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STATISTICAL MECHANICS OF THE RADIAL FOCUSING NONLINEAR SCHRÖDINGER EQUATION IN GENERAL TRAPS

VAN DUONG DINH, NICOLAS ROUGERIE, LEONARDO TOLOMEO, YUZHAO WANG

ABSTRACT. In this paper, we investigate the Gibbs measures associated with the focusing nonlinear Schrödinger equation with an anharmonic potential. We establish a dichotomy for normalizability and non-normalizability of the Gibbs measures in one dimension and higher dimensions with radial data. This extends a recent result of the third and fourth authors with Robert and Seong (2022), where the focusing Gibbs measures with a harmonic potential were addressed. Notably, in the case of a subharmonic potential, we identify a novel critical nonlinearity (below the usual mass-critical exponent) for which the Gibbs measures exhibit a phase transition. The primary challenge emerges from the limited understanding of eigenvalues and eigenfunctions of the Schrödinger operator with an anharmonic potential. We overcome the difficulty by employing techniques related to a recent work of the first two authors (2022).

CONTENTS

1. Introduction	2
1.1. Known results and motivation	3
1.2. Measure construction and main results	5
2. Radial Schrödinger operators with anharmonic potential	12
2.1. Properties of the resolvent	12
2.2. Weyl-type asymptotics for radial Schrödinger operators	22
3. Variational formulation	32
3.1. The Boué-Dupuis formula and its' use	33
3.2. Preliminary estimates	36
4. Subharmonic potential	39
4.1. Normalizability	39
4.2. Non-normalizability	48
5. Superharmonic potential	58
5.1. Normalizability	59
5.2. Non-normalizability	62
Appendix A. Reminder on coherent states and Husimi functions	68
Appendix B. Fractional Schrödinger operator	69
References	74

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1. INTRODUCTION

This paper concerns the statistical mechanics of the focusing nonlinear Schrödinger equation with anharmonic potential

$$i\partial_t u + (\Delta - |x|^s)u = -\alpha|u|^{p-2}u$$

on the Euclidean space \mathbb{R}^d , $d \geq 1$, restricted to radial functions for $d \geq 2$. We will assume that $s > 1$, $p > 2$ and $\alpha > 0$. More precisely, we study the integrability and non-integrability of the associated, formally time-invariant, Gibbs measures given by

$$d\rho_K(u) = \mathcal{Z}_K^{-1} \mathbf{1}_{\{|M(u)| \leq K\}} e^{\alpha R_p(u)} d\mu(u), \quad (1.1) \quad \boxed{\text{gibbs}}$$

where μ is the Gaussian measure associated with the anharmonic operator

$$\mathcal{L} = -\Delta + |x|^s \quad (1.2) \quad \boxed{\text{anharmonic}}$$

given formally by

$$d\mu(u) = \mathcal{Z}^{-1} e^{-\frac{1}{2}\langle \mathcal{L}u, u \rangle} du = \mathcal{Z}^{-1} e^{-\frac{1}{2} \int_{\mathbb{R}^d} (|\nabla u(x)|^2 + |x|^s |u(x)|^2) dx} du \quad (1.3) \quad \boxed{\text{gaussian}}$$

with a normalization constant \mathcal{Z} . The potential energy $R_p(u)$ is defined by

$$R_p(u) := \frac{1}{p} \int_{\mathbb{R}^d} |u(x)|^p dx. \quad (1.4) \quad \boxed{\text{Rp}}$$

Since we consider the focusing case $\alpha > 0$, we must impose a mass cut-off $|M(u)| \leq K$ for some parameter $K > 0$ in (1.1), where

$$M(u) = \int_{\mathbb{R}^d} |u(x)|^2 dx$$

is the (conserved) L^2 mass if $s > 2$. If $s \leq 2$, this is infinite μ -almost surely, and hence we consider the Wick-ordered version

$$M(u) = \int_{\mathbb{R}^d} :|u(x)|^2: dx$$

i.e. formally the L^2 -mass minus its' expectation value in the Gaussian measure.

Our main purpose is to identify the range of values for the parameters (s, p, α) that delineates whether the Gibbs measure (1.1) is well-defined or not.

This paper is a continuation of a recent work of the third and fourth authors with Robert and Seong [54] where the normalizability/non-normalizability of the focusing Gibbs measure with harmonic potential ($s = 2$) were investigated. More precisely, the following result was proved in [54].

THM:RSTW

Theorem 1.1 (The harmonic oscillator case, [54]).

Let $d \geq 1$, $s = 2$ and restrict the measures to radial functions when $d \geq 2$. Then, the following statements hold:

- (i) (subcritical case) If $2 < p < 2 + \frac{4}{d}$, then for any $K > 0$, the focusing Gibbs measure

$$d\rho_K(u) = \mathcal{Z}_K^{-1} \mathbf{1}_{\{\int_{\mathbb{R}^d} :|u(x)|^2: dx \leq K\}} e^{\frac{1}{p} \|u\|_{L^p(\mathbb{R}^d)}^p} d\mu(u)$$

is well-defined as a probability measure and it is absolutely continuous with respect to the base Gaussian free field μ .

- (ii) (critical/supercritical cases) If $p \geq 2 + \frac{4}{d}$ and $p < \frac{2d}{d-2}$ when $d \geq 3$, then for any $K > 0$, the focusing Gibbs measure

$$d\rho_K(u) = \mathcal{Z}_K^{-1} \mathbf{1}_{\{\int_{\mathbb{R}^d} :|u(x)|^2: dx \leq K\}} e^{\frac{1}{p} \|u\|_{L^p(\mathbb{R}^d)}^p} d\mu(u)$$

is not well-defined as a probability measure.

In the present paper we focus on the case where the parameter s exceeds 1 but is not equal to 2 (although our approach can lead to simplifications in the latter case). Our main findings can be divided into two distinct cases: the superharmonic case where $s > 2$ and the subharmonic case where $s < 2$.

In the case of superharmonic potentials, we identify the critical nonlinearity as

$$p_{s>2} := 2 + \frac{4}{d}.$$

The Gibbs measure (1.1) is normalizable for any $\alpha, K > 0$ provided that the power nonlinearity is subcritical, i.e., $p < p_{s>2}$. On the other hand, it is non-normalizable for any $\alpha, K > 0$ as long as the power nonlinearity is supercritical and below the energy-critical exponent, i.e., $p > p_{s>2}$ and $p < \frac{2d}{d-2}$ if $d \geq 3$. For the critical nonlinearity $p = p_{s>2}$ we observe a phase transition: the Gibbs measure is well-defined for $\alpha^{\frac{d}{2}} K < \|Q\|_{L^2(\mathbb{R}^d)}^2$, and it is not well-defined for $\alpha^{\frac{d}{2}} K > \|Q\|_{L^2(\mathbb{R}^d)}^2$. Here Q is an optimizer of the Gagliardo-Nirenberg-Sobolev inequality on \mathbb{R}^d .

In the case of subharmonic potentials, we identify a new critical nonlinearity

$$p_{s<2} := 2 + \frac{4s}{(d-1)s+2}$$

which is below the usual mass-critical one above. The Gibbs measure is again normalizable when the power nonlinearity is subcritical and is non-normalizable for supercritical power nonlinearity regardless of the values of $\alpha, K > 0$. For the critical nonlinearity, the phase transition is characterized in terms of the nonlinear strength α : for each $K > 0$, there exists $\alpha_0 = \alpha_0(K) > 0$ such that the Gibbs measure is well-defined for all $\alpha < \alpha_0$, and it is not well-defined for all $\alpha > \alpha_0$.

1.1. Known results and motivation. The construction of Gibbs measures associated with focusing nonlinear Schrödinger equations was initiated by Lebowitz, Rose and Speer [37]. They considered the focusing NLS on the one-dimensional torus and proposed to study the Gibbs measure with a mass cutoff, namely

$$d\rho_K(u) = \mathcal{Z}_K^{-1} \mathbf{1}_{\{\int_{\mathbb{T}} |u|^2 dx \leq K\}} e^{\frac{1}{p} \int_{\mathbb{T}} |u|^p dx} d\mu(u),$$

where \mathcal{Z}_K is the normalization constant (often referred to as the partition function). The imposition of a mass cutoff is reasonable since the mass is a conserved quantity under the NLS flow.

It was asserted in [37] that the aforementioned measure is normalizable under two distinct conditions: when $2 < p < 6$ for any positive mass cutoff K , and when $p = 6$ for K smaller than the mass of Q – the unique (up to symmetries) optimizer of the Gagliardo-Nirenberg-Sobolev inequality

$$\int_{\mathbb{R}} |u|^6 dx \leq C_{\text{GNS}} \left(\int_{\mathbb{R}} |u'|^2 dx \right) \left(\int_{\mathbb{R}} |u|^2 dx \right)^2$$

such that $C_{\text{GNS}} = 2\|Q\|_{L^2(\mathbb{R})}^2$. They also proved that the Gibbs measure is not well-defined when $p = 6$ and K exceeds the mass of Q , and when $p > 6$ regardless of the mass cutoff size K .

While the proof of normalizability presented in [37] uses an elegant probabilistic argument, it however contains a gap as pointed out and rectified for $p < 6$ in [13]. Later, Bourgain [6] gave an analytic proof for the normalizability for $2 < p < 6$ with any positive mass cutoff size K , and for $p = 6$ with sufficiently small K . He also proved the invariance of these measures under the NLS dynamics. Recently, the third author together with Oh and Sosoie in [47] proved the normalizability when $p = 6$ and K is smaller than the mass of Q , which resolves the issue in [37]. Remarkably, they were also able to prove the normalizability with the mass cutoff K exactly equal to the mass of Q . This result is rather surprising since the focusing quintic

NLS on the 1D torus admits blow-up solutions with this minimal mass, as found by Ogawa and Tsutsumi [44]. Essentially, this implies that the Gibbs measure lives on Sobolev spaces of low regularity on which there are no blow-up solutions at the critical mass threshold. The result of [47] shows a *phase transition* for the focusing Gibbs measure on the one-dimensional torus at the critical nonlinearity. Specifically, the partition function \mathcal{Z}_K is not analytic with respect to K when $p = 6$.

Similar results hold for the Gibbs measure

$$d\rho_K(u) = \mathcal{Z}_K^{-1} \mathbf{1}_{\{\int_{\mathbb{D}} |u|^2 dx \leq K\}} e^{\frac{1}{p} \int_{\mathbb{D}} |u|^p dx} d\mu(u).$$

associated with the focusing NLS on the two-dimensional unit disc with Dirichlet boundary condition, restricted to radial functions.

In [61], Tzvetkov constructed and proved the invariance of the focusing Gibbs measure with the subcritical nonlinearity $p < 4$ and any positive K . Later, Bourgain and Bulut [8] extended Tzvetkov's result to the critical nonlinearity $p = 4$ and sufficiently small mass cutoff K . In a subsequent work [47], the third author, in collaboration with Oh and Sosoë, demonstrated the normalizability of the Gibbs measure when $p = 4$ and K is less than the mass of Q – the (positive and radial) ground state solution of

$$-\Delta Q + Q - Q^3 = 0 \quad \text{in } \mathbb{R}^2.$$

They also established the non-normalizability of the Gibbs measure when $p = 4$ and K exceeds the mass of Q , and when $p > 4$ no matter the size of the mass cutoff K . Recently, Xian [63] proved the optimal mass normalizability in the critical case, that is, the above Gibbs measure is normalizable when $p = 4$ and K equals the mass of Q .

Concerning the Gibbs measures associated with focusing NLS with potential on the real line, the pioneering work was undertaken by Burq, Thomann and Tzvetkov [10], where they studied the focusing Gibbs measure for the cubic nonlinearity with harmonic potential $V(x) = |x|^2$. In this setting, the Gibbs measure is slightly different to the cases discussed earlier on the 1D torus and 2D unit disc:

$$d\rho_K(u) = \mathcal{Z}_K^{-1} \mathbf{1}_{\{\int_{\mathbb{R}^d} |u|^2 : dx \leq K\}} e^{\frac{1}{p} \int_{\mathbb{R}^d} |u|^p dx} d\mu(u).$$

Here a Wick-ordered renormalized mass is employed instead of the usual mass since the latter is infinite on the support of the Gaussian measure.

In one dimension, Burq, Thomann and Tzvetkov [10] demonstrated the normalizability for $p = 4$ and any positive K . They also proved the invariance of this measure under the dynamics of the nonlinear Schrödinger equation. In the two-dimensional case, the focusing Gibbs measure with harmonic potential was investigated by Deng [22] under a radial assumption. More precisely, he constructed the focusing Gibbs measure for any $2 < p < 4$ and any positive K and proved its invariance under the NLS flow. More recently, the last two authors, in collaboration with Robert and Seong, revisited the construction of focusing Gibbs measure with harmonic potential. By exploiting the so-called Boué-Dupuis variational formula and refined stochastic analysis, they proved the focusing measure is normalizable when $2 < p < 2 + \frac{4}{d}$, but non-normalizable when $p \geq 2 + \frac{4}{d}$ and $p < \frac{2d}{d-2}$ if $d \geq 3$ regardless of the mass cutoff size K (see Theorem 1.1). Their result completes the picture of focusing Gibbs measures with harmonic potential in one dimension and higher dimensions for radial data. Notably, a distinguishing feature from the cases of the 1D torus or 2D unit disc is the absence of a critical nonlinearity, thus precluding a phase transition.

One advantage of considering the harmonic potential is that eigenfunctions and eigenvalues of the linear operator $-\Delta + |x|^2$ are explicit, which allows one to obtain precise L^p -estimates of the eigenfunctions (see e.g., [36] and [33]). However, this advantageous feature is no longer accessible when dealing with a general anharmonic potential $|x|^s$ with $s \neq 2$.

In [23], the first two authors investigated the one-dimensional focusing Gibbs measure with a potential exhibiting a growth of $|x|^s$ at infinity. They successfully overcame the aforementioned difficulty by employing techniques from many-body quantum mechanics, specifically the application of a Lieb-Thirring type inequality and operator-inequalities in Schatten ideals, which originated from [41]. Instead of relying on L^p -estimates for eigenfunctions, which are unavailable in this context, they observed that the construction of the focusing Gibbs measure hinged on L^p -estimates for the (diagonal part of the) Green function of $-\Delta + V(x)$:

$$G(x, x) = \sum_{n \geq 0} \lambda_n^{-2} e_n^2(x),$$

where $(e_n, \lambda_n^2)_{n \geq 0}$ represent the (normalized) eigenfunctions and eigenvalues of $-\Delta + V(x)$. Acquiring this information proved to be considerably more tractable, thanks to the application of standard inequalities like Hölder and Kato-Seiler-Simon within Schatten spaces. With these methods, they could define the focusing Gibbs measure with a cubic nonlinearity $p = 4$ for any positive K , provided that $s > \frac{8}{5}$. Note that the potential energy $\frac{1}{4} \int_{\mathbb{R}} |u|^4 dx$ is finite almost surely with respect to the Gaussian measure as long as $s > 1$. Thus there is a gap between 1 and $\frac{8}{5}$ for the measure construction. This gap is technical due to the use of a fractional Gagliardo-Nirenberg inequality (see [23, Lemma 3.8]). In this paper, we will fill this gap and prove the normalizability of focusing Gibbs measure for all $s > 1$. We also aim at extending these result to higher dimensional radial NLS equations. Upon a suitable change of variables, the radial Schrödinger operator with an anharmonic potential on \mathbb{R}^d transforms into a one-dimensional Schrödinger operator with an inverse square potential plus the anharmonic potential on $(0, +\infty)$. The appearance of the inverse square potential necessitates new approaches, as standard inequalities in Schatten spaces, as used in [41, 23], are no longer applicable. Hence, novel arguments are required to address this potential term.

In addition to the previously discussed works on nonlinear Gibbs measures, we would like to mention related research on the probabilistic theory of Schrödinger operators with trapping potential ([1] and [53]). Additionally, there are notable works on the deterministic theory for the nonlinear Schrödinger equation with harmonic potential (see e.g., [50], [34], and a series of studies by Carles ([14, 15, 16, 17])).

Another motivation for this study comes from the mean-field approximation of Bose gases and Bose-Einstein condensates. More specifically, the Gibbs measure linked with the nonlinear Schrödinger equation was rigorously derived from many-body quantum mechanics (see [55, 56] for the 1D focusing NLS, [40, 41, 30, 58] for the 1D defocusing NLS and [39, 29, 31, 32] for higher dimensions). In [55], Rout and Sohinger rigorously derived the Gibbs measure associated with the focusing cubic NLS on the 1D torus, considering any mass cutoff size K . More recently, in their subsequent work [56], they successfully extended this result to the focusing quintic NLS provided that the mass cutoff size is small. It is expected that some form of phase transition, as proved in [47], should happen at the many-body level provided that the number of particles is close to a critical value. However, this remains an open question.

1.2. Measure construction and main results. In this subsection, we go over the construction of the Gibbs measures (1.1), and state our main results precisely.

Let us start by recalling some basic properties of the operator (1.2). It is known (under the radial assumption when $d \geq 2$) that \mathcal{L} has a sequence of eigenvalues λ_n^2 with

$$0 < \lambda_0 \leq \lambda_1 \leq \dots \leq \lambda_n \rightarrow \infty$$

and that the corresponding normalized eigenfunctions e_n , i.e.

$$\mathcal{L}e_n = \lambda_n^2 e_n,$$

form an orthonormal basis of $L^2(\mathbb{R}^d)$. The radial assumption is invoked hereafter whenever the spatial dimension is two or larger. We also use the liberty to choose a real-valued eigenbasis. More details on the radial Schrödinger operator are given in Section 2 below.

We define Sobolev spaces associated with the operator \mathcal{L} as follows.

Definition 1.2 (Sobolev spaces).

For $1 \leq q \leq \infty$ and $\sigma \in \mathbb{R}$, the Sobolev space $\mathcal{W}^{\sigma,q}(\mathbb{R}^d)$ is defined by the norm

$$\|u\|_{\mathcal{W}^{\sigma,q}(\mathbb{R}^d)} = \|\mathcal{L}^{\frac{\sigma}{2}}u\|_{L^q(\mathbb{R}^d)}.$$

When $p = 2$, we write $\mathcal{W}^{\sigma,2}(\mathbb{R}^d) = \mathcal{H}^\sigma(\mathbb{R}^d)$ and for $u = \sum_{n=0}^{\infty} u_n e_n$ we have

$$\|u\|_{\mathcal{H}^\sigma(\mathbb{R}^d)}^2 = \sum_{n=0}^{\infty} \lambda_n^{2\sigma} |u_n|^2.$$

With the above notation, we define the Hamiltonian as

$$H(u) = \frac{1}{2} \int_{\mathbb{R}^d} |\mathcal{L}^{\frac{1}{2}}u(x)|^2 dx - \frac{\alpha}{p} \int_{\mathbb{R}^d} |u(x)|^p dx. \quad (1.5) \quad \text{Hamil}$$

In particular, using the eigenbasis $\{e_n\}_{n \geq 0}$, we can decompose any $u \in S'(\mathbb{R}^d)$ as

$$u = \sum_{n=0}^{\infty} u_n e_n, \quad u_n = \langle u, e_n \rangle = \int_{\mathbb{R}^d} u(x) e_n(x) dx.$$

Then, in the coordinates $u = (u_n)_{n \geq 0}$, the Hamiltonian (1.5) has the form

$$H(u) = H\left(\sum_{n=0}^{\infty} u_n e_n\right) = \frac{1}{2} \sum_{n=0}^{\infty} \lambda_n^2 |u_n|^2 - \frac{\alpha}{p} \int_{\mathbb{R}^d} \left| \sum_{n=0}^{\infty} u_n e_n(x) \right|^p dx.$$

From the above computation, we may define the Gaussian measure with the Cameron-Martin space $\mathcal{H}^1(\mathbb{R}^d)$ formally given by

$$d\mu = \mathcal{Z}^{-1} e^{-\frac{1}{2}\|u\|_{\mathcal{H}^1}^2} du = \mathcal{Z}^{-1} \prod_{n=0}^{\infty} e^{-\frac{1}{2}\lambda_n^2 |u_n|^2} du_n d\bar{u}_n, \quad (1.6) \quad \text{Gaussian}$$

where $du_n d\bar{u}_n$ is the Lebesgue measure on \mathbb{C} . We note that this Gaussian measure μ is the induced probability measure under the map

$$\omega \in \Omega \mapsto u^\omega = \sum_{n=0}^{\infty} \frac{g_n(\omega)}{\lambda_n} e_n, \quad (1.7) \quad \text{maps}$$

where $\{g_n\}_{n \geq 0}$ is a sequence of independent standard complex-valued Gaussian random variables on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. From (1.7) and (2.61), we see that

$$\mathbb{E}[\|u^\omega\|_{L^2(\mathbb{R}^d)}^2] = \sum_{n=0}^{\infty} \lambda_n^{-2} \begin{cases} < \infty & \text{if } s > 2, \\ = \infty & \text{if } s < 2. \end{cases} \quad (1.8) \quad \text{L2}$$

This implies that a typical function u in the support of μ is not square integrable when $s < 2$. On the other hand, when $s < 2$, Corollary 3.4 (i) implies that $\mathbb{E}[\|u^\omega\|_{L^p(\mathbb{R}^d)}] < \infty$ when $p > \frac{4}{s}$ and $p < \frac{2d}{d-2}$ if $d \geq 3$. Therefore, the potential energy $\frac{1}{p} \int_{\mathbb{R}^d} |u(x)|^p dx$ does not require a renormalization if we confine ourselves to the range $p > \frac{4}{s}$ and $p < \frac{2d}{d-2}$ if $d \geq 3$.

To define the Gaussian measure μ in (1.6) rigorously, we start with a finite dimensional version. First, define the spectral projection \mathbf{P}_N by

$$\mathbf{P}_N u = \sum_{n=0}^N u_n e_n. \quad (1.9) \quad \boxed{\text{projN}}$$

The image of \mathbf{P}_N is the finite-dimensional space

$$E_N = \text{span}\{e_0, \dots, e_N\}.$$

Through the isometric map

$$(u_n)_{n=0}^N \mapsto \sum_{n=0}^N u_n e_n, \quad (1.10) \quad \boxed{\text{iso}}$$

from \mathbb{C}^{N+1} to E_N , we may identify E_N with \mathbb{C}^{N+1} . Consider a Gaussian measure on \mathbb{C}^{N+1} (or on \mathbb{R}^{2N+2}) given by

$$d\mu_N = \prod_{n=0}^N \frac{\lambda_n^2}{2\pi} e^{-\frac{\lambda_n^2}{2}|u_n|^2} du_n d\bar{u}_n.$$

This Gaussian measure defines a probability measure on the finite dimensional space E_N via the map (1.10), which will be also denoted by μ_N . The measure μ_N can also be viewed as the induced probability measure under the map

$$\omega \mapsto u_N^\omega := \sum_{n=0}^N \frac{g_n(\omega)}{\lambda_n} e_n. \quad (1.11) \quad \boxed{\text{RVN}}$$

Given any $\sigma > \frac{1}{s} - \frac{1}{2}$, the sequence (u_N^ω) is a Cauchy sequence in $L^2(\Omega; \mathcal{H}^{-\sigma}(\mathbb{R}^d))$ converging to u^ω given in (1.7). See Corollary 3.4 (ii). In particular, the distribution of the random variable $u^\omega \in \mathcal{H}^{-\sigma}(\mathbb{R}^d)$ is the Gaussian measure μ . The measure μ can be decomposed as

$$\mu = \mu_N \otimes \mu_N^\perp, \quad (1.12) \quad \boxed{\text{mu}}$$

where the measure μ_N^\perp is the distribution of the random variable given by

$$u_N^{\omega, \perp}(x) := \sum_{n=N+1}^{\infty} \frac{g_n(\omega)}{\lambda_n} e_n(x).$$

Recall from the discussion in the introduction that, to define the focusing Gibbs measure (1.1), a mass cut-off is necessary. As previously mentioned, note that $u^\omega \notin L^2(\mathbb{R}^d)$ μ -almost surely when $s < 2$. This motivates the introduction of a Wick-ordered renormalized L^2 -mass, similar to the approach used in [7, 10, 47, 54]. Given $x \in \mathbb{R}^d$, $u_N^\omega(x)$ in (1.11) is a mean-zero complex-valued Gaussian random variable with variance

$$\sigma_N(x) = \mathbb{E}[|u_N^\omega(x)|^2] = 2 \sum_{n=0}^N \frac{e_n^2(x)}{\lambda_n^2}, \quad (1.13) \quad \boxed{\text{variance}}$$

from which and Corollary 2.6 we have

$$\mathbb{E} \left[\|u_N^\omega\|_{L^2(\mathbb{R}^d)}^2 \right] = \int_{\mathbb{R}^d} \sigma_N(x) dx = \sum_{n=0}^N \frac{2}{\lambda_n^2} \sim \lambda_N^{-1+\frac{2}{s}} \rightarrow \infty$$

as $N \rightarrow \infty$ provided $s \in (1, 2)$. Here σ_N depends on $x \in \mathbb{R}^d$ as the random process u^ω given by (1.7) is not stationary. We can then define the Wick power $:|u_N|^2:$ via

$$:|u_N|^2: = |u_N|^2 - \sigma_N. \quad (1.14) \quad \boxed{\text{Wick}}$$

It is known (See Corollary 3.5 below) that

$$\int_{\mathbb{R}^d} :|u_N(x)|^2: dx \rightarrow \int_{\mathbb{R}^d} :|u(x)|^2: dx \quad (1.15) \quad \boxed{\text{Wick bis}}$$

μ -almost surely, which defines the renormalized (Wick-ordered) L^2 mass in the right-hand side.

The main purpose of this paper is to define the focusing Gibbs measure (1.1) with Wick-ordered L^2 -cutoff. We start with a finite dimensional approximation.

$$d\rho_{K,N}(u) = \mathcal{Z}_{K,N}^{-1} \mathbf{1}_{\{\int_{\mathbb{R}^d} :|u_N(x)|^2: dx \leq K\}} e^{\frac{\alpha}{p} \|u_N\|_{L^p(\mathbb{R}^d)}^p} d\mu_N(u_N) \otimes d\mu_N^\perp(u_N^\perp), \quad (1.16) \quad \boxed{\text{tru_rho}}$$

where $u_N = \mathbf{P}_N u$, $u_N^\perp = \mathbf{P}_N^\perp u := u - \mathbf{P}_N u$, and the partition function $\mathcal{Z}_{K,N}$ is given by

$$\mathcal{Z}_{K,N} = \int \mathbf{1}_{\{\int_{\mathbb{R}^d} :|u_N(x)|^2: dx \leq K\}} e^{\frac{\alpha}{p} \|u_N\|_{L^p(\mathbb{R}^d)}^p} d\mu(u). \quad (1.17) \quad \boxed{\text{partition}}$$

Our main findings are sharp criteria under which the above (1.16) converges to a probability measure as $N \rightarrow \infty$ (i.e. $0 < \mathcal{Z}_{K,N} < \infty$ uniformly in N). They are stated as follows:

THM:main

Theorem 1.3 (Gibbs measure construction, subharmonic case).

Let $d \geq 1$, $1 < s < 2$ and assume the radial condition when $d \geq 2$. Then the following statements hold:

(i) (subcritical case) If

$$\frac{4}{s} < p < 2 + \frac{4s}{(d-1)s+2},$$

then for any $\alpha, K > 0$, we have uniform exponential integrability of the density: given any finite $r \geq 1$,

$$\sup_{N \in \mathbb{N}} \left\| \mathbf{1}_{\{\int_{\mathbb{R}^d} :|u_N(x)|^2: dx \leq K\}} e^{\frac{\alpha}{p} \|u_N\|_{L^p(\mathbb{R}^d)}^p} \right\|_{L^r(\mu)} < \infty. \quad (1.18) \quad \boxed{\text{uniint_p}}$$

Moreover, we have

$$\lim_{N \rightarrow \infty} \mathbf{1}_{\{\int_{\mathbb{R}^d} :|u_N(x)|^2: dx \leq K\}} e^{\frac{\alpha}{p} \|u_N\|_{L^p(\mathbb{R}^d)}^p} = \mathbf{1}_{\{\int_{\mathbb{R}^d} :|u(x)|^2: dx \leq K\}} e^{\frac{\alpha}{p} \|u\|_{L^p(\mathbb{R}^d)}^p} \quad (1.19) \quad \boxed{\text{cov-lp}}$$

in $L^r(\mu)$. As a consequence, the Gibbs measure $\rho_{N,K}$ in (1.16) converges, in total variation, to the focusing Gibbs measure ρ_K defined by

$$d\rho_K(u) = \mathcal{Z}_K^{-1} \mathbf{1}_{\{\int_{\mathbb{R}^d} :|u(x)|^2: dx \leq K\}} e^{\frac{\alpha}{p} \|u\|_{L^p(\mathbb{R}^d)}^p} d\mu(u). \quad (1.20) \quad \boxed{\text{rho}}$$

Furthermore, the resulting measure ρ_K is absolutely continuous with respect to the base Gaussian free field μ in (1.12).

(ii) (critical case) If

$$p = 2 + \frac{4s}{(d-1)s+2},$$

we have the following phase transition for the Gibbs measure in (1.1). Then, for every $K > 0$, there exists $\alpha_0 = \alpha_0(K) \in (0, \infty)$ such that

(a) (weakly nonlinear regime). Let $\alpha < \alpha_0$. Then the Gibbs measure $\rho_{K,N}$ in (1.16) converges, in total variation, to the focusing Gibbs measure ρ_K defined by

$$d\rho_K(u) = \mathcal{Z}_K^{-1} \mathbf{1}_{\{\int_{\mathbb{R}^d} :|u(x)|^2: dx \leq K\}} e^{\frac{\alpha}{p} \|u\|_{L^p(\mathbb{R}^d)}^p} d\mu(u). \quad (1.21) \quad \boxed{\text{rho1}}$$

(b) (strongly nonlinear regime). When $\alpha > \alpha_0$, the focusing Gibbs measure (1.1), even with a Wick-ordered L^2 -cutoff, cannot be defined as a probability measure.

Moreover, we have that

$$0 < \inf_{K>0} \alpha_0(K) \leq \sup_{K>0} \alpha_0(K) < \infty,$$

so (a) and (b) hold for $\alpha \ll 1$ and $\alpha \gg 1$ (respectively), independently of the particular value of $K > 0$.

(iii) (supercritical case) Let

$$p > 2 + \frac{4s}{(d-1)s+2}$$

and further assume that

$$p < \frac{2d}{d-2} \text{ if } d \geq 3.$$

Then, for any $\alpha, K > 0$, we have

$$\sup_{N \in \mathbb{N}} \mathcal{Z}_{K,N} = \sup_{N \in \mathbb{N}} \left\| \mathbf{1}_{\{\int_{\mathbb{R}^d} |u_N(x)|^2 dx \leq K\}} e^{\frac{\alpha}{p} \|u_N\|_{L^p(\mathbb{R}^d)}^p} \right\|_{L^1(\mu)} = \infty, \quad (1.22) \quad \boxed{\text{non_int}}$$

where $\mathcal{Z}_{K,N}$ is the partition function given in (1.17). The same divergence holds for \mathcal{Z}_K , i.e.

$$\mathcal{Z}_K = \int \mathbf{1}_{\{\int_{\mathbb{R}^d} |u(x)|^2 dx \leq K\}} e^{\frac{\alpha}{p} \|u\|_{L^p(\mathbb{R}^d)}^p} d\mu = \infty. \quad (1.23) \quad \boxed{\text{non_int2}}$$

As a consequence, the focusing Gibbs measure (1.1), even with a Wick-ordered L^2 -cutoff, cannot be defined as a probability measure.

Remark 1.1. For the critical case, Theorem 1.3 (ii) claims the normalizability/non-normalizability of the Gibbs measure for $\alpha \ll 1$ and $\alpha \gg 1$ respectively, uniformly in the cut-off size K . Furthermore, when $\alpha \sim 1$, there exists a critical coupling constant $\alpha_0(K)$ for given $K > 0$. However, whether the critical coupling constant α_0 is independent of K is not clear.

Similar results also hold for superharmonic potentials. In this setting, we have a more precise description of the phase transition.

Theorem 1.4 (Gibbs measure construction, superharmonic case).

Let $d \geq 1$, $s > 2$ and assume the radial condition when $d \geq 2$. Given $\alpha, K > 0$, define the partition function \mathcal{Z}_K by

$$\mathcal{Z}_K = \mathbb{E}_\mu \left[e^{\frac{\alpha}{p} \int_{\mathbb{R}^d} |u|^p dx} \mathbf{1}_{\{\|u\|_{L^2(\mathbb{R}^d)}^2 \leq K\}} \right], \quad (1.24) \quad \boxed{Z1}$$

where \mathbb{E}_μ denotes an expectation with respect to the Gaussian measure μ . Then, the following statements hold:

- (i) (subcritical case) If $2 < p < 2 + \frac{4}{d}$, then $\mathcal{Z}_K < \infty$ for any $\alpha, K > 0$.
- (ii) (critical case) Let $p = 2 + \frac{4}{d}$. Then, $\mathcal{Z}_K < \infty$ if $\alpha^{\frac{d}{2}} K < \|Q\|_{L^2(\mathbb{R}^d)}^2$, and $\mathcal{Z}_K = \infty$ if $\alpha^{\frac{d}{2}} K > \|Q\|_{L^2(\mathbb{R}^d)}^2$. Here, Q is an optimizer of the Gagliardo-Nirenberg-Sobolev inequality on \mathbb{R}^d

$$\|u\|_{L^p(\mathbb{R}^d)}^p \leq C_{\text{GNS}} \|\nabla u\|_{L^2(\mathbb{R}^d)}^{\frac{d(p-2)}{2}} \|u\|_{L^2(\mathbb{R}^d)}^{\frac{4-(d-2)(p-2)}{2}}, \quad u \in H^1(\mathbb{R}^d). \quad (1.25) \quad \boxed{\text{GNS}}$$

- (iii) (supercritical case) If $p > 2 + \frac{4}{d}$ and $p < \frac{2d}{d-2}$ if $d \geq 3$, then $\mathcal{Z}_K = \infty$ for any $\alpha, K > 0$.

THM:main1

As previously mentioned, when $p = 4$, Theorems 1.3 and 1.4 represent an improvement over a recent work [23], where the normalizability of the focusing Gibbs measure was established only for $s > \frac{8}{5}$. Here we not only demonstrate its normalizability for all $s > 1$, but we also extend the result to other nonlinearities.

Furthermore, our main results also extend a recent work [54], where the normalizability and non-normalizability were established specifically for the case of the harmonic potential $s = 2$. The extension to anharmonic potentials with $s > 1$ is not a direct adaptation of the arguments presented in [54], primarily due to a lack of explicit knowledge concerning the eigenfunctions and eigenvalues of the Schrödinger operator with anharmonic potential.

An intriguing feature of our main results is the identification of a new critical nonlinearity (below the standard mass-critical nonlinearity $p = 2 + \frac{4}{d}$) that exhibits a phase transition in the subharmonic case $1 < s < 2$. It is noteworthy that, at this level of nonlinearity, the associated nonlinear Schrödinger equation always exhibits global dynamics given sufficiently high regularity data (so that the energy is finite). This leads to the suggestion that a new blow-up phenomenon may emerge for solutions of the NLS with low regularity (within the support of the Gibbs measure) due to the weak growth of the trapping potential. To the best of our knowledge, no such result is available in the existing literature.

As previously discussed, we address the challenge arising from the absence of an explicit formula for the eigenvalues and L^p -estimates of eigenfunctions by examining the Green function of the Schrödinger operator \mathcal{L} defined in (1.2).

First, in order to determine the regularity of the Gaussian measure, we need to establish an upper bound on the trace of \mathcal{L} raised to certain negative powers (see Lemma 2.2). In the one-dimensional case, this was proved in [41, Example 3.2] using a version of the Lieb-Thirring inequality originating in [24]. This does not apply to the radial case in higher dimensions, which leads to a Schrödinger operator with an inverse square potential. We circumvent this issue by employing the fundamental solution of the heat equation with an inverse-square potential, as provided by Ortner and Wagner [49]. This fundamental solution is expressed in terms of the modified Bessel function of the first kind. Taking advantage of the asymptotic behavior of this special function near the origin and at infinity, we establish a variant of the Lieb-Thirring inequality adapted to our context.

Next, we need an L^p -bound for the diagonal of the Green function (see Lemma 2.3). To achieve this, we first employ the asymptotics of the modified Bessel function to deduce a decay property near the origin. Subsequently, to capture decay behavior at infinity, we utilize the odd extension technique to extend the underlying operator to the entire real line and then use previously known results for the one-dimensional anharmonic oscillator by [41]. In the case of two dimensions, careful considerations are required due to the negative sign in front of the inverse-square potential, which is addressed through a refined Hardy inequality recently established by Frank and Merz [28].

Finally, we establish a Weyl-type asymptotic for the number of eigenvalues below a large threshold for the radial Schrödinger operator with an anharmonic potential. In one dimension, this type of estimate is known as the Cwikel-Lieb-Rozenbljum bound (see, for example, [23, Lemma D.1]). The proof is based on coherent states and semi-classical analysis on the phase space. Specifically, we define a quantum energy whose minimizer is attained by a fermionic density matrix. The problem of counting the number of eigenvalues below a large threshold is then reduced to computing the trace of this fermionic density matrix, which is determined by comparing lower and upper bounds.

The results of Theorem 1.3 and Theorem 1.4, together with the previous work on $s = 2$ in [54], give a complete characterization of the construction of regular¹ focusing Gibbs measures with trapping potentials.

¹We mean that the potential $|u|^p$ is well-defined.

We observe different critical phenomena for different values of s : when $s > 2$, the phase transition at the critical exponent depends on the cutoff parameter K ; when $s = 2$, there is no phase transition at the critical case [54]; when $s \in (1, 2)$, the critical exponent depends on s , and moreover, the phase transition depends on the coupling strength α . These differences reflect the distinct spectral properties of the Schrödinger operators for different s , and require different techniques for each case.

We apply the Barashkov-Gubinelli variational method [2] to prove Theorem 1.3 and Theorem 1.4. Specifically, we use the variational formula of Boué-Dupuis [5, 64], Lemma 3.1, to reformulate the Gibbs measure construction as a stochastic optimization problem. Then, to show the normalizability part of Theorems 1.3 and 1.4, we need to control the stochastic optimization problems uniformly; see Subsections 4.1 and 5.1. To show the non-normalizability parts, we need to find suitable sequences of appropriate drift terms, which drive the stochastic optimization problem to diverge; see Subsections 4.2 and 5.2. We remark that the asymptotic behaviour of the variational formulae, which depends on the behaviour of Schatten norms of trapped Laplacian in Corollary 2.6, determines the different criticality of different s .

For the subharmonic trapping case $s \in (1, 2)$, the proof of Theorem 1.3 is inspired by recent works [45, 46, 60]. In particular, in [45], the authors studied a focusing Φ_4^3 -model with a Hartree-type nonlinearity, where the potential for the Hartree nonlinearity is given by the Bessel potential of order β . They show that the case $\beta = 2$ is critical, leading to phase transition regarding the coupling strength. In [46], similar critical behaviour was shown for the Φ_3^3 measure. However, both the above-mentioned works only consider cases where the coupling strength is either very weak, i.e. weakly nonlinear regime $\alpha \ll 1$, or very strong, i.e. strongly nonlinear regime $\alpha \gg 1$; and leave the case $\alpha \sim 1$ open. Theorem 1.3 shows the existence of a critical coupling strength α_0 for normalizability. However, if $\alpha = \alpha_0$, then the normalizability is undetermined by Theorem 1.3.

For the superharmonic trapping case $s > 2$, the trap is strong enough to make the problem almost a local one, similar to the case on \mathbb{T} . Heuristically, this is because the trapping potential penalises functions which have nontrivial mass outside of a large ball. Indeed, when the trapping is strong enough, i.e. in the superharmonic case $s > 2$, we have $\|u\|_{L^2} < \infty$ μ -a.s., at which point the analysis becomes similar to the case of a bounded domain. Therefore, the Gibbs measure is less singular and the proof of Theorem 1.4 exploits ideas from the torus setting [42, 47]. In particular, the ground state, which is the minimizer of the Gagliardo-Nirenberg-Sobolev inequality, is essential to characterize the critical behaviour of the Gibbs measure, as in the torus cases in [42, 47]. Our proof also relies on the exponential decay of the ground state at infinity to eliminate the unbounded trapping by using a suitable scaling argument.

Remark 1.2.

- (i) The L^p -bound for the diagonal of the Green function, as established in Lemma 2.3, allows us to define the defocusing measure for all $p > \max\left\{\frac{4}{s}, 2\right\}$ and $p < \frac{2d}{d-2}$ if $d \geq 3$. However, when $p \geq \frac{2d}{d-2}$, the potential energy $R_p(u)$ (see (1.4)) becomes infinite almost surely on the support of the Gaussian measure μ , necessitating a renormalization. The question of constructing the defocusing Gibbs measure in this case remains an interesting open problem.
- (ii) In the absence of the radial assumption, a renormalization would be necessary for any (even) $p > 2$ as soon as $d \geq 2$. We are aware of the sole work by de Bouard and Debussche [21], where they established the construction of the Gibbs measure associated with the 2D defocusing cubic NLS with a harmonic potential. Extending this construction to other (even) power nonlinearities $p > 4$ and potentially exploring it with other (anharmonic) potentials would be a highly interesting problem. On the other hand, in the focusing case, even with a renormalization of the potential

energy, we expect that the Gibbs measure would remain non-normalizable, similar to the work of Brydges and Slade [18].

The structure of this paper is as follows. In Section 2, we study the radial Schrödinger operator with anharmonic potential including some properties of the resolvent and a Weyl-type asymptotic. In Section 3, we recall the Boué-Dupuis variational formula, which plays a vital role in our proof. We also give some applications of the resolvent estimates derived in the preceding section. Section 4 is devoted to demonstrating the normalizability and non-normalizability for the subharmonic potential, while Section 5 addresses the case of the superharmonic potential. Finally, we recall in the appendices some essential tools used in proving Weyl-type asymptotics, and extend our result to the case of fractional Schrödinger operators with anharmonic potentials.

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2. RADIAL SCHRÖDINGER OPERATORS WITH ANHARMONIC POTENTIAL

sec:Schro

In this section, we collect some properties of the Schrödinger operator with anharmonic potential. In dimensions larger than 1, we restrict its' domain to radial functions.

Let $d \geq 1$, $s > 0$ and consider the Schrödinger operator

$$\mathcal{L} := -\Delta + |x|^s \quad \text{on } \mathbb{R}^d.$$

When $d \geq 2$, we restrict our consideration to radial functions, that is we look at

$$\mathcal{L} := -\partial_r^2 - \frac{d-1}{r}\partial_r + r^s \quad \text{on } (0, +\infty). \quad (2.1) \quad \text{eq:RadLap}$$

We may define the above as the Friedrichs extension of the associated quadratic form, starting from the domain of C^∞ functions vanishing at infinity.

We will rely on the change of variable

$$U : f(r) \mapsto r^{\frac{d-1}{2}} f(r) \quad (2.2) \quad \text{eq:changerad}$$

to reduce our study to the operator

$$\mathcal{L}_1 = -\partial_r^2 + \frac{(d-1)(d-3)}{4r^2} + r^s \quad (2.3) \quad \text{L1}$$

acting on $L^2((0, +\infty), dr)$ with a Dirichlet boundary condition at 0. More precisely, the radial Laplacian corresponds to the Friedrichs extension of (2.3) (see below).

2.1. Properties of the resolvent.

LEM:main1

Lemma 2.1 (Definition of the radial Laplacian with trap).

The follow properties hold:

- (i) *When $d \geq 2$, the self-adjoint extension of \mathcal{L} (still denoted by \mathcal{L}) has domain $\mathcal{D}(\mathcal{L})$ such that $U\mathcal{D}(\mathcal{L})$ is a subset of continuous functions vanishing at the origin.*
- (ii) *$(\mathcal{L} + 1)^{-1}$ is compact.*

(iii) *There exists a sequence of eigenvalues $(\lambda_n^2)_{n \geq 0}$ of \mathcal{L} satisfying*

$$0 < \lambda_0 \leq \lambda_1 \leq \dots \leq \lambda_n \rightarrow +\infty$$

and the corresponding eigenfunctions $(e_n)_{n \geq 0}$ form an orthonormal basis of $L^2(\mathbb{R}^d)$ (radial functions of $L^2(\mathbb{R}^d)$ when $d \geq 2$). In addition, eigenfunctions can be chosen to be real-valued.

The Dirichlet condition at the origin inherited by the domain of \mathcal{L}_1 will be crucial to employ results from [49] in the sequel (see (2.14) and (2.15) below).

Proof. The 1D case is standard (see e.g., [3, Theorem 3.1]). Let us consider the case $d \geq 2$ where the radial assumption is imposed. Consider the quadratic form associated to \mathcal{L}_1 in (2.3), namely

$$\mathcal{Q}(f, f) := \int_0^{+\infty} |\partial_r f(r)|^2 + \frac{(d-1)(d-3)}{4r^2} |f(r)|^2 + r^s |f(r)|^2 dr, \quad f \in C_0^\infty((0, +\infty)). \quad (2.4)$$

eq:RadForm

When $d \geq 3$, it is clear that \mathcal{Q} is non-negative. When $d = 2$, we use the following Hardy inequality (see e.g., [20]):

$$\int_0^{+\infty} \frac{1}{4r^2} |f(r)|^2 dr \leq \int_0^{+\infty} |\partial_r f(r)|^2 dr, \quad f \in C_0^\infty((0, +\infty)) \quad (2.5)$$

Hardy-ineq

to deduce that \mathcal{Q} is also non-negative. The Friedrichs extension theorem (see e.g., [51, Theorem X.23]) ensures that \mathcal{L}_1 admits a self-adjoint realization [49], still denoted by \mathcal{L}_1 , whose core is $C_0^\infty((0, +\infty))$. Observe that the map U defined in (2.2) maps the quadratic form domain of the radial Laplacian (2.1) to the domain of (2.4). Hence the Friedrichs extension, whose domain includes the quadratic form domain, is the appropriate one. Other self-adjoint extensions exist (see [9]), but we shall prove below that they are not selected by conjugating (2.1) with (2.2).

When $d \geq 3$, it is immediate that the quadratic form domain of (2.4) is embedded in

$$H^1((0, +\infty)) \cap L^2((0, +\infty), r^s dr),$$

which is well-known to be compactly embedded in $L^2((0, +\infty))$. Hence the Friedrichs realization of \mathcal{L}_1 has compact resolvent. Since the above also continuously embeds into $C^0((0, +\infty))$ and the second term in (2.4) comes with a positive sign, it is also clear that functions of the domain must vanish at the origin.

When $d = 2$, we need the following refined Hardy inequality² proved recently by Frank and Merz [28]: for $0 < \theta < 1$,

$$\int_0^{+\infty} |(-\partial_r^2)^{\theta/2} f(r)|^2 dr \leq C \int_0^{+\infty} \left| \left(-\partial_r^2 - \frac{1}{4r^2} \right)^{\theta/2} f(r) \right|^2 dr, \quad \forall f \in C_0^\infty((0, +\infty)). \quad (2.6)$$

Hardy-ineq-2

Since $x \mapsto x^\theta$ with $0 < \theta < 1$ is concave, Jensen's inequality for operators (see³ e.g. [57, Theorem 8.9]) yields

$$\mathcal{Q}(f, f) \geq C \left(\int_0^{+\infty} |(-\partial_r^2)^{\theta/2} f(r)|^2 dr \right)^{\frac{1}{\theta}} + \int_0^{+\infty} r^s |f(r)|^2 dr,$$

which implies that the bottom of the spectrum of \mathcal{L}_1 is positive. Also

$$(-\partial_r^2)^\theta \leq C \left(-\partial_r^2 - \frac{1}{4r^2} \right)^\theta$$

²The inverse inequality holds for all $0 < \theta < \frac{3}{2}$. In particular, we have the equivalence of norms for all $0 < \theta < 1$.

³In [57, Theorem 8.9], Jensen's inequality was proved in a finite dimensional setting. However, the same proof applies for the infinite dimensional case using the multiplication operator form of the spectral theorem for unbounded self-adjoint operators (see e.g. [51, Theorem VII.4]).

as operators. By the operator monotonicity of $x \mapsto x^\theta$ with $0 < \theta < 1$ (see [12, Theorem 2.6]), we infer that

$$\begin{aligned} (-\partial_r^2)^\theta &\leq C \left(-\partial_r^2 - \frac{1}{4r^2} + \frac{1}{2}r^s \right)^\theta \\ &\leq C \left(-\partial_r^2 - \frac{1}{4r^2} + \frac{1}{2}r^s \right) \end{aligned} \quad (2.7) \quad \boxed{\text{est-A-theta}}$$

or

$$-\partial_r^2 - \frac{1}{4r^2} + r^s \geq C^{-1}(-\partial_r^2)^\theta + \frac{1}{2}r^s$$

for some constant $C > 0$ depending only on the bottom of the spectrum of \mathcal{L}_1 . Here the constant C varies line-by-line. Hence, for all $0 < \theta < 1$ the quadratic form domain of \mathcal{L}_1 continuously embeds into

$$H^\theta((0, +\infty)) \cap L^2((0, +\infty), r^s dr),$$

which is compactly embedded in $L^2((0, +\infty))$ when $\theta > 1/2$. Hence \mathcal{L}_1 has compact resolvent. Since the above space also continuously embeds into continuous functions and (2.2) maps regular functions to functions vanishing at the origin, we also deduce that functions from the Friedrichs domain of \mathcal{L}_1 must vanish at the origin.

Applying the spectral theorem for compact operators (see e.g., [52, Theorem XIII.64]), there exists a sequence of eigenfunctions f_n of $(\mathcal{L}_1 + 1)^{-1}$ which forms an orthonormal basis of $L^2((0, +\infty), dr)$. Moreover, the corresponding eigenvalues satisfy $\mu_n \rightarrow 0^+$ as $n \rightarrow \infty$. Note that

$$(\mathcal{L}_1 + 1)^{-1} f_n = \mu_n f_n \Leftrightarrow \mathcal{L}_1 f_n = (\mu_n^{-1} - 1) f_n.$$

In particular, f_n is also an eigenfunction of \mathcal{L}_1 with the corresponding eigenvalue $\mu_n^{-1} - 1 \rightarrow +\infty$ as $n \rightarrow \infty$. The rest of the lemma follows by setting $e_n = r^{\frac{d-1}{2}} f_n$ and $\lambda_n^2 = \mu_n^{-1} - 1$. \square

We will now specify which Schatten space the resolvent of the operator we just constructed belongs to:

LEM:main2 **Lemma 2.2 (Schatten-norm bounds for the resolvent).**

Let $s > 0$ and $(\lambda_n^2)_{n \geq 0}$ be the eigenvalues of \mathcal{L} given in Lemma 2.1. Then we have

$$\text{Tr}[\mathcal{L}^{-\alpha}] = \sum_{n \geq 0} \lambda_n^{-2\alpha} < \infty \quad (2.8) \quad \boxed{\text{traceEst}}$$

provided $\alpha > \frac{1}{2} + \frac{1}{s}$.

Proof. The 1D case was proved in [40, Example 3.2] and [23, Lemma A.1]. In the following, we only consider the higher dimensional cases. To this end, we shall show the following Lieb–Thirring type inequality for the radial Schrödinger operator $\mathcal{L} = -\Delta + V(r)$: for $\alpha > \frac{1}{2}$,

$$\text{Tr}[\mathcal{L}^{-\alpha}] \leq C(\alpha) \int_0^{+\infty} (V(r))^{\frac{1}{2}-\alpha} dr. \quad (2.9) \quad \boxed{\text{Lieb-Thir-in}}$$

Assume (2.9) for the moment, let us prove (2.8). Since $\mathcal{L} \geq \lambda_0$, we infer that

$$\text{Tr}[\mathcal{L}^{-\alpha}] \leq 2^\alpha \text{Tr}[(\mathcal{L} + \lambda_0)^{-\alpha}],$$

where $\lambda_0 > 0$ is the first eigenvalue of \mathcal{L} . Applying (2.9) with $V(r) = r^s + \lambda_0$, we have

$$\text{Tr}[\mathcal{L}^{-\alpha}] \leq C \int_0^{+\infty} (r^s + \lambda_0)^{\frac{1}{2}-\alpha} dr < \infty$$

provided $s\alpha - \frac{s}{2} > 1$ or $\alpha > \frac{1}{2} + \frac{1}{s}$. This proves (2.8).

It remains to prove (2.9). Let us start with the following observation. Let (λ^2, f) be an eigenpair of $\mathcal{L} = -\Delta + V(r)$, namely

$$\left(-\partial_r^2 - \frac{d-1}{r}\partial_r + V(r)\right) f(r) = \lambda^2 f(r).$$

By the change of variable

$$g(r) = r^{\frac{d-1}{2}} f(r), \quad (2.10) \quad \text{chan-vari}$$

it becomes

$$\left(-\partial_r^2 + \frac{(d-1)(d-3)}{4r^2} + V(r)\right) g(r) = \lambda^2 g(r)$$

or (λ^2, g) is an eigenpair of

$$\mathcal{L}_1 := -\partial_r^2 + \frac{(d-1)(d-3)}{4r^2} + V(r)$$

acting on $L^2((0, +\infty), dr)$ with the Dirichlet boundary condition at 0. In particular, we have

$$\text{Tr}[\mathcal{L}^{-\alpha}] = \sum_{n \geq 0} \lambda_n^{-2\alpha} = \text{Tr}[\mathcal{L}_1^{-\alpha}] \quad (2.11) \quad \text{TraceL1}$$

hence the proof of (2.9) is reduced to proving

$$\text{Tr}[\mathcal{L}_1^{-\alpha}] \leq C(\alpha) \int_0^{+\infty} (V(r))^{\frac{1}{2}-\alpha} dr. \quad (2.12) \quad \text{LT-proof}$$

To show this, we rely on an idea of Dolbeault et al. [24] which is done in two steps.

Step 1. A Golden–Thompson type inequality. We first prove that: for $t > 0$,

$$\text{Tr}[e^{-t\mathcal{L}_1}] \leq Ct^{-1/2} \int_0^{+\infty} e^{-tV(r)} dr. \quad (2.13) \quad \text{Gold-Thom-in}$$

To prove (2.13), we recall the following result of Ortner and Wagner [49] concerning the fundamental solution to the heat equation with an inverse-square potential on the half line with Dirichlet condition at 0. More precisely, the unique solution to

$$\begin{cases} \left(\partial_t - \partial_r^2 + \frac{(d-1)(d-3)}{4r^2}\right) u(t, r) = 0, & t > 0, \quad r > 0, \\ u(t = 0, r) = g(r), & r > 0, \\ u(t, r = 0) = 0, & t > 0. \end{cases} \quad (2.14) \quad \text{heat-equa}$$

is given by

$$u(t, r) = \int_0^{+\infty} G(t, r, \tau) g(\tau) d\tau,$$

where

$$G(t, r, \tau) = \frac{\sqrt{r\tau}}{2t} \exp\left(-\frac{r^2 + \tau^2}{4t}\right) I_\nu\left(\frac{r\tau}{2t}\right) \quad (2.15) \quad \text{heat-kern}$$

with $\nu = \frac{d-2}{2}$ and I_ν the modified Bessel function of the first kind with index ν , namely

$$I_\nu(x) = \sum_{j \geq 0} \frac{1}{\Gamma(j + \nu + 1)j!} \left(\frac{x}{2}\right)^{2j + \nu}.$$

We have the following asymptotic behaviors of the modified Bessel function I_ν (see e.g., [48, Section 10.30]):

$$I_\nu(x) \sim \frac{1}{\Gamma(\nu + 1)} \left(\frac{x}{2}\right)^\nu \quad \text{as } x \rightarrow 0 \quad (2.16) \quad \text{I-nu-zero}$$

and

$$I_\nu(x) \sim \frac{1}{\sqrt{2\pi x}} e^x \quad \text{as } x \rightarrow +\infty. \quad (2.17) \quad \boxed{\text{I-nu-infi}}$$

Using Trotter's formula⁴ (see [35]), we have

$$e^{-t\mathcal{L}_1} = s - \lim_{n \rightarrow \infty} \left(e^{-\frac{t}{n}H} e^{-\frac{t}{n}V(r)} \right)^n,$$

where

$$H := -\partial_r^2 + \frac{(d-1)(d-3)}{4r^2}. \quad (2.18) \quad \boxed{\text{H}}$$

The integral kernel of $\left(e^{-\frac{t}{n}H} e^{-\frac{t}{n}V(r)} \right)^n$ is written as

$$K(t, r, \tau) = \int_{(0, +\infty)^{n-1}} G\left(\frac{t}{n}, r, \tau_1\right) e^{-\frac{t}{n}V(\tau_1)} G\left(\frac{t}{n}, \tau_1, \tau_2\right) e^{-\frac{t}{n}V(\tau_2)} \\ \dots G\left(\frac{t}{n}, \tau_{n-1}, \tau\right) e^{-\frac{t}{n}V(\tau)} d\tau_1 d\tau_2 \dots d\tau_{n-1}.$$

Thus $\text{Tr} \left[\left(e^{-\frac{t}{n}H} e^{-\frac{t}{n}V(r)} \right)^n \right]$ is

$$\int_0^{+\infty} K(t, \tau, \tau) d\tau = \int_{(0, +\infty)^n} G\left(\frac{t}{n}, \tau, \tau_1\right) e^{-\frac{t}{n}V(\tau_1)} G\left(\frac{t}{n}, \tau_1, \tau_2\right) e^{-\frac{t}{n}V(\tau_2)} \\ \dots G\left(\frac{t}{n}, \tau_{n-1}, \tau\right) e^{-\frac{t}{n}V(\tau)} d\tau_1 d\tau_2 \dots d\tau_{n-1} d\tau.$$

Set $\tau_0 = \tau$. We rewrite this as

$$\int_{(0, +\infty)^n} G\left(\frac{t}{n}, \tau_0, \tau_1\right) G\left(\frac{t}{n}, \tau_1, \tau_2\right) \dots G\left(\frac{t}{n}, \tau_{n-1}, \tau_0\right) \\ e^{-\frac{t}{n}(V(\tau_0)+V(\tau_1)+\dots+V(\tau_{n-1}))} d\tau_0 d\tau_1 d\tau_2 \dots d\tau_{n-1}.$$

By the convexity of $x \mapsto e^{-x}$, we have

$$e^{-\frac{t}{n}(V(\tau_0)+V(\tau_1)+\dots+V(\tau_{n-1}))} \leq \frac{1}{n} \sum_{k=0}^{n-1} e^{-tV(\tau_k)}.$$

Thus we get

$$\text{Tr}[e^{-t\mathcal{L}_1}] \leq \frac{1}{n} \sum_{k=0}^{n-1} \int_{(0, +\infty)^n} G\left(\frac{t}{n}, \tau_0, \tau_1\right) G\left(\frac{t}{n}, \tau_1, \tau_2\right) \dots G\left(\frac{t}{n}, \tau_{n-1}, \tau_0\right) e^{-tV(\tau_k)} d\tau_0 d\tau_1 \dots d\tau_{n-1}.$$

The right hand side can be rewritten as

$$\frac{1}{n} \sum_{k=0}^{n-1} \int_0^{+\infty} F(\tau_k) e^{-tV(\tau_k)} d\tau_k,$$

⁴For A, B two positive operators on a Hilbert space \mathcal{H} , then for all $t > 0$, we have

$$e^{-t(A+B)} = s - \lim_{n \rightarrow \infty} (e^{-tA/n} e^{-tB/n})^n.$$

where

$$F(\tau_k) := \int_{(0,+\infty)^{n-1}} G\left(\frac{t}{n}, \tau_0, \tau_1\right) G\left(\frac{t}{n}, \tau_1, \tau_2\right) \dots G\left(\frac{t}{n}, \tau_{n-1}, \tau_0\right) d\tau_0 d\tau_1 \dots d\tau_{k-1} d\tau_{k+1} \dots d\tau_{n-1}.$$

We recall that the heat kernel (2.15) is symmetric in r, τ , and satisfies

$$\int_0^{+\infty} G(t, r, \tau') G(s, \tau', \tau) d\tau' = G(t+s, r, \tau),$$

where the latter comes from the fact that $e^{-tH} e^{-sH} = e^{-(t+s)H}$. We thus deduce

$$\begin{aligned} F(\tau_k) &= G(t, \tau_k, \tau_k) \\ &= \frac{1}{\sqrt{2t}} \sqrt{\frac{\tau_k^2}{2t}} \exp\left(-\frac{\tau_k^2}{2t}\right) I_\nu\left(\frac{\tau_k^2}{2t}\right) \\ &\leq Ct^{-1/2}, \quad k = 0, \dots, n-1, \end{aligned}$$

where we have used (2.16) and (2.17) to get the last estimate. This shows that

$$\begin{aligned} \text{Tr}[e^{-t\mathcal{L}_1}] &\leq \frac{1}{n} \sum_{k=0}^{n-1} \int_0^{+\infty} Ct^{-1/2} e^{-tV(\tau_k)} d\tau_k \\ &\leq Ct^{-1/2} \int_0^{+\infty} e^{-tV(r)} dr \end{aligned}$$

which is (2.13).

Step 2. A Lieb–Thirring type inequality. Using the Gamma function

$$\lambda^{-\alpha} = \frac{1}{\Gamma(\alpha)} \int_0^{+\infty} e^{-t\lambda} t^{\alpha-1} dt, \quad \alpha > 0, \quad \lambda > 0, \quad (2.19) \quad \boxed{\text{Gamma}}$$

the functional calculus gives

$$\text{Tr}[\mathcal{L}_1^{-\alpha}] = \frac{1}{\Gamma(\alpha)} \int_0^{+\infty} \text{Tr}[e^{-t\mathcal{L}_1}] t^{\alpha-1} dt.$$

Using (2.13), we get

$$\begin{aligned} \text{Tr}[\mathcal{L}_1^{-\alpha}] &\leq \frac{1}{\Gamma(\alpha)} \int_0^{+\infty} \left(Ct^{-1/2} \int_0^{+\infty} e^{-tV(r)} dr \right) t^{\alpha-1} dt \\ &= \frac{C}{\Gamma(\alpha)} \int_0^{+\infty} \left(\int_0^{+\infty} e^{-tV(r)} t^{\alpha-\frac{1}{2}-1} dt \right) dr \\ &= \frac{C\Gamma\left(\alpha - \frac{1}{2}\right)}{\Gamma(\alpha)} \int_0^{+\infty} (V(r))^{\frac{1}{2}-\alpha} dr \end{aligned}$$

which proves (2.12). □

We next turn to integrability properties of the diagonal part of the resolvent's integral kernel:

LEM:main3 **Lemma 2.3** (*L^p bounds for the resolvent's integral kernel*).

Let $s > 0$ and $(\lambda_n^2, e_n)_{n \geq 0}$ be the eigenpairs of \mathcal{L} given in Lemma 2.1. Then, the Green function of \mathcal{L} satisfies

$$\mathcal{L}^{-1}(x, x) = \sum_{n \geq 0} \frac{e_n^2(x)}{\lambda_n^2} \in L^p(\mathbb{R}^d) \quad (2.20) \quad \boxed{\text{GreenEst}}$$

provided

$$\max \left\{ 1, \frac{2}{s} \right\} < p < \begin{cases} \infty & \text{if } d = 1, 2, \\ \frac{d}{d-2} & \text{if } d \geq 3. \end{cases}$$

Proof. For $d = 1$, this is contained in [41, Lemma 3.2] and [23, Lemma 3.3]. By the change of variable (2.10), we observe that

$$\mathcal{L}^{-1}(x, x) = \sum_{n \geq 0} \frac{e_n^2(x)}{\lambda_n^2} \in L^p(\mathbb{R}^d)$$

is equivalent to

$$r^{-(d-1)\left(1-\frac{1}{p}\right)} \mathcal{L}_1^{-1}(r, r) = r^{-(d-1)\left(1-\frac{1}{p}\right)} \sum_{n \geq 0} \frac{g_n^2(r)}{\lambda_n^2} \in L^p((0, +\infty), dr), \quad (2.21)$$

GreenEst-pro

where $g_n(r) = r^{\frac{d-1}{2}} e_n(r)$ are eigenfunctions of \mathcal{L}_1 .

We proceed in several steps.

Step 1. A decay property near the origin. We first show the following estimate of the Green function: for any $0 < \beta < 1$, there exists $C_\beta > 0$ such that for all $r > 0$,

$$r^{-\beta} \mathcal{L}_1^{-1}(r, r) \leq C_\beta. \quad (2.22)$$

beta

In fact, since $\mathcal{L}_1 \geq C(H + 1)$ with H as in (2.18), the operator monotonicity of $x \mapsto x^{-1}$ gives $\mathcal{L}_1^{-1} \leq C^{-1}(H + 1)^{-1}$, hence

$$\mathcal{L}_1^{-1}(r, r) \leq C^{-1}(H + 1)^{-1}(r, r). \quad (2.23)$$

boun-H1-L

We will use the Gamma function (2.19) to express the integral kernel of $(H + 1)^{-1}$ in terms of the heat kernel of e^{-tH} . More precisely, we have

$$(H + 1)^{-1} = \frac{1}{\Gamma(1)} \int_0^{+\infty} e^{-tH} e^{-t} dt$$

which implies

$$\begin{aligned} (H + 1)^{-1}(r, \tau) &= \frac{1}{\Gamma(1)} \int_0^{+\infty} G(t, r, \tau) e^{-t} dt \\ &= \frac{1}{\Gamma(1)} \int_0^{+\infty} \frac{\sqrt{r\tau}}{2t} \exp\left(-\frac{r^2 + \tau^2}{4t}\right) I_\nu\left(\frac{r\tau}{2t}\right) e^{-t} dt, \end{aligned}$$

where G is as in (2.15). We write

$$(H + 1)^{-1}(r, r) = \frac{1}{\Gamma(1)} \int_0^{+\infty} \left(\frac{r^2}{2t}\right)^{\frac{1}{2}} \exp\left(-\frac{r^2}{2t}\right) I_\nu\left(\frac{r^2}{2t}\right) e^{-t} (2t)^{-\frac{1}{2}} dt$$

and use (2.16), (2.17) to deduce

$$(H + 1)^{-1}(r, r) \leq C \int_0^{+\infty} e^{-t} t^{-\frac{1}{2}} dt < \infty.$$

In particular, we have

$$\mathcal{L}_1^{-1}(r, r) \in L^\infty((0, +\infty), dr). \quad (2.24)$$

L-infnty

Now we write

$$\begin{aligned} r^{-\beta}(H+1)^{-1}(r,r) &= \frac{1}{\Gamma(1)} \int_0^{+\infty} r^{-\beta} \left(\frac{r^2}{2t}\right)^{\frac{1}{2}} \exp\left(-\frac{r^2}{2t}\right) I_\nu\left(\frac{r^2}{2t}\right) e^{-t(2t)^{-\frac{1}{2}}} dt \\ &= \frac{1}{\Gamma(1)} \int_0^{+\infty} \left(\frac{r^2}{2t}\right)^{\frac{1}{2}-\frac{\beta}{2}} \exp\left(-\frac{r^2}{2t}\right) I_\nu\left(\frac{r^2}{2t}\right) e^{-t(2t)^{-\frac{1}{2}-\frac{\beta}{2}}} dt. \end{aligned}$$

According to the asymptotic behaviors of the modified Bessel function (see (2.16) and (2.17)), we split the integral into three parts

$$\begin{aligned} \Omega_1 &= \left\{ t \in (0, +\infty) : \frac{r^2}{2t} \leq C_1 \right\}, \\ \Omega_2 &= \left\{ t \in (0, +\infty) : C_1 \leq \frac{r^2}{2t} \leq C_2 \right\}, \\ \Omega_3 &= \left\{ t \in (0, +\infty) : \frac{r^2}{2t} \geq C_2 \right\}, \end{aligned}$$

for some $C_1 \ll 1$ and $C_2 \gg 1$. On Ω_2 , it is clear that

$$\left(\frac{r^2}{2t}\right)^{\frac{1}{2}-\frac{\beta}{2}} \exp\left(-\frac{r^2}{2t}\right) I_\nu\left(\frac{r^2}{2t}\right) \lesssim 1.$$

On Ω_1 , we use (2.16) to get

$$\left(\frac{r^2}{2t}\right)^{\frac{1}{2}-\frac{\beta}{2}} \exp\left(-\frac{r^2}{2t}\right) I_\nu\left(\frac{r^2}{2t}\right) \lesssim \left(\frac{r^2}{2t}\right)^{\frac{1}{2}-\frac{\beta}{2}+\nu} \exp\left(-\frac{r^2}{2t}\right) \lesssim 1.$$

On Ω_3 , by (2.17), we have

$$\left(\frac{r^2}{2t}\right)^{\frac{1}{2}-\frac{\beta}{2}} \exp\left(-\frac{r^2}{2t}\right) I_\nu\left(\frac{r^2}{2t}\right) \lesssim \left(\frac{r^2}{2t}\right)^{-\frac{\beta}{2}} \lesssim 1.$$

Thus we obtain

$$r^{-\beta}(H+1)^{-1}(r,r) \lesssim \int_0^{+\infty} e^{-t} t^{-\frac{1}{2}-\frac{\beta}{2}} dt < \infty$$

as $0 < \beta < 1$. This proves (2.22).

Step 2. The superharmonic case $s > 2$. By (2.8) and (2.11), we see that $\text{Tr}[\mathcal{L}_1^{-1}] < \infty$. Hence $\mathcal{L}_1^{-1}(r,r) \in L^1((0, +\infty), dr)$, which together with (2.24) implies

$$\mathcal{L}_1^{-1}(r,r) \in L^p((0, +\infty), dr), \quad \forall p \in [1, \infty]. \quad (2.25) \quad \boxed{\text{Lp-L1}}$$

We now estimate for $p > 1$,

$$\begin{aligned} \|(\cdot)^{-(d-1)\left(1-\frac{1}{p}\right)} \mathcal{L}_1^{-1}(\cdot, \cdot)\|_{L^p((0, +\infty), dr)}^p &= \int_0^{+\infty} \left(r^{-(d-1)\left(1-\frac{1}{p}\right)} \mathcal{L}_1^{-1}(r,r) \right)^p dr \\ &= \int_0^1 \left(r^{-(d-1)\left(1-\frac{1}{p}\right)} \mathcal{L}_1^{-1}(r,r) \right)^p dr \\ &\quad + \int_1^{+\infty} \left(r^{-(d-1)\left(1-\frac{1}{p}\right)} \mathcal{L}_1^{-1}(r,r) \right)^p dr. \end{aligned} \quad (2.26) \quad \boxed{\text{Lp-est-proof}}$$

The integration on $(1, +\infty)$ is finite using (2.25). On the other hand, using (2.22), we have

$$\begin{aligned} \int_0^1 \left(r^{-(d-1)\left(1-\frac{1}{p}\right)} \mathcal{L}_1^{-1}(r, r) \right)^p dr &= \int_0^1 \left(r^{-(d-1)\left(1-\frac{1}{p}\right)+\beta} r^{-\beta} \mathcal{L}_1^{-1}(r, r) \right)^p dr \\ &\lesssim \int_0^1 r^{-\left((d-1)\left(1-\frac{1}{p}\right)-\beta\right)p} dr \end{aligned}$$

which is finite as long as

$$\left((d-1) \left(1 - \frac{1}{p} \right) - \beta \right) p < 1$$

which we guarantee by picking

$$p < \frac{d}{d-1-\beta}.$$

Since β can be chosen arbitrarily in $(0, 1)$, we infer that the integration on $(0, 1)$ is ensured to be finite provided

$$1 < p < \frac{d}{d-2}. \quad (2.27)$$

Lp-est-01

Step 3. The (sub)-harmonic case $1 < s \leq 2$. It remains to show the boundedness of the integration on $(1, +\infty)$ in (2.26). To do this, we consider separately $d \geq 3$ and $d = 2$.

$d \geq 3$ We observe that

$$\mathcal{L}_1 \geq -\partial_r^2 + r^s$$

so

$$\mathcal{L}_1^{-1}(r, r) \leq (-\partial_r^2 + r^s)^{-1}(r, r). \quad (2.28)$$

est-Green-d-

To compute the Green function of $-\partial_r^2 + r^s$ on the half-line $(0, +\infty)$, we take $\varphi \in C_0^\infty((0, +\infty))$. The unique solution to

$$\begin{cases} (-\partial_r^2 + r^s)u = \varphi, & r > 0, \\ u(0) = 0, \end{cases} \quad (2.29)$$

equ-As

is given by

$$u(r) = (-\partial_r^2 + r^s)^{-1}\varphi(r) = \int_0^{+\infty} (-\partial_r^2 + r^s)^{-1}(r, \tau)\varphi(\tau)d\tau. \quad (2.30)$$

As

We use the odd extension technique to extend (2.29) to the whole line. More precisely, we denote

$$u_{\text{odd}}(r) := \begin{cases} u(r) & \text{if } r > 0, \\ -u(-r) & \text{if } r < 0, \\ 0 & \text{if } r = 0, \end{cases} \quad \varphi_{\text{odd}}(r) := \begin{cases} \varphi(r) & \text{if } r > 0, \\ -\varphi(-r) & \text{if } r < 0, \\ 0 & \text{if } r = 0, \end{cases} \quad (2.31)$$

odd-exte-u

the odd extensions of u and φ respectively. Then u_{odd} solves

$$(-\partial_r^2 + |r|^s)u_{\text{odd}} = \varphi_{\text{odd}}, \quad r \in (-\infty, +\infty)$$

which admits a unique solution given by

$$u_{\text{odd}}(r) = (-\partial_r^2 + |r|^s)^{-1}\varphi_{\text{odd}}(r) = \int_{-\infty}^{+\infty} K(r, \tau)\varphi_{\text{odd}}(\tau)d\tau,$$

where $K(r, \tau)$ is the Green function of $-\partial_r^2 + |r|^s$ on the whole line \mathbb{R} . Restricting to $(0, +\infty)$, we get for $r > 0$,

$$\begin{aligned} u(r) &= \int_{-\infty}^{+\infty} K(r, \tau) \varphi_{\text{odd}}(\tau) d\tau \\ &= \int_{-\infty}^0 K(r, \tau) (-\varphi(-\tau)) d\tau + \int_0^{+\infty} K(r, \tau) \varphi(\tau) d\tau \\ &= \int_0^{+\infty} (K(r, \tau) - K(r, -\tau)) \varphi(\tau) d\tau. \end{aligned}$$

Comparing to (2.30), we obtain

$$(-\partial_r^2 + r^s)^{-1}(r, \tau) = K(r, \tau) - K(r, -\tau), \quad r, \tau \in (0, +\infty).$$

Thanks to the fact that $K(r, r) \in L^p((-\infty, +\infty), dr)$ for all $\frac{2}{s} < p \leq \infty$ due to [41, Lemma 3.2] and

$$K(r, \tau) = \sum_{n \geq 1} \mu_n^{-2} l_n(r) l_n(\tau),$$

where (μ_n^2, l_n) is the eigenpair of $-\partial_r^2 + |r|^s$ on \mathbb{R} , we infer that

$$(-\partial_r^2 + r^s)^{-1}(r, r) \in L^p((0, +\infty), dr), \quad \forall p \in \left(\frac{2}{s}, \infty\right].$$

This together with (2.28) show that the integration on $(1, +\infty)$ is finite for all $\frac{2}{s} < p \leq \infty$. Combining with (2.27) yields $r^{-(d-1)\left(1-\frac{1}{p}\right)} H_1^{-1}(r, r) \in L^p((0, +\infty), dr)$ for all

$$\frac{2}{s} < p < \frac{d}{d-2}.$$

$d = 2$ In this case, we recall from the proof of Lemma 2.1 that for $0 < \theta < 1$

$$-\partial_r^2 - \frac{1}{4r^2} + r^s \geq C^{-1}(-\partial_r^2)^\theta + \frac{1}{2}r^s$$

for some constant $C > 0$. In particular, we have

$$\mathcal{L}_1^{-1}(r, r) \leq \left(C^{-1}(-\partial_r^2)^\theta + \frac{1}{2}r^s\right)^{-1}(r, r). \quad (2.32)$$

Using an odd extension as above, we have

$$\left(C^{-1}(-\partial_r^2)^\theta + \frac{1}{2}r^s\right)^{-1}(r, \tau) = K(r, \tau) - K(r, -\tau), \quad \forall r, \tau \in (0, +\infty),$$

where $K(r, \tau)$ is the integral kernel of $\left(C^{-1}(-\partial_r^2)^\theta + \frac{1}{2}|r|^s\right)^{-1}$ defined on \mathbb{R} . Thanks to an estimate on the Green function of the fractional Schrödinger operator with an anharmonic oscillator (see Proposition B.2), we have for $\theta \in (1/2, 1)$, $K(r, r) \in L^p(\mathbb{R})$ for all $\frac{2\theta}{s(2\theta-1)} < p \leq \infty$. This implies that

$$\left(C^{-1}(-\partial_r^2)^\theta + \frac{1}{2}r^s\right)^{-1}(r, r) \in L^p((0, +\infty), dr), \quad \forall p \in \left(\frac{2\theta}{s(2\theta-1)}, \infty\right].$$

Since θ can be taken arbitrarily in $(1/2, 1)$, we deduce from (2.32) that

$$\mathcal{L}_1^{-1}(r, r) \in L^p((0, +\infty), dr), \quad \forall p \in \left(\frac{2}{s}, \infty\right].$$

From this, we can conclude as in the case $d \geq 3$. The proof is complete. □

Lemmas 2.2 and 2.3 are crucial in the proof of Theorem 1.3 (i), i.e. the normalizability in the subcritical cases. To deal with the critical/supercritical cases, we also need tighter estimates, in particular estimates on the number of eigenvalues below a given threshold.

2.2. Weyl-type asymptotics for radial Schrödinger operators. Let

$$N(\mathcal{L}, \Lambda) := \#\{\lambda_n^2 : \lambda_n^2 \leq \Lambda\},$$

where λ_n^2 are the eigenvalues of \mathcal{L} given in Lemma 2.1. We start with the following, whose equivalent in 1D is known as the ‘‘Cwikel-Lieb-Rozenbljum’’ bound (see e.g. [23, Lemma D.1]).

Theorem 2.4 (Weyl’s law for radial Schrödinger operators).

Let $d \geq 2$ and $s > 0$. Then, for two constants $c, C > 0$

$$c\Lambda^{\frac{1}{2}+\frac{1}{s}} \leq N(\mathcal{L}, \Lambda) \leq C\Lambda^{\frac{1}{2}+\frac{1}{s}} \quad \text{as } \Lambda \rightarrow \infty. \quad (2.33)$$

We use the method of coherent states and semi-classical analysis on the phase space $\mathbb{R} \times \mathbb{R}$, whose basic ingredients we recall in Appendix A. In the following, the single integral sign stands for the integral over the configuration space \mathbb{R} and the double integral one is for the integration over the phase space $\mathbb{R} \times \mathbb{R}$.

Proof of Theorem 2.4. We have $N(\mathcal{L}, \Lambda) = N(\mathcal{L}_1, \Lambda)$, where \mathcal{L}_1 is given in (2.3). Thus, the analysis is reduced to study \mathcal{L}_1 , which we do in several steps.

Step 1. An odd extension. It is convenient to extend the operator \mathcal{L}_1 to the whole line. More precisely, let u be an eigenfunction of \mathcal{L}_1 with the eigenvalue λ^2 , i.e.,

$$-\partial_r^2 u + \frac{(d-1)(d-3)}{4r^2} u + r^s u = \lambda^2 u$$

with $u \in L^2((0, +\infty), dr)$ satisfying $u(0) = 0$. We extend u to the whole line using the odd extension, denoted u_{odd} (see (2.31)). In particular, u_{odd} is a solution to

$$-\partial_x^2 u + \frac{(d-1)(d-3)}{4|x|^2} u + |x|^s u = \lambda^2 u,$$

where $u \in L^2(\mathbb{R})$ is an odd function. In particular, $(\lambda^2, u_{\text{odd}})$ is an eigenpair of

$$\bar{\mathcal{L}}_1 = -\partial_x^2 + \frac{(d-1)(d-3)}{4|x|^2} + |x|^s$$

acting on $L^2_{\text{odd}}(\mathbb{R})$ the space of odd L^2 -functions. In addition, we have

$$N(\mathcal{L}_1, \Lambda) = N(\bar{\mathcal{L}}_1, \Lambda). \quad (2.34)$$

Step 2. A semiclassical reduction. To estimate $N(\bar{\mathcal{L}}_1, \Lambda)$, we use a suitable scaling to reduce the problem to a semiclassical one that counts number of eigenvalues of a semiclassical operator below 1. More precisely, if u is an eigenfunction of $\bar{\mathcal{L}}_1$ with the eigenvalue λ^2 , then $v(x) := u(\Lambda^{\frac{1}{s}} x)$ solves

$$\Lambda^{-1-\frac{2}{s}} \left(-\partial_x^2 + \frac{(d-1)(d-3)}{4|x|^2} \right) v + |x|^s v = \lambda^2 \Lambda^{-1} v.$$

Set

$$\hbar := \Lambda^{-\frac{1}{2}-\frac{1}{s}}, \quad \mu^2 = \lambda^2 \Lambda^{-1}. \quad (2.35)$$

Then v solves the semiclassical equation

$$\bar{\mathcal{L}}_{1,\hbar} v = \mu^2 v, \quad \bar{\mathcal{L}}_{1,\hbar} := \hbar^2 \left(-\partial_x^2 + \frac{(d-1)(d-3)}{4|x|^2} \right) + |x|^s.$$

subsec:Weyl

theo-Weyl-rad

Weyl-rad

odd-exte

hbar

In particular,

$$N(\bar{\mathcal{L}}_1, \Lambda) = N(\bar{\mathcal{L}}_{1, \hbar}, 1). \quad (2.36) \quad \boxed{\text{scaling}}$$

The study of $N(\bar{\mathcal{L}}_{1, \hbar}, 1)$ is now a semiclassical problem. To proceed further, we consider the following quantum energy:

$$E_{\hbar}^{\text{qu}} := \inf \{ \mathcal{E}_{\hbar}^{\text{qu}}[\gamma] = \text{Tr}[(\bar{\mathcal{L}}_{1, \hbar} - 1)\gamma] : \gamma \in \mathfrak{S}^1(L_{\text{odd}}^2), 0 \leq \gamma \leq \mathbf{1} \}, \quad (2.37) \quad \boxed{\text{eq: quantum e}}$$

where $\mathfrak{S}^1(L_{\text{odd}}^2(\mathbb{R}))$ is the space of trace-class operators on $L_{\text{odd}}^2(\mathbb{R})$. Let $(u_n^{\hbar})_{n \geq 0}$ be an orthonormal basis of $L_{\text{odd}}^2(\mathbb{R})$ consisting of eigenfunctions of $\bar{\mathcal{L}}_{1, \hbar}$, with associated eigenvalues $(\lambda_n^{\hbar})^2$. One readily sees that the quantum energy in (2.37) is achieved by the following fermionic density matrix

$$\gamma_{\hbar} = \sum_{n \geq 0} \mathbf{1}_{(\lambda_n^{\hbar})^2 \leq 1} |u_n^{\hbar}\rangle \langle u_n^{\hbar}|. \quad (2.38) \quad \boxed{\text{gamma-h}}$$

In particular, we have

$$\text{Tr}[\gamma_{\hbar}] = \sum_{(\lambda_n^{\hbar})^2 \leq 1} 1 = N(\bar{\mathcal{L}}_{1, \hbar}, 1).$$

Our goal is to show that

$$\sup_{\hbar \rightarrow 0} \hbar \text{Tr}[\gamma_{\hbar}] \in (0, \infty) \quad (2.39) \quad \boxed{\text{limi-trac-h}}$$

which together with the scaling (2.35) and (2.36) yield (2.33).

Step 3. A lower bound. We next aim at proving the following lower bound

$$\liminf_{\hbar \rightarrow 0} \hbar \text{Tr}[\gamma_{\hbar}] \geq C \quad (2.40) \quad \boxed{\text{lowe-boun-tr}}$$

for some constant $C > 0$.

Fix a constant $K \in (0, 1)$ and define the trial state

$$\gamma^{\text{test}} = \frac{1}{2\pi\hbar} \iint m_K(x, p) |f_{x,p}^{\hbar}\rangle \langle f_{x,p}^{\hbar}| dx dp,$$

where

$$m_K(x, p) := \mathbf{1}_{\{|p|^2 + |x|^s - 1 \leq 0\} \cap \{|x| \geq K\}} \quad (2.41) \quad \boxed{\text{m0-test}}$$

and $f_{x,p}^{\hbar}$ is the coherent state (see Appendix A for the definition). By (A.1), we see that $0 \leq \gamma^{\text{test}} \leq \mathbf{1}$ and

$$\text{Tr}[\gamma^{\text{test}}] = \frac{1}{2\pi\hbar} \iint m_K(x, p) \langle f_{x,p}^{\hbar}, f_{x,p}^{\hbar} \rangle dx dp = \frac{1}{2\pi\hbar} \iint m_K(x, p) dx dp \leq C(K, \hbar).$$

Note that

$$\begin{aligned} \iint m_K(x, p) dx dp &= \int \left(\int \mathbf{1}_{\{|p|^2 + |x|^s - 1 \leq 0\} \cap \{|x| \geq K\}} dp \right) dx \\ &= \int_{|x| \geq K} \left(\int_{\{|p| \leq (|x|^s - 1)_+^{1/2}\}} dp \right) dx \\ &= 2 \int_{|x| \geq K} (|x|^s - 1)_+^{1/2} dx < \infty, \end{aligned} \quad (2.42) \quad \boxed{\text{inte-m0}}$$

where $V_-(x) = \max\{-V(x), 0\}$ is the negative part of $V(x)$.

We will use the energy of γ^{test} as an upper bound to the quantum energy:

$$E_{\hbar}^{\text{qu}} \leq \mathcal{E}_{\hbar}^{\text{qu}}[\gamma^{\text{test}}].$$

By the Plancherel identity, we compute

$$\begin{aligned}
\hbar \text{Tr}[-\hbar^2 \partial_x^2 \gamma^{\text{test}}] &= \frac{1}{2\pi} \iint m_K(x, p) \langle f_{x,p}^\hbar, -\hbar^2 \partial_y^2 f_{x,p}^\hbar \rangle dx dp \\
&= \frac{1}{2\pi} \iint m_K(x, p) \left(\int |q|^2 |\mathcal{F}_\hbar[f_{x,p}^\hbar](q)|^2 dq \right) dx dp \\
&= \frac{1}{2\pi} \iint m_K(x, p) \left(\hbar^{-1/2} \int |q|^2 \left| \hat{f} \left(\frac{q-p}{\sqrt{\hbar}} \right) \right|^2 dq \right) dx dp \\
&= \frac{1}{2\pi} \iint m_K(x, p) \left(\int |p + \sqrt{\hbar}q|^2 |\hat{f}(q)|^2 dq \right) dx dp \\
&= \frac{1}{2\pi} \iint m_K(x, p) \left(\int (|p|^2 + 2\sqrt{\hbar}pq + \hbar|q|^2) |\hat{f}(q)|^2 dq \right) dx dp,
\end{aligned}$$

where \mathcal{F}_\hbar is the semi-classical Fourier transform (see Appendix A for the definition). We observe that

$$\int |p|^2 |\hat{f}(q)|^2 dq = |p|^2 \|\hat{f}\|_{L^2}^2 = |p|^2$$

and

$$\int |q|^2 |\hat{f}(q)|^2 dq = \|f'\|_{L^2}^2.$$

Since f is an odd function, so is \hat{f} , hence for p fixed,

$$\int pq |\hat{f}(q)|^2 dq = 0.$$

In particular, we obtain

$$\hbar \text{Tr}[-\hbar^2 \partial_x^2 \gamma^{\text{test}}] = \frac{1}{2\pi} \iint m_K(x, p) (|p|^2 + \hbar \|f'\|_{L^2}^2) dx dp. \quad (2.43) \quad \boxed{\text{iden-1}}$$

We next have

$$\hbar \text{Tr} \left[\hbar^2 \frac{(d-1)(d-3)}{4|x|^2} \gamma^{\text{test}} \right] = \frac{(d-1)(d-3)\hbar^2}{8\pi} \iint m_K(x, p) \left\langle f_{x,p}^\hbar, \frac{1}{|y|^2} f_{x,p}^\hbar \right\rangle dx dp.$$

It suffices to estimate the bracket on the region $\{|x| \geq K\}$ (see (2.41)). We have

$$\begin{aligned}
\left\langle f_{x,p}^\hbar, \frac{1}{|y|^2} f_{x,p}^\hbar \right\rangle &= \int \frac{1}{|y|^2} |f_{x,p}^\hbar(y)|^2 dy \\
&= \hbar^{-1/2} \int \frac{1}{|y|^2} \left| f \left(\frac{y-x}{\sqrt{\hbar}} \right) \right|^2 dy \\
&= \int \frac{1}{|x + \sqrt{\hbar}y|^2} |f(y)|^2 dy.
\end{aligned}$$

Since $|x| \geq K$ and f has compact support, there exists $\hbar_0 \in (0, 1]$ small such that

$$|x + \sqrt{\hbar}y| \geq |x| - \sqrt{\hbar}|y| \geq \frac{K}{2}$$

for all $\hbar \in (0, \hbar_0)$ and all $y \in \text{supp}(f)$. It follows that

$$\left\langle f_{x,p}^\hbar, \frac{1}{|y|^2} f_{x,p}^\hbar \right\rangle \leq \frac{4}{K^2} \|f\|_{L^2}^2 = \frac{4}{K^2},$$

hence

$$\hbar \text{Tr} \left[\hbar^2 \frac{(d-1)(d-3)}{4|x|^2} \gamma^{\text{test}} \right] \leq C_K \hbar^2 \iint m_K(x, p) dx dp \quad (2.44) \quad \boxed{\text{iden-2}}$$

for all $\hbar \in (0, \hbar_0)$. We also have

$$\begin{aligned} \hbar \text{Tr}[|x|^s \gamma^{\text{test}}] &= \frac{1}{2\pi} \iint m_K(x, p) \langle f_{x,p}^\hbar, |y|^s f_{x,p}^\hbar \rangle dx dp \\ &= \frac{1}{2\pi} \iint m_K(x, p) \left(\int |y|^s |f_{x,p}^\hbar(y)|^2 dy \right) dx dp \\ &= \frac{1}{2\pi} \iint m_K(x, p) \left(\hbar^{-1/2} \int |y|^s \left| f\left(\frac{y-x}{\sqrt{\hbar}}\right) \right|^2 dy \right) dx dp \\ &= \frac{1}{2\pi} \iint m_K(x, p) \left(\int |x + \sqrt{\hbar}y|^s |f(y)|^2 dy \right) dx dp. \end{aligned}$$

Using the inequality

$$|x + \sqrt{\hbar}y|^s \leq \begin{cases} |x|^s + C|\sqrt{\hbar}y|^s + C\sqrt{\hbar}|x|^{s-1}|y| & \text{if } s \geq 1, \\ |x|^s + C|\sqrt{\hbar}y|^s & \text{if } 0 < s < 1, \end{cases} \quad (2.45) \quad \boxed{\text{diff-est}}$$

we get

$$\int |x + \sqrt{\hbar}y|^s |f(y)|^2 dy \leq |x|^s + C\hbar^{s/2} \int |y|^s |f(y)|^2 dy + C\hbar^{1/2}|x|^{s-1} \int |y| |f(y)|^2 dy$$

for $s \geq 1$ and

$$\int |x + \sqrt{\hbar}y|^s |f(y)|^2 dy \leq |x|^s + C\hbar^{s/2} \int |y|^s |f(y)|^2 dy$$

for $0 < s < 1$. In the following, we consider only the case $s \geq 1$ since the one for $0 < s < 1$ is similar. In particular, we obtain for $s \geq 1$,

$$\begin{aligned} \hbar \text{Tr}[|x|^s \gamma^{\text{test}}] &\leq \frac{1}{2\pi} \iint |x|^s m_K(x, p) dx dp + C\hbar^{1/2} \| |y|^{1/2} f \|_{L^2}^2 \iint |x|^{s-1} m_K(x, p) dx dp \\ &\quad + C\hbar^{s/2} \| |y|^{s/2} f \|_{L^2}^2 \iint m_K(x, p) dx dp. \end{aligned} \quad (2.46) \quad \boxed{\text{iden-3}}$$

Collecting (2.43), (2.44), (2.46), and using the resolution of the identity (A.1) yields

$$\begin{aligned} \hbar E_\hbar^{\text{qu}} &\leq \hbar \mathcal{E}_\hbar^{\text{qu}}[\gamma^{\text{test}}] \\ &= \hbar \text{Tr}[(\bar{\mathcal{L}}_{1,\hbar} - 1)\gamma^{\text{test}}] \\ &\leq \frac{1}{2\pi} \iint (|p|^2 + |x|^s - 1) m_K(x, p) dx dp \\ &\quad + C \left(\hbar^2 + \hbar^{s/2} \| |y|^{s/2} f \|_{L^2}^2 \right) \iint m_K(x, p) dx dp \\ &\quad + C\hbar^{1/2} \| |y|^{1/2} f \|_{L^2}^2 \iint |x|^{s-1} m_K(x, p) dx dp \end{aligned} \quad (2.47) \quad \boxed{\text{trial-boun}}$$

for all $\hbar \in (0, \hbar_0)$ and some constant $C > 0$ independent of \hbar . Since (see (2.42))

$$\iint m_K(x, p) dx dp, \quad \iint |x|^{s-1} m_K(x, p) dx dp < \infty,$$

letting $\hbar \rightarrow 0$, we obtain

$$\limsup_{\hbar \rightarrow 0} \hbar E_{\hbar}^{\text{qu}} \leq \frac{1}{2\pi} \iint (|p|^2 + |x|^s - 1) m_K(x, p) dx dp. \quad (2.48) \quad \boxed{\text{uppe-boun-en}}$$

Now let γ_{\hbar} be a minimizer for E_{\hbar}^{qu} , i.e., $E_{\hbar}^{\text{qu}} = \mathcal{E}_{\hbar}^{\text{qu}}[\gamma_{\hbar}]$. Since $\bar{\mathcal{L}}_{1, \hbar} \geq 0$, we have

$$\hbar E_{\hbar}^{\text{qu}} = \hbar \mathcal{E}_{\hbar}^{\text{qu}}[\gamma_{\hbar}] \geq -\hbar \text{Tr}[\gamma_{\hbar}].$$

It follows that

$$\liminf_{\hbar \rightarrow 0} \hbar \text{Tr}[\gamma_{\hbar}] \geq -\limsup_{\hbar \rightarrow 0} \hbar E_{\hbar}^{\text{qu}} \geq -\frac{1}{2\pi} \iint (|p|^2 + |x|^s - 1) m_K(x, p) dx dp.$$

This proves (2.40) since

$$-\frac{1}{2\pi} \iint (|p|^2 + |x|^s - 1) m_K(x, p) dx dp = \frac{2}{3\pi} \int_{|x| \geq K} (|x|^s - 1)_-^{3/2} dx \in (0, \infty).$$

Step 4. An upper bound. This step is devoted to the following upper bound

$$\limsup_{\hbar \rightarrow 0} \hbar \text{Tr}[\gamma_{\hbar}] \leq C \quad (2.49) \quad \boxed{\text{uppe-boun-tr}}$$

for some constant $C > 0$. Let

$$\gamma_{\hbar} = \sum_{n \geq 0} \mu_n^{\hbar} |u_n^{\hbar}\rangle \langle u_n^{\hbar}|$$

be a minimizer for E_{\hbar}^{qu} . We first denote

$$\rho_{\hbar}(x) := \sum_n \mu_n^{\hbar} |u_n^{\hbar}(x)|^2$$

and claim that, for \hbar small enough,

$$\hbar \int |x|^s \rho_{\hbar}(x) dx \leq \hbar \int \rho_{\hbar}(x) dx. \quad (2.50) \quad \boxed{\text{densi-gamma}}$$

Indeed, thanks to the energy upper bound (2.48), there exists $\hbar_0 \in (0, 1]$ such that

$$\hbar E_{\hbar}^{\text{qu}} \leq 0, \quad \forall \hbar \in (0, \hbar_0). \quad (2.51) \quad \boxed{\text{nega-energy}}$$

Since $-\partial_x^2 + \frac{(d-1)(d-3)}{4|x|^2} \geq 0$, we have

$$\begin{aligned} 0 &\geq \hbar E_{\hbar}^{\text{qu}} = \hbar \mathcal{E}_{\hbar}^{\text{qu}}[\gamma_{\hbar}] \\ &= \hbar \text{Tr}[(\bar{\mathcal{L}}_{1, \hbar} - 1)\gamma_{\hbar}] \\ &\geq \hbar \text{Tr}[(|x|^s - 1)\gamma_{\hbar}] \\ &= \hbar \int (|x|^s - 1) \rho_{\hbar}(x) dx \end{aligned}$$

which gives the desired estimate.

We next define the Husimi function associated to γ_{\hbar} by setting

$$m_{\hbar}(x, p) := \langle f_{x,p}^{\hbar}, \gamma_{\hbar} f_{x,p}^{\hbar} \rangle$$

and refer to Appendix A for its' properties.

Now let us consider separately two cases: $d \geq 3$ and $d = 2$.

$d \geq 3$ We have

$$\hbar E_{\hbar}^{\text{qu}} = \hbar \mathcal{E}_{\hbar}^{\text{qu}}[\gamma_{\hbar}] = \hbar \text{Tr}[(\bar{\mathcal{L}}_{1, \hbar} - 1)\gamma_{\hbar}] \geq \hbar \text{Tr}[(-\hbar^2 \partial_x^2 + |x|^s - 1)\gamma_{\hbar}]. \quad (2.52) \quad \boxed{\text{boun-d-geq3-}}$$

By Plancherel's identity, we compute

$$\mathrm{Tr}[-\hbar^2 \partial_x^2 \gamma_\hbar] = \sum_{n \geq 0} \mu_n^\hbar \langle u_n^\hbar, -\hbar^2 \partial_x^2 u_n^\hbar \rangle = \int |q|^2 t_\hbar(q) dq,$$

where

$$t_\hbar(q) = \sum_{n \geq 1} \mu_n^\hbar |\mathcal{F}_\hbar[u_n^\hbar](q)|^2.$$

We also have

$$\mathrm{Tr}[(|x|^s - 1)\gamma_\hbar] = \sum_{n \geq 0} \mu_n^\hbar \int (|x|^s - 1) |u_n^\hbar(x)|^2 dx = \int (|x|^s - 1) \rho_\hbar(x) dx.$$

Thus we get

$$\hbar \mathrm{Tr}[(-\hbar^2 \partial_x^2 + |x|^s - 1)\gamma_\hbar] = \hbar \left(\int |q|^2 t_\hbar(q) dq + \int (|x|^s - 1) \rho_\hbar(x) dx \right). \quad (2.53) \quad \boxed{\text{bound-geq3}}$$

On the other hand, we have (see Lemma A.2)

$$\begin{aligned} \frac{1}{2\pi} \iint m_\hbar(x, p) |p|^2 dx dp &= \hbar \int |p|^2 t_\hbar * |g^\hbar|^2(p) dp \\ &= \hbar \int t_\hbar(q) \left(\int |p|^2 |g^\hbar(p - q)|^2 dp \right) dq \\ &= \hbar \int t_\hbar(q) \left(\hbar^{-1/2} \int |p|^2 \left| \hat{f}\left(\frac{p - q}{\sqrt{\hbar}}\right) \right|^2 dp \right) dq \\ &= \hbar \int t_\hbar(q) \left(\int (|q|^2 + 2\sqrt{\hbar}pq + \hbar|p|^2) |\hat{f}(p)|^2 dp \right) dq \\ &= \hbar \int t_\hbar(q) \left(|q|^2 + \hbar \|f'\|_{L^2}^2 \right) dq \\ &= \hbar \int t_\hbar(q) |q|^2 dq + \hbar^2 \|f'\|_{L^2}^2 \int t_\hbar(q) dq, \end{aligned}$$

where we have performed a similar calculation as in Step 3. We also have

$$\begin{aligned} \frac{1}{2\pi} \iint m_\hbar(x, p) (|x|^s - 1) dx dp &= \hbar \int (|x|^s - 1) \rho_\hbar * |f^\hbar|^2(x) dx \\ &= \hbar \int \rho_\hbar(y) \left(\int (|x|^s - 1) |f^\hbar(x - y)|^2 dx \right) dy \\ &= \hbar \int \rho_\hbar(y) \left(\int (|y + \sqrt{\hbar}x|^s - 1) |f(x)|^2 dx \right) dy \\ &= \hbar \int \rho_\hbar(y) (|y|^s - 1) dy \\ &\quad + \hbar \int \rho_\hbar(y) \left(\int |y + \sqrt{\hbar}x|^s |f(x)|^2 dx - |y|^s \right) dy, \end{aligned}$$

where we used $\|f\|_{L^2} = 1$. To estimate the second term, we use (2.45) to have

$$\begin{aligned} \left| \int |y + \sqrt{\hbar}x|^s |f(x)|^2 dx - |y|^s \right| &\leq C \hbar^{1/2} |y|^{s-1} \int |x| |f(x)|^2 dx + C \hbar^{s/2} \int |x|^s |f(x)|^2 dx \\ &\leq C \hbar^{1/2} |y|^{s-1} \| |x|^{1/2} f \|_{L^2}^2 + C \hbar^{s/2} \| |x|^{s/2} f \|_{L^2}^2. \end{aligned}$$

This shows that

$$\begin{aligned} \left| \hbar \int \rho_{\hbar}(y) \left(\int |y + \sqrt{\hbar}x|^s |f(x)|^2 dx - |y|^s \right) dy \right| &\leq C \hbar^{3/2} \| |x|^{1/2} f \|_{L^2}^2 \int |y|^{s-1} \rho_{\hbar}(y) dy \\ &\quad + C \hbar^{s/2+1} \| |x|^{s/2} f \|_{L^2}^2 \int \rho_{\hbar}(y) dy. \end{aligned}$$

Using the inequality $|y|^{s-1} \leq C(|y|^s + 1)$ and $s \geq 1$ (there is no such term if $0 < s < 1$), we infer from (2.50) that

$$\left| \hbar \int \rho_{\hbar}(y) \left(\int |y + \sqrt{\hbar}x|^s |f(x)|^2 dx - |y|^s \right) dy \right| = O(\hbar^{3/2}) \int \rho_{\hbar}(y) dy$$

for all $\hbar \in (0, \hbar_0)$, where \hbar_0 is as in (2.51) and $A_{\hbar} = O(\hbar^\alpha)$ means that $|A_{\hbar}| \leq C\hbar^\alpha$ for some constant $C > 0$ independent of \hbar . Collecting the above estimates, we have

$$\begin{aligned} \frac{1}{2\pi} \iint m_{\hbar}(x, p) (|p|^2 + |x|^s - 1) dx dp &= \hbar \left(\int t_{\hbar}(q) |q|^2 dq + \int \rho_{\hbar}(x) (|x|^s - 1) dx \right) \\ &\quad + O(\hbar^2) \int t_{\hbar}(q) dq + O(\hbar^{3/2}) \int \rho_{\hbar}(x) dx. \end{aligned} \quad (2.54)$$

boun-d-geq3-

From (2.51), (2.52), (2.53), and (2.54), we obtain

$$\begin{aligned} 0 &\geq \hbar E_{\hbar}^{\text{qu}} \geq \hbar \left(\int |q|^2 t_{\hbar}(q) dq + \int (|x|^s - 1) \rho_{\hbar}(x) dx \right) \\ &\geq \frac{1}{2\pi} \iint m_{\hbar}(x, p) (|p|^2 + |x|^s - 1) dx dp - C\hbar^2 \int t_{\hbar}(q) dq - C\hbar^{3/2} \int \rho_{\hbar}(x) dx \\ &\geq \frac{1}{2\pi} \iint m_{\hbar}(x, p) (|p|^2 + |x|^s - 1) dx dp - C\hbar \int t_{\hbar}(q) dq - C\hbar \int \rho_{\hbar}(x) dx \\ &= \frac{1}{2\pi} \iint m_{\hbar}(x, p) (|p|^2 + |x|^s - 1 - 2C) dx dp \end{aligned} \quad (2.55)$$

boun-d-geq3-

for all $\hbar \in (0, \hbar_0)$, where we used (A.2) to get the last identity.

Set

$$W(x, p) := |p|^2 + |x|^s - 1 - 2C \geq -1 - 2C, \quad \forall x, p \in \mathbb{R}.$$

From (2.55), we have

$$\begin{aligned} 0 &\geq \frac{1}{2\pi} \iint m_{\hbar}(x, p) W(x, p) dx dp \\ &= \frac{1}{2\pi} \iint_{\{W \leq 1\}} m_{\hbar}(x, p) W(x, p) dx dp + \frac{1}{2\pi} \iint_{\{W \geq 1\}} m_{\hbar}(x, p) W(x, p) dx dp \\ &\geq -\frac{1+2C}{2\pi} \iint_{\{W \leq 1\}} m_{\hbar}(x, p) dx dp + \frac{1}{2\pi} \iint_{\{W \geq 1\}} m_{\hbar}(x, p) dx dp \\ &= \frac{1}{2\pi} \iint m_{\hbar}(x, p) dx dp - \frac{1+C}{\pi} \iint_{\{W \leq 1\}} m_{\hbar}(x, p) dx dp \\ &\geq \frac{1}{2\pi} \iint m_{\hbar}(x, p) dx dp - \frac{1+C}{\pi} \iint_{\{W \leq 1\}} dx dp, \end{aligned}$$

where we used $0 \leq m_{\hbar}(x, p) \leq 1$ for all $x, p \in \mathbb{R}$. Therefore, we obtain

$$\frac{1}{2\pi} \iint m_{\hbar}(x, p) dx dp \leq \frac{1+C}{\pi} \iint_{\{W \leq 1\}} dx dp = \text{constant} \quad (2.56)$$

boun-m

for some $C > 0$ since $\{W \leq 1\}$ is a non-empty compact set of \mathbb{R}^2 . This together with (A.2) proves (2.49) when $d \geq 3$.

$\boxed{d=2}$ In this case, we first extend the refined Hardy inequality (2.6) to the whole line, i.e., for $0 < \theta < 1$,

$$\int |(-\partial_x^2)^{\theta/2} f(x)|^2 dx \leq C \int \left| \left(-\partial_x^2 - \frac{1}{4|x|^2} \right)^{\theta/2} f(x) \right|^2 dx, \quad \forall f \in C_0^\infty(\mathbb{R}) \text{ odd functions.}$$

As operators, we have from the operator monotonicity of $x \mapsto x^\theta$ with $0 < \theta < 1$ that

$$\begin{aligned} (-\hbar^2 \partial_x^2)^\theta &\leq C \left[\hbar^2 \left(-\partial_x^2 - \frac{1}{4|x|^2} \right) \right]^\theta \\ &\leq C \left[\hbar^2 \left(-\partial_x^2 - \frac{1}{4|x|^2} \right) + 1 \right]^\theta \\ &\leq C \left[\hbar^2 \left(-\partial_x^2 - \frac{1}{4|x|^2} \right) + 1 \right] \end{aligned}$$

which gives

$$\hbar^2 \left(-\partial_x^2 - \frac{1}{4|x|^2} \right) \geq C(-\hbar^2 \partial_x^2)^\theta - 1.$$

In particular, we have

$$\bar{H}_{1,\hbar} \geq C(-\hbar^2 \partial_x^2)^\theta + |x|^s - 1.$$

for some constant $C > 0$ which may change from line to line. By taking $0 < \theta < 1/2$, we get

$$\hbar E_\hbar^{\text{qu}} = \hbar \mathcal{E}_\hbar^{\text{qu}}[\gamma_\hbar] = \hbar \text{Tr}[(\bar{\mathcal{L}}_{1,\hbar} - 1)\gamma_\hbar] \geq \hbar \text{Tr}[(C(-\hbar^2 \partial_x^2)^\theta + |x|^s - 2)\gamma_\hbar]. \quad (2.57)$$

uppe-boun-2d

The same argument as in the case $d \geq 3$ applies here. The only difference is the term

$$\frac{1}{2\pi} \iint m_\hbar(x, p) |p|^{2\theta} dx dp$$

which can be treated as follows. We have

$$\begin{aligned} \frac{1}{2\pi} \iint m_\hbar(x, p) |p|^{2\theta} dx dp &= \hbar \int t_\hbar(q) \left(\int |p|^{2\theta} |g^\hbar(p-q)|^2 dp \right) dq \\ &= \hbar \int t_\hbar(q) \left(\int |q + \sqrt{\hbar} p|^{2\theta} |\hat{f}(p)|^2 dp \right) dq \\ &= \hbar \int t_\hbar(q) |q|^{2\theta} dq + O(\hbar^{1+\theta}) \int t_\hbar(q) dq. \end{aligned}$$

We now can repeat the same reasoning as in the case $d \geq 3$ (i.e., (2.55) with $|p|^{2\theta}$ in place of $|p|^2$ and -2 instead of -1) to get the upper bound (2.49) when $d = 2$.

Finally, by the lower bound (2.40) and the upper bound (2.49), we show (2.39). This completes the proof of Theorem 2.4. \square

Remark 2.1 (Precise Weyl asymptotics when $d \geq 3$).

When $d \geq 3$, the above proof actually gives the asymptotic behavior

$$\lim_{\hbar \rightarrow 0} \hbar \text{Tr}[\gamma_\hbar] = \frac{1}{2\pi} \iint m_0(x, p) dx dp, \quad (2.58)$$

preci-limi-t

where

$$m_0(x, p) = \mathbf{1}_{\{|p|^2 + |x|^s - 1 \leq 0\}}.$$

m-rad-d-geq3

To see this, we define the classical energy

$$E^{\text{cl}} := \inf \left\{ \mathcal{E}^{\text{cl}}[m] := \frac{1}{2\pi} \iint m(x, p)(|p|^2 + |x|^s - 1) dx dp : m \in L^1(\mathbb{R} \times \mathbb{R}), 0 \leq m \leq 1 \right\}.$$

By the bathtub principle (see e.g. [43, Theorem 1.14]), the unique minimizer for E^{cl} is given by the function m_0 defined above. By (2.48) and letting $K \rightarrow 0$, we have

$$\begin{aligned} \limsup_{\hbar \rightarrow 0} \hbar E_{\hbar}^{\text{qu}} &\leq \frac{1}{2\pi} \iint (|p|^2 + |x|^s - 1) m_K(x, p) dx dp \\ &\leq \frac{1}{2\pi} \iint (|p|^2 + |x|^s - 1) m_0(x, p) dx dp = E^{\text{cl}}, \end{aligned}$$

where m_0 is as in (2.41). On the other hand, from (A.2), (2.54), and (2.56), we have

$$\hbar E_{\hbar}^{\text{qu}} = \hbar \mathcal{E}_{\hbar}^{\text{qu}}[\gamma_{\hbar}] = \frac{1}{2\pi} \iint m_{\hbar}(x, p)(|p|^2 + |x|^s - 1) dx dp + O(\hbar^{1/2}) \geq E^{\text{cl}} - C\hbar^{1/2}.$$

Taking the liminf, we obtain

$$\liminf_{\hbar \rightarrow 0} \hbar E_{\hbar}^{\text{qu}} \geq E^{\text{cl}}.$$

This shows that

$$\lim_{\hbar} \hbar E_{\hbar}^{\text{qu}} = E^{\text{cl}}$$

and

$$\iint m_{\hbar}(x, p)(|p|^2 + |x|^s) dx dp \leq C$$

for all $\hbar \rightarrow 0$. In particular, m_{\hbar} is a tight minimizing sequence for E^{cl} . From this, we deduce that $m_{\hbar} \rightarrow m_0$ strongly in $L^1(\mathbb{R} \times \mathbb{R})$ and (2.58) follows.

As a corollary, we deduce an estimate on the number of eigenvalues in some spectral windows.

LEM:CLR

Corollary 2.5. *There exist $C_0, c_0 > 0$ and $k_0 > 1$ such that for all $\Lambda \geq \lambda_0^2$,*

$$c_0 \Lambda^{\frac{1}{2} + \frac{1}{s}} \leq \#\{\lambda_n^2 : \Lambda < \lambda_n^2 \leq k_0 \Lambda\} \leq C_0 \Lambda^{\frac{1}{2} + \frac{1}{s}}. \quad (2.59)$$

CLR2

Proof. From (2.33) or its' equivalent in 1D, there exist $C, c > 0$ and $\Lambda_0 > 0$ large such that

$$c \Lambda^{\frac{1}{2} + \frac{1}{s}} \leq N(\mathcal{L}, \Lambda) \leq C \Lambda^{\frac{1}{2} + \frac{1}{s}} \quad (2.60)$$

Weyl-rad-app

for all $\Lambda \geq \Lambda_0$. The same estimate still holds for $\lambda_0^2 \leq \Lambda \leq \Lambda_0$ by adjusting the constants C, c accordingly. In fact, we have

$$N(\mathcal{L}, \Lambda) \geq 1 \geq \Lambda_0^{-\frac{1}{2} - \frac{1}{s}} \Lambda^{\frac{1}{2} + \frac{1}{s}}$$

and

$$N(\mathcal{L}, \Lambda) \leq N(\mathcal{L}, \Lambda_0) \leq C \Lambda_0^{\frac{1}{2} + \frac{1}{s}} = C \left(\frac{\Lambda_0}{\Lambda} \right)^{\frac{1}{2} + \frac{1}{s}} \Lambda^{\frac{1}{2} + \frac{1}{s}} \leq C \left(\frac{\Lambda_0}{\lambda_0^2} \right)^{\frac{1}{2} + \frac{1}{s}} \Lambda^{\frac{1}{2} + \frac{1}{s}}.$$

To see (2.59), we use (2.60) to have

$$\#\{\lambda_n^2 : \Lambda < \lambda_n^2 \leq k_0 \Lambda\} \leq N(\mathcal{L}, k_0 \Lambda) \leq C k_0^{\frac{1}{2} + \frac{1}{s}} \Lambda^{\frac{1}{2} + \frac{1}{s}}$$

and

$$\begin{aligned}
\#\{\lambda_n^2 : \Lambda < \lambda_n^2 \leq k_0\Lambda\} &= N(\mathcal{L}, k_0\Lambda) - N(\mathcal{L}, \Lambda) \\
&\geq c(k_0\Lambda)^{\frac{1}{2} + \frac{1}{s}} - C\Lambda^{\frac{1}{2} - \frac{1}{s}} \\
&= \left(ck_0^{\frac{1}{2} + \frac{1}{s}} - C\right)\Lambda^{\frac{1}{2} - \frac{1}{s}} \\
&= c_0\Lambda^{\frac{1}{2} - \frac{1}{s}}
\end{aligned}$$

provided that $k_0 > 1$ is taken sufficiently large depending on C, c . \square

Let $\mathcal{L}_N = \mathbf{P}_N \mathcal{L}$, where \mathbf{P}_N is given in (1.9). Then, we have

$$\mathrm{Tr}[\mathcal{L}_N^{-p}] = \sum_{n=0}^N \lambda_n^{-2p}.$$

Similarly, we define $\mathcal{L}_N^\perp = \mathcal{L} - \mathbf{P}_N \mathcal{L}$ and thus

$$\mathrm{Tr}[(\mathcal{L}_N^\perp)^{-p}] = \sum_{n=N+1}^{\infty} \lambda_n^{-2p}.$$

The following corollaries are crucial in proving the non-normalizability.

COR:CLR

Corollary 2.6 (Behavior of truncated Schatten norms).

Let $d \geq 1, s > 0$ and $p > 0$. We have

$$\mathrm{Tr}[\mathcal{L}_N^{-p}] = \sum_{n=0}^N \lambda_n^{-2p} \sim \begin{cases} 1 & \text{if } p > \frac{1}{2} + \frac{1}{s}, \\ (\log \lambda_N)^2 & \text{if } p = \frac{1}{2} + \frac{1}{s}, \\ \lambda_N^{-2p+1+\frac{2}{s}} & \text{if } p < \frac{1}{2} + \frac{1}{s}. \end{cases} \quad (2.61) \quad \text{traceN}$$

Proof. When $p > \frac{1}{2} + \frac{1}{s}$, (2.61) follows from Lemma 2.2 and (2.8). In the following, we assume that $p \leq \frac{1}{2} + \frac{1}{s}$. We first decompose the interval $[\lambda_0, \lambda_N]$ dyadically and then apply the Weyl bound (2.59) to get

$$\begin{aligned}
\mathrm{Tr}[\mathcal{L}_N^{-p}] &= \sum_{n=0}^N \lambda_n^{-2p} = \sum_{l=1}^{c \log \lambda_N} \sum_{k_0^{-l} \lambda_N^2 < \lambda_n^2 \leq k_0^{-l+1} \lambda_N^2} \lambda_n^{-2p} \\
&\sim \sum_{l=1}^{c \log \lambda_N} (k_0^{-l} \lambda_N^2)^{-p} \cdot \#\{\lambda_n^2 : k_0^{-l} \lambda_N^2 < \lambda_n^2 \leq k_0^{-l+1} \lambda_N^2\} \\
&\sim \sum_{l=1}^{c \log \lambda_N} (k_0^{-l} \lambda_N^2)^{-p} (k_0^{-l} \lambda_N^2)^{\frac{1}{2} + \frac{1}{s}} \\
&\sim \begin{cases} (\log \lambda_N)^2 & \text{if } p = \frac{1}{2} + \frac{1}{s}, \\ \left(\sum_{l=1}^{c \log \lambda_N} k_0^{-l(\frac{1}{2} + \frac{1}{s} - p)}\right) \cdot \lambda_N^{-2p+1+\frac{2}{s}} & \text{if } p < \frac{1}{2} + \frac{1}{s}, \end{cases}
\end{aligned}$$

which gives the desired estimate. \square

We also need the following:

COR:CLR1

Corollary 2.7 (Tail estimate).

Let $d \geq 1$, $s > 0$ and $p > \frac{1}{2} + \frac{1}{s}$. Then, we have

$$\mathrm{Tr}[(\mathcal{L}_N^\perp)^{-p}] = \sum_{n=N+1}^{\infty} \lambda_n^{-2p} \sim \lambda_N^{-2p+1+\frac{2}{s}}. \quad (2.62) \quad \text{traceN1}$$

Proof. We have from (2.59) that

$$\begin{aligned} \mathrm{Tr}[(\mathcal{L}_N^\perp)^{-p}] &= \sum_{n=N+1}^{\infty} \lambda_n^{-2p} \sim \sum_{l=0}^{\infty} \sum_{k_0^l \lambda_N^2 < \lambda_n^2 \leq k_0^{l+1} \lambda_N^2} \lambda_n^{-2p} \\ &\sim \sum_{l=0}^{\infty} (k_0^l \lambda_N^2)^{-p} \cdot \#\{\lambda_n^2 : k_0^l \lambda_N^2 < \lambda_n^2 \leq k_0^{l+1} \lambda_N^2\} \\ &\sim \sum_{l=0}^{\infty} (k_0^l \lambda_N^2)^{-p} (k_0^l \lambda_N^2)^{\frac{1}{2}+\frac{1}{s}} \\ &\sim \lambda_N^{-2p+1+\frac{2}{s}}, \end{aligned} \quad (2.63)$$

provided $p > \frac{1}{2} + \frac{1}{s}$. □

We finally state the following observation.

LEM:asym

Lemma 2.8 (Polynomial growth of eigenvalues).

Let $d \geq 1$, $s > 0$ and $k \in \mathbb{N}$. Then, there exists $c(s) > 0$ such that

$$\lim_{N \rightarrow \infty} \frac{\lambda_{kN}}{\lambda_N^{c(s)}} = 0.$$

Proof. It comes from the fact that

$$N = N(\mathcal{L}, \lambda_N^2) \sim \lambda_N^{1+\frac{2}{s}} \text{ or } \lambda_N \sim N^{\frac{s}{2+s}} \text{ as } N \rightarrow \infty. \quad \square$$

3. VARIATIONAL FORMULATION

Most of our arguments proceed from a variational formula for the partition function that we recall from [5]. This relies on the Boué-Dupuis variational formula [5, 62, 64, 4, 38], which has already been used extensively to define and characterize nonlinear Gibbs measures in related contexts, see e.g. [2, 45, 46].

We wish to determine whether the partition function

$$\mathcal{Z}_K := \int \mathbf{1}_{\{|M(u)| \leq K\}} e^{\alpha R_p(u)} d\mu(u)$$

is finite or not, where μ is the Gaussian measure with covariance \mathcal{L}^{-1} , restricted to radial functions when $d \geq 2$. Since (see e.g. [5, Proposition 2.5])

$$-\log \mathcal{Z}_K =$$

$$\inf \left\{ \int (-\alpha R_p(u) + \log(f(u))) f(u) d\mu_K(u) \mid f \geq 0, f \in L^1(d\mu_K), \int f(u) d\mu_K(u) = 1 \right\} \quad (3.1) \quad \text{eq:freeener}$$

with $d\mu_K(u) = \mathbf{1}_{\{|M(u)| \leq K\}} d\mu(u)$, the finiteness of \mathcal{Z}_K is related to some free energy being bounded from below. Namely, the question is whether the positive entropy term (relative to the Gaussian measure) in

the above is sufficient to compensate the possibly very negative potential energy term. The Boué-Dupuis formula will help us decide that question by providing a more wieldy formulation of the entropy term in the variational principle above. This requires introducing an extra time variable and seeing the Gaussian measure μ as the law of a Brownian motion at time 1. Then, Girsanov's theorem provides a description of random processes whose laws are absolutely continuous with respect to that of the Brownian motion, and the Boué-Dupuis formula an expression of the latter's entropies relative to the Wiener measure. This turns (3.1) into a control problem. We set this up briefly in Section 3.1 below, sketching how this is of use for our problem. In Section 3.2, we collect some consequences of the bounds of Section 2 which legitimate the application of the formula to our context.

sec:BouDup1

3.1. **The Boué-Dupuis formula and its' use.** Let $W(t)$ denote a cylindrical Brownian motion in $L^2(\mathbb{R}^d)$ defined by

$$W(t) = \sum_{n \geq 0} B_n(t) e_n, \tag{3.2} \quad \text{Bro}$$

where $\{e_n\}_{n \geq 0}$ is the sequence of normalized eigenfunctions of the operator \mathcal{L} given in Lemma 2.1, and $\{B_n\}_{n \geq 0}$ is a sequence of mutually independent complex-valued Brownian⁵ motions. We define a centered Gaussian process $Y(t)$ by

$$Y(t) = \mathcal{L}^{-\frac{1}{2}} W(t) = \sum_{n \geq 0} \frac{B_n(t)}{\lambda_n} e_n. \tag{3.3} \quad \text{Yt}$$

Then, $Y(t)$ is well-defined in $\mathcal{H}^{-\sigma}(\mathbb{R}^d)$ for any $\sigma > \frac{1}{s} - \frac{1}{2}$, see Corollary 3.4 below. But $Y(t)$ is not in $L^2(\mathbb{R}^d)$ when $1 < s \leq 2$, since we have

$$\mathbb{E}[\|Y(t)\|_{L^2(\mathbb{R}^d)}^2] \sim \sum_{n \geq 0} \frac{\mathbb{E}[|B_n(t)|^2]}{\lambda_n^2} = 2t \sum_{n \geq 0} \lambda_n^{-2} \begin{cases} < \infty & \text{if } s > 2, \\ = \infty & \text{if } s \leq 2 \text{ (unless } t = 0). \end{cases}$$

From (3.3), we see that

$$\text{Law}(Y(1)) = \mu, \tag{3.4} \quad \text{law}$$

where μ is the Gaussian free field given in (1.12).

Let \mathbb{H}_a be the space of drifts, which consists of progressively measurable processes belonging to $L^2([0, 1]; L^2(\mathbb{R}^d))$, \mathbb{P} -almost surely. One of the key tools in this paper is the following Boué-Dupuis variational formula [5, Theorem 5.1] (see also [62] and [64]).

LEM:var

Lemma 3.1 (Boué-Dupuis variational formula).

Let $Y(t)$ be as in (3.3). For $\sigma > \frac{1}{s} - \frac{1}{2}$, let $F : \mathcal{H}^{-\sigma}(\mathbb{R}^d) \rightarrow \mathbb{R}$ be a Borel measurable function that is bounded from above. Then, we have

$$-\log \mathbb{E} \left[e^{-F(Y(1))} \right] = \inf_{\theta \in \mathbb{H}_a} \mathbb{E} \left[F(Y(1) + I(\theta)(1)) + \frac{1}{2} \int_0^1 \|\theta(t)\|_{L^2(\mathbb{R}^d)}^2 dt \right], \tag{3.5} \quad \text{var}$$

where $I(\theta)$ is defined by

$$I(\theta)(t) = \int_0^t \mathcal{L}^{-\frac{1}{2}} \theta(\tau) d\tau$$

and the expectation $\mathbb{E} = \mathbb{E}_{\mathbb{P}}$ is with respect to the underlying probability measure \mathbb{P} .

⁵Essentially, we have $B_n(t) \sim \mathcal{N}_{\mathbb{C}}(0, 2t)$, its' density is given by $\frac{1}{2\pi t} e^{-\frac{|z|^2}{2t}}$.

Proof. The fact that (3.5) holds when

$$\inf_{\theta \in \mathbb{H}_a} \mathbb{E} \left[F(Y(1) + I(\theta)(1)) + \frac{1}{2} \int_0^1 \|\theta(t)\|_{L^2(\mathbb{R}^d)}^2 dt \right] > -\infty \quad (3.6) \quad \boxed{\text{varRHS}}$$

follows with same proof as [60, Proposition A.2]. Therefore, we only need to show equality when (3.6) = $-\infty$ instead. Let $M > 0$. By monotone convergence, there exists $\theta_M \in \mathbb{H}_a$ and $L > 0$ such that

$$\mathbb{E} \left[\max \left(F(Y(1) + I(\theta_M)(1)), -L \right) + \frac{1}{2} \int_0^1 \|\theta_M(t)\|_{L^2(\mathbb{R}^d)}^2 dt \right] < -M.$$

Therefore, we have that

$$\begin{aligned} -\log \mathbb{E} \left[e^{-F(Y(1))} \right] &\leq -\log \mathbb{E} \left[e^{-\max(F(Y(1)), L)} \right] \\ &= \inf_{\theta \in \mathbb{H}_a} \mathbb{E} \left[\max \left(F(Y(1) + I(\theta)(1)), -L \right) + \frac{1}{2} \int_0^1 \|\theta(t)\|_{L^2(\mathbb{R}^d)}^2 dt \right] \\ &\leq \mathbb{E} \left[\max \left(F(Y(1) + I(\theta_M)(1)), -L \right) + \frac{1}{2} \int_0^1 \|\theta_M(t)\|_{L^2(\mathbb{R}^d)}^2 dt \right] \\ &< -M. \end{aligned}$$

Since M is arbitrary, we conclude that

$$-\log \mathbb{E} \left[e^{-F(Y(1))} \right] = -\infty = \inf_{\theta \in \mathbb{H}_a} \mathbb{E} \left[F(Y(1) + I(\theta)(1)) + \frac{1}{2} \int_0^1 \|\theta(t)\|_{L^2(\mathbb{R}^d)}^2 dt \right].$$

□

We shall use the above with

$$F(Y(1)) = -\alpha R_p(Y_N(1)) \mathbf{1}_{\{|M(Y_N(1))| \leq K\}} \quad (3.7) \quad \boxed{\text{eq:usevar}}$$

or a close variant, where we set (see (1.9) for the definition of \mathbf{P}_N)

$$Y_N(1) = \mathbf{P}_N Y(1). \quad (3.8) \quad \boxed{\text{eq:YN}}$$

This is a legitimate choice, as per the estimates provided in Section 3.2 below. A lower bound to the infimum in (3.5), uniform in N , will imply finiteness of \mathcal{Z}_K and hence normalizability of the interacting Gibbs measure, while an upper bound diverging to $-\infty$ when $N \rightarrow \infty$ will imply $\mathcal{Z}_K = +\infty$, hence non-normalizability.

Proving normalizability. To bound (3.5) from below, we must show that the second term is, for any drift, large enough to compensate the first one. Clearly, from Minkowski's inequality, we have that $I(\theta)(1)$ enjoys the following pathwise regularity bound:

LEM:bounds

Lemma 3.2 (Pathwise regularity).

For any $\theta \in \mathbb{H}_a$, we have

$$\|I(\theta)(1)\|_{\mathcal{H}^1(\mathbb{R}^d)}^2 \leq \int_0^1 \|\theta(t)\|_{L^2(\mathbb{R}^d)}^2 dt. \quad (3.9) \quad \boxed{\text{I}}$$

Hence the general scheme is essentially to bound

$$F(Y_N(1) + \mathbf{P}_N I(\theta)(1)) \geq -\frac{1}{2} \|I(\theta)(1)\|_{\mathcal{H}^1(\mathbb{R}^d)}^2 - G(Y_N(1))$$

where $G(Y_N(1))$ is an expression involving different quantities (e.g. Lebesgue or Sobolev norms) only related to the Gaussian process $Y_N(1)$. This uses a (case-dependent) suitable mix of functional inequalities and triangle inequalities. Inserting this and Lemma 3.2 in (3.5) will yield normalizability provided

$$\mathbb{E} [G(Y_N(1))] < \infty$$

uniformly in N , i.e. essentially

$$\int G(u) d\mu(u) < \infty$$

with μ the original Gaussian measure. We obtain such conclusions with a suitable selection of estimates from Section 2.

Proving non-normalizability. To bound (3.5) from above, a suitable drift $\theta(t)$ is constructed, making the first term diverge to $-\infty$, while keeping the second one under control. In practice both terms will diverge, so that one needs a construction making the first one diverge faster. Under our assumptions, there exists blow-up profiles f_N such that

$$F(f_N) \xrightarrow{N \rightarrow \infty} -\infty.$$

We pick one and construct a drift θ such that

$$Y_N(1) + \mathbf{P}_N I(\theta)(1) \simeq f_N + \text{lower order}.$$

In particular, it is tempting to let

$$\theta(t) = \mathcal{L}^{1/2} (-\partial_t Y_N(t) + f_N)$$

so that

$$I(\theta)(t) = -Y_N(t) + f_N.$$

Of course, the time-derivative of the Brownian motion in the above is problematic, leading to the second term in (3.5) being $+\infty$ for such a choice. The idea is thus to approximate $Y_N(t)$ by a smoother process $Z_N(t)$ and let

$$\theta(t) := \mathcal{L}^{1/2} (-\partial_t Z_N(t) + f_N)$$

be our trial drift. With a suitable construction we ensure that, for this choice of θ , the first term of (3.5) diverges to $-\infty$ faster than the second term diverges to $+\infty$. Namely we introduce, following ideas from [60].

Definition 3.3 (Approximate Brownian motion).

Given $N \geq M \gg 1$, we define the process Z_M by its coefficients in the eigenfunction expansion $(e_n)_{n \geq 0}$ of (the radial restriction of) \mathcal{L} .

For $n \leq M$ let $\tilde{Z}_M(n, t)$ be a solution of the following differential equation:

$$\begin{cases} d\tilde{Z}_M(n, t) = c \lambda_n^{-1} \lambda_M (\tilde{Y}_N(n, t) - \tilde{Z}_M(n, t)) dt \\ \tilde{Z}_M|_{t=0} = 0, \end{cases} \quad (3.10) \quad \boxed{\text{ZZZ}}$$

where $c > 0$ is a constant to be chosen later on and⁶

$$\tilde{Y}_N(n, t) = \int_{\mathbb{R}^d} Y_N(t, x) e_n(x) dx.$$

We set $\tilde{Z}_M(n, t) \equiv 0$ for $n > M$ and define

$$Z_M(t, x) := \sum_{n \leq M} \tilde{Z}_M(n, t) e_n(x)$$

⁶We use our liberty of choosing a real eigenbasis for \mathcal{L} .

:gaussapprox

which is a centered Gaussian process in $L^2(\mathbb{R}^d)$ and satisfies $\mathbf{P}_N Z_M = Z_M$.

Let us briefly explain why the above yields a suitable approximation to the Gaussian process $Y(t)$ defined in (3.3). For illustration, we only consider a one-dimensional Gaussian variable

$$Y(t) = \sigma W(t),$$

where $W(t)$ is the standard Brownian motion. Thus $Y = Y(1) \sim \mathcal{N}(0, 2\sigma^2)$ is a Gaussian variable with variance $2\sigma^2$. Using a one-dimensional version of Lemma 3.1, we have

$$-\log(\mathbb{E}[e^{-F(Y)}]) = \sup_{Z \in \mathbb{H}_a^1} \mathbb{E} \left[F(Y(1) - Z(1)) - \frac{\sigma^2}{2} \int_0^1 |\dot{Z}(s)|^2 ds \right],$$

where \mathbb{H}_a^1 is a set of stochastic processes defined as

$$\mathbb{H}_a^1 = \{ Z : Z(0) = 0, \dot{Z} \in L^2([0, 1] \times \Omega), \text{ and } Z \text{ is progressively measurable} \}.$$

We want to construct a stochastic process $Z \in \mathbb{H}_a^1$ such that $Y(1) - Z(1)$ is small, while keeping $\int_0^1 |\dot{Z}(s)|^2 ds$ under control. For this purpose, we postulate an ansatz for the difference

$$X(t) = Y(t) - Z(t) = \sigma W(t) - Z(t)$$

being the Itô process

$$dX = -AX dt + \sigma dW, \tag{3.11} \quad \text{app}$$

or

$$X(t) = \int_0^t e^{-A(t-s)} \sigma dW(s)$$

with $A \gg 1$. Then X is “small” in the sense that

$$\mathbb{E}[|X(t)|^2] \sim \sigma^2/A. \tag{3.12} \quad \text{eq: good}$$

On the other hand

$$\mathbb{E} \left[\int_0^1 |\dot{Z}(s)|^2 ds \right] = A^2 \mathbb{E} \left[\int_0^1 |\dot{X}(s)|^2 ds \right] \sim \sigma^2 A \tag{3.13} \quad \text{eq: bad}$$

Rewriting (3.11) as

$$dZ = -A(Y - Z)dt.$$

serves as inspiration for our choice in (3.10). The parameters A and σ (or their variants) should be determined by our later analysis, to balance the competing effects of (3.12) and (3.13) when they are inserted in the variational principle.

3.2. Preliminary estimates. With the above notation, we have the following consequence of Lemmas 2.2 and 2.3. They will help us vindicating that the choice (3.7) can indeed be inserted in Lemma 3.1.

Corollary 3.4 (Regularity and integrability of the Gaussian process).

Let $d \geq 1$, $s > 0$ and assume the radial condition when $d \geq 2$. The following statements hold:

(i) Let $1 \leq q < \infty$ and

$$\max \left\{ 2, \frac{4}{s} \right\} < p < \begin{cases} \infty & \text{if } d = 1, 2, \\ \frac{2d}{d-2} & \text{if } d \geq 3. \end{cases}$$

Then, we have

$$\mathbb{E}[\|Y_N(1)\|_{L^p(\mathbb{R}^d)}^q] \lesssim_{d,p,q} 1,$$

sec:BouDup2

COR:intp

where the constant depends only on d, p, q . In particular, $Y_N(1)$ is a Cauchy sequence in $L^q(\Omega, L^p(\mathbb{R}^d))$ and

$$\mathbb{E} \left[\|Y(1) - Y_N(1)\|_{L^p(\mathbb{R}^d)}^q \right] \rightarrow 0 \text{ as } N \rightarrow \infty.$$

(ii) Let $\delta > -\frac{1}{2} + \frac{1}{s}$ and $1 \leq q < \infty$. Then, we have

$$\mathbb{E} \left[\|Y_N(1)\|_{\mathcal{H}^{-\delta}(\mathbb{R}^d)}^q \right] \lesssim_{d,q} 1.$$

In addition, $Y_N(1)$ is a Cauchy sequence in $L^p(\Omega, \mathcal{H}^{-\delta}(\mathbb{R}^d))$ and

$$\mathbb{E} \left[\|Y(1) - Y_N(1)\|_{\mathcal{H}^{-\delta}(\mathbb{R}^d)}^q \right] \lesssim_{d,q} \lambda_N^{-\left(\frac{1}{2} + \delta - \frac{1}{s}\right)q}.$$

Proof. (i) We only consider the case $q \geq p$ since the one where $1 \leq q < p$ follows from Hölder's inequality. By the Minkowski with $q \geq p$ and the Khintchine inequality (see e.g., [11, Lemma 4.2]) and Lemma 2.3, we have

$$\begin{aligned} \mathbb{E} \left[\|Y_N(1)\|_{L^p(\mathbb{R}^d)}^q \right] &= \left\| \left\| \sum_{n=0}^N \frac{B_n(1)}{\lambda_n} e_n \right\|_{L^p(\mathbb{R}^d)} \right\|_{L^q(\Omega)}^q \\ &\leq \left\| \left\| \sum_{n=0}^N \frac{B_n(1)}{\lambda_n} e_n \right\|_{L^q(\Omega)} \right\|_{L^p(\mathbb{R}^d)}^q \\ &\leq \left(C(q) \left\| \left\| \sum_{n=0}^N \frac{B_n(1)}{\lambda_n} e_n \right\|_{L^2(\Omega)} \right\|_{L^p(\mathbb{R}^d)} \right)^q \\ &= \left(C(q) \left\| \left(\sum_{n=0}^N \frac{e_n^2}{\lambda_n^2} \right)^{1/2} \right\|_{L^p(\mathbb{R}^d)} \right)^q \\ &= \left(C(q) \left(\int_{\mathbb{R}^d} \left(\sum_{n=0}^N \frac{e_n^2(x)}{\lambda_n^2} \right)^{p/2} dx \right)^{1/p} \right)^q \\ &\leq \left(C(q) \left(\int_{\mathbb{R}^d} (\mathcal{L}^{-1}(x, x))^{p/2} dx \right)^{1/p} \right)^q \\ &\leq C(d, p, q) \end{aligned}$$

provided

$$\max \left\{ 1, \frac{2}{s} \right\} < p < \begin{cases} \infty & \text{if } d = 1, 2, \\ \frac{d}{d-2} & \text{if } d \geq 3. \end{cases}$$

For $M > N$, we have

$$\begin{aligned} \mathbb{E} \left[\|Y_M(1) - Y_N(1)\|_{L^p(\mathbb{R}^d)}^q \right] &\leq \left(C(q) \left(\int_{\mathbb{R}^d} \left(\sum_{n=N+1}^M \frac{e_n^2(x)}{\lambda_n^2} \right)^{p/2} dx \right)^{1/p} \right)^q \\ &\leq \left(C(q) \left(\int_{\mathbb{R}^d} \left(\sum_{n=N+1}^{\infty} \frac{e_n^2(x)}{\lambda_n^2} \right)^{p/2} dx \right)^{1/p} \right)^q \end{aligned}$$

which converges to zero when $N \rightarrow \infty$ by the dominated convergence theorem.

(ii) It suffices to consider $q \geq 2$ since the one where $1 \leq q < 2$ follows from Hölder's inequality and the case $q = 2$. By the Minkowski inequality and the Khintchine inequality, we have from Lemma 2.2 that

$$\begin{aligned} \mathbb{E} \left[\|Y_N(1)\|_{H^{-\delta}(\mathbb{R}^d)}^q \right] &= \mathbb{E} \left[\left\| \sum_{n=0}^N \frac{B_n(1)}{\lambda_n^{1+\delta}} e_n \right\|_{L^2(\mathbb{R}^d)}^q \right] \\ &\leq \left\| \left\| \sum_{n=0}^N \frac{B_n(1)}{\lambda_n^{1+\delta}} e_n \right\|_{L^q(\Omega)} \right\|_{L^2(\mathbb{R}^d)}^q \\ &\leq C(q) \left\| \left(\sum_{n=0}^N \frac{e_n^2(x)}{\lambda_n^{2+2\delta}} \right)^{1/2} \right\|_{L^2(\mathbb{R}^d)}^q \\ &= C(q) \left(\sum_{n=0}^N \lambda_n^{-2(1+\delta)} \right)^{q/2} \\ &\leq C(d, q) \end{aligned}$$

provided $1 + \delta > \frac{1}{2} + \frac{1}{s}$ or $\delta > -\frac{1}{2} + \frac{1}{s}$. In a similar manner, for $M > N$, we have

$$\begin{aligned} \mathbb{E} \left[\|Y_M(1) - Y_N(1)\|_{H^{-\delta}(\mathbb{R}^d)}^q \right] &\leq C(q) \left(\sum_{n=N+1}^M \lambda_n^{-2(1+\delta)} \right)^{q/2} \\ &\leq C(q) \left(\sum_{n=N+1}^{\infty} \lambda_n^{-2(1+\delta)} \right)^{q/2} \\ &\leq C(q) \lambda_N^{-\left(\frac{1}{2} + \delta - \frac{1}{s}\right)q}, \end{aligned}$$

where we have used Corollary 2.7 to get the last estimate. \square

COR: WCE

Corollary 3.5 (Integrability of Wick renormalized mass).

Let $d \geq 1$, $s > \frac{2}{3}$, $p \geq 1$ and assume the radial condition when $d \geq 2$. Then, we have

$$\left\| \int_{\mathbb{R}^d} :|Y_N(1)|^2: dx \right\|_{L^p(\Omega)} \lesssim_{d,p} 1.$$

where the constant depends only on d, p . Moreover, the sequence $\int_{\mathbb{R}^d} :|Y_N(1)|^2: dx$ is Cauchy in $L^p(\Omega)$ and

$$\left\| \int_{\mathbb{R}^d} :|Y(1)|^2: dx - \int_{\mathbb{R}^d} :|Y_N(1)|^2: dx \right\|_{L^p(\Omega)} \lesssim_{d,p} \lambda_N^{-\frac{3}{2} + \frac{1}{s}}.$$

Proof. By the Hölder inequality, it suffices to consider the case $p \geq 2$. Using the Wiener chaos estimate for $p \geq 2$ (see e.g., [57, Theorem I.22] or [59, Proposition 2.4]), we have

$$\left\| \int_{\mathbb{R}^d} :|Y_N(1)|^2: dx \right\|_{L^p(\Omega)} \lesssim_p \left\| \int_{\mathbb{R}^d} :|Y_N(1)|^2: dx \right\|_{L^2(\Omega)}.$$

Then from the definition of Wick order (1.14) and (1.13), it follows that

$$\begin{aligned} \int_{\mathbb{R}^d} :|Y_N(1)|^2: dx &= \int_{\mathbb{R}^d} |Y_N(1)|^2 dx - \int_{\mathbb{R}^d} \sigma_N(x) dx \\ &= \sum_{n=0}^N \frac{|B_n(1)|^2 - 2}{\lambda_n^2}. \end{aligned} \tag{3.14} \quad \boxed{\text{wick}Y}$$

Since the random variable $\{B_n(1)\}$ are normalized and independent, we have

$$\mathbb{E}[(|B_n(1)|^2 - 2)^2] = \mathbb{E}[|B_n(1)|^4] - 4[|B_n(1)|^2] + 4 = 4$$

and

$$\mathbb{E}[(|B_{n_1}(1)|^2 - 2)(|B_{n_2}(1)|^2 - 2)] = 0$$

for all $n_1 \neq n_2$. Therefore, using Corollary 2.6, we have

$$\begin{aligned} \mathbb{E} \left[\left| \int_{\mathbb{R}^d} :|Y_N(1)|^2 : dx \right|^2 \right] &= \sum_{n=0}^N \frac{\mathbb{E}[(|B_n(1)|^2 - 2)^2]}{\lambda_n^4} \\ &= \sum_{n=0}^N \frac{4}{\lambda_n^4} < \infty \end{aligned}$$

provided $2 > \frac{1}{2} + \frac{1}{s}$ or $s > \frac{2}{3}$.

We now move to the second part. Proceeding similarly, for $M > N$, we have from Corollary 2.7 that

$$\left\| \int_{\mathbb{R}^d} :|Y_M(1)|^2 : dx - \int_{\mathbb{R}^d} :|Y_N(1)|^2 : dx \right\|_{L^p(\Omega)}^p \lesssim_p \left(\sum_{n=N+1}^M \lambda_n^{-4} \right)^{\frac{p}{2}} \lesssim \left(\lambda_N^{-\frac{3}{2} + \frac{1}{s}} \right)^p,$$

which shows that the sequence $\int_{\mathbb{R}^d} :|Y_N(1)|^2 : dx$ is Cauchy in $L^p(\Omega)$ for $s > \frac{2}{3}$. We conclude the proof by taking a limit as $M \rightarrow \infty$. \square

4. SUBHARMONIC POTENTIAL

This section concerns the normalizability and non-normalizability of the focusing Gibbs measure with subharmonic potential $s < 2$.

SEC:nor

4.1. Normalizability. In this subsection, we show the integrability part of Theorem 1.3. The conclusion will be obtained in Section 4.1.3 below from the main estimate we now state, in the form of a bound on $\mathcal{Z}_{K,N}$:

Lemma 4.1 (Exponential Integrability).

Let $s < 2$ and assume

$$\begin{aligned} \text{either (i) } \frac{4}{s} < p < 2 + \frac{4s}{(d-1)s+2}; \\ \text{or (ii) } p = 2 + \frac{4s}{(d-1)s+2} \text{ in the weakly nonlinear regime.} \end{aligned} \tag{4.1} \quad \text{condition1}$$

Then

$$\sup_N \mathbb{E}_\mu \left[\exp(\alpha R_p(u_N)) \cdot \mathbf{1}_{\{ \int_{\mathbb{R}^d} :|u_N(x)|^2 : dx \leq K \}} \right] < \infty, \tag{4.2} \quad \text{var1d}$$

where $u_N = \mathbf{P}_N u$ and $R_p(u_N)$ is the potential energy denoted by

$$R_p(u_N) := \frac{1}{p} \int_{\mathbb{R}^d} |u_N(x)|^p dx. \tag{4.3} \quad \text{RpN}$$

Observing that

$$\mathbb{E}_\mu \left[\exp(\alpha R_p(u_N)) \cdot \mathbf{1}_{\{ \int_{\mathbb{R}^d} :|u_N(x)|^2 : dx \leq K \}} \right] \leq \mathbb{E}_\mu \left[\exp \left(\alpha R_p(u_N) \cdot \mathbf{1}_{\{ \int_{\mathbb{R}^d} :|u_N(x)|^2 : dx \leq K \}} \right) \right],$$

:subharmonic

lem:exp int

the bound (4.2) follows once we have

$$\sup_N \mathbb{E}_\mu \left[\exp \left(\alpha R_p(u_N) \cdot \mathbf{1}_{\{|\int_{\mathbb{R}^d} :|u_N(x)|^2 : dx| \leq K\}} \right) \right] < \infty. \quad (4.4) \quad \boxed{\text{var3}}$$

To prove (4.4), we apply the Boué-Dupuis variational formula Lemma 3.1 with $F(Y(1))$ as in (3.7) and use the fact that

$$\text{Law}(Y_N(1)) = (\mathbf{P}_N)_* \mu$$

to get

$$\begin{aligned} & -\log \mathbb{E}_\mu \left[\exp \left(\alpha R_p(u_N) \cdot \mathbf{1}_{\{|\int_{\mathbb{R}^d} :|u_N|^2 : dx| \leq K\}} \right) \right] \\ &= \inf_{\theta \in \mathbb{H}_a} \mathbb{E} \left[-\alpha R_p(Y_N(1) + \mathbf{P}_N I(\theta)(1)) \cdot \mathbf{1}_{\{|\int_{\mathbb{R}^d} :|Y_N(1) + \mathbf{P}_N I(\theta)(1)|^2 : dx| \leq K\}} \right. \\ & \quad \left. + \frac{1}{2} \int_0^1 \|\theta(t)\|_{L^2(\mathbb{R}^d)}^2 dt \right]. \end{aligned} \quad (4.5) \quad \boxed{\text{var4}}$$

Here, \mathbb{E}_μ and \mathbb{E} denote expectations with respect to the Gaussian field μ and the underlying probability measure \mathbb{P} respectively. In what follows, we will denote

$$Y_N = Y_N(1) = \mathbf{P}_N Y(1), \quad \Theta_N = \mathbf{P}_N I(\theta)(1) \quad (4.6) \quad \boxed{\text{nots}}$$

for simplicity. In the rest of this subsection, we show that the right hand side of (4.5) has a finite lower bound under (4.1), separating cases (i) and (ii).

4.1.1. Subcritical cases. We first consider the easier subcritical case

$$\frac{4}{s} < p < 2 + \frac{4s}{(d-1)s+2},$$

where the Wick-ordered renormalized mass is of any finite size $K > 0$, and the nonlinearity is of any magnitude $\alpha > 0$. Note that the condition on p coupled with $s < 2$ implies

$$\frac{d-2+\sqrt{d^2+8}}{d+1} < s < 2.$$

Proof of (4.2). By duality and Young's inequality, we have

$$\begin{aligned} & \left| \int_{\mathbb{R}^d} :|Y_N + \Theta_N|^2 : dx \right| \\ &= \left| \int_{\mathbb{R}^d} :|Y_N|^2 : dx + 2 \int_{\mathbb{R}^d} \text{Re}(Y_N \overline{\Theta_N}) dx + \int_{\mathbb{R}^d} |\Theta_N|^2 dx \right| \\ &\geq - \left| \int_{\mathbb{R}^d} :|Y_N|^2 : dx \right| - 2 \|Y_N\|_{\mathcal{H}^{-\delta}(\mathbb{R}^d)} \|\Theta_N\|_{\mathcal{H}^{\delta}(\mathbb{R}^d)} + \int_{\mathbb{R}^d} |\Theta_N|^2 dx \\ &\geq - \left| \int_{\mathbb{R}^d} :|Y_N|^2 : dx \right| - C_\varepsilon \|Y_N\|_{\mathcal{H}^{-\delta}(\mathbb{R}^d)}^{p_1} - \varepsilon \|\Theta_N\|_{\mathcal{H}^{\delta}(\mathbb{R}^d)}^{q_1} + \int_{\mathbb{R}^d} |\Theta_N|^2 dx. \end{aligned} \quad (4.7) \quad \boxed{\text{cutoff0-d}}$$

where $\frac{1}{s} - \frac{1}{2} < \delta < 1$ and $p_1, q_1 > 1$ are such that

$$\frac{1}{p_1} + \frac{1}{q_1} = 1.$$

Furthermore, by interpolation, we have

$$\|\Theta_N\|_{\mathcal{H}^{\delta}(\mathbb{R}^d)} \leq C \|\Theta_N\|_{L^2(\mathbb{R}^d)}^{(1-\delta)p_2} + C \|\Theta_N\|_{\mathcal{H}^1(\mathbb{R}^d)}^{\delta q_2},$$

where $p_2, q_2 > 1$ are such that

$$\frac{1}{p_2} + \frac{1}{q_2} = 1.$$

We may then choose $\delta = \frac{2-s}{2s} + \epsilon$, $q_1 = 1 + \epsilon$ and $p_2 = \frac{4s}{3s-2} + \epsilon$ so that

$$q_1(1 - \delta)p_2 = 2. \quad (4.8)$$

choiexpo-d

It follows that

$$\|\Theta_N\|_{\mathcal{H}^\delta(\mathbb{R}^d)}^{q_1} \leq C\|\Theta_N\|_{L^2(\mathbb{R}^d)}^2 + C\|\Theta_N\|_{\mathcal{H}^1(\mathbb{R}^d)}^{\delta q_1 q_2}. \quad (4.9)$$

Lqbound-d

Here the constant C is independent of N and may vary from line to line. By choosing $\epsilon C < \frac{1}{2}$, from (4.7) and (4.9), we conclude that

$$\begin{aligned} & \left\{ \left| \int_{\mathbb{R}^d} :|Y_N + \Theta_N|^2 : dx \right| \leq K \right\} \\ & \subset \left\{ \|\Theta_N\|_{L^2(\mathbb{R}^d)}^2 \leq K + \left| \int_{\mathbb{R}^d} :|Y_N|^2 : dx \right| + C_\epsilon \|Y_N\|_{\mathcal{H}^{-\delta}(\mathbb{R}^d)}^{p_1} + \frac{1}{2} \|\Theta_N\|_{L^2(\mathbb{R}^d)}^2 + \epsilon C \|\Theta_N\|_{\mathcal{H}^1(\mathbb{R}^d)}^{\delta q_1 q_2} \right\} \\ & = \left\{ \|\Theta_N\|_{L^2(\mathbb{R}^d)}^2 \leq 2K + 2 \left| \int_{\mathbb{R}^d} :|Y_N|^2 : dx \right| + 2C_\epsilon \|Y_N\|_{\mathcal{H}^{-\delta}(\mathbb{R}^d)}^{p_1} + 2\epsilon C \|\Theta_N\|_{\mathcal{H}^1(\mathbb{R}^d)}^{\delta q_1 q_2} \right\} \\ & =: \Omega_K. \end{aligned} \quad (4.10)$$

cutoff1-d

We then recall an elementary inequality, which is a direct consequence of the mean value theorem and the Young's inequality. Given $p > 2$ and $\epsilon > 0$, there exists C_ϵ such that

$$|z_1 + z_2|^p \leq (1 + \epsilon)|z_1|^p + C_\epsilon |z_2|^p \quad (4.11)$$

Young-d

holds uniformly in $z_1, z_2 \in \mathbb{C}$. Here C_ϵ , which may differ from line to line, denotes a constant depending only on ϵ . We conclude from (1.4), (4.11) with $\epsilon = 1$, (4.10), the sharp Gagliardo-Nirenberg-Sobolev inequality (1.25) and

$$\|\nabla u\|_{L^2(\mathbb{R}^d)}^2 \leq \|\nabla u\|_{L^2(\mathbb{R}^d)}^2 + \|\langle x \rangle^{s/2} u\|_{L^2(\mathbb{R}^d)}^2 = \|u\|_{\mathcal{H}^1(\mathbb{R}^d)}^2$$

that

$$\begin{aligned} & \alpha R_p(Y_N + \Theta_N) \cdot \mathbf{1}_{\left\{ \int_{\mathbb{R}^d} :|Y_N + \Theta_N|^2 : dx \leq K \right\}} \\ & \leq 2\alpha R_p(\Theta_N) \cdot \mathbf{1}_{\left\{ \int_{\mathbb{R}^d} :|Y_N + \Theta_N|^2 : dx \leq K \right\}} + C\alpha R_p(Y_N) \\ & \leq 2\alpha R_p(\Theta_N) \cdot \mathbf{1}_{\Omega_K} + C\alpha R_p(Y_N) \\ & \leq \frac{2\alpha}{p} C_{\text{GNS}} \|\Theta_N\|_{\mathcal{H}^1(\mathbb{R}^d)}^{\frac{d(p-2)}{2}} \|\Theta_N\|_{L^2(\mathbb{R}^d)}^{\frac{4-(d-2)(p-2)}{2}} \cdot \mathbf{1}_{\Omega_K} + C\alpha R_p(Y_N) \end{aligned} \quad (4.12)$$

var5-1-d

where C_{GNS} is the implicit constant (depending on the dimension and p) and the set Ω_K is given in (4.10). Noting that $\frac{d(p-2)}{2} < 2$ when $p < 2 + \frac{4s}{(d-1)s+2}$, we apply Young's inequality to continue with

$$\leq C\|\Theta_N\|_{L^2(\mathbb{R}^d)}^{\frac{2(4-(d-2)(p-2))}{4-d(p-2)}} \cdot \mathbf{1}_{\Omega_K} + \frac{1}{4} \|\Theta_N\|_{\mathcal{H}^1(\mathbb{R}^d)}^2 + C R_p(Y_N), \quad (4.13)$$

var5-d

where the constant C depends only on d, p, α . Then from (4.10), interpolation and Young's inequality, we have

$$\begin{aligned}
& \|\Theta_N\|_{L^2(\mathbb{R}^d)}^{\frac{2(4-(d-2)(p-2))}{4-d(p-2)}} \cdot \mathbf{1}_{\Omega_K} \\
& \lesssim K^{\frac{4-(d-2)(p-2)}{4-d(p-2)}} + \left| \int_{\mathbb{R}^d} :|Y_N|^2: dx \right|^{\frac{4-(d-2)(p-2)}{4-d(p-2)}} + C_\varepsilon \|Y_N\|_{\mathcal{H}^{-\delta}(\mathbb{R}^d)}^{p_1 \frac{4-(d-2)(p-2)}{4-d(p-2)}} + \left(\varepsilon \|\Theta_N\|_{\mathcal{H}^1(\mathbb{R}^d)}^{\delta q_1 q_2} \right)^{\frac{4-(d-2)(p-2)}{4-d(p-2)}} \\
& \leq C_K + C \left| \int_{\mathbb{R}^d} :|Y_N|^2: dx \right|^{\frac{4-(d-2)(p-2)}{4-d(p-2)}} + C_\varepsilon \|Y_N\|_{\mathcal{H}^{-\delta}(\mathbb{R}^d)}^{p_1 \frac{4-(d-2)(p-2)}{4-d(p-2)}} + \varepsilon C \|\Theta_N\|_{\mathcal{H}^1(\mathbb{R}^d)}^2,
\end{aligned} \tag{4.14} \quad \boxed{\text{cutoff2-d}}$$

provided

$$\delta q_1 q_2 \frac{4 - (d-2)(p-2)}{4 - d(p-2)} < 2.$$

Since $\delta = \frac{2-s}{2s} +$, $q_1 = 1 +$ and $q_2 = \frac{4s}{s+2} -$ (as $p_2 = \frac{4s}{3s-2} +$, see (4.8)), the above condition requires

$$p < 2 + \frac{4s}{(d-1)s+2}. \tag{4.15} \quad \boxed{\text{cond-p}}$$

Here we remark that the constants C, C_K in (4.14), which may differ from line to line, are independent of ε . By collecting (4.5), (3.9), (4.13) and (4.14), we arrive at

$$\begin{aligned}
& -\log \mathbb{E}_\mu \left[\exp \left(\alpha R_p(u_N) \cdot \mathbf{1}_{\{\int_{\mathbb{R}^d} :|u_N|^2: dx \leq K\}} \right) \right] \\
& \geq \inf_{\theta \in \mathbb{H}_a} \mathbb{E} \left[-C \|\Theta_N\|_{L^2(\mathbb{R}^d)}^{\frac{2(4-(d-2)(p-2))}{4-d(p-2)}} \cdot \mathbf{1}_{\Omega_K} - C R_p(Y_N) \right. \\
& \quad \left. - \frac{1}{4} \|\Theta_N\|_{\mathcal{H}^1(\mathbb{R}^d)}^2 + \frac{1}{2} \int_0^1 \|\theta(t)\|_{L^2(\mathbb{R}^d)}^2 dt \right] \\
& \geq \inf_{\theta \in \mathbb{H}_a} \mathbb{E} \left[-C \|\Theta_N\|_{L^2(\mathbb{R}^d)}^{\frac{2(4-(d-2)(p-2))}{4-d(p-2)}} \cdot \mathbf{1}_{\Omega_K} + \frac{1}{4} \int_0^1 \|\theta(t)\|_{L^2(\mathbb{R}^d)}^2 dt - C R_p(Y_N) \right] \\
& \geq \inf_{\theta \in \mathbb{H}_a} \mathbb{E} \left[-CC_K - C \left| \int_{\mathbb{R}^d} :|Y_N|^2: dx \right|^{\frac{4-(d-2)(p-2)}{4-d(p-2)}} - CC_\varepsilon \|Y_N\|_{\mathcal{H}^{-\delta}(\mathbb{R}^d)}^{p_1 \frac{4-(d-2)(p-2)}{4-d(p-2)}} \right. \\
& \quad \left. + \left(\frac{1}{4} - C\varepsilon \right) \int_0^1 \|\theta(t)\|_{L^2(\mathbb{R}^d)}^2 dt - C R_p(Y_N) \right].
\end{aligned} \tag{4.16} \quad \boxed{\text{var70-d}}$$

Then, by choosing $\varepsilon > 0$ sufficiently small such that $C\varepsilon < \frac{1}{4}$, we obtain

$$\begin{aligned}
& -\log \mathbb{E}_\mu \left[\exp \left(\alpha R_p(u_N) \cdot \mathbf{1}_{\{\int_{\mathbb{R}^d} :|u_N|^2: dx \leq K\}} \right) \right] \\
& \geq -CC_K - C \mathbb{E} [R_p(Y_N)] - C \mathbb{E} \left[\left| \int_{\mathbb{R}^d} :|Y_N|^2: dx \right|^{\frac{4-(d-2)(p-2)}{4-d(p-2)}} \right] \\
& \quad - CC_\varepsilon \mathbb{E} \left[\|Y_N\|_{\mathcal{H}^{-\delta}(\mathbb{R}^d)}^{p_1 \frac{4-(d-2)(p-2)}{4-d(p-2)}} \right].
\end{aligned} \tag{4.17} \quad \boxed{\text{var71-d}}$$

Since $\frac{4}{s} < p < 2 + \frac{4s}{(d-1)s+2}$, Corollaries 3.4 and 3.5 give

$$\sup_N \mathbb{E} [R_p(Y_N)] + \sup_N \mathbb{E} \left[\left| \int_{\mathbb{R}^d} :|Y_N|^2: dx \right|^{\frac{4-(d-2)(p-2)}{4-d(p-2)}} \right] + \sup_N \mathbb{E} \left[\|Y_N\|_{\mathcal{H}^{-\delta}(\mathbb{R}^d)}^{p_1 \frac{4-(d-2)(p-2)}{4-d(p-2)}} \right] < \infty.$$

This proves (4.4), hence (4.2) in the subcritical case. \square

SUB:cri

4.1.2. **Critical case.**

This section focuses on the critical case

$$p = 2 + \frac{4s}{(d-1)s+2}.$$

We shall prove Theorem 1.3 (ii) - (a) as well as the case when $\alpha \ll 1$. The argument here is inspired by [45, 46]. We first show the uniform exponential integrability

$$\sup_{N \in \mathbb{N}} \left\| \mathbf{1}_{\{\int_{\mathbb{R}^d} :|u_N(x)|^2: dx \leq K\}} e^{\frac{\alpha}{p} \|u_N\|_{L^p(\mathbb{R}^d)}^p} \right\|_{L^1(\mu)} < \infty, \quad (4.18) \quad \text{uniint_pc-d}$$

for all $0 < \alpha \ll 1$ uniformly in $K > 0$. Then, given $K > 0$, it follows that the set

$$A(K) = \left\{ \alpha > 0 : \sup_{N \in \mathbb{N}} \left\| \mathbf{1}_{\{\int_{\mathbb{R}^d} :|u_N(x)|^2: dx \leq K\}} e^{\frac{\alpha}{p} \|u_N\|_{L^p(\mathbb{R}^d)}^p} \right\|_{L^1(\mu)} < \infty \right\},$$

is non-empty. We define

$$\alpha_0(K) = \sup A(K). \quad (4.19) \quad \text{threshold-d}$$

Then, given $K > 0$, it is easy to see that (4.18) holds for all $0 < \alpha < \alpha_0(K)$. We will establish the convergence of the truncated Gibbs measure in the next subsection.

Proceeding as in (4.13), we have

$$\begin{aligned} R_p(Y_N + \Theta_N) \cdot \mathbf{1}_{\{\int_{\mathbb{R}^d} :|Y_N + \Theta_N|^2: dx \leq K\}} \\ \leq 2R_p(\Theta_N) \cdot \mathbf{1}_{\{\int_{\mathbb{R}^d} :|Y_N + \Theta_N|^2: dx \leq K\}} + CR_p(Y_N). \end{aligned} \quad (4.20) \quad \text{CNO-d}$$

By Gagliardo-Nirenberg-Sobolev inequality and Young's inequality, we have

$$\begin{aligned} R_p(\Theta_N) &= \frac{1}{p} \int_{\mathbb{R}^d} |\Theta_N|^p dx \\ &\lesssim \|\Theta_N\|_{L^2(\mathbb{R}^d)}^{\frac{4-(d-2)(p-2)}{2}} \|\Theta_N\|_{H^1(\mathbb{R}^d)}^{\frac{d(p-2)}{2}} \\ &\lesssim \|\Theta_N\|_{L^2(\mathbb{R}^d)}^{\frac{2(4-(d-2)(p-2))}{4-d(p-2)}} + \|\Theta_N\|_{H^1(\mathbb{R}^d)}^2. \end{aligned} \quad (4.21) \quad \text{CN1_1-d}$$

Also, we notice that

$$\left| \int_{\mathbb{R}^d} :|Y_N + \Theta_N|^2: dx \right| \geq \left| \int_{\mathbb{R}^d} 2 \operatorname{Re}(Y_N \overline{\Theta_N}) + |\Theta_N|^2 dx \right| - \left| \int_{\mathbb{R}^d} :|Y_N|^2: dx \right| \quad (4.22) \quad \text{CN1-d}$$

Now, we consider two cases.

Case 1. We first assume

$$\|\Theta_N\|_{L^2(\mathbb{R}^d)}^2 \gg \left| \int_{\mathbb{R}^d} \operatorname{Re}(Y_N \overline{\Theta_N}) dx \right|. \quad (4.23)$$

Then, together with (4.22) we have

$$\begin{aligned} &\|\Theta_N\|_{L^2(\mathbb{R}^d)}^{\frac{2(4-(d-2)(p-2))}{4-d(p-2)}} \mathbf{1}_{\{\int_{\mathbb{R}^d} :|Y_N + \Theta_N|^2: dx \leq K\}} \\ &\sim \left| \int_{\mathbb{R}^d} 2 \operatorname{Re}(Y_N \overline{\Theta_N}) + |\Theta_N|^2 dx \right|^{\frac{4-(d-2)(p-2)}{4-d(p-2)}} \mathbf{1}_{\{\int_{\mathbb{R}^d} :|Y_N + \Theta_N|^2: dx \leq K\}} \\ &\lesssim K^{\frac{4-(d-2)(p-2)}{4-d(p-2)}} + \left| \int_{\mathbb{R}^d} :|Y_N|^2: dx \right|^{\frac{4-(d-2)(p-2)}{4-d(p-2)}}, \end{aligned} \quad (4.24) \quad \text{CN1_2-d}$$

which will be sufficient for our purpose.

Case 2. Let us assume that

$$\|\Theta_N\|_{L^2(\mathbb{R}^d)}^2 \lesssim \left| \int_{\mathbb{R}^d} \operatorname{Re}(Y_N \overline{\Theta_N}) dx \right|. \quad (4.25) \quad \boxed{\text{CN2-d}}$$

We write

$$Y_N = \sum_{n=0}^N Y_{N,n} e_n, \quad \Theta_N = \sum_{n=0}^N \Theta_{N,n} e_n.$$

For each n , we decompose

$$\Theta_{N,n} = a_n Y_{N,n} + w_n, \quad (4.26) \quad \boxed{\text{Tdecom-d}}$$

where

$$a_n := \begin{cases} \frac{\operatorname{Re}(\Theta_{N,n} \overline{Y_{N,n}})}{|Y_{N,n}|^2} & \text{if } n \leq N \\ 0 & \text{otherwise} \end{cases} \quad \text{and } w_n = \Theta_{N,n} - a_n Y_{N,n}.$$

From the above definition, we have

$$\|\Theta_N\|_{L^2(\mathbb{R}^d)}^2 = \sum_{n=0}^N (a_n^2 |Y_{N,n}|^2 + |w_n|^2), \quad (4.27) \quad \boxed{\text{CN3-d}}$$

$$\int_{\mathbb{R}^d} \operatorname{Re}(Y_N \overline{\Theta_N}) dx = \sum_{n=0}^N a_n |Y_{N,n}|^2. \quad (4.28) \quad \boxed{\text{CN4-d}}$$

By collecting (4.25), (4.27) and (4.28), we have

$$\sum_{n=0}^N a_n^2 |Y_{N,n}|^2 \lesssim \left| \sum_{n=0}^N a_n |Y_{N,n}|^2 \right|. \quad (4.29) \quad \boxed{\text{CN5-d}}$$

We fix $n_0 < N$ to be chosen later. Then, by the orthogonal decomposition (4.26), we see that $a_n^2 |Y_{N,n}|^2 \leq |\Theta_{N,n}|^2$, and thus

$$\begin{aligned} \left| \sum_{n=n_0+1}^N a_n |Y_{N,n}|^2 \right| &\leq \left(\sum_{n=0}^N a_n^2 \lambda_n^2 |Y_{N,n}|^2 \right)^{\frac{1}{2}} \left(\sum_{n=n_0+1}^N \lambda_n^{-2} |Y_{N,n}|^2 \right)^{\frac{1}{2}} \\ &\leq \left(\sum_{n=0}^N \lambda_n^2 |\Theta_{N,n}|^2 \right)^{\frac{1}{2}} \left(\sum_{n=n_0+1}^N \lambda_n^{-2} |Y_{N,n}|^2 \right)^{\frac{1}{2}} \\ &= \|\Theta_N\|_{\mathcal{H}^1(\mathbb{R}^d)} \|\mathbf{P}_{n_0}^\perp Y_N\|_{\mathcal{H}^{-1}(\mathbb{R}^d)}. \end{aligned} \quad (4.30) \quad \boxed{\text{CN6-d}}$$

For $n \leq n_0$, from (4.29) and then (4.28), we have

$$\begin{aligned}
\left| \sum_{n=0}^{n_0} a_n |Y_{N,n}|^2 \right| &\leq \left(\sum_{n=0}^N a_n^2 |Y_{N,n}|^2 \right)^{\frac{1}{2}} \left(\sum_{n=0}^{n_0} |Y_{N,n}|^2 \right)^{\frac{1}{2}} \\
&\leq C \left| \sum_{n=0}^N a_n |Y_{N,n}|^2 \right|^{\frac{1}{2}} \left(\sum_{n=0}^{n_0} |Y_{N,n}|^2 \right)^{\frac{1}{2}} \\
&\leq \frac{1}{2} \left| \sum_{n=0}^N a_n |Y_{N,n}|^2 \right| + C' \|\mathbf{P}_{n_0} Y\|_{L^2(\mathbb{R}^d)}^2 \\
&= \frac{1}{2} \left| \int_{\mathbb{R}^d} \operatorname{Re}(Y_N \overline{\Theta_N}) dx \right| + C' \|\mathbf{P}_{n_0} Y\|_{L^2(\mathbb{R}^d)}^2.
\end{aligned} \tag{4.31} \quad \boxed{\text{CN7-d}}$$

By collecting (4.28), (4.30) and (4.31), we arrive at

$$\left| \int_{\mathbb{R}^d} \operatorname{Re}(Y_N \overline{\Theta_N}) dx \right| \lesssim \|\Theta_N\|_{H^1(\mathbb{R}^d)} \|\mathbf{P}_{n_0}^\perp Y_N\|_{H^{-1}(\mathbb{R}^d)} + \|\mathbf{P}_{n_0} Y\|_{L^2(\mathbb{R}^d)}^2,$$

which together with (4.25) yields

$$\|\Theta_N\|_{L^2(\mathbb{R}^d)}^2 \lesssim \|\Theta_N\|_{H^1(\mathbb{R}^d)} \|\mathbf{P}_{n_0}^\perp Y_N\|_{H^{-1}(\mathbb{R}^d)} + \|\mathbf{P}_{n_0} Y\|_{L^2(\mathbb{R}^d)}^2. \tag{4.32} \quad \boxed{\text{CN8-d}}$$

By denoting

$$Z_{N,n_0} = \mathcal{L}^{-1/2} \mathbf{P}_{n_0}^\perp Y_N, \quad \tilde{\sigma}_{N,n_0} = \mathbb{E}[\|Z_{N,n_0}\|_{L^2(\mathbb{R}^d)}^2],$$

we have

$$\begin{aligned}
\|\mathbf{P}_{n_0}^\perp Y_N\|_{H^{-1}(\mathbb{R}^d)}^2 &= \|\mathcal{L}^{-1/2} \mathbf{P}_{n_0}^\perp Y_N\|_{L^2(\mathbb{R}^d)}^2 \\
&= \int_{\mathbb{R}^d} |\mathcal{L}^{-1/2} \mathbf{P}_{n_0}^\perp Y_N|^2 - \mathbb{E}[|\mathcal{L}^{-1/2} \mathbf{P}_{n_0}^\perp Y_N|^2] + \mathbb{E}[|\mathcal{L}^{-1/2} \mathbf{P}_{n_0}^\perp Y_N|^2] dx \\
&= \int_{\mathbb{R}^d} |Z_{N,n_0}|^2 - \mathbb{E}[|Z_{N,n_0}|^2] dx + \tilde{\sigma}_{N,n_0}.
\end{aligned} \tag{4.33} \quad \boxed{\text{B0e-d}}$$

Then by using Corollary 2.7 we obtain

$$\tilde{\sigma}_{N,n_0} = \sum_{n=n_0+1}^N \frac{2}{\lambda_n^4} \lesssim \lambda_{n_0}^{-3+\frac{2}{s}}, \tag{4.34} \quad \boxed{\text{B0-d}}$$

and

$$\mathbb{E} \left[\left(\int_{\mathbb{R}^d} |Z_{N,n_0}|^2 - \mathbb{E}[|Z_{N,n_0}|^2] dx \right)^2 \right] = \sum_{n=n_0+1}^N \frac{4}{\lambda_n^8} \lesssim \lambda_{n_0}^{-7+\frac{2}{s}}. \tag{4.35} \quad \boxed{\text{B01-d}}$$

Now we define a non-negative random variable $B_1(\omega)$ by

$$B_1(\omega) = \lambda_{n_0}^{\frac{7}{2}-\frac{1}{s}} \left| \int_{\mathbb{R}^d} |Z_{N,n_0}|^2 - \mathbb{E}[|Z_{N,n_0}|^2] dx \right|. \tag{4.36} \quad \boxed{\text{B1-d}}$$

By the Wiener chaos estimate (see e.g., [57, Theorem I.22] or [59, Proposition 2.4]), we have

$$\mathbb{E}[B_1^q] \leq C(q) \left(\lambda_{n_0}^{\frac{7-2}{s}} \mathbb{E} \left[\left(\int_{\mathbb{R}^d} |Z_{N,n_0}|^2 - \mathbb{E}[|Z_{N,n_0}|^2] dx \right)^2 \right] \right)^{\frac{q}{2}} < \infty \tag{4.37} \quad \boxed{\text{B1e-d}}$$

for any finite $q \geq 2$. The same bound holds for $1 \leq q < 2$ by Hölder's inequality. From (4.33), (4.34), (4.35), (4.36) and (4.37), we have

$$\|\mathbf{P}_{n_0}^\perp Y_N\|_{H^{-1}(\mathbb{R}^d)}^2 \lesssim \lambda_{n_0}^{-\frac{7}{2}+\frac{1}{s}} B_1(\omega) + \lambda_{n_0}^{-3+\frac{2}{s}}. \tag{4.38} \quad \boxed{\text{B1e2-d}}$$

Now we set

$$B_2(\omega) = \left| \int_{\mathbb{R}^d} |Y_N|^2 - \mathbb{E}[|Y_N|^2] dx \right|. \quad (4.39) \quad \boxed{\text{B2-d}}$$

Then it follows that

$$\begin{aligned} \|\mathbf{P}_{n_0} Y_N\|_{L^2(\mathbb{R}^d)}^2 &= \int_{\mathbb{R}^d} |\mathbf{P}_{n_0} Y_N|^2 - \mathbb{E}[|\mathbf{P}_{n_0} Y_N|^2] dx + \mathbb{E}[\|\mathbf{P}_{n_0} Y_N\|_{L^2(\mathbb{R}^d)}^2] \\ &\leq B_2(\omega) + \sum_{n=0}^{n_0} \frac{2}{\lambda_n^2} \\ &\lesssim B_2(\omega) + \lambda_{n_0}^{-1+\frac{2}{s}}, \end{aligned} \quad (4.40) \quad \boxed{\text{B2e-d}}$$

where we used Corollary 2.6. Furthermore, from computations similar to those in the proof of Corollary 3.5, and an application of Corollary 2.7, we obtain

$$\begin{aligned} \mathbb{E}[B_2^q] &\leq C(q) \left(\mathbb{E} \left[\left(\int_{\mathbb{R}^d} |Y_N|^2 - \mathbb{E}[|Y_N|^2] dx \right)^2 \right] \right)^{q/2} \\ &\leq C(q) \left(\sum_{n=0}^N \frac{4}{\lambda_n^4} \right)^{q/2} \\ &\leq C(q) (\lambda_N^{-3+\frac{2}{s}})^{q/2} < \infty, \end{aligned} \quad (4.41) \quad \boxed{\text{B2ea-d}}$$

uniformly, since $-3 + \frac{2}{s} < 0$ and $\lambda_N \geq \lambda_0$.

From (4.32), (4.38), and (4.40), we have

$$\begin{aligned} \|\Theta_N\|_{L^2(\mathbb{R}^d)}^{\frac{2(4-(d-2)(p-2))}{4-d(p-2)}} &\lesssim \left(\|\Theta_N\|_{H^1(\mathbb{R}^d)} \|\mathbf{P}_{n_0}^\perp Y_N\|_{H^{-1}(\mathbb{R}^d)} + \|\mathbf{P}_{n_0} Y\|_{L^2(\mathbb{R}^d)}^2 \right)^{\frac{4-(d-2)(p-2)}{4-d(p-2)}} \\ &\lesssim \left(\lambda_{n_0}^{-\frac{7}{2}+\frac{1}{s}} B_1(\omega) + \lambda_{n_0}^{-3+\frac{2}{s}} \right)^{\frac{4-(d-2)(p-2)}{4-d(p-2)}} \|\Theta_N\|_{H^1(\mathbb{R}^d)}^{\frac{4-(d-2)(p-2)}{4-d(p-2)}} + \left(B_2(\omega) + \lambda_{n_0}^{\frac{2-s}{s}} \right)^{\frac{4-(d-2)(p-2)}{4-d(p-2)}} \\ &\lesssim \left(\lambda_{n_0}^{-\frac{7}{2}+\frac{1}{s}} B_1(\omega) \|\Theta_N\|_{H^1(\mathbb{R}^d)} \right)^{\frac{4-(d-2)(p-2)}{4-d(p-2)}} + \left(\lambda_{n_0}^{-3+\frac{2}{s}} \|\Theta_N\|_{H^1(\mathbb{R}^d)} \right)^{\frac{4-(d-2)(p-2)}{4-d(p-2)}} \\ &\quad + B_2(\omega)^{\frac{4-(d-2)(p-2)}{4-d(p-2)}} + \lambda_{n_0}^{\frac{2-s}{s} \cdot \frac{4-(d-2)(p-2)}{4-d(p-2)}}, \end{aligned} \quad (4.42) \quad \boxed{\text{CN9-d}}$$

We next choose n_0 such that

$$\lambda_{n_0}^{\frac{2(2-s)}{s} \cdot \frac{4-(d-2)(p-2)}{4-d(p-2)}} \sim \|\Theta_N\|_{H^1(\mathbb{R}^d)}^2. \quad (4.43) \quad \boxed{\text{sizeM-d}}$$

Recall that in the critical case $p = 2 + \frac{4s}{(d-1)s+2}$ and thus

$$\frac{4-(d-2)(p-2)}{4-d(p-2)} = \frac{2+s}{2-s},$$

which together with (4.43) gives

$$\left(\lambda_{n_0}^{-3+\frac{2}{s}} \|\Theta_N\|_{H^1(\mathbb{R}^d)} \right)^{\frac{4-(d-2)(p-2)}{4-d(p-2)}} = \|\Theta_N\|_{H^1(\mathbb{R}^d)}^2. \quad (4.44) \quad \boxed{\text{CN10-d}}$$

Then it follows that

$$\begin{aligned}
& \left(\lambda_{n_0}^{-\frac{7}{2} + \frac{1}{s}} B_1(\omega) \|\Theta_N\|_{\mathcal{H}^1(\mathbb{R}^d)} \right)^{\frac{4-(d-2)(p-2)}{4-d(p-2)}} \\
&= \lambda_{n_0}^{\left(-\frac{7}{2} + \frac{1}{s}\right) \frac{4-(d-2)(p-2)}{4-d(p-2)}} B_1(\omega)^{\frac{4-(d-2)(p-2)}{4-d(p-2)}} \|\Theta_N\|_{\mathcal{H}^1(\mathbb{R}^d)} \\
&\lesssim \lambda_{n_0}^{\left(-\frac{7}{2} + \frac{1}{s}\right) \frac{2(4-(d-2)(p-2))}{4-d(p-2)}} B_1(\omega)^{\frac{2(4-(d-2)(p-2))}{4-d(p-2)}} + \|\Theta_N\|_{\mathcal{H}^1(\mathbb{R}^d)}^2 \\
&\lesssim B_1(\omega)^{\frac{2(4-(d-2)(p-2))}{4-d(p-2)}} + \|\Theta_N\|_{\mathcal{H}^1(\mathbb{R}^d)}^2,
\end{aligned} \tag{4.45} \quad \text{CN11-d}$$

where we used the fact that $-\frac{7}{2} + \frac{1}{s} < 0$. By collecting (4.42), (4.44) and (4.45), it follows that

$$\|\Theta_N\|_{L^2(\mathbb{R}^d)}^{\frac{2(4-(d-2)(p-2))}{4-d(p-2)}} \lesssim \|\Theta_N\|_{\mathcal{H}^1(\mathbb{R}^d)}^2 + B_1(\omega)^{\frac{2(4-(d-2)(p-2))}{4-d(p-2)}} + B_2(\omega)^{\frac{4-(d-2)(p-2)}{4-d(p-2)}} + C \tag{4.46} \quad \text{CN12-d}$$

for some constant $C > 0$ independent of N .

Finally, we are ready to prove (4.18). By collecting (4.5), (4.20), (4.37) (4.41), (4.45) and (4.46), we arrive at

$$\begin{aligned}
& -\log \mathbb{E}_\mu \left[\exp \left(\alpha R_p(u_N) \cdot \mathbf{1}_{\{|\int_{\mathbb{R}^d} |u_N(x)|^2 : dx| \leq K\}} \right) \right] \\
&= \inf_{\theta \in \mathbb{H}_a} \mathbb{E} \left[-\alpha R_p(Y_N + \Theta_N) \cdot \mathbf{1}_{\{|\int_{\mathbb{R}^d} |Y_N + \Theta_N|^2 : dx| \leq K\}} + \frac{1}{2} \int_0^1 \|\theta(t)\|_{L^2(\mathbb{R}^d)}^2 dt \right] \\
&\geq \inf_{\theta \in \mathbb{H}_a} \mathbb{E} \left[-C\alpha \|\Theta_N\|_{L^2(\mathbb{R}^d)}^{\frac{2(4-(d-2)(p-2))}{4-d(p-2)}} \cdot \mathbf{1}_{\{|\int_{\mathbb{R}^d} |Y_N + \Theta_N|^2 : dx| \leq K\}} \right. \\
&\quad \left. + \left(\frac{1}{2} - C\alpha \right) \int_0^1 \|\theta(t)\|_{L^2(\mathbb{R}^d)}^2 dt - C\alpha R_p(Y_N) \right] \\
&\geq \inf_{\theta \in \mathbb{H}_a} \mathbb{E} \left[-C\alpha \|\Theta_N\|_{\mathcal{H}^1(\mathbb{R}^d)}^2 - C B_1(\omega)^{\frac{2(4-(d-2)(p-2))}{4-d(p-2)}} - C B_2(\omega)^{\frac{4-(d-2)(p-2)}{4-d(p-2)}} - C_K \right. \\
&\quad \left. - \left| \int_{\mathbb{R}^d} |Y_N|^2 : dx \right|^{\frac{4-(d-2)(p-2)}{4-d(p-2)}} + \left(\frac{1}{2} - C\alpha \right) \int_0^1 \|\theta(t)\|_{L^2(\mathbb{R}^d)}^2 dt - C\alpha R_p(Y_N) \right] \\
&\geq \inf_{\theta \in \mathbb{H}_a} \mathbb{E} \left[-C + \left(\frac{1}{2} - 2C\alpha \right) \int_0^1 \mathbb{E} [\|\theta(t)\|_{L^2(\mathbb{R}^d)}^2] dt \right] > -C,
\end{aligned} \tag{4.47} \quad \text{CN13-d}$$

provided $\alpha \ll 1$. Here the constant C may vary from line to line. Thus, we finish the proof of the uniform exponential integrability (4.18) uniformly in $K > 0$.

SUB:nor

4.1.3. Normalizability. In this subsection, we complete the proof of Theorem 1.3 (i) and (ii)-(a), i.e. the convergence (1.19) provided (4.18). Before proceeding with the proof, we recall a useful lemma.

A set of functions $\{f_n\}_{n \geq 1} \subset L^1(\mu)$ is called uniformly integrable if

$$\lim_{M \rightarrow \infty} \sup_n \int_{\{|f_n| > M\}} |f_n| d\mu = 0.$$

Then we have the following lemma

Lemma 4.2 (Vitali).

The sequence (f_n) converges in $L^1(\mu)$ if and only if (f_n) converges in μ -measure and $(|f_n|)$ is uniformly integrable.

Now we are ready to prove Theorem 1.3.

LEM:vitali

Proof of Theorem 1.3 (i) and (ii)-(a). According to Lemma 4.2, we need to show that the density

$$f_N = \mathbf{1}_{\{\int_{\mathbb{R}^d} : |u_N(x)|^2 : dx \leq K\}} e^{\frac{\alpha}{p} \|u_N\|_{L^p(\mathbb{R}^d)}^p} \quad (4.48) \quad \boxed{\text{density}}$$

is convergent in μ -measure and is uniformly integrable.

We first note that from Corollary 3.4 we have

$$\lim_N e^{\frac{\alpha}{p} \|u_N\|_{L^p(\mathbb{R}^d)}^p} = e^{\frac{\alpha}{p} \|u\|_{L^p(\mathbb{R}^d)}^p}$$

in μ -measure. The μ -measure convergence of $\mathbf{1}_{\{\int_{\mathbb{R}^d} : |u_N(x)|^2 : dx \leq K\}}$ follows from a similar argument as in [54, Lemma 2.7].

It remains to show that f_N given in (4.48) is uniformly integrable. To see this, we distinguish two cases. For the subcritical case, i.e. $\frac{4}{s} < p < 2 + \frac{4s}{(d-1)s+2}$, we have

$$\sup_N \int_{|f_N| > M} |f_N| d\mu \leq M^{-1} \sup_N \int |f_N|^2 d\mu \lesssim M^{-1}, \quad (4.49) \quad \boxed{\text{uni_sub}}$$

where in the last step we used (1.18) with $r = 2$. This then implies that (f_N) is uniformly integrable. Now we turn to the critical case, i.e. $p = 2 + \frac{4s}{(d-1)s+2}$. Let α_0 be as in Subsection 4.1.2 such that (4.18) holds for all $0 < \alpha < \alpha_0$. Given $\alpha \in (0, \alpha_0)$, there exists $\varepsilon > 0$ such that $\alpha(1 + \varepsilon) < \alpha_0$. In particular, we have

$$\sup_{N \in \mathbb{N}} \|f_N\|_{L^{1+\varepsilon}(\mu)} = \sup_{N \in \mathbb{N}} \left\| \mathbf{1}_{\{\int_{\mathbb{R}^d} : |u_N(x)|^2 : dx \leq K\}} e^{\frac{\alpha(1+\varepsilon)}{p} \|u_N\|_{L^p(\mathbb{R}^d)}^p} \right\|_{L^1(\mu)} < \infty. \quad (4.50) \quad \boxed{\text{uniint_pc1}}$$

Then we have

$$\sup_N \int_{|f_N| > M} |f_N| d\mu \leq M^{-\varepsilon} \sup_N \int |f_N|^{1+\varepsilon} d\mu \lesssim M^{-\varepsilon}, \quad (4.51) \quad \boxed{\text{uni_cri}}$$

which implies that (f_N) is uniformly integrable. \square

SEC:non

4.2. Non-normalizability. In this section, we prove the second part of Theorem 1.3, i.e. (ii)-(b) and (iii). We follow the strategy of [54] but with some necessary modifications. The main estimate is as follows:

lem:div

Lemma 4.3 (Divergence of the partition function).

Let $s < 2$ and assume

$$\begin{aligned} & \text{either (i) } p \geq 2 + \frac{4s}{(d-1)s+2} \text{ and } \alpha \gg 1 \text{ when } p = 2 + \frac{4s}{(d-1)s+2}; \\ & \text{or (ii) } K > 0, \alpha > \alpha_0(K) \text{ when } p = 2 + \frac{4s}{(d-1)s+2}, \end{aligned} \quad (4.52) \quad \boxed{\text{condition2}}$$

where $\alpha_0(K)$ is given in (4.19). Then

$$\limsup_{N \rightarrow \infty} \mathbb{E}_\mu \left[\exp(\alpha R_p(u_N)) \cdot \mathbf{1}_{\{\int_{\mathbb{R}^d} : |u_N(x)|^2 : dx \leq K\}} \right] = \infty. \quad (4.53) \quad \boxed{\text{pax}}$$

First, we notice that

$$\begin{aligned} & \mathbb{E}_\mu \left[\exp(\alpha R_p(u_N)) \cdot \mathbf{1}_{\{\int_{\mathbb{R}^d} : |u_N|^2 : dx \leq K\}} \right] \\ & \geq \mathbb{E}_\mu \left[\exp(\alpha R_p(u_N)) \cdot \mathbf{1}_{\{\int_{\mathbb{R}^d} : |u_N|^2 : dx \leq K\}} \right] - 1. \end{aligned} \quad (4.54) \quad \boxed{\text{pax1}}$$

Therefore, the divergence (4.53) follows once we prove

$$\limsup_{N \rightarrow \infty} \mathbb{E}_\mu \left[\exp(\alpha R_p(u_N)) \cdot \mathbf{1}_{\{\int_{\mathbb{R}^d} : |u_N|^2 : dx \leq K\}} \right] = \infty. \quad (4.55) \quad \boxed{\text{pa0}}$$

By the Boué-Dupuis variational formula Lemma 3.1, we have

$$\begin{aligned} & -\log \mathbb{E}_\mu \left[\exp \left(\alpha R_p(u_N) \cdot \mathbf{1}_{\{|f_{\mathbb{R}^d}|^2 : dx| \leq K\}} \right) \right] \\ &= \inf_{\theta \in \mathbb{H}_a} \mathbb{E} \left[-\alpha R_p(Y_N + \Theta_N) \cdot \mathbf{1}_{\{|f_{\mathbb{R}^d}|^2 : +2 \operatorname{Re}(Y_N \overline{\Theta_N}) + |\Theta_N|^2 dx| \leq K\}} \right. \\ & \quad \left. + \frac{1}{2} \int_0^1 \|\theta(t)\|_{L^2(\mathbb{R}^d)}^2 dt \right], \end{aligned} \quad (4.56) \quad \boxed{\text{DPf}}$$

where Y_N and Θ_N are as in (4.6). Here, \mathbb{E}_μ and \mathbb{E} denote expectations with respect to the Gaussian field μ and the underlying probability measure \mathbb{P} , respectively. We prove that (4.56) diverges to $-\infty$ for large N by exhibiting a suitable trial state for the variational problem.

4.2.1. Construction of the drift. We follow the plan outlined in Section 3.1. We first construct a profile which stays bounded in $L^2(\mathbb{R}^d)$ but grows in $L^p(\mathbb{R}^d)$ with $p > 2$. Fix a large parameter $M \gg 1$. Let $f : \mathbb{R}^d \rightarrow \mathbb{R}$ be a real-valued radial Schwartz function with $\|f\|_{L^2(\mathbb{R}^d)} = 1$ such that the Fourier transform \hat{f} is a smooth function supported on $\{\frac{1}{2} < |\xi| \leq 1\}$. Define a function f_M on \mathbb{R} by

$$f_M(x) = M^{-\frac{1}{2}} \int_{\mathbb{R}^d} e^{2\pi i x \cdot \xi} \hat{f}\left(\frac{\xi}{M}\right) d\xi = M^{\frac{1}{2}} f(Mx), \quad (4.57) \quad \boxed{\text{fMdef}}$$

where \hat{f} denotes the Fourier transform on \mathbb{R}^d defined by

$$\hat{f}(\xi) = \int_{\mathbb{R}^d} f(x) e^{-2\pi i \xi \cdot x} dx.$$

Then, a direct computation yields the following lemma (see e.g., [54, Lemma 4.1]).

LEM:soliton **Lemma 4.4 (Blow-up profile).**

Let $s \in \mathbb{R}$. Then, we have

$$\int_{\mathbb{R}^d} f_M^2 dx = 1, \quad (4.58) \quad \boxed{\text{fM0}}$$

$$\int_{\mathbb{R}^d} (\mathcal{L}^{\frac{s}{2}} f_M)^2 dx \lesssim M^{2s}, \quad (4.59) \quad \boxed{\text{fm2}}$$

$$\int_{\mathbb{R}^d} |f_M|^p dx \sim M^{\frac{dp}{2}-d} \quad (4.60) \quad \boxed{\text{fM1}}$$

for any $p > 0$ and $M \gg 1$.

Next, for some $1 \ll M \leq N$ we construct an approximation $Z_M(t)$ to $Y_N(t)$ as in Definition 3.3. We also set

$$\alpha_{M,N} = \frac{\mathbb{E} \left[2 \int_{\mathbb{R}^d} \operatorname{Re}(Y_N \overline{Z_M}) dx - \int_{\mathbb{R}^d} |Z_M|^2 dx \right]}{\int_{\mathbb{R}^d} |\mathbf{P}_N f_{\lambda_M}|^2 dx}. \quad (4.61) \quad \boxed{\text{fmb1}}$$

for $N \geq M \gg 1$. Our trial drift is now:

def:trial 1 **Definition 4.5 (Trial drift in the subharmonic case).**

For $M \gg 1$, we set f_{λ_M} , Z_M , and $\alpha_{M,N}$ as above and define a drift $\theta = \theta^0$ by

$$\theta^0(t) = \mathcal{L}^{\frac{1}{2}} \left(-\frac{d}{dt} Z_M(t) + \sqrt{\alpha_{M,N}} f_{\lambda_M} \right) \quad (4.62) \quad \boxed{\text{drift}}$$

and

$$\Theta^0 = I(\theta^0)(1) = \int_0^1 \mathcal{L}^{-\frac{1}{2}} \theta^0(t) dt = -Z_M + \sqrt{\alpha_{M,N}} f_{\lambda_M}. \quad (4.63) \quad \text{paa0}$$

We remark that $\sqrt{\alpha_{M,N}}(\mathbf{P}_N f_{\lambda_M})$ acts as a blow-up profile in our analysis, and $\theta^0 \in \mathbb{H}_a$ is the stochastic drift such that $Y_N + \Theta_N^0$ approximates $\sqrt{\alpha_{M,N}}(\mathbf{P}_N f_{\lambda_M})$, which drives the potential energy $R_p(Y_N + \Theta_N^0)$ to blow up. Remarkably, due to the construction, the Wick-ordered L^2 norm of this approximation $Y_N + \Theta_N^0$ can be made as small as possible, i.e. the cutoff in the Wick-ordered L^2 norm does not exclude the blow-up profiles.

Before inserting this choice in the variational formula (4.56) to obtain an upper bound, we vindicate that Definition 3.3 indeed efficiently approximates the Brownian motion $Y_N(t)$ in the case at hand:

Lemma 4.6 (Approximating the Brownian motion, subharmonic case).

Let $d \geq 1$, $1 < s < 2$ and restrict to radial functions when $d \geq 2$. Given $N \geq M \gg 1$, we define Z_M by its coefficients in the eigenfunction expansion of \mathcal{L} as in Definition 3.3.

If c is chosen large enough in (3.10), we have the following estimates:

$$\mathbb{E}[\|Z_M\|_{L^2(\mathbb{R}^d)}^2] \sim \lambda_M^{\frac{2}{s}-1}, \quad (4.64) \quad \text{NRZ0}$$

$$\mathbb{E}\left[2 \operatorname{Re} \int_{\mathbb{R}^d} Y_N \overline{Z_M} dx - \int_{\mathbb{R}^d} |Z_M|^2 dx\right] \sim \lambda_M^{\frac{2}{s}-1}, \quad (4.65) \quad \text{NRZ1}$$

$$\mathbb{E}\left[\left|\|Y_N - Z_M\|_{L^2(\mathbb{R}^d)}\right|^2\right] \lesssim \lambda_M^{-3+\frac{2}{s}}, \quad (4.66) \quad \text{NRZ3}$$

$$\mathbb{E}\left[\left|\int_{\mathbb{R}^d} Y_N f_{\lambda_M} dx\right|^2\right] + \mathbb{E}\left[\left|\int_{\mathbb{R}^d} Z_M f_{\lambda_M} dx\right|^2\right] \lesssim \lambda_M^{-2}, \quad (4.67) \quad \text{NRZ5}$$

$$\mathbb{E}\left[\int_0^1 \left\|\frac{d}{d\tau} Z_M(\tau)\right\|_{H^1(\mathbb{R}^d)}^2 d\tau\right] \lesssim \lambda_M^{\frac{2}{s}} \quad (4.68) \quad \text{NRZ6}$$

for any $N \geq M \gg 1$, where $Z_M = Z_M(1)$ and

$$\|Y_N - Z_M\|_{L^2(\mathbb{R}^d)}^2 \stackrel{\text{def}}{=} \|Y_N - Z_M\|_{L^2(\mathbb{R}^d)}^2 - \mathbb{E}[\|Y_N - Z_M\|_{L^2(\mathbb{R}^d)}^2]. \quad (4.69) \quad \text{ZZZ2}$$

Proof. Let

$$X_n(t) = \tilde{Y}_N(n, t) - \tilde{Z}_M(n, t), \quad 0 \leq n \leq M. \quad (4.70) \quad \text{ZZ1}$$

Then, from (3.10), we see that $X_n(t)$ satisfies the following stochastic differential equation:

$$\begin{cases} dX_n(t) = -c\lambda_n^{-1}\lambda_M X_n(t)dt + \lambda_n^{-1}dB_n(t) \\ X_n(0) = 0 \end{cases}$$

for $0 \leq n \leq M$, where $c \gg 1$ is a constant. By solving this stochastic differential equation, we have

$$X_n(t) = \lambda_n^{-1} \int_0^t e^{-c\lambda_n^{-1}\lambda_M(t-\tau)} dB_n(\tau). \quad (4.71) \quad \text{ZZ2}$$

Then, from (4.70) and (4.71), we have

$$\tilde{Z}_M(n, t) = \tilde{Y}_N(n, t) - \lambda_n^{-1} \int_0^t e^{-c\lambda_n^{-1}\lambda_M(t-\tau)} dB_n(\tau) \quad (4.72) \quad \text{SDE1}$$

for $n \leq M$. Hence, from (4.72), the independence of $\{B_n\}_{n \in \mathbb{N}}$, Ito's isometry (see [25, Section 4.2]) and Corollary 2.6 with $p = 1$ and $p = \frac{1}{2}$, we have

$$\begin{aligned} \mathbb{E}[\|Z_M\|_{L^2(\mathbb{R}^d)}^2] &= \sum_{n \leq M} \left(\mathbb{E}[|\tilde{Y}_N(n)|^2] - 2\lambda_n^{-2} \int_0^1 e^{-c\lambda_n^{-1}\lambda_M(1-\tau)} d\tau \right. \\ &\quad \left. + \lambda_n^{-2} \int_0^1 e^{-2c\lambda_n^{-1}\lambda_M(1-\tau)} d\tau \right) \\ &\sim \sum_{n=0}^M \lambda_n^{-2} + O\left(c^{-1} \sum_{n \leq M} \lambda_n^{-1} \lambda_M^{-1}\right) \\ &\sim \lambda_M^{\frac{2}{s}-1} + O(c^{-1} \lambda_M^{\frac{2}{s}-1}) \sim \lambda_M^{\frac{2}{s}-1}, \end{aligned} \tag{4.73} \quad \boxed{\text{ZZ3}}$$

for any $M \gg 1$, $c \gg 1$ and $s \in (1, 2)$. This proves (4.64).

By the L^2 orthogonality of $\{e_n\}_{n \in \mathbb{N}}$, (4.72), (4.64), and proceeding as in (4.73), we have

$$\begin{aligned} \mathbb{E}\left[2 \operatorname{Re} \int_{\mathbb{R}^d} Y_N \overline{Z_M} dx - \int_{\mathbb{R}^d} |Z_M|^2 dx\right] &= \mathbb{E}\left[2 \operatorname{Re} \sum_{n \leq M} \tilde{Y}_N(n) \overline{\tilde{Z}_M(n)} - \sum_{n \leq M} |\tilde{Z}_M(n)|^2\right] \\ &= \mathbb{E}\left[\sum_{n \leq M} |\tilde{Z}_M(n)|^2 + \sum_{n \leq M} \operatorname{Re} \left(2\lambda_n^{-1} \int_0^1 e^{-c\lambda_n^{-1}\lambda_M(1-\tau)} dB_n(\tau)\right) \overline{\tilde{Z}_M(n)}\right] \\ &\sim \lambda_M^{\frac{2}{s}-1} + O\left(c^{-1} \sum_{n \leq M} \lambda_n^{-1} \lambda_M^{-1}\right) \\ &\sim \lambda_M^{\frac{2}{s}-1} \end{aligned}$$

for any $N \geq M \gg 1$ and $c \gg 1$. Here we used Corollary 2.6 with $p = \frac{1}{2}$ and $s \in (1, 2)$. This proves (4.65).

Note that $\tilde{Y}_N(n) - \tilde{Z}_M(n)$ is a mean-zero Gaussian random variable. Then, from (4.72) and Ito's isometry, we have

$$\begin{aligned} &\mathbb{E}\left[\left(|\tilde{Y}_N(n) - \tilde{Z}_M(n)|^2 - \mathbb{E}[|\tilde{Y}_N(n) - \tilde{Z}_M(n)|^2]\right)^2\right] \\ &= 7\left(\mathbb{E}[|\tilde{Y}_N(n) - \tilde{Z}_M(n)|^2]\right)^2 \\ &= 7\lambda_n^{-4} \left(\int_0^1 e^{-2c\lambda_n^{-1}\lambda_M(1-\tau)} d\tau\right)^2 \\ &\sim \lambda_n^{-2} \lambda_M^{-2}, \end{aligned} \tag{4.74} \quad \boxed{\text{ZZ4}}$$

for $0 \leq n \leq M$, where in the second step we used that $\mathbb{E}[|X|^4] = 8\sigma^4$ for complex random variable $X \sim \mathcal{N}(0, 2\sigma^2)$. Hence, from Plancherel's theorem, (4.69), the independence of $\{B_n\}_{n \in \mathbb{N}}$, the independence of $\{|\tilde{Y}_N(n)|^2 - \mathbb{E}[|\tilde{Y}_N(n)|^2]\}_{M < n \leq N}$ and

$$\{|\tilde{Y}_N(n) - \tilde{Z}_M(n)|^2 - \mathbb{E}[|\tilde{Y}_N(n) - \tilde{Z}_M(n)|^2]\}_{n \leq M},$$

and (4.74), we have

$$\begin{aligned}
& \mathbb{E} \left[\left| \|Y_N - Z_M\|_{L^2(\mathbb{R}^d)}^2 \right|^2 \right] \\
&= \sum_{M < n \leq N} \mathbb{E} \left[\left(|\tilde{Y}_N(n)|^2 - \mathbb{E}[|\tilde{Y}_N(n)|^2] \right)^2 \right] \\
&\quad + \sum_{n \leq M} \mathbb{E} \left[\left(|\tilde{Y}_N(n) - \tilde{Z}_M(n)|^2 - \mathbb{E}[|\tilde{Y}_N(n) - \tilde{Z}_M(n)|^2] \right)^2 \right] \\
&\lesssim \sum_{M < n \leq N} \lambda_n^{-4} + \sum_{n \leq M} \lambda_n^{-2} \lambda_M^{-2} \lesssim \lambda_M^{-3 + \frac{2}{s}},
\end{aligned}$$

where we used Corollary 2.6 with $p = 2$ or $p = 1$ and $s \in (1, 2)$. This proves (4.66).

From (4.59) and the definition of Y_N , we have

$$\begin{aligned}
\mathbb{E} \left[\left| \int_{\mathbb{R}^d} Y_N f_{\lambda_M} dx \right|^2 \right] &= \mathbb{E} \left[\left| \sum_{n \leq N} \tilde{Y}_N(n) \langle f_{\lambda_M}, e_n \rangle \right|^2 \right] = \sum_{n \leq N} \lambda_n^{-2} |\langle f_{\lambda_M}, e_n \rangle|^2 \\
&\leq \int_{\mathbb{R}^d} \left| \mathcal{L}^{-\frac{1}{2}} f_{\lambda_M}(x) \right|^2 dx \lesssim \lambda_M^{-2},
\end{aligned} \tag{4.75} \quad \boxed{\text{app4}}$$

where in the last step we used Lemma 4.4. From (4.71), Ito's isometry, and (4.59), we have

$$\begin{aligned}
\mathbb{E} \left[\left| \sum_{n \leq M} X_n(1) \overline{\langle f_{\lambda_M}, e_n \rangle} \right|^2 \right] &= \mathbb{E} \left[\left| \sum_{n \leq M} \left(\lambda_n^{-1} \int_0^1 e^{-c\lambda_n^{-1}\lambda_M(1-\tau)} dB_n(\tau) \right) \langle f_{\lambda_M}, e_n \rangle \right|^2 \right] \\
&\lesssim \lambda_M^{-1} \sum_{n \leq M} \lambda_n^{-1} |\langle f_{\lambda_M}, e_n \rangle|^2 \\
&\lesssim \lambda_M^{-1} \|\mathcal{L}^{-\frac{1}{4}} f_{\lambda_M}\|_{L^2(\mathbb{R}^d)}^2 \\
&\lesssim \lambda_M^{-2}.
\end{aligned} \tag{4.76} \quad \boxed{\text{app5}}$$

Hence, (4.67) follows from (4.75) and (4.76) with (4.72).

Lastly, from (3.10), (4.70), (4.71), and Ito's isometry, we have

$$\begin{aligned}
\mathbb{E} \left[\int_0^1 \left\| \frac{d}{d\tau} Z_M(\tau) \right\|_{\mathcal{H}^1(\mathbb{R}^d)}^2 d\tau \right] &= \lambda_M \mathbb{E} \left[\int_0^1 \left\| \mathbf{P}_M(Y_N(\tau)) - Z_M(\tau) \right\|_{L^2(\mathbb{R}^d)}^2 d\tau \right] \\
&= \lambda_M \mathbb{E} \left[\int_0^1 \sum_{n \leq M} |X_n(\tau)|^2 d\tau \right] \\
&= \lambda_M \sum_{n \leq M} \int_0^1 \mathbb{E}[|X_n(\tau)|^2] d\tau \\
&= \lambda_M \sum_{n \leq M} \lambda_n^{-2} \int_0^1 \int_0^\tau e^{-2c\lambda_n^{-1}\lambda_M(\tau-\tau')} d\tau' d\tau \\
&\lesssim \lambda_M \sum_{n \leq M} \lambda_n^{-1} \lambda_M^{-1} \\
&\lesssim \lambda_M^{\frac{2}{s}},
\end{aligned}$$

yielding (4.68). This completes the proof of Lemma 4.6. \square

As a consequence of Lemma 4.6, we have a control on the second term of (4.56).

LEM:bddrift

Lemma 4.7 (Entropy of a the test drift).

Let θ^0 be as in (4.62), then we have

$$\int_0^1 \mathbb{E} \left[\|\theta^0(t)\|_{L^2(\mathbb{R}^d)}^2 \right] dt \lesssim \lambda_M^{\frac{2}{s}+1},$$

uniformly in $N \geq N_0(M) \gg M \geq 1$.

Proof. From (4.68) and (4.62), it suffices to show that

$$\alpha_{M,N} \|f_{\lambda_M}\|_{H^1(\mathbb{R}^d)}^2 \lesssim \lambda_M^{\frac{2}{s}+1}.$$

However, from (4.58) and (4.65), we have

$$\alpha_{M,N} \sim \lambda_M^{\frac{2}{s}-1} \quad (4.77) \quad \boxed{\text{logM}}$$

provided $N \gg M$ and N is sufficiently large. The conclusion follows from (4.59) and (4.77) provided $N \gg M$. \square

In what follows, we abuse notation by denoting

$$\Theta_N^0 = \mathbf{P}_N \Theta^0 = -Z_M + \sqrt{\alpha_{M,N}} (\mathbf{P}_N f_{\lambda_M}) \quad (4.78) \quad \boxed{\text{YY0a}}$$

for $N \geq M \geq 1$. Now we vindicate our claim about the Wick-ordered L^2 mass of our trial state.

LEM:key

Lemma 4.8 (Mass of the test drift).

For any $K > 0$, there exists $M_0 = M_0(K) \geq 1$ such that

$$\mathbb{P} \left(\left| \int_{\mathbb{R}^d} |Y_N(x)|^2 dx + \int_{\mathbb{R}^d} 2 \operatorname{Re}(Y_N \overline{\Theta_N^0}) + |\Theta_N^0|^2 dx \right| \leq K \right) \geq \frac{1}{2}, \quad (4.79) \quad \boxed{\text{pa5}}$$

uniformly in $N \geq N_0(M) \gg M \geq M_0$.

Proof. First, from (4.58), we note that the condition $N \geq N_0(M) \gg M$ guarantees that

$$\int_{\mathbb{R}^d} |\mathbf{P}_N f_{\lambda_M}|^2 dx \gtrsim 1, \quad (4.80) \quad \boxed{\text{N}_0(M)}$$

which further implies that $\alpha_{M,N} \sim \lambda_M^{\frac{2}{s}-1}$. From (4.63), we have

$$\begin{aligned} & \mathbb{E} \left[\left| \int_{\mathbb{R}^d} |Y_N(x)|^2 dx + \int_{\mathbb{R}^d} 2 \operatorname{Re}(Y_N \overline{\Theta_N^0}) + |\Theta_N^0|^2 dx \right|^2 \right] \\ &= \mathbb{E} \left[\left| \int_{\mathbb{R}^d} |Y_N(x)|^2 dx - 2 \int_{\mathbb{R}^d} \operatorname{Re}(Y_N \overline{Z_M}) dx + \int_{\mathbb{R}^d} |Z_M|^2 dx \right. \right. \\ & \quad \left. \left. + \alpha_{M,N} \int_{\mathbb{R}^d} |\mathbf{P}_N f_{\lambda_M}|^2 dx + 2\sqrt{\alpha_{M,N}} \int_{\mathbb{R}^d} \operatorname{Re}((Y_N - Z_M) f_{\lambda_M}) dx \right|^2 \right]. \end{aligned} \quad (4.81) \quad \boxed{\text{Pr1}}$$

From (4.77) and (4.67) in Lemma 4.6, we have

$$\mathbb{E} \left[\left| \sqrt{\alpha_{M,N}} \int_{\mathbb{R}^d} (Y_N - Z_M) f_{\lambda_M} dx \right|^2 \right] \lesssim \lambda_M^{\frac{2}{s}-3}. \quad (4.82) \quad \boxed{\text{Pr4}}$$

On the other hand, from (4.61) and (4.69), we have

$$\begin{aligned} & \int_{\mathbb{R}^d} :|Y_N(x)|^2: dx - 2 \int_{\mathbb{R}^d} \operatorname{Re}(Y_N \overline{Z_M}) dx + \int_{\mathbb{R}^d} |Z_M|^2 dx + \alpha_{M,N} \int_{\mathbb{R}^d} |\mathbf{P}_N f_{\lambda_M}|^2 dx \\ &= \int_{\mathbb{R}^d} |Y_N - Z_M|^2 - \mathbb{E}[|Y_N - Z_M|^2] dx \\ &=: \|Y_N - Z_M\|_{L^2(\mathbb{R}^d)}^2. \end{aligned} \quad (4.83) \quad \boxed{\text{Pr2}}$$

Hence, from (4.81), (4.82), and (4.83) with (4.66) in Lemma 4.6, we obtain

$$\mathbb{E} \left[\left| \int_{\mathbb{R}^d} :|Y_N(x)|^2: dx + \int_{\mathbb{R}^d} 2 \operatorname{Re}(Y_N \overline{\Theta_N^0}) + |\Theta_N^0|^2 dx \right|^2 \right] \lesssim \lambda_M^{-3+\frac{2}{s}}.$$

Therefore, by Chebyshev's inequality, given any $K > 0$, there exists $M_0 = M_0(K) \geq 1$ such that

$$\mathbb{P} \left(\left| \int_{\mathbb{R}^d} :|Y_N(x)|^2: dx + \int_{\mathbb{R}^d} 2 \operatorname{Re}(Y_N \overline{\Theta_N^0}) + |\Theta_N^0|^2 dx \right| > K \right) \leq C \frac{\lambda_M^{-3+\frac{2}{s}}}{K^2} < \frac{1}{2} \quad (4.84) \quad \boxed{\text{M}_0(K)}$$

for any $M \geq M_0(K)$ and $s \in (1, 2)$. This proves (4.79). \square

4.2.2. Divergence of the partition function. We are now ready to prove:

Lemma 4.9 (The test drift leads to divergences).

Given $K > 0$, let M and N as in Lemma 4.8. Recall the choice $\Theta^0 = -Z_M + \sqrt{\alpha_{M,N}} f_{\lambda_M}$ from Definition 4.5. Then

$$\mathbb{E} \left[-\alpha R_p(Y_N + \Theta^0) \cdot \mathbf{1}_{\left\{ \int_{\mathbb{R}^d} :|Y_N|^2: + 2 \operatorname{Re}(Y_N \overline{\Theta_N^0}) + |\Theta_N^0|^2 dx \leq K \right\}} + \frac{1}{2} \int_0^1 \|\theta^0(t)\|_{L^2(\mathbb{R}^d)}^2 dt \right] \xrightarrow{M \rightarrow \infty} -\infty.$$

Remark 4.1. Here $M \leq N$ are chosen such that Lemma 4.7 and Lemma 4.8 hold. In particular, given $K > 0$, there exists $M_0(K) \gg 1$ such that (4.84) holds for all $M \geq M_0(K)$. With $M \geq M_0(K)$ chosen, there exists $N_0(M) \geq M$ such that (4.80) (or equivalently (4.77)) holds for all $N \geq N_0(M)$.

Proof. Using the mean value theorem and Young's inequality, we have for any $\varepsilon > 0$,

$$\begin{aligned} & \left| R_p(Y_N + \Theta^0) - R_p(\sqrt{\alpha_{M,N}} f_{\lambda_M}) \right| \\ & \leq C \int_{\mathbb{R}^d} |Y_N - Z_M| (|Y_N - Z_M| + |\sqrt{\alpha_{M,N}} f_{\lambda_M}|)^{p-1} dx \\ & \leq \varepsilon R_p(\sqrt{\alpha_{M,N}} f_{\lambda_M}) + C_\varepsilon R_p(Y_N - Z_M). \end{aligned} \quad (4.85) \quad \boxed{\text{paa}}$$

Moreover, we have

$$\begin{aligned} \mathbb{E} [R_p(Y_N - Z_M)] &= \frac{1}{p} \int_{\mathbb{R}^d} \mathbb{E} \left[\left| \sum_{M < n \leq N} \frac{B_n(1)e_n}{\lambda_n} + \sum_{n \leq M} X_n(1)e_n \right|^p \right] dx \\ &\lesssim \int_{\mathbb{R}^d} \left(\mathbb{E} \left[\left| \sum_{M < n \leq N} \frac{B_n(1)e_n}{\lambda_n} + \sum_{n \leq M} X_n(1)e_n \right|^2 \right] \right)^{\frac{p}{2}} dx \\ &\lesssim \int_{\mathbb{R}^d} \left(\sum_{M < n \leq N} \frac{e_n^2(x)}{\lambda_n^2} + \sum_{n \leq M} \frac{e_n^2(x)}{\lambda_n \lambda_M} \right)^{\frac{p}{2}} dx \\ &\lesssim \int_{\mathbb{R}^d} \left(\sum_{n \leq N} \frac{e_n^2(x)}{\lambda_n^2} \right)^{\frac{p}{2}} dx \\ &\lesssim 1, \end{aligned} \quad (4.86) \quad \boxed{\text{paa2}}$$

provided $p > \frac{4}{s}$, uniformly in M and N . Here we use the fact that $X_n(1)$ is a Gaussian random variable with variance $\sim (\lambda_n \lambda_M)^{-1}$.

We are now ready to put everything together. It follows from (4.56), (4.85), (4.86), and (4.78) that there exists $C > 0$ such that

$$\begin{aligned}
& -\log \mathbb{E}_\mu \left[\exp \left(\alpha R_p(u_N) \cdot \mathbf{1}_{\{|\int_{\mathbb{R}^d} |u_N|^2 : dx| \leq K\}} \right) \right] \\
& \leq \mathbb{E} \left[-\alpha R_p(Y_N + \Theta^0) \cdot \mathbf{1}_{\{|\int_{\mathbb{R}^d} |Y_N|^2 : dx + \int_{\mathbb{R}^d} 2 \operatorname{Re}(Y_N \overline{\Theta_N^0}) + |\Theta_N^0|^2 : dx| \leq K\}} + \frac{1}{2} \int_0^1 \|\theta^0(t)\|_{L^2(\mathbb{R}^d)}^2 dt \right] \\
& \leq \mathbb{E} \left[-\frac{1}{2} R_p(\sqrt{\alpha_{M,N}} f_{\lambda_M}) \cdot \mathbf{1}_{\{|\int_{\mathbb{R}^d} |Y_N|^2 : dx + \int_{\mathbb{R}^d} 2 \operatorname{Re}(Y_N \overline{\Theta_N^0}) + |\Theta_N^0|^2 : dx| \leq K\}} \right. \\
& \quad \left. + C R_p(Y_N - Z_M) + \frac{1}{2} \int_0^1 \|\theta^0(t)\|_{L^2(\mathbb{R}^d)}^2 dt \right] \\
& \leq -\frac{1}{2} R_p(\sqrt{\alpha_{M,N}} f_{\lambda_M}) \cdot \mathbb{P} \left(\left| \int_{\mathbb{R}^d} |Y_N|^2 : dx + \int_{\mathbb{R}^d} 2 \operatorname{Re}(Y_N \overline{\Theta_N^0}) + |\Theta_N^0|^2 : dx \right| \leq K \right) \\
& \quad + C \mathbb{E} [R_p(Y_N - Z_M)] + \frac{1}{2} \int_0^1 \mathbb{E} [\|\theta^0(t)\|_{L^2(\mathbb{R}^d)}^2] dt
\end{aligned}$$

then from Lemma 4.8, (4.86), and Lemma 4.7, we may continue with

$$\leq -C_1^p \alpha \lambda_M^{\frac{dp-d}{2}} (\alpha_{M,N})^{\frac{p}{2}} + C_2^2 \lambda_M^2 \alpha_{M,N} + C_3$$

for some constants $C_1, C_2, C_3 > 0$, provided $N \geq N_0(M) \gg M \geq M_0(K)$. Therefore, when $p > 2 + \frac{4s}{(d-1)s+2}$, it follows that

$$\begin{aligned}
& \liminf_{N \rightarrow \infty} \mathbb{E}_\mu \left[\exp \left(R_p(u_N) \cdot \mathbf{1}_{\{|\int_{\mathbb{R}^d} |u_N|^2 : dx| \leq K\}} \right) \right] \\
& \geq \exp \left(C_1^p \alpha \lambda_M^{\frac{dp-d}{2}} (\lambda_M^s)^{\frac{2-p}{2}} - C_2^2 \lambda_M^2 \lambda_M^{\frac{2-p}{2}} - C_3 \right),
\end{aligned}$$

which diverges to infinity as $M \rightarrow \infty$ provided $p > 2 + \frac{4s}{(d-1)s+2}$ and $K, \alpha > 0$.

It remains to consider the critical case when $p = 2 + \frac{4s}{(d-1)s+2}$. From the above computation, we see that

$$\begin{aligned}
& \sup_{N \in \mathbb{N}} \mathbb{E}_\mu \left[\exp \left(\alpha R_p(u_N) \right) \cdot \mathbf{1}_{\{|\int_{\mathbb{R}^d} |u_N|^2 : dx| \leq K\}} \right] \\
& = \sup_{N \in \mathbb{N}} \left\| \mathbf{1}_{\{|\int_{\mathbb{R}^d} |u_N(x)|^2 : dx| \leq K\}} e^{\frac{\alpha}{p} \|u_N\|_{L^p(\mathbb{R}^d)}^p} \right\|_{L^1(\mu)} \\
& = \infty
\end{aligned} \tag{4.87}$$

div_cri

provided $\alpha \gg 1$ such that $C_1^p \alpha > C_2^2$. In particular, this shows when $p = 2 + \frac{4s}{(d-1)s+2}$ the number $\alpha_0(K)$ defined in (4.19) is bounded for given $K > 0$. From the definition (4.19), given $K > 0$, we see that (4.87) holds for all $\alpha > \alpha_0(K)$. Thus, we finish the proof of (4.53) for $p \geq 2 + \frac{4s}{(d-1)s+2}$, and $\alpha > \alpha_0(K)$ for any given $K > 0$ when $p = 2 + \frac{4s}{(d-1)s+2}$. \square

4.2.3. Intermediate cases. This subsection considers the proof of Theorem 1.3 (ii) - (b). From the previous subsection, we have (4.53). Recall the decomposition (1.12)

$$d\mu(u) = d\mu_N(u_N) \otimes d\mu_N^\perp(u_N^\perp),$$

where $u_N = \mathbf{P}_N u$ and $u_N^\perp = u - \mathbf{P}_N u$. Moreover, by (3.4), we have that

$$\text{Law}(Y_N(1), Y(1) - Y_N(1)) = \mu_N \otimes \mu_N^\perp.$$

Define the set

$$\Omega_K^\perp = \left\{ u_N^\perp : \left| \int_{\mathbb{R}^d} :|u_N^\perp|^2 : dx \right| \leq \frac{K}{2}, \frac{1}{p} \int_{\mathbb{R}^d} |u_N^\perp|^p dx \leq 1 \right\}, \quad (4.88)$$

muperpdef

where we defined

$$\int_{\mathbb{R}^d} :|u_N^\perp|^2 : dx = \sum_{n=N+1}^{\infty} \frac{|g_n|^2 - 2}{\lambda_n^2},$$

which is chosen so that

$$\int_{\mathbb{R}^d} :|u_N^\perp|^2 : dx = \int_{\mathbb{R}^d} :|u|^2 : dx - \int_{\mathbb{R}^d} :|u_N|^2 : dx.$$

The proof of Theorem 1.3 (ii) - (b) requires a delicate analysis of the cut-off function when it is slightly perturbed. To overcome the challenges that arise, we introduce a crucial lemma that allows us to preserve the cut-off size K in the approximation process.

LEM:sameK

Lemma 4.10. *Consider the sets*

$$\begin{aligned} \Omega_+^\perp &:= \left\{ u_N^\perp \in \Omega_K^\perp : \int_{\mathbb{R}^d} :|u_N^\perp|^2 : dx \geq 0 \right\}, \\ \Omega_-^\perp &:= \left\{ u_N^\perp \in \Omega_K^\perp : \int_{\mathbb{R}^d} :|u_N^\perp|^2 : dx \leq 0 \right\}. \end{aligned}$$

Then there exists $\varepsilon_0 > 0$ such that

$$\min(\mu_N^\perp(\Omega_+^\perp), \mu_N^\perp(\Omega_-^\perp)) \geq \varepsilon_0 \quad (4.89)$$

sameK_1

for every $N \gg 1$ large enough.

Before we prove Lemma 4.10, we prepare a technical lemma.

LEM:tech

Lemma 4.11. *Let $Y \geq 0$ be a random variable such that $0 < c(\mathbb{E}[Y^2])^{\frac{1}{2}} \leq \mathbb{E}[Y] < \infty$ for some positive constant c . Then, we have*

$$\mathbb{P}(Y > 0) \geq \frac{c^2}{4}.$$

Proof. We first note that up to multiplying Y by a constant, we can assume that $\mathbb{E}[Y^2] = 1$. For $M > 0$, we have

$$\mathbb{E}[Y \mathbf{1}_{Y > M}] \leq \mathbb{E}\left[\frac{Y^2}{M} \mathbf{1}_{Y > M}\right] \leq \frac{1}{M}.$$

Therefore, by choosing $M = \frac{2}{c}$, we obtain that

$$\frac{c}{2} \leq \mathbb{E}[Y] - \mathbb{E}[Y \mathbf{1}_{Y > M}] \leq \mathbb{E}[Y \mathbf{1}_{Y \leq M}] \leq M \mathbb{P}(Y > 0).$$

Therefore,

$$\mathbb{P}(Y > 0) \geq \frac{1}{M} \cdot \frac{c}{2} = \frac{c^2}{4}.$$

We thus finish the proof. \square

We are ready to prove Lemma 4.10.

Proof of Lemma 4.10. Define the random variables

$$Y = \int_{\mathbb{R}^d} :|u_N^\perp|^2: dx \text{ and } Y_\pm = \left(\int_{\mathbb{R}^d} :|u_N^\perp|^2: dx \right)_\pm,$$

where u_N^\perp is distributed according to μ_N^\perp , $a_+ = \max(0, a)$, and $a_- = |\min(0, a)|$. By Corollary 3.4 and Corollary 3.5, we have that $\mu_N^\perp(\Omega_K^\perp) \rightarrow 1$ as $N \rightarrow \infty$. As

$$\Omega_\pm^\perp = \Omega_K \cap \{\omega : Y_\pm > 0\},$$

so for (4.89), it is enough to show that

$$\min(\mathbb{P}(Y_+ > 0), \mathbb{P}(Y_- > 0)) > 2\varepsilon_0, \quad (4.90) \quad \boxed{\text{same_K2}}$$

for some ε_0 independent of N . From Corollary 3.5 and the fact

$$\mathbb{E} \left[\int_{\mathbb{R}^d} :|u_N^\perp|^2: dx \right] = 0,$$

we have that

$$\mathbb{E}[Y_+] = \mathbb{E}[Y_-] = \frac{1}{2} \mathbb{E}[|Y|] < \infty. \quad (4.91) \quad \boxed{\text{Ypm}}$$

Moreover, by a direct computation (as in the proof of Corollary 3.5 and (3.14)), we have

$$\begin{aligned} \mathbb{E}[Y^4] &= \mathbb{E} \left[\left(\sum_{n=N+1}^{\infty} \frac{|B_n|^2 - 2}{\lambda_n^2} \right)^4 \right] \\ &= -2 \left(\sum_{n=N+1}^{\infty} \frac{\mathbb{E}[(|B_n|^2 - 2)^4]}{\lambda_n^8} \right) + 3 \left(\sum_{n=N+1}^{\infty} \frac{\mathbb{E}[(|B_n|^2 - 2)^2]}{\lambda_n^4} \right)^2 \\ &\leq C \left(\sum_{n=N+1}^{\infty} \frac{\mathbb{E}[(|B_n|^2 - 2)^2]}{\lambda_n^4} \right)^2 = C (\mathbb{E}[Y^2])^2 \end{aligned} \quad (4.92) \quad \boxed{\text{Y4}}$$

for some universal constant $C > 0$. Therefore, by Hölder inequality and (4.92), we have

$$\mathbb{E}[Y^2] \leq \mathbb{E}[|Y|]^{\frac{2}{3}} \mathbb{E}[Y^4]^{\frac{1}{3}} \leq C^{\frac{1}{3}} \mathbb{E}[|Y|]^{\frac{2}{3}} \mathbb{E}[Y^2]^{\frac{2}{3}}.$$

The above also reads

$$\mathbb{E}[|Y|] \geq C^{-\frac{1}{2}} \mathbb{E}[Y^2]^{\frac{1}{2}},$$

which together with (4.91) implies

$$\mathbb{E}[Y_\pm] \geq \frac{1}{2} C^{-\frac{1}{2}} \mathbb{E}[Y^2]^{\frac{1}{2}} \geq \frac{1}{2} C^{-\frac{1}{2}} \mathbb{E}[Y_\pm^2]^{\frac{1}{2}}. \quad (4.93) \quad \boxed{\text{Ypm2}}$$

Then (4.89) follows from (4.93) and Lemma 4.11 by taking $\varepsilon_0 = \frac{1}{32C}$. \square

From the elementary inequality

$$|a + b|^p \geq (1 - \varepsilon)|a|^p - C_\varepsilon |b|^p$$

for some constant $C_\varepsilon > 0$, we obtain that

$$\exp \left(\frac{\alpha}{p} \int_{\mathbb{R}^d} |u|^p dx \right) \geq \exp \left(-\frac{C_\varepsilon}{p} \int_{\mathbb{R}^d} |u_N^\perp|^p dx \right) \exp \left(\frac{\alpha - \varepsilon}{p} \int_{\mathbb{R}^d} |u_N|^p dx \right). \quad (4.94) \quad \boxed{\text{elementary}}$$

Therefore, by (4.88), (4.94), Lemma 4.10, and (4.53) with $r = 1$, we obtain

$$\begin{aligned}
\mathcal{Z}_K &= \int \exp\left(\frac{\alpha}{p} \int_{\mathbb{R}^d} |u|^p dx\right) \mathbf{1}_{\{|f_{\mathbb{R}^d} : |u(x)|^2 : dx| \leq K\}} d\mu(u) \\
&\geq \int \exp\left(-\frac{C_\varepsilon}{p} \int_{\mathbb{R}^d} |u_N^\perp|^p dx\right) \exp\left(\frac{\alpha - \varepsilon}{p} \int_{\mathbb{R}^d} |u_N|^p dx\right) \\
&\quad \times \mathbf{1}_{\{|f_{\mathbb{R}^d} : |u_N(x)|^2 : dx + f_{\mathbb{R}^d} : |u_N^\perp(x)|^2 : dx| \leq K\}} \mathbf{1}_{\Omega_{\text{sgn}(f_{\mathbb{R}^d} : |u_N(x)|^2 : dx)}}^\perp (u_N^\perp) d\mu_N(u_N) d\mu_N^\perp(u_N^\perp) \\
&\geq \int e^{-C_\varepsilon} \exp\left(\frac{\alpha - \varepsilon}{p} \int_{\mathbb{R}^d} |u_N|^p dx\right) \mathbf{1}_{\{|f_{\mathbb{R}^d} : |u_N(x)|^2 : dx| \leq K\}} \mathbf{1}_{\Omega_{\text{sgn}(f_{\mathbb{R}^d} : |u_N(x)|^2 : dx)}}^\perp (u_N^\perp) d\mu_N(u_N) d\mu_N^\perp(u_N^\perp) \\
&\geq e^{-C_\varepsilon \varepsilon_0} \int \exp\left(\frac{\alpha - \varepsilon}{p} \int_{\mathbb{R}^d} |u_N|^p dx\right) \mathbf{1}_{\{|f_{\mathbb{R}^d} : |u_N(x)|^2 : dx| \leq K\}} d\mu_N(u_N) \\
&\geq e^{-C_\varepsilon \varepsilon_0} \int \exp\left(\frac{\alpha - \varepsilon}{p} \int_{\mathbb{R}^d} |u_N|^p dx\right) \mathbf{1}_{\{|f_{\mathbb{R}^d} : |u_N(x)|^2 : dx| \leq K\}} d\mu_N(u_N),
\end{aligned}$$

provided $N \gg 1$, which together with (4.19) (or (4.53)) implies that

$$\mathcal{Z}_K \geq \limsup_{N \rightarrow \infty} e^{-C_\varepsilon \varepsilon_0} \int \exp\left(\frac{\alpha - \varepsilon}{p} \int_{\mathbb{R}^d} |u_N|^p dx\right) \mathbf{1}_{\{|f_{\mathbb{R}^d} : |u_N(x)|^2 : dx| \leq K\}} d\mu_N(u_N) = \infty$$

provided $\varepsilon \ll 1$ such that $\alpha - \varepsilon > \alpha_0(K)$ for given $K > 0$. This concludes the proof of (1.23), and hence Theorem 1.3 (ii) - (b).

5. SUPERHARMONIC POTENTIAL

In this section, we see how to extend the previous results for cases of $s \in (1, 2)$ to cases of $s > 2$. The argument of this section is inspired by [42]. One of the key advantages of the superharmonic case is that $\|u\|_{L^2(\mathbb{R}^d)} < \infty$ almost surely with respect to μ , i.e.

$$\mathbb{E}_\mu[\|u\|_{L^2(\mathbb{R}^d)}^2] = \mathbb{E}[\|Y(1)\|_{L^2(\mathbb{R}^d)}^2] = \sum_{n=0}^{\infty} \frac{2}{\lambda_n^2} < \infty, \quad (5.1) \quad \boxed{\text{sup_L2}}$$

where $Y(1)$ is defined in (3.3), in view of Lemma 2.2. Furthermore, for $q \geq p$ we have

$$\begin{aligned}
(\mathbb{E}_\mu[\|u\|_{L^p(\mathbb{R}^d)}^q])^{\frac{1}{q}} &\leq \left\| \|Y(1)\|_{L^q(\Omega)} \right\|_{L^p(\mathbb{R}^d)} \\
&\leq C(q) \left\| \|Y(1)\|_{L^2(\Omega)} \right\|_{L^p(\mathbb{R}^d)} \\
&= C(q) \left\| \left(\sum_{n \geq 0} \frac{e_n^2}{\lambda_n^2} \right)^{\frac{1}{2}} \right\|_{L^p(\mathbb{R}^d)} \\
&= C(q) \|\mathcal{L}^{-1}(x, x)\|_{L^{\frac{p}{2}}(\mathbb{R}^d)}^{\frac{1}{2}} < \infty,
\end{aligned}$$

provided $p > 2$ and $p < \frac{2d}{d-2}$ when $d \geq 3$, by using Lemma 2.3. The Hölder inequality then implies

$$\mathbb{E}_\mu[\|u\|_{L^p(\mathbb{R}^d)}^q] = \mathbb{E}[\|Y(1)\|_{L^p(\mathbb{R}^d)}^q] < \infty \quad (5.2) \quad \boxed{\text{sup_Lp}}$$

provided $1 \leq q < \infty$, $p > 2$ and $p < \frac{2d}{d-2}$ when $d \geq 3$. We also need the following consequence of Fernique's theorem [26]. See also Theorem 2.7 in [19] and Lemma 3.3 in [54].

LEM:Fer

Lemma 5.1 (Fernique-type bounds).

There exists a constant $c > 0$ such that if X is a mean-zero Gaussian process with values in a separable Banach space B with $\mathbb{E}[\|X\|_B] < \infty$, then

$$\int e^{\frac{c\|X\|_B^2}{(\mathbb{E}\|X\|_B)^2}} d\mathbb{P} < \infty.$$

In particular, we have

$$\mathbb{P}(\|X\|_B \geq t) \lesssim \exp\left[-\frac{ct^2}{(\mathbb{E}\|X\|_B)^2}\right]$$

for any $t > 1$.

SEC:nor1

5.1. Normalizability. In this subsection, we provide the proof of the integrability part of Theorem 1.4. Namely, we prove:

Lemma 5.2 (Integrability for superharmonic potentials).

Let $s > 2$. Assume either one of the following conditions:

$$(i) \text{ subcritical nonlinearity: } 2 < p < 2 + \frac{4}{d} \text{ and } K > 0;$$

$$(ii) \text{ critical nonlinearity: } p = 2 + \frac{4}{d} \text{ and } K < \|Q\|_{L^2(\mathbb{R}^d)}^2.$$

(5.3) cond_int

Then

$$\mathcal{Z}_K = \mathbb{E}_\mu \left[\exp(\alpha R_p(u)) \cdot \mathbf{1}_{\{\|u\|_{L^2(\mathbb{R}^d)}^2 \leq K\}} \right] < \infty,$$

(5.4) var1

where $R_p(u)$ is given in (1.4).

Proof. Step 1. Preliminaries. Observing that

$$\mathbb{E}_\mu \left[\exp(\alpha R_p(u)) \cdot \mathbf{1}_{\{\|u\|_{L^2(\mathbb{R}^d)}^2 \leq K\}} \right] \leq \mathbb{E}_\mu \left[\exp \left(\alpha R_p(u) \cdot \mathbf{1}_{\{\|u\|_{L^2(\mathbb{R}^d)}^2 \leq K\}} \right) \right],$$

the bound (5.4) follows once we have

$$\mathbb{E}_\mu \left[\exp \left(\alpha R_p(u) \cdot \mathbf{1}_{\{\|u\|_{L^2(\mathbb{R}^d)}^2 \leq K\}} \right) \right] < \infty.$$

(5.5) var3_1

One can observe that the equation (5.5) does not require any frequency truncation \mathbf{P}_N unlike the subharmonic case (4.2). The main reason is that the L^2 mass does not involve the Wick renormalization. On the other hand, the equation (4.2) needs the frequency truncation \mathbf{P}_N to deal with the Wick power $|u_N|^2$: defined by (1.14) and (1.15).

Using $\text{Law}(Y(1)) = \mu$, we apply the Boué-Dupuis variational formula⁷, Lemma 3.1 to

$$F(Y(1)) = -\alpha R_p(Y(1)) \cdot \mathbf{1}_{\{\|Y(1)\|_{L^2(\mathbb{R}^d)}^2 \leq K\}}$$

⁷See [64, Theorem 3.2] for a version of the non-singular case, where the frequency cut-off is not needed. Also, see [60, Proposition A.1] or [42, Lemma 3.1] for similar results.

and get

$$\begin{aligned}
& -\log \mathbb{E}_\mu \left[\exp \left(\alpha R_p(u) \cdot \mathbf{1}_{\{\|u\|_{L^2(\mathbb{R}^d)}^2 \leq K\}} \right) \right] \\
&= -\log \mathbb{E} \left[e^{-F(Y(1))} \right] \\
&= \inf_{\theta \in \mathbb{H}_a} \mathbb{E} \left[-\alpha R_p(Y(1) + I(\theta)(1)) \cdot \mathbf{1}_{\{\|Y(1) + I(\theta)(1)\|_{L^2(\mathbb{R}^d)}^2 \leq K\}} \right. \\
&\quad \left. + \frac{1}{2} \int_0^1 \|\theta(t)\|_{L^2(\mathbb{R}^d)}^2 dt \right],
\end{aligned} \tag{5.6} \quad \boxed{\text{var4_1}}$$

where $Y(1)$ is given in (3.3). Here, \mathbb{E}_μ and \mathbb{E} denote expectations with respect to the Gaussian field μ and the underlying probability measure \mathbb{P} respectively. In the following, we show that the right hand side of (5.6) has a finite lower bound.

Step 2. Subcritical case. In the case

$$2 < p < 2 + \frac{4}{d},$$

we prove (5.5) with a mass cut-off of any finite size K . Then, by using (4.11) with $\varepsilon = 1$ and the sharp Gagliardo-Nirenberg-Sobolev inequality, we obtain

$$\begin{aligned}
& \alpha R_p(Y(1) + I(\theta)(1)) \cdot \mathbf{1}_{\{\|Y(1) + I(\theta)(1)\|_{L^2(\mathbb{R}^d)}^2 \leq K\}} \\
& \leq 2\alpha R_p(I(\theta)(1)) \cdot \mathbf{1}_{\{\|I(\theta)(1)\|_{L^2(\mathbb{R}^d)} \leq \sqrt{K} + \|Y(1)\|_{L^2(\mathbb{R}^d)}\}} + C\alpha R_p(Y(1)) \\
& \leq \frac{2\alpha}{p} C_{\text{GNS}} (\sqrt{K} + \|Y(1)\|_{L^2(\mathbb{R}^d)})^{\frac{4-(d-2)(p-2)}{2}} \|I(\theta)(1)\|_{\mathcal{H}^1(\mathbb{R}^d)}^{\frac{d(p-2)}{2}} + C\alpha R_p(Y(1)), \\
& \leq C + C\|Y(1)\|_{L^2(\mathbb{R}^d)}^{\frac{2(4-(d-2)(p-2))}{4-d(p-2)}} + \frac{1}{4} \|I(\theta)(1)\|_{\mathcal{H}^1(\mathbb{R}^d)}^2 + C R_p(Y(1)),
\end{aligned} \tag{5.7} \quad \boxed{\text{var5_1}}$$

where C_{GNS} is the sharp Gagliardo-Nirenberg-Sobolev constant and the constant C depends only on d, p, α . By collecting (5.6), (5.7) and Lemma 3.2, we arrive at

$$\begin{aligned}
& -\log \mathbb{E}_\mu \left[\exp \left(\alpha R_p(u) \cdot \mathbf{1}_{\{\|u\|_{L^2(\mathbb{R}^d)}^2 \leq K\}} \right) \right] \\
& \geq \inf_{\theta \in \mathbb{H}_a} \mathbb{E} \left[-C - C\|Y(1)\|_{L^2(\mathbb{R}^d)}^{\frac{2(4-(d-2)(p-2))}{4-d(p-2)}} - C R_p(Y(1)) \right. \\
&\quad \left. - \frac{1}{4} \|I(\theta)(1)\|_{\mathcal{H}^1(\mathbb{R}^d)}^2 + \frac{1}{2} \int_0^1 \|\theta(t)\|_{L^2(\mathbb{R}^d)}^2 dt \right] \\
& \geq \inf_{\theta \in \mathbb{H}_a} \mathbb{E} \left[-C - C\|Y(1)\|_{L^2(\mathbb{R}^d)}^{\frac{2(4-(d-2)(p-2))}{4-d(p-2)}} - C\|Y(1)\|_{L^p(\mathbb{R}^d)}^p \right. \\
&\quad \left. + \frac{1}{4} \int_0^1 \|\theta(t)\|_{L^2(\mathbb{R}^d)}^2 dt \right] \\
& \geq \mathbb{E} \left[-C - C\|Y(1)\|_{L^p(\mathbb{R}^d)}^p - C\|Y(1)\|_{L^2(\mathbb{R}^d)}^{\frac{2(4-(d-2)(p-2))}{4-d(p-2)}} \right] \\
& > -\infty,
\end{aligned}$$

where we used (5.1) and (5.2) in the second to last step, i.e.

$$\mathbb{E} \left[\|Y(1)\|_{L^p(\mathbb{R}^d)}^p + \|Y(1)\|_{L^2(\mathbb{R}^d)}^{\frac{2(4-(d-2)(p-2))}{4-d(p-2)}} \right] < \infty.$$

Here C is a constant that may vary from line to line. Thus we finish the proof of (5.5) in the subcritical case.

Step 3. Critical case. Let now

$$p = 2 + \frac{4}{d}.$$

We shall prove (5.5) under the assumption $\alpha^{\frac{d}{2}} K < \|Q\|_{L^2(\mathbb{R}^d)}^2$.

Since $s > 2$, from (5.1), it follows that

$$\lim_{N \rightarrow \infty} \|\mathbf{P}_N^\perp Y(1)\|_{L^2(\mathbb{R}^d)} = 0,$$

almost surely. Therefore, given small $\varepsilon > 0$, for $\omega \in \Omega$ almost sure, there exists a unique $N_\varepsilon := N_\varepsilon(\omega)$ such that $N_\varepsilon = 1$ for $\omega \in \{\omega : \|\mathbf{P}_1^\perp Y(1)\|_{L^2(\mathbb{R}^d)} \leq \varepsilon\}$; otherwise

$$N_\varepsilon = \inf \left\{ N \text{ is dyadic} : N \geq 2 \text{ such that } \|\mathbf{P}_{\frac{N}{2}}^\perp Y(1)\|_{L^2(\mathbb{R}^d)} > \varepsilon \text{ and } \|\mathbf{P}_N^\perp Y(1)\|_{L^2(\mathbb{R}^d)} \leq \varepsilon \right\}. \quad (5.8) \quad \boxed{\text{Neps}}$$

Similar argument as before combined with (4.11) and (5.8) yield that

$$\begin{aligned} & \alpha R_p(Y(1) + I(\theta)(1)) \cdot \mathbf{1}_{\{\|Y(1)+I(\theta)(1)\|_{L^2(\mathbb{R}^d)}^2 \leq K\}} \\ & \leq \alpha(1 + \varepsilon) R_p(\mathbf{P}_{N_\varepsilon} Y(1) + I(\theta)(1)) \cdot \mathbf{1}_{\{\|\mathbf{P}_{N_\varepsilon} Y(1)+I(\theta)(1)\|_{L^2(\mathbb{R}^d)} \leq \sqrt{K} + \varepsilon\}} + C_\varepsilon \alpha R_p(\mathbf{P}_{N_\varepsilon}^\perp Y(1)) \\ & \leq \alpha \frac{1 + \varepsilon}{p} C_{\text{GNS}}(\sqrt{K} + \varepsilon)^{p-2} \left(\|\mathbf{P}_{N_\varepsilon} Y(1)\|_{\mathcal{H}^1(\mathbb{R}^d)} + \|I(\theta)(1)\|_{\mathcal{H}^1(\mathbb{R}^d)} \right)^2 + C_\varepsilon \alpha R_p(\mathbf{P}_{N_\varepsilon}^\perp Y(1)) \\ & \leq \alpha \frac{(1 + \varepsilon)^2}{p} C_{\text{GNS}}(\sqrt{K} + \varepsilon)^{p-2} \|I(\theta)(1)\|_{\mathcal{H}^1(\mathbb{R}^d)}^2 + C_\varepsilon \|\mathbf{P}_{N_\varepsilon} Y(1)\|_{\mathcal{H}^1(\mathbb{R}^d)}^2 + C_\varepsilon \alpha R_p(\mathbf{P}_{N_\varepsilon}^\perp Y(1)). \end{aligned} \quad (5.9) \quad \boxed{\text{var5a}}$$

Since $p = 2 + \frac{4}{d}$, $C_{\text{GNS}} = \frac{p}{2} \|Q\|_{L^2(\mathbb{R}^d)}^{2-p}$ and $\alpha^{\frac{d}{2}} K < \|Q\|_{L^2(\mathbb{R}^d)}^2$, there exist $\eta, \varepsilon > 0$ such that

$$\alpha \frac{(1 + \varepsilon)^2}{p} C_{\text{GNS}}(\sqrt{K} + \varepsilon)^{p-2} < \frac{1 - \eta}{2}. \quad (5.10) \quad \boxed{\text{eta}}$$

By collecting (5.6), (5.9), (5.10) and Lemma 3.2, we get

$$\begin{aligned} & -\log \mathbb{E}_\mu \left[\exp \left(\alpha R_p(u) \cdot \mathbf{1}_{\{\|u\|_{L^2(\mathbb{R}^d)}^2 \leq K\}} \right) \right] \\ & \geq \inf_{\theta \in \mathbb{H}_a} \mathbb{E} \left[-\frac{1 - \eta}{2} \|I(\theta)(1)\|_{\mathcal{H}^1(\mathbb{R}^d)}^2 - C_\varepsilon \|\mathbf{P}_{N_\varepsilon} Y(1)\|_{\mathcal{H}^1(\mathbb{R}^d)}^2 - C_\varepsilon R_p(Y(1)) + \frac{1}{2} \int_0^1 \|\theta(t)\|_{L^2(\mathbb{R}^d)}^2 dt \right] \\ & \geq \inf_{\theta \in \mathbb{H}_a} \mathbb{E} \left[-C_\varepsilon R_p(\mathbf{P}_{N_\varepsilon}^\perp Y(1)) - C_\varepsilon \|\mathbf{P}_{N_\varepsilon} Y(1)\|_{\mathcal{H}^1(\mathbb{R}^d)}^2 + \frac{\eta}{2} \int_0^1 \|\theta(t)\|_{L^2(\mathbb{R}^d)}^2 dt \right] \\ & \geq \mathbb{E} \left[-C_\varepsilon R_p(\mathbf{P}_{N_\varepsilon}^\perp Y(1)) - C_\varepsilon \|\mathbf{P}_{N_\varepsilon} Y(1)\|_{\mathcal{H}^1(\mathbb{R}^d)}^2 \right] \\ & \geq -C_\varepsilon - C_\varepsilon \mathbb{E} \left[\|\mathbf{P}_{N_\varepsilon} Y(1)\|_{\mathcal{H}^1(\mathbb{R}^d)}^2 \right]. \end{aligned}$$

We remark that $Y(1) \notin \mathcal{H}^1(\mathbb{R})$ almost surely. Therefore, to prove (5.5), there remains to show that

$$\mathbb{E} \left[\|\mathbf{P}_{N_\varepsilon} Y(1)\|_{\mathcal{H}^1(\mathbb{R}^d)}^2 \right] < \infty, \quad (5.11) \quad \boxed{\text{Hsbound}}$$

where N_ε is a random variable given by (5.8).

Noting that $Y(1)$ is a mean-zero random variable, we may decompose Ω (by ignoring a zero-measure set) as

$$\Omega = \bigcup_{N \geq 1} \Omega_N, \quad (5.12) \quad \boxed{\text{decom}}$$

where

$$\Omega_N = \{\omega \in \Omega : N_\varepsilon(\omega) = N\}. \quad (5.13) \quad \boxed{\text{ON}}$$

By (5.12) and Hölder's inequality, we have

$$\begin{aligned} \mathbb{E}[\|\mathbf{P}_{N_\varepsilon} Y(1)\|_{\mathcal{H}^1(\mathbb{R}^d)}^2] &\leq \sum_{N \geq 1} \mathbb{E}[\|\mathbf{P}_N Y(1)\|_{\mathcal{H}^1(\mathbb{R}^d)}^2 \cdot \mathbf{1}_{\Omega_N}] \\ &\leq \sum_{N \geq 1} \lambda_N^2 \mathbb{E}[\|\mathbf{P}_N Y(1)\|_{L^2(\mathbb{R}^d)}^2 \cdot \mathbf{1}_{\Omega_N}] \\ &\leq \sum_{N \geq 1} \lambda_N^2 \left(\mathbb{E}[\|Y(1)\|_{L^2(\mathbb{R}^d)}^4] \right)^{\frac{1}{2}} \cdot \mathbb{P}(\Omega_N)^{\frac{1}{2}} \\ &\leq C \sum_{N \geq 1} \lambda_N^2 \mathbb{P}(\Omega_N)^{\frac{1}{2}}. \end{aligned} \quad (5.14) \quad \boxed{\text{Hsbound1}}$$

By using Corollary 2.7, we also have

$$\mathbb{E}[\|\mathbf{P}_{\frac{N}{4}}^\perp Y(1)\|_{L^2(\mathbb{R}^d)}^2] = \sum_{n=\lfloor \frac{N}{4} \rfloor + 1}^{\infty} \lambda_n^{-2} \lesssim \lambda_{\lfloor \frac{N}{4} \rfloor}^{-1 + \frac{2}{s}}. \quad (5.15) \quad \boxed{\text{L2bound}}$$

It then follows from (5.8), (5.13), Hölder's inequality, Lemma 5.1 and (5.15), that

$$\begin{aligned} \mathbb{P}(\Omega_N) &\leq \mathbb{P}(\{\|\mathbf{P}_{\frac{N}{4}}^\perp Y(1)\|_{L^2(\mathbb{R}^d)} > \varepsilon\}) \\ &\lesssim \exp \left\{ -c \left(\frac{\varepsilon}{\mathbb{E}[\|\mathbf{P}_{\frac{N}{4}}^\perp Y(1)\|_{L^2(\mathbb{R}^d)}]} \right)^2 \right\} \\ &\lesssim \exp \left\{ -\frac{c\varepsilon^2}{\mathbb{E}[\|\mathbf{P}_{\frac{N}{4}}^\perp Y(1)\|_{L^2(\mathbb{R}^d)}^2]} \right\} \\ &\lesssim e^{-\tilde{c}\varepsilon^2 \lambda_{\lfloor \frac{N}{4} \rfloor}^{-\frac{2}{s} + 1}}, \end{aligned} \quad (5.16) \quad \boxed{\text{Hsbound2}}$$

where c and \tilde{c} are constants. By collecting (5.14), (5.16) and Lemma 2.8, we conclude that

$$\mathbb{E}[\|\mathbf{P}_{N_\varepsilon} Y(1)\|_{\mathcal{H}^1(\mathbb{R}^d)}^2] \leq \sum_{N \geq 1} \lambda_N^2 e^{-\frac{\tilde{c}}{2}\varepsilon^2 \lambda_{\lfloor \frac{N}{4} \rfloor}^{-\frac{2}{s} + 1}} < \infty,$$

where we used $s > 2$, which finishes the proof of (5.11), and thus (5.5) in the critical case. \square

SEC:non2

5.2. Non-normalizability. In this subsection, we prove the rest of Theorem 1.4, i.e. the non-integrability part of (ii) and (iii):

Lemma 5.3 (Divergence of the partition function).

Let $s > 2$ and assume either of the following conditions

- (i) critical nonlinearity: $p = 2 + \frac{4}{d}$ and $\alpha^{\frac{d}{2}} K > \|Q\|_{L^2(\mathbb{R}^d)}^2$;
 - (ii) supercritical nonlinearity: $p > 2 + \frac{4}{d}$ and any $\alpha, K > 0$.
- (5.17) $\boxed{\text{conditions}}$

non:blow super

where Q is an optimizer of the GNS inequality. Then

$$\mathcal{Z}_K = \mathbb{E}_\mu \left[\exp(\alpha R_\rho(u)) \mathbf{1}_{\{\|u\|_{L^2(\mathbb{R}^d)}^2 \leq K\}} \right] = \infty. \quad (5.18) \quad \text{part}$$

We construct a test drift giving a $-\infty$ upper bound in the Boué-Dupuis variational principle, as sketched in Section 3.1. We use the following blow-up profiles:

Lemma 5.4 (Blow-up profiles).

Assume (5.17) holds. Let

$$W_\rho = \beta \rho^{-\frac{d}{2}} Q(\rho^{-1}x). \quad (5.19) \quad \text{W}$$

Then, in the limit $\rho \rightarrow 0$ we have

$$\begin{aligned} \text{(i)} \quad & H(W_\rho) \leq -A_1 \rho^{-\frac{dp}{2}+d}, \\ \text{(ii)} \quad & \|W_\rho\|_{L^p(\mathbb{R}^d)}^p \leq A_2 \rho^{-\frac{dp}{2}+d}, \\ \text{(iii)} \quad & \|W_\rho\|_{L^2(\mathbb{R}^d)}^2 \leq K - \eta, \end{aligned} \quad (5.20) \quad \text{soliton}$$

where H is the Hamiltonian functional given in (1.5), and $A_1, A_2, \eta > 0$ are constants independent of $\rho > 0$.

Proof. There exists $\beta > 0$ such that

$$K > \|W_\rho\|_{L^2(\mathbb{R}^d)}^2 = \beta^2 \|Q\|_{L^2(\mathbb{R}^d)}^2.$$

In fact, since $\alpha^{\frac{d}{2}} K > \|Q\|_{L^2(\mathbb{R}^d)}^2$ in the critical case, we take β so that

$$\beta^2 \alpha^{\frac{d}{2}} > 1; \quad (5.21) \quad \text{choi-be}$$

while in the supercritical case, we can take β small. By choosing $\eta \in \left(0, \alpha^{\frac{d}{2}} K - \|W_\rho\|_{L^2(\mathbb{R}^d)}^2\right]$, we have (5.20)-(iii).

The rest follows from a similar computation as in [42, Lemma 3.4] by taking into account (5.21) and

$$\lim_{\rho \rightarrow 0} \int_{\mathbb{R}^d} |x|^s |W_\rho(x)|^2 dx = 0. \quad (5.22) \quad \text{poten_int}$$

Note that (5.22) comes from

$$\int_{\mathbb{R}^d} |x|^s |W_\rho(x)|^2 dx = \beta^2 \rho^s \int_{\mathbb{R}^d} |x|^s |Q(x)|^2 dx$$

and the exponential decay at infinity of Q . See [47, Proposition 3.1]. \square

Remark 5.1 (Refined blow-up profiles).

We could use the blow-up profiles from Lemma 4.4 to show (5.18) under (5.17) (ii), and (i) with $\alpha^{\frac{d}{2}} K \gg \|Q\|_{L^2(\mathbb{R}^d)}^2$. But this cannot explain the sharp phase transition (5.17) (i) when $\alpha^{\frac{d}{2}} K \sim \|Q\|_{L^2(\mathbb{R}^d)}^2$. For this, we build a sequence of scalings of ground state Q , namely $\{W_\rho\}$, which are new blow-up profiles that accurately capture the critical mass.

We construct a series of drift terms as follows. Let $\rho > 0$, $M = \rho^{-1}$, W_ρ be as in Lemma 5.4 and $Z_M(t)$ as in Definition 3.3. We set

$$\theta_\rho(t) := -\mathcal{L}^{\frac{1}{2}} \frac{d}{dt} Z_M(t) + \mathcal{L}^{\frac{1}{2}} W_\rho, \quad (5.23) \quad \text{theta}$$

where $\rho \ll 1$ and $\lambda_M \sim \rho^{-1}$.

From (5.23), we have

$$\begin{aligned}
I(\theta)(1) &= \int_0^1 \mathcal{L}^{-\frac{1}{2}} \theta(t) dt \\
&= \int_0^1 (W_\rho - \frac{d}{dt} Z_M(t)) dt \\
&= W_\rho - Z_M(1).
\end{aligned} \tag{5.24} \quad \boxed{\text{Itheta}}$$

We need the following properties of the approximate Brownian motion.

Lemma 5.5 (Approximating the Brownian motion, superharmonic case).

Given $s > 2$ and a dyadic number $M \sim \rho^{-1} \gg 1$, let $Z_M(t)$ be as in Definition 3.3. The following holds:

$$\mathbb{E}[\|Y(1) - Z_M\|_{L^2(\mathbb{R}^d)}^2] \sim \lambda_M^{\frac{2}{s}-1}, \tag{5.25} \quad \boxed{\text{L2}_1}$$

$$\mathbb{E}[\|Z_M(1) - Y(1)\|_{L^p(\mathbb{R}^d)}^p] \lesssim \left(\lambda_M^{\frac{2}{s}-1}\right)^{\frac{p}{2}} \text{ for } p \geq 1, \tag{5.26} \quad \boxed{\text{Lp}}$$

$$\mathbb{E}\left[\int_0^1 \left\|\frac{d}{d\tau} Z_M(\tau)\right\|_{H^1(\mathbb{R}^d)}^2 d\tau\right] \lesssim \lambda_M^{\frac{2}{s}}, \tag{5.27} \quad \boxed{\text{dZL2}}$$

for any $M \gg 1$.

Proof. Let

$$X_n(t) = \tilde{Y}(n, t) - \tilde{Z}_M(n, t), \quad 0 < n \leq M. \tag{5.28} \quad \boxed{\text{ZZ1}_1}$$

Then, from (3.10), we see that $X_n(t)$ satisfies the following stochastic differential equation:

$$\begin{cases} dX_n(t) = -c\lambda_n^{-1}\lambda_M X_n(t)dt + \lambda_n^{-1}dB_n(t) \\ X_n(0) = 0 \end{cases}$$

for $0 < n \leq M$, where $c \gg 1$ is a constant. By solving this stochastic differential equation, we have

$$X_n(t) = \lambda_n^{-1} \int_0^t e^{-c\lambda_n^{-1}\lambda_M(t-\tau)} dB_n(\tau). \tag{5.29} \quad \boxed{\text{ZZ2}_1}$$

Then, from (5.28) and (5.29), we have

$$\tilde{Z}_M(n, t) = \tilde{Y}(n, t) - \lambda_n^{-1} \int_0^t e^{-c\lambda_n^{-1}\lambda_M(t-\tau)} dB_n(\tau) \tag{5.30} \quad \boxed{\text{SDE12}}$$

for $n \leq M$. Hence, from (5.30), the independence of $\{B_n\}_{n \in \mathbb{N}}$, Ito's isometry and Corollary 2.6 with $p = 1$ and $p = \frac{1}{2}$, we have

$$\begin{aligned}
\mathbb{E}[\|Z_M\|_{L^2(\mathbb{R}^d)}^2] &= \sum_{n \leq M} \left(\mathbb{E}[|\tilde{Y}(n)|^2] - 2\lambda_n^{-2} \int_0^1 e^{-c\lambda_n^{-1}\lambda_M(1-\tau)} d\tau \right. \\
&\quad \left. + \lambda_n^{-2} \int_0^1 e^{-2c\lambda_n^{-1}\lambda_M(1-\tau)} d\tau \right) \\
&\sim 1 + O\left(c^{-1} \sum_{n \leq M} \lambda_n^{-1} \lambda_M^{-1}\right) \\
&\sim 1,
\end{aligned} \tag{5.31} \quad \boxed{\text{ODE3}}$$

for any $M \gg 1$, $c \gg 1$ and $s > 2$. Similarly, we have

$$\begin{aligned}
\mathbb{E}[\|Y(1) - Z_M\|_{L^2(\mathbb{R}^d)}^2] &= \sum_{n \leq M} \mathbb{E}[|X_n(t)|^2] + \sum_{n > M} \mathbb{E}[|\tilde{Y}(n)|^2] \\
&\sim \sum_{n \leq M} \lambda_n^{-2} \int_0^1 e^{-2c\lambda_n^{-1}\lambda_M(1-\tau)} d\tau + O\left(\sum_{n > M} \lambda_n^{-2}\right) \\
&\sim O\left(c^{-1} \sum_{n \leq M} \lambda_n^{-1} \lambda_M^{-1}\right) + \lambda_M^{\frac{2}{s}-1} \\
&\sim \lambda_M^{\frac{2}{s}-1},
\end{aligned} \tag{5.32}$$

ODE31

where we used Corollaries 2.6 and 2.7. This proves (5.25). Then, (5.26) follows from (5.25) and the Khintchine inequality (see e.g., [11, Lemma 4.2]).

By the L^2 orthogonality of $\{e_n\}_{n \in \mathbb{N}}$, (5.30), (4.64) and proceeding as in (4.73), we have

$$\begin{aligned}
\mathbb{E}\left[2 \operatorname{Re} \int_{\mathbb{R}^d} Y_N \overline{Z_M} dx - \int_{\mathbb{R}^d} |Z_M|^2 dx\right] &= \mathbb{E}\left[2 \operatorname{Re} \sum_{n \leq M} \tilde{Y}_N(n) \overline{\tilde{Z}_M(n)} - \sum_{n \leq M} |\tilde{Z}_M(n)|^2\right] \\
&= \mathbb{E}\left[\sum_{n \leq M} |\tilde{Z}_M(n)|^2 + \sum_{n \leq M} \operatorname{Re} \left(2\lambda_n^{-1} \int_0^1 e^{-c\lambda_n^{-1}\lambda_M(1-\tau)} dB_n(\tau)\right) \overline{\tilde{Z}_M(n)}\right] \\
&\sim \lambda_M^{\frac{2}{s}-1} + O\left(c^{-1} \sum_{n \leq M} \lambda_n^{-1} \lambda_M^{-1}\right) \\
&\sim \lambda_M^{\frac{2}{s}-1}
\end{aligned}$$

for any $N \geq M \gg 1$ and $c \gg 1$.

Similarly, from (3.10), (5.28), (5.29), and Ito's isometry, we have

$$\begin{aligned}
\mathbb{E}\left[\int_0^1 \left\|\frac{d}{d\tau} Z_M(\tau)\right\|_{\mathcal{H}^1(\mathbb{R}^d)}^2 d\tau\right] &= \lambda_M \mathbb{E}\left[\int_0^1 \left\|\mathbf{P}_M(Y_N(\tau)) - Z_M(\tau)\right\|_{L^2(\mathbb{R}^d)}^2 ds\right] \\
&= \lambda_M \mathbb{E}\left[\int_0^1 \sum_{n \leq M} |X_n(\tau)|^2 d\tau\right] \\
&= \lambda_M \sum_{n \leq M} \int_0^1 \mathbb{E}[|X_n(\tau)|^2] d\tau \\
&= \lambda_M \sum_{n \leq M} \lambda_n^{-2} \int_0^1 \int_0^\tau e^{-2c\lambda_n^{-1}\lambda_M(\tau-\tau')} d\tau' d\tau \\
&\lesssim \lambda_M \sum_{n \leq M} \lambda_n^{-1} \lambda_M^{-1} \\
&\lesssim \lambda_M^{\frac{2}{s}},
\end{aligned}$$

yielding (5.27). This completes the proof of Lemma 5.5. \square

Now we are ready to prove the rest of Theorem 1.3.

Proof of Theorem 1.4 - the second half of (ii) and (iii). We shall prove (5.18) under conditions (5.17). Observing that

$$\mathbb{E}_\mu \left[\exp(\alpha R_p(u)) \cdot \mathbf{1}_{\{\|u\|_{L^2(\mathbb{R}^d)}^2 \geq K\}} \right] \geq \mathbb{E}_\mu \left[\exp \left(\alpha R_p(u) \cdot \mathbf{1}_{\{\|u\|_{L^2(\mathbb{R}^d)}^2 \leq K\}} \right) \right] - 1,$$

then (5.18) follows from

$$\mathbb{E}_\mu \left[\exp \left(\alpha R_p(u) \cdot \mathbf{1}_{\{\|u\|_{L^2(\mathbb{R}^d)}^2 \leq K\}} \right) \right] = \infty. \quad (5.33) \quad \boxed{\text{var10}}$$

We apply Lemma 3.1, together with (5.23) and (5.24), to get

$$\begin{aligned} & -\log \mathbb{E}_\mu \left[\exp \left(\alpha R_p(u) \cdot \mathbf{1}_{\{\|u\|_{L^2(\mathbb{R}^d)}^2 \leq K\}} \right) \right] \\ &= \inf_{\theta \in \mathbb{H}_a} \mathbb{E} \left[\left(-\alpha R_p(Y(1) + I(\theta)(1)) \cdot \mathbf{1}_{\{\|Y(1) + I(\theta)(1)\|_{L^2(\mathbb{R}^d)}^2 \leq K\}} + \frac{1}{2} \int_0^1 \|\theta(t)\|_{L^2(\mathbb{R}^d)}^2 dt \right) \right] \\ &\leq \inf_{0 < \rho \ll 1} \mathbb{E} \left[\left(-\alpha R_p(Y(1) - Z_M(1) + W_\rho) \cdot \mathbf{1}_{\{\|Y(1) - Z_M(1) + W_\rho\|_{L^2(\mathbb{R}^d)}^2 \leq K\}} \right. \right. \\ &\quad \left. \left. + \frac{1}{2} \int_0^1 \left\| -\frac{d}{dt} Z_M(t) + W_\rho \right\|_{\mathcal{H}^1(\mathbb{R}^d)}^2 dt \right) \right] \\ &\leq \inf_{0 < \rho \ll 1} \mathbb{E} \left[\left(-\alpha R_p(W_\rho) + \frac{1}{2} \|W_\rho\|_{\mathcal{H}^1(\mathbb{R}^d)}^2 \right) \right. \\ &\quad \left. + \left(\alpha R_p(W_\rho) - \alpha R_p(Y(1) - Z_M(1) + W_\rho) \right) \cdot \mathbf{1}_{\{\|Y(1) - Z_M(1) + W_\rho\|_{L^2(\mathbb{R}^d)}^2 \leq K\}} \right. \\ &\quad \left. + \alpha R_p(W_\rho) \cdot \mathbf{1}_{\{\|Y(1) - Z_M(1) + W_\rho\|_{L^2(\mathbb{R}^d)}^2 > K\}} \right. \\ &\quad \left. + \frac{1}{2} \int_0^1 \left\| -\frac{d}{dt} Z_M(t) \right\|_{\mathcal{H}^1(\mathbb{R}^d)}^2 - 2 \left\langle \frac{d}{dt} Z_M(t), W_\rho \right\rangle_{\mathcal{H}^1(\mathbb{R}^d)} dt \right] \\ &= \inf_{0 < \rho \ll 1} (A + B + C + D), \end{aligned} \quad (5.34) \quad \boxed{\text{var13}}$$

by inserting the test drift (5.23) in the Boué-Dupuis variational principle. In what follows, we consider the terms in the right-hand side one by one.

For the term (A), from Lemma 5.4, we have

$$A = -\alpha R_p(W_\rho) + \frac{1}{2} \|W_\rho\|_{\mathcal{H}^1(\mathbb{R}^d)}^2 = H(W_\rho) \lesssim -\rho^{-\frac{d}{2} + d}, \quad (5.35) \quad \boxed{\text{term1}}$$

where the Hamiltonian H is given in (1.5).

For the term (B), by using the mean value theorem we see that

$$\begin{aligned} & \int_{\mathbb{R}^d} (|W_\rho|^p - |Y(1) - Z_M(1) + W_\rho|^p) dx \\ & \lesssim \int_{\mathbb{R}^d} (|Y(1) - Z_M(1)|^p + |Y(1) - Z_M(1)| |W_\rho|^{p-1}) dx, \end{aligned}$$

which together with Lemma 5.5, Lemma 5.4, and Young's inequality, gives

$$\begin{aligned}
\mathbf{B} &= \mathbb{E} \left[\left(\alpha R_p(W_\rho) - \alpha R_p(Y(1) - Z_M(1) + W_\rho) \right) \cdot \mathbf{1}_{\{\|Y(1) - Z_M(1) + W_\rho\|_{L^2(\mathbb{R}^d)}^2 \leq K\}} \right] \\
&\lesssim \int_{\mathbb{R}^d} (\mathbb{E}[|Y(1) - Z_M(1)|^p] + \mathbb{E}[|Y(1) - Z_M(1)|] |W_\rho|^{p-1}) dx \\
&\leq C_\varepsilon \mathbb{E}[\|Y(1) - Z_M(1)\|_{L^p(\mathbb{R}^d)}^p] + \varepsilon \|W_\rho\|_{L^p(\mathbb{R}^d)}^p \\
&\leq C_\varepsilon \left(\lambda_M^{\frac{2}{s}-1} \right)^{\frac{p}{2}} + \varepsilon \rho^{-\frac{dp}{2}+d} \\
&\leq 2\varepsilon \rho^{-\frac{dp}{2}+d},
\end{aligned} \tag{5.36} \quad \boxed{\text{term3}}$$

provided $M \gg 1$. Now we turn to the term (C), by using Chebyshev's inequality, we have

$$\begin{aligned}
\mathbf{C} &= \mathbb{E} \left[\alpha R_p(W_\rho) \cdot \mathbf{1}_{\{\|Y(1) - Z_M(1) + W_\rho\|_{L^2(\mathbb{R}^d)}^2 > K\}} \right] \\
&\leq \alpha R_p(W_\rho) \cdot \mathbb{E} \left[\mathbf{1}_{\{\|Y(1) - Z_M(1)\|_{L^2(\mathbb{R}^d)} > \sqrt{K} - \|W_\rho\|_{L^2(\mathbb{R}^d)}\}} \right] \\
&\leq \alpha R_p(W_\rho) \frac{\mathbb{E}[\|Y(1) - Z_M(1)\|_{L^2(\mathbb{R}^d)}^2]}{\left(\sqrt{K} - \|W_\rho\|_{L^2(\mathbb{R}^d)} \right)^2} \\
&\lesssim \eta^{-2} \rho^{-\frac{dp}{2}+d} \lambda_M^{\frac{2}{s}-1} = o(1) \rho^{-\frac{dp}{2}+d}.
\end{aligned} \tag{5.37} \quad \boxed{\text{term4}}$$

where in the last step we use the fact $\lambda_M \rightarrow \infty$ as $M \rightarrow \infty$ and $s > 2$. For term (D), from (5.27), we have

$$\mathbf{D} = \frac{1}{2} \int_0^1 \mathbb{E} \left[\left\| -\frac{d}{dt} Z_M(t) \right\|_{H^1(\mathbb{R}^d)}^2 \right] dt \leq C \lambda_M^{\frac{2}{s}} \ll \rho^{-\frac{2}{s}}, \tag{5.38} \quad \boxed{\text{term5}}$$

where we used the fact that $\lambda_M \ll \rho^{-1}$. By collecting estimates (5.35), (5.36), (5.37) and (5.38), we conclude that

$$\mathbf{A} + \mathbf{B} + \mathbf{C} + \mathbf{D} \lesssim -\rho^{-\frac{dp}{2}+d} + c\rho^{-\frac{2}{s}} \rightarrow -\infty, \tag{5.39} \quad \boxed{\text{abcd}}$$

provided $c \ll 1$, $p \geq 6$ and $s > 2$.

Finally, the desired estimate (5.33) follows from (5.34) and (5.39). We thus finish the proof of Theorem 1.4. \square

APPENDIX A. REMINDER ON COHERENT STATES AND HUSIMI FUNCTIONS

SEC:Weyl

Let $f \in C_0^\infty(\mathbb{R})$ be an odd function satisfying $\|f\|_{L^2(\mathbb{R})}^2 = 1$. For $\hbar \in (0, 1]$ and $x, p \in \mathbb{R}$, we define the coherent state

$$f_{x,p}^\hbar(y) := \hbar^{-1/4} f\left(\frac{y-x}{\sqrt{\hbar}}\right) e^{i\frac{py}{\hbar}}.$$

Denote

$$f^\hbar(y) := \hbar^{-1/4} f\left(\frac{y}{\sqrt{\hbar}}\right), \quad g^\hbar(q) := \hbar^{-1/4} \hat{f}\left(\frac{q}{\sqrt{\hbar}}\right),$$

where

$$\hat{f}(q) := \frac{1}{\sqrt{2\pi}} \int e^{-iqy} f(y) dy$$

is the standard Fourier transform. We also define the semiclassical Fourier transform

$$\mathcal{F}_\hbar[f](q) := \frac{1}{\sqrt{2\pi\hbar}} \int e^{-i\frac{qy}{\hbar}} f(y) dy.$$

We have the following observation (see e.g., [43, Chapter 12] or [27, Section 2.1]).

lem:coherent

Lemma A.1 (Coherent state formalism).

- (Plancherel identity)

$$\langle \varphi, \psi \rangle_{L^2} = \langle \mathcal{F}_\hbar \varphi, \mathcal{F}_\hbar \psi \rangle_{L^2} \text{ for all } \varphi, \psi \in L^2(\mathbb{R}).$$

- (Localization of coherent states)

$$\begin{aligned} |f_{x,p}^\hbar(y)|^2 &= \hbar^{-1/2} \left| f\left(\frac{y-x}{\sqrt{\hbar}}\right) \right|^2 \rightarrow \delta_{\{y=x\}} \text{ as } \hbar \rightarrow 0, \\ |\mathcal{F}_\hbar[f_{x,p}^\hbar](q)|^2 &= \hbar^{-1/2} \left| \hat{f}\left(\frac{q-p}{\sqrt{\hbar}}\right) \right|^2 \rightarrow \delta_{\{q=p\}} \text{ as } \hbar \rightarrow 0. \end{aligned}$$

- (Resolution of identity)

$$\frac{1}{\sqrt{2\pi\hbar}} \iint |f_{x,p}^\hbar\rangle \langle f_{x,p}^\hbar| dx dp = \mathbf{1}_{L^2(\mathbb{R})}.$$

(A.1) reso-iden

We define the Husimi function associated to a non-negative trace-class operator γ_\hbar as

$$m_\hbar(x, p) := \langle f_{x,p}^\hbar, \gamma_\hbar f_{x,p}^\hbar \rangle$$

Here are some of its' basic properties, see again (see e.g., [43, Chapter 12] or [27, Section 2.1]).

lem:Husimi

Lemma A.2 (Properties of Husimi functions).

Write the spectral decomposition of γ_\hbar as

$$\gamma_\hbar = \sum_{n \geq 1} \mu_n^\hbar |u_n^\hbar\rangle \langle u_n^\hbar|.$$

We have

$$\begin{aligned} 0 &\leq m_{\hbar}(x, p) \leq 1, \quad \forall x, p \in \mathbb{R}, \\ \frac{1}{2\pi\hbar} \iint m_{\hbar}(x, p) dx dp &= \text{Tr}[\gamma_{\hbar}], \\ \frac{1}{2\pi\hbar} \int m_{\hbar}(x, p) dp &= \rho_{\hbar} * |f^{\hbar}|^2(x), \\ \frac{1}{2\pi\hbar} \int m_{\hbar}(x, p) dx &= t_{\hbar} * |g^{\hbar}|^2(p), \end{aligned}$$

where

$$\rho_{\hbar}(x) = \sum_{n \geq 1} \mu_n^{\hbar} |u_n^{\hbar}(x)|^2, \quad t_{\hbar}(p) = \sum_{n \geq 1} \mu_n^{\hbar} |\mathcal{F}_{\hbar}[u_n^{\hbar}](p)|^2$$

are the density and momentum functions associated to γ_{\hbar} . In addition, we have

$$\hbar \int \rho_{\hbar}(x) dx = \hbar \int t_{\hbar}(p) dp = \hbar \text{Tr}[\gamma_{\hbar}] = \frac{1}{2\pi} \iint m_{\hbar}(x, p) dx dp. \quad (\text{A.2})$$

momen-densi-

APPENDIX B. FRACTIONAL SCHRÖDINGER OPERATOR

In this appendix, we prove some estimates for the fractional Schrödinger operator, which enable us to prove normalizability of Gibbs measure associated with fractional Schrödinger operators.

Let H_{α} be the fractional Schrödinger operator

$$H_{\alpha} = (-\Delta)^{\alpha} + V(x)$$

defined on \mathbb{R}^d , where $V : \mathbb{R}^d \rightarrow \mathbb{R}_+$ is a trapping potential, i.e., $V(x) \rightarrow +\infty$ as $|x| \rightarrow \infty$. In particular, we are interested in the anharmonic potential $V(x) = |x|^s$ with $s > 0$. The operator H_{α} has a sequence of eigenvalues λ_n^2 with

$$0 < \lambda_0 \leq \lambda_1 \leq \dots \leq \lambda_n \rightarrow \infty$$

and the corresponding eigenfunctions $(e_n)_{n \geq 0}$ form an orthonormal basis of $L^2(\mathbb{R}^d)$. We emphasize that we do not assume the radial condition here. Let us start with the following result.

PROP:LS

Proposition B.1 (Schatten-norm bounds for the resolvent).

Let $d \geq 1$, $\alpha > 0$, $s > 0$ and $\gamma > \frac{d}{2\alpha}$. Then we have

$$\text{Tr}[H_{\alpha}^{-\gamma}] = \sum_{n=0}^{\infty} \lambda_n^{-2\gamma} \leq C \int_{\mathbb{R}^d} (V(x))^{\frac{d}{2\alpha}-\gamma} dx. \quad (\text{B.1})$$

LT

In particular, if $V(x) = |x|^s$, then

$$\text{Tr}[H_{\alpha}^{-\gamma}] < \infty$$

provided $\gamma > \frac{d}{2\alpha} + \frac{d}{s}$.

Proof. Let $G(t, x)$ be the fundamental solution to the fractional heat equation

$$\partial_t u + (-\Delta)^{\alpha} u = 0$$

such that the solution to the above equation with initial data $u(0) = f$ can be written as

$$u(t, x) = [e^{-t(-\Delta)^{\alpha}} f](x) := [G(t, \cdot) * f](x).$$

In particular, we have

$$G(t, x) = t^{-\frac{d}{2\alpha}} G(1, t^{-\frac{1}{2\alpha}} x). \quad (\text{B.2})$$

Green1

By Trotter's formula, we have

$$\begin{aligned}
\mathrm{Tr}[e^{-t((-\Delta)^\alpha + V)}] &= \lim_n \mathrm{Tr}[(e^{-\frac{t}{n}(-\Delta)^\alpha} e^{-\frac{t}{n}V})^n] \\
&= \lim_n \int_{(\mathbb{R}^d)^n} G\left(\frac{t}{n}, x - x_1\right) e^{-\frac{t}{n}V(x_1)} G\left(\frac{t}{n}, x_1 - x_2\right) e^{-\frac{t}{n}V(x_2)} \\
&\quad \cdots G\left(\frac{t}{n}, x_{n-1} - x\right) e^{-\frac{t}{n}V(x)} dx dx_1 \cdots dx_{n-1} \\
&= \lim_n \int_{(\mathbb{R}^d)^n} \prod_{j=0}^{n-1} G\left(\frac{t}{n}, x_j - x_{j+1}\right) e^{-\frac{t}{n} \sum_{k=0}^{n-1} V(x_k)} dx_0 dx_1 \cdots dx_{n-1} \\
&\leq \lim_n \frac{1}{n} \sum_{k=0}^{n-1} \int_{(\mathbb{R}^d)^n} \prod_{j=0}^{n-1} G\left(\frac{t}{n}, x_j - x_{j+1}\right) e^{-tV(x_k)} dx_0 dx_1 \cdots dx_{n-1} \\
&= \lim_n \frac{1}{n} \sum_{k=0}^{n-1} \int_{\mathbb{R}^d} G(t, x_k - x_k) e^{-tV(x_k)} dx_k \\
&= G(t, 0) \int_{\mathbb{R}^d} e^{-tV(x)} dx \\
&= C_\alpha t^{-\frac{d}{2\alpha}} \int_{\mathbb{R}^d} e^{-tV(x)} dx,
\end{aligned} \tag{B.3}$$

Green2

where in the last step we used (B.2) and

$$C_\alpha = c_d \int_{\mathbb{R}^d} e^{-|\xi|^{2\alpha}} d\xi.$$

Then by using (B.3) and a similar argument as in [24], we get

$$\begin{aligned}
\mathrm{Tr}[((-\Delta)^\alpha + V)^{-\gamma}] &= \frac{1}{\Gamma(\gamma)} \int_0^\infty \mathrm{Tr}[e^{-t((-\Delta)^\alpha + V)}] t^{\gamma-1} dt \\
&\leq C_\alpha \frac{1}{\Gamma(\gamma)} \int_0^\infty \int_{\mathbb{R}^d} t^{-\frac{d}{2\alpha}} e^{tV(x)} t^{\gamma-1} dt \\
&= C_\alpha \frac{\Gamma(\gamma - \frac{d}{2\alpha})}{\Gamma(\gamma)} \int_{\mathbb{R}^d} (V(x))^{\frac{d}{2\alpha}-\gamma} dx.
\end{aligned} \tag{B.4}$$

Since $H_\alpha \geq \lambda_0$, we have

$$\mathrm{Tr}[H_\alpha^{-\gamma}] \leq 2^\gamma \mathrm{Tr}[(H_\alpha + \lambda_0)^{-\gamma}],$$

where $\lambda_0 > 0$ is the first eigenvalue of H_α . It follows that

$$\mathrm{Tr}[H_\alpha^{-\gamma}] \leq C \int_{\mathbb{R}^d} (|x|^s + \lambda_0)^{\frac{d}{2\alpha}-\gamma} dx < \infty$$

provided $\gamma > \frac{d}{2\alpha} + \frac{d}{s}$. Thus we finish the proof. □

PROP:GreenLp

Proposition B.2 (*L^p bounds for the resolvent's integral kernel*).

Let $d \geq 1$, $\alpha > \frac{d}{2}$, $s > 0$ and $V(x) = |x|^s$. Then we have

$$\|H_\alpha^{-1}(x, x)\|_{L^p(\mathbb{R}^d)} = \left\| \sum_{n \in \mathbb{N}} \frac{e_n^2(x)}{\lambda_n^2} \right\|_{L^p(\mathbb{R}^d)} < \infty$$

provided

$$p > \max \left\{ 1, \frac{2\alpha d}{s(2\alpha - d)} \right\}.$$

Proof. The following proof is similar to that of [23]. It suffices to show

$$\int_{\mathbb{R}^d} H_\alpha^{-1}(x, x) g^2(x) dx \lesssim \|g^2\|_{L^q(\mathbb{R}^d)}, \quad (\text{B.5}) \quad \boxed{\text{G1}}$$

where $\frac{1}{p} + \frac{1}{q} = 1$, for all $g \geq 0$ and $g^2 \in L^q(\mathbb{R}^d)$. With a slight abuse of notation, we still use g to denote the multiplication operator of $g(x)$. Then we can rewrite

$$\int_{\mathbb{R}^d} H_\alpha^{-1}(x, x) g^2(x) dx = \text{Tr}[g H_\alpha^{-1} g] = \left\| H_\alpha^{-\frac{1}{2}} P_N g \right\|_{\mathfrak{S}^2}^2, \quad (\text{B.6}) \quad \boxed{\text{G2}}$$

where \mathfrak{S}^p is the p -Schatten space. Then we apply the Hölder inequality in Schatten spaces to continue with

$$\begin{aligned} \left\| H_\alpha^{-\frac{1}{2}} g \right\|_{\mathfrak{S}^2}^2 &= \left\| H_\alpha^{\sigma-\frac{1}{2}} H_\alpha^{-\sigma} (1 + (-\Delta)^\alpha)^\sigma (1 + (-\Delta)^\alpha)^{-\sigma} g \right\|_{\mathfrak{S}^2}^2 \\ &\leq \left\| H_\alpha^{\sigma-\frac{1}{2}} P_N \right\|_{\mathfrak{S}^{2p}}^2 \left\| H_\alpha^{-\sigma} (1 + (-\Delta)^\alpha)^\sigma \right\|_{\mathfrak{S}^\infty}^2 \left\| (1 + (-\Delta)^\alpha)^{-\sigma} g \right\|_{\mathfrak{S}^{2q}}^2 \end{aligned} \quad (\text{B.7}) \quad \boxed{\text{G3}}$$

We shall estimate the three factors on the right-hand side of (B.7) one by one.

For the first factor in (B.7), we have

$$\begin{aligned} \left\| H_\alpha^{\sigma-\frac{1}{2}} \right\|_{\mathfrak{S}^{2p}}^2 &= \left(\text{Tr}[(H_\alpha^{2\sigma-1})^p] \right)^{\frac{1}{p}} \\ &= \left(\sum_{n \in \mathbb{N}} \lambda_n^{4\sigma p - 2p} \right)^{\frac{1}{p}} \\ &< \infty \end{aligned} \quad (\text{B.8}) \quad \boxed{\text{G4}}$$

provided $p - 2\sigma p > \frac{d}{2\alpha} + \frac{d}{s}$.

We turn to the second factor in (B.7). Since $H_\alpha \geq (-\Delta)^\alpha + \lambda_0$ with $\lambda_0 > 0$ being the lowest eigenvalue of H_α , we obtain that $\mathcal{L} \geq C(1 + (-\Delta)^\alpha)$ for some $C > 0$. We also note the operator monotonicity of $x \mapsto x^{2\sigma}$ for $\sigma < \frac{1}{2}$ gives

$$H_\alpha^{2\sigma} \gtrsim (1 + (-\Delta)^\alpha)^{2\sigma}$$

or

$$H_\alpha^{-\sigma} (1 + (-\Delta)^\alpha)^{2\sigma} H_\alpha^{-\sigma} \lesssim 1.$$

Therefore, we conclude that the operator $\mathcal{L}^{-\sigma} (1 + (-\Delta)^\alpha)^{2\sigma} \mathcal{L}^{-\sigma}$ is bounded for $0 < \sigma < \frac{1}{2}$, i.e.

$$\left\| H_\alpha^{-\sigma} (1 + (-\Delta)^\alpha)^\sigma \right\|_{\mathfrak{S}^\infty}^2 \lesssim 1. \quad (\text{B.9}) \quad \boxed{\text{G5}}$$

For the third factor in (B.7), we apply the Kato-Seiler-Simon inequality to get

$$\left\| (1 + (-\Delta)^\alpha)^{-\sigma} g \right\|_{\mathfrak{S}^{2q}}^2 \leq \|\langle \xi \rangle^{-2\alpha\sigma}\|_{L^{2q}(\mathbb{R}^d)}^2 \|g\|_{L^{2q}(\mathbb{R}^d)}^2 \lesssim \|g^2\|_{L^q(\mathbb{R}^d)}, \quad (\text{B.10}) \quad \boxed{\text{G6}}$$

provided $4\alpha\sigma q > d$ and $1 \leq q < \infty$.

Finally, by collecting (B.6), (B.7), (B.8), (B.9), and (B.10), we arrive at

$$\int_{\mathbb{R}^d} H_\alpha^{-1}(x, x) g^2(x) dx < \infty \quad (\text{B.11}) \quad \boxed{\text{G7}}$$

provided

$$\left\{ \begin{array}{l} p - 2\sigma p > \frac{d}{2\alpha} + \frac{d}{s} \\ \frac{1}{p} + \frac{1}{q} = 1 \\ \sigma < \frac{1}{2} \\ 4\alpha\sigma q > d \\ 1 \leq q < \infty. \end{array} \right. \quad (\text{B.12}) \quad \boxed{\text{G8}}$$

Since $\alpha > \frac{d}{2}$, we choose $p > \frac{2\alpha d}{s(2\alpha-d)}$ and

$$\frac{d}{4\alpha q} < \sigma < \frac{1}{2} - \frac{1}{2p} \left(\frac{d}{2\alpha} + \frac{d}{s} \right)$$

so that all the conditions in (B.12) are satisfied. Thus we finish the proof. \square

Proposition B.3 (Weyl's law for fractional Schrödinger operators).

Let $d \geq 1$, $\alpha > 0$, and $s > 0$. Then

$$N(H_\alpha, \Lambda) \sim \Lambda^{\frac{d}{2\alpha} + \frac{d}{s}} \quad \text{as } \Lambda \rightarrow \infty, \quad (\text{B.13}) \quad \boxed{\text{CLR-fraction}}$$

where

$$N(H_\alpha, \Lambda) := \#\{\lambda_n^2 : \lambda_n^2 \leq \Lambda\}.$$

The proof of this result follows the same argument as in Subsection 2.2. The coherent state is now defined by

$$f_{x,p}^\hbar(y) := \hbar^{-d/4} f\left(\frac{y-x}{\sqrt{\hbar}}\right) e^{i\frac{py}{\hbar}}, \quad f^\hbar(y) = \hbar^{-d/4} f\left(\frac{y}{\sqrt{\hbar}}\right), \quad g^\hbar(q) := \hbar^{-d/4} \hat{f}\left(\frac{q}{\sqrt{\hbar}}\right)$$

for some function $f \in C_0^\infty(\mathbb{R}^d)$ satisfying $\|f\|_{L^2(\mathbb{R}^d)} = 1$. The semiclassical Fourier transform is

$$\mathcal{F}_\hbar[f](q) := \frac{1}{(2\pi\hbar)^{d/2}} \int e^{-i\frac{qy}{\hbar}} f(y) dy.$$

The upper energy upper bound is proved by using the trial state

$$\gamma^{\text{test}} = \frac{1}{(2\pi\hbar)^d} \iint m_0(x, p) |f_{x,p}^\hbar\rangle \langle f_{x,p}^\hbar| dx dp$$

with

$$m_0(x, p) := \mathbf{1}_{\{|p|^{2\alpha} + |x|^s - 1 \leq 0\}}. \quad (\text{B.14}) \quad \boxed{\text{m1-fraction}}$$

From the energy upper bound, we deduce the lower bound and the upper bound on the trace. As in Remark 2.1, we can prove that

$$\lim_{\hbar \rightarrow 0} \hbar^d \text{Tr}[\gamma_\hbar] = \frac{1}{(2\pi)^d} \iint m_0(x, p) dx dp.$$

Since most of the estimates are similar to the radial case, we omit the details.

COR:LS

Corollary B.4 (Behavior of truncated Schatten norms).

Let $d \geq 1$, $\alpha > \frac{d}{2}$, $s > 0$ and $V(x) = |x|^s$. Then, we have

$$\text{Tr}[(\mathbf{P}_N H_\alpha)^{-\gamma}] = \sum_{n=0}^N \lambda_n^{-2\gamma} \lesssim \begin{cases} 1 & \text{if } \gamma > \frac{d}{2\alpha} + \frac{d}{s}, \\ (\log \lambda_N)^2 & \text{if } \gamma = \frac{d}{2\alpha} + \frac{d}{s}, \\ \lambda_N^{-2\gamma + \frac{d}{\alpha} + \frac{d}{2s}} & \text{if } \gamma < \frac{d}{2\alpha} + \frac{d}{s}. \end{cases}$$

The proof is similar to that of Corollary 2.6. We omit the details.

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