_{S_f} - (\ell - 1)d_{\Omega_f} + d_{P_f} & \text{otherwise} \end{cases}$$

On the other hand

$$\deg(D_{2\ell+1}) = \begin{cases} 2\ell d_{P_f} & \text{if } d_{S_f} \leq d_{\Omega_f} + 2d_{P_f} \\ \ell d_{S_f} - \ell d_{\Omega_f} & \text{otherwise} \end{cases}$$

Finally

$$\deg(g^{2\ell+1}) = \begin{cases} \max\left((2\ell + 1)d_{P_f} + d_{\Omega_f} + 1, 2\ell d_{P_f} + d_{\Omega_f} + d_{S_f}, 2\ell d_{P_f} + 1\right) & \text{if } d_{S_f} \leq d_{\Omega_f} + 2d_{P_f} \\ \max\left(\ell d_{S_f} - (\ell - 1)d_{\Omega_f} + d_{P_f} + 1, (\ell + 1)d_{S_f} - (\ell - 1)d_{\Omega_f}, \ell d_{S_f} - \ell d_{\Omega_f} + 1\right) & \text{otherwise} \end{cases}$$

We conclude with the equality $\mu(g) = 2 \lim_{k \rightarrow +\infty} \frac{\deg(g^k)}{k}$. \square

Lemma 2.3. *Let $\Omega_f = \gcd(P, F)$ and write P (resp. F) as $\Omega_f P_f$ (resp. $\Omega_f S_f$). Suppose that $\Omega_f = S_f^p T_f$ with $p \geq 1$ and $\gcd(T_f, S_f) = 1$. Then*

$$\mu(g) = 2d_{P_f} + d_{\Omega_f} - d_{S_f}.$$

Proof. (a) Assume k even, write k as 2ℓ . We obtain

$$\Upsilon_{2\ell} = \frac{\sum_{j=0}^{\ell} \binom{2\ell}{2j} \Omega_f^{\ell-j} P_f^{2(\ell-j)} S_f^j}{P_f \sum_{j=0}^{\ell-1} \binom{2\ell}{2j+1} \Omega_f^{\ell-1-j} P_f^{2(\ell-1-j)} S_f^j} = \frac{S_f \sum_{j=0}^{\ell} \binom{2\ell}{2j} S_f^{j(p-1)} P_f^{2j} T_f^j}{P_f \sum_{j=0}^{\ell-1} \binom{2\ell}{2j+1} S_f^{j(p-1)} P_f^{2j} T_f^j}$$

Recall that $\gcd(S_f, T_f) = 1$; one has

$$\deg(N_{2\ell}) = (p\ell - \ell + 1)d_{S_f} + 2\ell d_{P_f} + \ell d_{T_f}$$

and

$$\deg(D_{2\ell}) = (\ell - 1)(p - 1)d_{S_f} + (2\ell - 1)d_{P_f} + (\ell - 1)d_{T_f}$$

Finally

$$\begin{aligned} \deg(g^{2\ell}) = \max \left((p\ell - \ell + 1)d_{S_f} + 2\ell d_{P_f} + \ell d_{T_f} + 1, \right. \\ \left. (\ell(p - 1) + 2)d_{S_f} + (2\ell - 1)d_{P_f} + \ell d_{T_f}, \right. \\ \left. (\ell - 1)(p - 1)d_{S_f} + (2\ell - 1)d_{P_f} + (\ell - 1)d_{T_f} + 2 \right) \end{aligned}$$

(b) Suppose k odd, write k as $2\ell + 1$. We get

$$\Upsilon_{2\ell+1} = \frac{\sum_{j=0}^{\ell} \binom{2\ell+1}{2j} P^{2\ell+1-2j} F^j \quad S_f^p T_f P_f \sum_{j=0}^{\ell} \binom{2\ell+1}{2(\ell-j)} S_f^{j(p-1)} T_f^j P_f^{2j}}{\sum_{j=0}^{\ell} \binom{2\ell+1}{2j+1} P^{2\ell-2j} F^j \quad \sum_{j=0}^{\ell} \binom{2\ell+1}{2j} S_f^{j(p-1)} T_f^j P_f^{2j}}.$$

On the one hand

$$\deg(N_{2\ell+1}) = (p + \ell(p - 1))d_{S_f} + (\ell + 1)d_{T_f} + (2\ell + 1)d_{P_f},$$

and on the other hand

$$\deg(D_{2\ell+1}) = 2\ell d_{P_f} + \ell d_{T_f} + \ell(p - 1)d_{S_f}.$$

Finally

$$\begin{aligned} \deg(g^{2\ell+1}) = \max \left((p + \ell(p - 1))d_{S_f} + (\ell + 1)d_{T_f} + (2\ell + 1)d_{P_f} + 1, \right. \\ \left. (p + 1 + \ell(p - 1))d_{S_f} + 2\ell d_{P_f} + (\ell + 1)d_{T_f}, \right. \\ \left. 2\ell d_{P_f} + \ell d_{T_f} + \ell(p - 1)d_{S_f} + 2 \right) \end{aligned}$$

We conclude with the equality $\mu(g) = 2 \lim_{k \rightarrow +\infty} \frac{\deg(g^k)}{k}$. □

3. EXAMPLES

In this section we will give examples that illustrate Theorem B; more precisely §3.1 (resp. §3.2) illustrates Theorem B.1. (resp. Theorem B.2.)

3.1. Example that illustrates Theorem B.1.

3.1.1. *First example.* Consider the birational map of J given in the affine chart $x = 1$ by $f = (y, (1 - y)yz)$. The matrix associated to f is

$$M_f = \begin{pmatrix} (1-y)y & 0 \\ 0 & 1 \end{pmatrix},$$

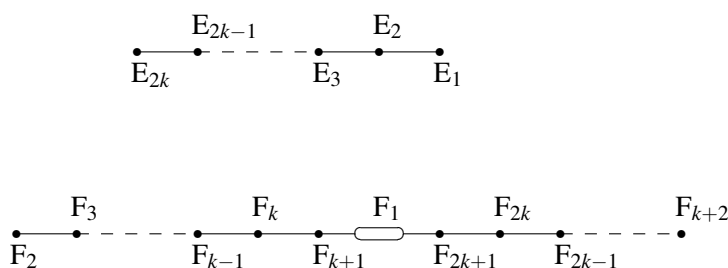
and the Baum Bott index $\text{BB}(f)$ of f is $\frac{((1-y)y+1)^2}{(1-y)y}$; in particular f belongs to \mathcal{J} (Proposition 1.1). The characteristic polynomial of M_f is

$$\chi_f(X) = (X - (1 - y)y)(X - 1).$$

According to Theorem B.1. one has $\mu(f) = 4 \leq 2 \max(\deg(2), \deg(2(1 - y)y)) = 4$.

We can see it another way: [CD12] asserts that $\deg(f^k) = k \deg(f) - k + 1 = 3k - k + 1 = 2k + 1$. Consequently $\mu(f) = 2 \lim_{k \rightarrow +\infty} \frac{\deg(f^k)}{k} = 2 \lim_{k \rightarrow +\infty} \frac{2k + 1}{k} = 4$.

A third way to see this is to look at the configuration of the exceptional divisors. For any $k \geq 1$ one has $f^k = (x^{2k+1} : x^{2k}y : (x - y)^k y^k z)$. The configuration of the exceptional divisors of f^k is



where

- ◊ two curves are related by an edge if their intersection is positive;
- ◊ the self-intersections correspond to the shape of the vertices;
- ◊ the point means self-intersection -1 , the rectangle means self-intersection $-2k$.

In particular the number of base-points of f^k is $2k + 2k + 1 = 4k + 1$ and

$$\mu(f) = \lim_{k \rightarrow +\infty} \frac{\#\mathfrak{b}(f^k)}{k} = 4.$$

3.1.2. *Second example.* Consider the birational map of J given in the affine chart $z = 1$ by $f = (x, xy + x(x - 1))$. The matrix associated to f is

$$M_f = \begin{pmatrix} x & x(x-1) \\ 0 & 1 \end{pmatrix};$$

according to Proposition 1.1 the map f is a Jonquière's twist (indeed $\text{BB}(f) = \frac{(1+x)^2}{x} \in \mathbb{C}(x) \setminus \mathbb{C}$). The characteristic polynomial of M_f is $\chi_f(X) = (X-x)(X-1)$. and f is conjugate to $g = (x, xy)$. According to Theorem B.1. one has

$$\mu(f) = \mu(g) = 2(\deg(g) - 1) = 2 \leq 2 \max(\deg(2), \deg(2(1-y)y)) = 2.$$

We can see it another way: for any $k \geq 1$ one has $f^k = (x, x^k y + x^{k+1} - x)$ and thus $\deg(f^k) = k+1$. As a result $\mu(f) = 2 \lim_{k \rightarrow +\infty} \frac{\deg(f^k)}{k} = 2 \times 1 = 2$.

3.2. Examples that illustrate Theorem B.2.

3.2.1. *First example.* Consider the map of J given in the affine chart $y = 1$ by

$$f = \left(x, \frac{x(1-xz)}{z} \right).$$

The matrix associated to f is

$$M_f = \begin{pmatrix} -x^2 & x \\ 1 & 0 \end{pmatrix},$$

the Baum Bott index $\text{BB}(f)$ of f is $-x^3$ and f belongs to \mathcal{J} (Proposition 1.1).

Theorem B.2.a. asserts that $\mu(f) = 3$. We can see it another way: a computation gives $\deg(f^{2k}) = 3k+1$ and $\deg(f^{2k+1}) = 3(k+1)$ for any $k \geq 0$. Since $\mu(f) = 2 \lim_{k \rightarrow +\infty} \frac{\deg(f^k)}{k}$ one gets $\mu(f) = 3$.

3.2.2. *Second example.* Consider the map f of J associated to the matrix

$$M_f = \begin{pmatrix} y & 2y^8 \\ y & 1 \end{pmatrix}.$$

The Baum Bott index $\text{BB}(f)$ of f is $\frac{(y+1)^2}{y(1-2y^8)}$ and f belongs to \mathcal{J} (Proposition 1.1). Theorem B.2.a. asserts that $\mu(f) = 9$. We can see it another way: a computation gives $\deg(f^{2k}) = 9k+1$ and $\deg(f^{2k+1}) = 9k+8$ for any $k \geq 0$. Since $2 \lim_{k \rightarrow +\infty} \frac{\deg(f^k)}{k} = \mu(f)$ one gets $\mu(f) = 9$.

3.2.3. *Third example.* Let us consider the Jonquière's map of $\mathbb{P}_{\mathbb{C}}^2$ given in the affine chart $z = 1$ by

$$f = \left(\frac{y(y+2)x + y^5}{x + y(y+2)}, y \right).$$

The matrix associated to f is

$$M_f = \begin{pmatrix} y(y+2) & y^5 \\ 1 & y(y+2) \end{pmatrix}$$

and the Baum Bott index $\text{BB}(f)$ of f is $\frac{4(y+2)^2}{(y+2)^2 - y^5}$. In particular f is a Jonquière's twist (Proposition 1.1).

According to Theorem B.2.b. one has $\mu(f) = 3$. An other way to see that is to compute $\deg f^k$ for any k : for any $\ell \geq 1$ one has

$$\deg(f^{2\ell}) = 3(\ell + 1), \quad \deg(f^{2\ell+1}) = 3\ell + 5.$$

Then we find again $\mu(f) = 2 \lim_{k \rightarrow +\infty} \frac{\deg(f^k)}{k} = 3$.

3.2.4. *Fourth example.* Consider the map f of \mathbb{J} associated to the matrix

$$M_f = \begin{pmatrix} y(y+2)^8 & y^5 \\ 1 & y(y+2)^8 \end{pmatrix}.$$

The Baum Bott index $\text{BB}(f)$ of f is $\frac{4(y+2)^{16}}{(y+2)^{16}-y^3}$ and f belongs to \mathcal{J} (Proposition 1.1). According to Theorem B.2.b. one has $\mu(f) = 16$. An other way to see that is to compute $\deg f^k$ for any k : for any $k \geq 1$ one has $\deg f^k = 8k + 2$. Then we find again $\mu(f) = 2 \lim_{k \rightarrow +\infty} \frac{\deg(f^k)}{k} = 2 \times 8 = 16$.

3.2.5. *Fifth example.* Let us consider the Jonquière map of $\mathbb{P}_{\mathbb{C}}^2$ given in the affine chart $z = 1$ by

$$f = \left(\frac{y(y+1)(y+2)x + y^2}{(y+2)x + y(y+1)(y+2)}, y \right).$$

The matrix associated to f is

$$M_f = \begin{pmatrix} y(y+1)(y+2) & y^2 \\ y+2 & y(y+1)(y+2) \end{pmatrix}$$

and the Baum-Bott index $\text{BB}(f)$ of f is $\frac{4(y+1)^2(y+2)}{(y+1)^2(y+2)-1}$; in particular f is a Jonquière twist (Proposition 1.1).

Theorem B.2.c. asserts that $\mu(f) = 3$. An other way to see that is to compute $\deg f^k$ for any k : for any $k \geq 1$

$$\deg(f^{2k}) = 3k + 2 \quad \deg(f^{2k+1}) = 3k + 4$$

so $2 \lim_{k \rightarrow +\infty} \frac{\deg(f^k)}{k} = 3$ and we find again $\mu(f) = 3$.

3.3. Families.

3.3.1. *First family.* Let us consider the family $(f_t)_t$ of elements of \mathbb{J} given by $f_t = (x+t, y \frac{x}{x+1})$. A straightforward computation yields to

$$f_t^n = \left(x+nt, y \frac{x}{x+1} \frac{x+t}{x+t+1} \cdots \frac{x+(n-1)t}{x+(n-1)t+1} \right)$$

The birational map f_t belongs to \mathcal{J} if some multiple of t is equal to 1, and to $\mathbb{J} \setminus \mathcal{J}$ otherwise. Furthermore,

- ◇ if no multiple of t is equal to 1, then $\mu(f_t) = 2$ (because $\lim_{k \rightarrow +\infty} \frac{\deg f_t^k}{k} = 1$);
- ◇ otherwise $\mu(f_t) = 0$.

3.3.2. *Second family, illustration of Theorem A.3.* Let us recall a result of [CDX21]: let f be any element of $\mathrm{PGL}_3(\mathbb{C})$, or any elliptic element of $\mathrm{Bir}(\mathbb{P}_{\mathbb{C}}^2)$ of infinite order; then f is a limit of pairwise conjugate loxodromic elements (resp. Jonquières twists) in the Cremona group. Hence there exist families $(f_n)_n$ of birational self-maps of the complex projective plane such that

- ◇ $\mu(f_n) > 0$ for any $n \in \mathbb{N}$;
- ◇ $\mu\left(\lim_{n \rightarrow +\infty} f_n\right) = 0$.

3.3.3. *Third family, illustration of Theorem A.4.* Let us recall a construction given in [CDX21]. Consider a pencil of cubic curves with nine distinct base points p_i in $\mathbb{P}_{\mathbb{C}}^2$. Given a point m in $\mathbb{P}_{\mathbb{C}}^2$, draw the line $(p_1 m)$ and denote by m' the third intersection point of this line with the cubic of our pencil that contains m : the map $m \mapsto \sigma_1(m) = m'$ is a birational involution. Replacing p_1 by p_2 , we get a second involution and, for a very general pencil, $\sigma_1 \circ \sigma_2$ is a Halphen twist that preserves our cubic pencil. At the opposite range, consider the degenerate cubic pencil, the members of which are the union of a line through the origin and the circle $C = \{x^2 + y^2 = z^2\}$. Choose $p_1 = (1 : 0 : 1)$ and $p_2 = (0 : 1 : 1)$ as our distinguished base points. Then, $\sigma_1 \circ \sigma_2$ is a Jonquières twist preserving the pencil of lines through the origin; if the plane is parameterized by $(s, t) \mapsto (st, t)$, this Jonquières twist is conjugate to $(s, t) \mapsto \left(s, \frac{(s-1)t+1}{(s^2+1)t+s-1}\right)$. Now, if we consider a family of general cubic pencils converging towards this degenerate pencil, we obtain a sequence of Halphen twists converging to a Jonquières twist. So there exists a sequence $(f_n)_n$ of birational self-maps of $\mathbb{P}_{\mathbb{C}}^2$ whose limit is also a birational self-map of $\mathbb{P}_{\mathbb{C}}^2$ and such that

- ◇ $\mu(f_n) = 0$ for any $n \in \mathbb{N}$;
- ◇ $\mu\left(\lim_{n \rightarrow +\infty} f_n\right) > 0$.

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