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ABSTRACT. We prove that any holomorphic codimension 1 foliation on the complex projective plane has at most one singularity up to the action of an ad-hoc birational map. Consequently, any algebraic foliation on the affine plane has no singularities up to the action of a suitable birational self map of the complex projective plane into itself.

Let $\mathscr{F}$ be a codimension one holomorphic foliation of degree N on $\mathbb{P}^2_{\mathbb{C}}$. Denote by $\pi\colon \mathbb{C}^3\smallsetminus\{0\}\to\mathbb{P}^2_{\mathbb{C}}$ the canonical projection. The homogeneous foliation $\pi^{-1}\mathscr{F}$ extends to $\mathbb{C}^3$ and is defined by a 1-form

$$\omega = a(x,y,z)\,\mathrm{d}x + b(x,y,z)\,\mathrm{d}y + c(x,y,z)\,\mathrm{d}z$$

where a, b, c are homogeneous polynomials of degree $N+1$ without common component satisfying the Euler identity: $ax+by+cz=0$; it is the Chow theorem for foliations. The singular set $\mathrm{Sing}(\mathscr{F})$ of $\mathscr{F}$ is given by

$$\pi(\{a=b=c=0\}\smallsetminus\{0\}).$$

Let us recall what is the Milnor number $\mu(\mathscr{F},m)$ of a foliation $\mathscr{F}$ at a singular point m. Let us fix a local chart (u,v) such that $m=(0,0)$. The germ of $\mathscr{F}$ at m is defined, up to multiplication by a unit at 0, by a 1-form $E\,\mathrm{d}u+F\,\mathrm{d}v$. Denote by $\langle E,F\rangle$ the ideal generated by E and F, then

$$\mu(\mathscr{F},m)=\dim\frac{\mathbb{C}\{u,v\}}{\langle E,F\rangle};$$

it is also the multiplicity of intersection of the germs of curves $(E=0)$ and $(F=0)$. Let us recall that ([Bru15, Chapter 2, Section 3, Example 2])

$$\sum_{m\in\mathrm{Sing}(\mathscr{F})}\mu(\mathscr{F},m)=N^2+N+1;$$

in particular, $\mathscr{F}$ has a least one singular point.

2020 *Mathematics Subject Classification.* 32S65.

Key words and phrases. Singularities of holomorphic foliations.

Example. Let us consider the diagonal linear foliation $\mathscr{F}$ given by the 1-form

$$\omega = \lambda yz\,\mathrm{d}x + xz\,\mathrm{d}y - (1+\lambda)xy\,\mathrm{d}z$$

with $\lambda \in \mathbb{C} \setminus \{0, -1, -2\}$. Note that $\mathrm{Sing}(\mathscr{F}) = \big\{(0:0:1), (1:0:0), (0:1:0)\big\}$; moreover, $x^{\lambda} yz^{-(1+\lambda)}$ is a first integral of $\mathscr{F}$.

The birational involution from $\mathbb{P}^2_{\mathbb{C}}$ into itself given by

$$\mathscr{I}_1 \colon (x:y:z) \dashrightarrow (xy : y^2 : x^2 - yz),$$

defines an isomorphism of $\{y \neq 0\}$. Remark that the foliation $\mathscr{I}_1^{-1}\mathscr{F}$ is described by the 1-form

$$-y\big((2+\lambda)x^2 + \lambda yz\big)\,\mathrm{d}x + x\big((2+\lambda)x^2 - yz\big)\,\mathrm{d}y + (1+\lambda)xy^2\,\mathrm{d}z;$$

in particular, $\mathrm{Sing}(\mathscr{I}_1^{-1}\mathscr{F}) = \big\{(0:0:1), (0:1:0)\big\}$.

Let us now consider the birational involution $\mathscr{I}_2$ from $\mathbb{P}^2_{\mathbb{C}}$ into itself given by

$$\mathscr{I}_2 \colon (x:y:z) \dashrightarrow (x^2 : -xy + z^2 : xz);$$

note that $\mathscr{I}_2$ defines an isomorphism of $\{x \neq 0\}$. Furthermore, the restriction of $\mathscr{I}_2^{-1}\mathscr{I}_1^{-1}\mathscr{F}$ on $(x=0)$ is described by the 1-form $(1-\lambda)z^5\,\mathrm{d}x$; consequently, $\mathrm{Sing}(\mathscr{I}_2^{-1}\mathscr{I}_1^{-1}\mathscr{F}) = \big\{(0:1:0)\big\}$.

Can we generalize this to all holomorphic foliations on $\mathbb{P}^2_{\mathbb{C}}$? The answer is yes:

Theorem 1. *Let $\mathscr{F}$ be a holomorphic foliation on $\mathbb{P}^2_{\mathbb{C}}$. There exists a birational self map ϕ from $\mathbb{P}^2_{\mathbb{C}}$ into itself such that $\phi^{-1}\mathscr{F}$ has at most one singular point.*

Note that many properties are preserved by conjugacy by birational maps, for instance the existence of invariant algebraic curves of genus ≥ 2, but also the existence of dense leaves.

Corollary 2. *Let $\mathscr{F}$ be an algebraic foliation on $\mathbb{C}^2$. There exists a birational map ϕ from $\mathbb{C}^2$ into itself such that $\phi^{-1}\mathscr{F}$ has no singular point.*

In a certain sense Corollary 2 indicates that algebraic foliations of $\mathbb{C}^2$ have no special properties.

Note that the situation is completely different on $\mathbb{P}^1_{\mathbb{C}} \times \mathbb{P}^1_{\mathbb{C}}$; in that space there exist foliations without singularities: *a foliation on $\mathbb{P}^1_{\mathbb{C}} \times \mathbb{P}^1_{\mathbb{C}}$ birationally conjugate to a foliation without singularities has a rational first integral* ([Bru15]).

To prove Theorem 1 we will use the following statement:

Lemma 3. *Let $\mathscr{F}$ be a holomorphic codimension one foliation on $\mathbb{P}^2_{\mathbb{C}}$. Assume that $\mathscr{F}$ has N singular points, and among them $n \geq 2$ singular points on a line L. There exists a birational self map ϕ from $\mathbb{P}^2_{\mathbb{C}}$ into itself such that $\phi^{-1}\mathscr{F}$ has $N-n+1$ singular points, one on L and $N-n$ on $\mathbb{P}^2_{\mathbb{C}} \setminus L$.*

Proof. ◇ Assume first that L is not invariant by $\mathscr{F}$. Let us choose coordinates such that

$$\begin{cases} L=(z=0) \\ (0:1:0)\notin \mathrm{Sing}(\mathscr{F}) \\ L \text{ is not tangent to the leaf of } \mathscr{F} \text{ through } (0:1:0) \text{ at } (0:1:0) \end{cases} \quad (\star)$$

The foliation $\mathscr{F}$ is defined by

$$\omega = A(x,y,z)\,\mathrm{d}x + B(x,y,z)\,\mathrm{d}y + C(x,y,z)\,\mathrm{d}z.$$

Let us recall the Euler condition: $xA+yB+zC=0$; in particular note that $B(0,1,0)=0$. The tangency condition $(\star)$ says that $A(0,1,0)\neq 0$.

Let us consider the birational involution ϕ of $\mathbb{P}^2_{\mathbb{C}}$ into itself given by

$$\phi\colon (x:y:z) \dashrightarrow (xz:-yz+x^2:z^2);$$

note that

$$\mathrm{Ind}(\phi)=\{(0:1:0)\}, \qquad \mathrm{Exc}(\phi)=\{(z=0)\}$$

where $\mathrm{Ind}(\phi)$ (resp. $\mathrm{Exc}(\phi)$) denotes the set of indeterminacy (resp. the exceptional set) of ϕ. In particular, ϕ defines an isomorphism of $\{z\neq 0\}$. One has

$$\begin{aligned} \phi^*\omega &= A(xz,-yz+x^2,z^2)(x\,\mathrm{d}z+z\,\mathrm{d}x) \\ &\quad +B(xz,-yz+x^2,z^2)(-y\,\mathrm{d}z-z\,\mathrm{d}y+2x\,\mathrm{d}x)+2C(xz,-yz+x^2,z^2)z\,\mathrm{d}z, \end{aligned}$$

and

$$\phi^*\omega\big|_{z=0} = xA(0,x^2,0)\,\mathrm{d}x + \underbrace{B(0,x^2,0)}_{=0}\big(-y\,\mathrm{d}z+2x\,\mathrm{d}x\big) = xA(0,x^2,0)\,\mathrm{d}x;$$

in particular $\phi^*\omega\big|_{z=0}$ vanishes only at $x=0$ since $A(0,1,0)\neq 0$. As a consequence, $\#\big(\mathrm{Sing}(\phi^{-1}\mathscr{F})\cap(z=0)\big)=1$ and $\#\mathrm{Sing}\big(\phi^{-1}\mathscr{F}\big)=N-n+1$.

◇ Suppose now that L is invariant by $\mathscr{F}$. Let us choose coordinates such that $\begin{cases} L=(z=0) \\ (0:1:0)\notin \mathrm{Sing}(\mathscr{F}) \end{cases}$

The foliation $\mathscr{F}$ is defined by

$$\omega = zA(x,y,z)\,\mathrm{d}x + zB(x,y,z)\,\mathrm{d}y + C(x,y,z)\,\mathrm{d}z.$$

Let us still consider the birational involution ϕ of $\mathbb{P}^2_{\mathbb{C}}$ into itself given by

$$\phi\colon (x:y:z) \dashrightarrow (xz:-yz+x^2:z^2)$$

that defines an isomorphism of $\{z\neq 0\}$. Then

$$\begin{aligned} \phi^*\omega &= z\Big(zA(xz,-yz+x^2,z^2)(x\,\mathrm{d}z+z\,\mathrm{d}x) \\ &\quad +zB(xz,-yz+x^2,z^2)(-y\,\mathrm{d}z-z\,\mathrm{d}y+2x\,\mathrm{d}x)+2C(xz,-yz+x^2,z^2)\,\mathrm{d}z\Big), \end{aligned}$$

and the restriction of $\phi^{-1}\mathscr{F}$ on $(z=0)$ is described by the 1-form $2C(0,x^2,0)\,\mathrm{d}z$. But $C(0,1,0)\neq 0$ (because $(0:1:0)\notin \mathrm{Sing}(\mathscr{F})$) so $C(0,x^2,0)=0$ if and only if $x=0$. In particular, $\#\big(\mathrm{Sing}(\phi^{-1}\mathscr{F})\cap (z=0)\big)=1$ and $\#\mathrm{Sing}\big(\phi^{-1}\mathscr{F}\big)=N-n+1$.

□

Proof of Theorem 1. We get the result by iteration using Lemma 3. □

Assume that $\mathscr{F}$ is an algebraic foliation of any codimension of $\mathbb{C}^n$; is it true that there exists a birational self map of $\mathbb{P}^n_{\mathbb{C}}$ into itself such that $\phi^{-1}\mathscr{F}$ has no singularity ? In the real case (for example in $\mathbb{P}^2_{\mathbb{R}}$) what is the best result we can expect ?

Remark that following [CDGBM10] in which we classify, up to automorphisms of $\mathbb{P}^2_{\mathbb{C}}$, the quadratic foliations having a unique singularity there is a lot of activity around the foliations of $\mathbb{P}^2_{\mathbb{C}}$ having a unique singularity (*for instance* [FPR22, Alc22, Cou22, CJ21, CF21, CF20]).

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