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SINGULARITIES OF HOLOMORPHIC CODIMENSION ONE FOLIATIONS OF THE COMPLEX PROJECTIVE PLANE

DOMINIQUE CERVEAU AND JULIE DÉSERTI

ABSTRACT. We prove that any holomorphic codimension 1 foliation on the complex projective plane has at most one singularity up to the action of an ad-hoc birational map. Consequently, any algebraic foliation on the affine plane has no singularities up to the action of a suitable birational self map of the complex projective plane into itself.

Let \mathcal{F} be a codimension one holomorphic foliation of degree N on $\mathbb{P}_{\mathbb{C}}^2$. Denote by $\pi: \mathbb{C}^3 \setminus \{0\} \rightarrow \mathbb{P}_{\mathbb{C}}^2$ the canonical projection. The homogeneous foliation $\pi^{-1}\mathcal{F}$ extends to \mathbb{C}^3 and is defined by a 1-form

$$\omega = a(x, y, z) dx + b(x, y, z) dy + c(x, y, z) dz$$

where a, b, c are homogeneous polynomials of degree $N + 1$ without common component satisfying the Euler identity: $ax + by + cz = 0$; it is the Chow theorem for foliations. The singular set $\text{Sing}(\mathcal{F})$ of \mathcal{F} is given by

$$\pi(\{a = b = c = 0\} \setminus \{0\}).$$

Let us recall what is the Milnor number $\mu(\mathcal{F}, m)$ of a foliation \mathcal{F} at a singular point m . Let us fix a local chart (u, v) such that $m = (0, 0)$. The germ of \mathcal{F} at m is defined, up to multiplication by a unit at 0, by a 1-form $E du + F dv$. Denote by $\langle E, F \rangle$ the ideal generated by E and F , then

$$\mu(\mathcal{F}, m) = \dim \frac{\mathbb{C}\{u, v\}}{\langle E, F \rangle};$$

it is also the multiplicity of intersection of the germs of curves $(E = 0)$ and $(F = 0)$. Let us recall that ([Bru15, Chapter 2, Section 3, Example 2])

$$\sum_{m \in \text{Sing}(\mathcal{F})} \mu(\mathcal{F}, m) = N^2 + N + 1;$$

in particular, \mathcal{F} has a least one singular point.

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Example. Let us consider the diagonal linear foliation \mathcal{F} given by the 1-form

$$\omega = \lambda yz dx + xz dy - (1 + \lambda)xy dz$$

with $\lambda \in \mathbb{C} \setminus \{0, -1, -2\}$. Note that $\text{Sing}(\mathcal{F}) = \{(0:0:1), (1:0:0), (0:1:0)\}$; moreover, $x^\lambda yz^{-(1+\lambda)}$ is a first integral of \mathcal{F} .

The birational involution from $\mathbb{P}_{\mathbb{C}}^2$ into itself given by

$$\mathcal{I}_1: (x:y:z) \dashrightarrow (xy:y^2:x^2-yz),$$

defines an isomorphism of $\{y \neq 0\}$. Remark that the foliation $\mathcal{I}_1^{-1}\mathcal{F}$ is described by the 1-form

$$-y((2+\lambda)x^2 + \lambda yz) dx + x((2+\lambda)x^2 - yz) dy + (1+\lambda)xy^2 dz;$$

in particular, $\text{Sing}(\mathcal{I}_1^{-1}\mathcal{F}) = \{(0:0:1), (0:1:0)\}$.

Let us now consider the birational involution \mathcal{I}_2 from $\mathbb{P}_{\mathbb{C}}^2$ into itself given by

$$\mathcal{I}_2: (x:y:z) \dashrightarrow (x^2:-xy+z^2:xz);$$

note that \mathcal{I}_2 defines an isomorphism of $\{x \neq 0\}$. Furthermore, the restriction of $\mathcal{I}_2^{-1}\mathcal{I}_1^{-1}\mathcal{F}$ on $(x=0)$ is described by the 1-form $(1-\lambda)z^5 dx$; consequently, $\text{Sing}(\mathcal{I}_2^{-1}\mathcal{I}_1^{-1}\mathcal{F}) = \{(0:1:0)\}$.

Can we generalize this to all holomorphic foliations on $\mathbb{P}_{\mathbb{C}}^2$? The answer is yes:

Theorem 1. *Let \mathcal{F} be a holomorphic foliation on $\mathbb{P}_{\mathbb{C}}^2$. There exists a birational self map ϕ from $\mathbb{P}_{\mathbb{C}}^2$ into itself such that $\phi^{-1}\mathcal{F}$ has at most one singular point.*

Note that many properties are preserved by conjugacy by birational maps, for instance the existence of invariant algebraic curves of genus ≥ 2 , but also the existence of dense leaves.

Corollary 2. *Let \mathcal{F} be an algebraic foliation on \mathbb{C}^2 . There exists a birational map ϕ from \mathbb{C}^2 into itself such that $\phi^{-1}\mathcal{F}$ has no singular point.*

In a certain sense Corollary 2 indicates that algebraic foliations of \mathbb{C}^2 have no special properties.

Note that the situation is completely different on $\mathbb{P}_{\mathbb{C}}^1 \times \mathbb{P}_{\mathbb{C}}^1$; in that space there exist foliations without singularities: a foliation on $\mathbb{P}_{\mathbb{C}}^1 \times \mathbb{P}_{\mathbb{C}}^1$ birationally conjugate to a foliation without singularities has a rational first integral ([Bru15]).

To prove Theorem 1 we will use the following statement:

Lemma 3. *Let \mathcal{F} be a holomorphic codimension one foliation on $\mathbb{P}_{\mathbb{C}}^2$. Assume that \mathcal{F} has N singular points, and among them $n \geq 2$ singular points on a line L . There exists a birational self map ϕ from $\mathbb{P}_{\mathbb{C}}^2$ into itself such that $\phi^{-1}\mathcal{F}$ has $N - n + 1$ singular points, one on L and $N - n$ on $\mathbb{P}_{\mathbb{C}}^2 \setminus L$.*

Proof. \diamond Assume first that L is not invariant by \mathcal{F} . Let us choose coordinates such that

$$\begin{cases} L = (z = 0) \\ (0 : 1 : 0) \notin \text{Sing}(\mathcal{F}) \\ L \text{ is not tangent to the leaf of } \mathcal{F} \text{ through } (0 : 1 : 0) \end{cases} \quad (\star)$$

The foliation \mathcal{F} is defined by

$$\omega = A(x, y, z) dx + B(x, y, z) dy + C(x, y, z) dz.$$

Let us recall the Euler condition: $xA + yB + zC = 0$; in particular note that $B(0, 1, 0) = 0$. The tangency condition (\star) says that $A(0, 1, 0) \neq 0$.

Let us consider the birational involution ϕ of $\mathbb{P}_{\mathbb{C}}^2$ into itself given by

$$\phi : (x : y : z) \dashrightarrow (xz : -yz + x^2 : z^2);$$

note that

$$\text{Ind}(\phi) = \{(0 : 1 : 0)\}, \quad \text{Exc}(\phi) = \{(z = 0)\}$$

where $\text{Ind}(\phi)$ (resp. $\text{Exc}(\phi)$) denotes the set of indeterminacy (resp. the exceptional set) of ϕ . In particular, ϕ defines an isomorphism of $\{z \neq 0\}$. One has

$$\begin{aligned} \phi^* \omega &= A(xz, -yz + x^2, z^2)(x dz + z dx) \\ &\quad + B(xz, -yz + x^2, z^2)(-y dz - z dy + 2x dx) + 2C(xz, -yz + x^2, z^2)z dz, \end{aligned}$$

and

$$\phi^* \omega|_{z=0} = xA(0, x^2, 0) dx + \underbrace{B(0, x^2, 0)}_{=0} (-y dz + 2x dx) = xA(0, x^2, 0) dx;$$

in particular $\phi^* \omega|_{z=0}$ vanishes only at $x = 0$ since $A(0, 1, 0) \neq 0$. As a consequence, $\#(\text{Sing}(\phi^{-1} \mathcal{F}) \cap (z = 0)) = 1$ and $\#\text{Sing}(\phi^{-1} \mathcal{F}) = N - n + 1$.

\diamond Suppose now that L is invariant by \mathcal{F} . Let us choose coordinates such that $\begin{cases} L = (z = 0) \\ (0 : 1 : 0) \notin \text{Sing}(\mathcal{F}) \end{cases}$

The foliation \mathcal{F} is defined by

$$\omega = zA(x, y, z) dx + zB(x, y, z) dy + C(x, y, z) dz.$$

Let us still consider the birational involution ϕ of $\mathbb{P}_{\mathbb{C}}^2$ into itself given by

$$\phi : (x : y : z) \dashrightarrow (xz : -yz + x^2 : z^2)$$

that defines an isomorphism of $\{z \neq 0\}$. Then

$$\begin{aligned} \phi^* \omega &= z \left(zA(xz, -yz + x^2, z^2)(x dz + z dx) \right. \\ &\quad \left. + zB(xz, -yz + x^2, z^2)(-y dz - z dy + 2x dx) + 2C(xz, -yz + x^2, z^2) dz \right), \end{aligned}$$

and the restriction of $\phi^{-1}\mathcal{F}$ on $(z=0)$ is described by the 1-form $2C(0, x^2, 0) dz$. But $C(0, 1, 0) \neq 0$ (because $(0:1:0) \notin \text{Sing}(\mathcal{F})$) so $C(0, x^2, 0) = 0$ if and only if $x = 0$. In particular, $\#(\text{Sing}(\phi^{-1}\mathcal{F}) \cap (z=0)) = 1$ and $\#\text{Sing}(\phi^{-1}\mathcal{F}) = N - n + 1$. □

Proof of Theorem 1. We get the result by iteration using Lemma 3. □

Assume that \mathcal{F} is an algebraic foliation of any codimension of \mathbb{C}^n ; is it true that there exists a birational self map of $\mathbb{P}_{\mathbb{C}}^n$ into itself such that $\phi^{-1}\mathcal{F}$ has no singularity ? In the real case (for example in $\mathbb{P}_{\mathbb{R}}^2$) what is the best result we can expect ?

Remark that following [CDGBM10] in which we classify, up to automorphisms of $\mathbb{P}_{\mathbb{C}}^2$, the quadratic foliations having a unique singularity there is a lot of activity around the foliations of $\mathbb{P}_{\mathbb{C}}^2$ having a unique singularity (for instance [FPR22, Alc22, Cou22, CJ21, CF21, CF20]).

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