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# Sampled-Data Output Feedback Control of Nonlinear ODE-PDE Systems

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## Abstract

The aim of this paper is to propose an extension of the high-gain observed-based control design obtained in Ahmed-Ali et al. (2023) to the case of sampled-data systems by considering a Zero-Order-Hold (ZOH) device at the input. Moreover the class of nonlinear PDE is also enlarged by adding a nonlinear term in the heat equation. The stability of the overall closed systems is analyzed by using a suitable Lyapunov function.

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*Keywords:* Sampled- data control, High-gain observers, Cascade ODE-PDE systems.

## 1. INTRODUCTION

In this paper, we consider the design of sampled-data output feedback control for a class of nonlinear cascade ODE-PDE systems. This topic is receiving an important attention in the literature. Let us mention for instance the works by Krstic (2009) and by Wu (2013) for a linear ODE and the work by Wu (2013) where a nonlinear ODE is considered. Recently Ahmed-Ali et al. (2023) proposed to use the high-gain observer-based output feedback control for nonlinear systems developed by Khalil and Praly (2014) in order to design a new scheme of output feedback of nonlinear ODEs triangular systems in cascade with a PDE (heat equation). This algorithm, which can be viewed as an extension of Khalil and Praly (2014) to a class of cascade ODE-PDE systems, offers us a simple parameters design and a kind of separation principle for this class of systems. Another important advantage of using the high-gain observers is that they can recover the performances of state feedback control in the sense that, for instance, the trajectories of the system under output feedback are approaching those under state feedback as soon as the observer gain increases.

In the present contribution we propose to extend the result of Ahmed-Ali et al. (2023) to a more complex case by considering a nonlinear term in the PDE part and by adding the presence at the input of a Zero-Order-Hold (ZOH) device. A new Lyapunov functional is introduced in order to derive new stability conditions of the the overall closed loop system.

### Notations and preliminaries

Throughout the paper the superscript  $T$  stands for matrix transposition,  $\mathbb{R}^n$  denotes the  $n$ -dimensional Euclidean

space with vector norm  $|\cdot|$ ,  $\mathbf{I}_n$  is the  $n \times n$  identity matrix,  $\mathbb{R}^{n \times m}$  is the set of all  $n \times m$  real matrices, and the notation  $P > 0$ , for  $P \in \mathbb{R}^{n \times n}$ , means that  $P$  is symmetric and positive definite. In matrices, symmetric terms are denoted  $*$ ;  $\lambda_{\min}(P)$  (resp.  $\lambda_{\max}(P)$ ) denotes the smallest (resp. largest) eigenvalue.  $L_2(0, D)$  is the Hilbert space of square integrable functions  $z : \rightarrow \mathbf{R}$  with the corresponding norm  $\|z\|_{L_2} = \sqrt{\int_0^D z^2(x)dx}$ .  $\mathcal{H}^k(0, D)$  is the Sobolev space of functions  $z : [0, D] \rightarrow \mathbf{R}$  having  $k$  square integrable weak derivatives. Given a two-argument function  $u(x, t)$ , then its partial derivatives are denoted  $u_t = \frac{\partial u}{\partial t}$ ,  $u_{xx} = \frac{\partial^2 u}{\partial x^2}$ .  $u[t]$  and  $u_x[t]$  refer to the functions defined on  $0 \leq x \leq D$  by  $(u[t])(x) = u(x, t)$  and  $(u_x[t])(x) = \frac{\partial u(x, t)}{\partial x}$ .

## 2. PROBLEM STATEMENT

Let us consider the following class of systems in the state space  $\mathbb{R}^n \times L_2(0, D)$ .

$$\begin{cases} \dot{X}_i = X_{i+1}, & i = 1, \dots, n-1 \\ \dot{X}_n = f(X, v(t_k)) & t \in [t_k, t_{k+1}) \\ u(D, t) = X_1 \\ u_x(0, t) = 0 \\ u_t = u_{xx} + g(u), & x \in [0, D] \\ y(t) = u(0, t) \end{cases} \quad (1)$$

where  $v$  and  $y$  represent respectively the input and the output of the above system. The sampling instants  $t_k$  are an increasing sequence defined as follows :

$$0 = t_0 < t_1 < \dots, < t_k < \dots, \lim_{i \rightarrow \infty} t_k = \infty$$

with  $t_{k+1} - t_k \leq h$ , where  $h > 0$  is the maximum allowable sampling period.

Compared to Ahmed-Ali et al. (2023), we need to consider that: i) the control  $v(t_k)$  is constant between  $[t_k, t_{k+1})$ , and ii) the PDE part contains a new nonlinear term  $g(u)$ .

Throughout the paper, we assume the following hypotheses:

**H1:** The function  $f$  is continuous globally Lipschitz in both  $X$  and  $v$  with a Lipschitz constant  $K_0$  and  $f(X, 0) = 0$ .

**H2:** There exists a continuously differentiable and globally Lipschitz function  $\alpha(X)$ , such that the following dynamical system

$$\begin{cases} \dot{X}_i = X_{i+1} \\ \dot{X}_n = f(X, \alpha(X)) \end{cases}$$

is globally exponentially stable.

Using the converse Lyapunov Theorem Khalil (1996) we can say that there exists a function  $V_0(X) > 0$  and positive parameters  $c_i, i = 1, \dots, 4$  such that:

$$\begin{cases} c_1|X|^2 \leq V_0(X) \leq c_2|X|^2 \\ \left| \frac{\partial V_0}{\partial X_i} \right| \leq c_3|X| \\ \sum_{i=1}^{n-1} \frac{\partial V_0}{\partial X_i} X_{i+1} + \frac{\partial V_0}{\partial X_n} f(X, \alpha(X)) \leq -c_4|X|^2 \end{cases} \quad (2)$$

for all  $X$ .

**H3:** The function  $g$  is globally Lipschitz function in  $u$  with a Lipschitz constant  $K_g$ .

If we consider the changes of coordinates  $p(x, t) = u(x, t) - X_1$  then the above system can be written as follows

$$\begin{cases} \dot{X}_i = X_{i+1}, & i = 1, \dots, n-1 \\ \dot{X}_n = f(X, v(t_k)) & t \in [t_k, t_{k+1}) \\ p(D, t) = 0 \\ p_x(0, t) = 0 \\ p_t = p_{xx} + g(p(x, t) + X_1) - X_2, & x \in [0, D] \\ y(t) = p(0, t) + X_1 \end{cases} \quad (3)$$

### 3. OUTPUT FEEDBACK DESIGN

Based on the above hypotheses, we propose the following high-gain observer-based output feedback control:

$$\begin{cases} \dot{Z}_i = Z_{i+1} - l_i \theta^i (\hat{u}(0, t) - y), & i = 1, \dots, n-1 \\ \dot{Z}_n = f(Z, v(t_k)) - l_n \theta^n (\hat{u}(0, t) - y) & t \in [t_k, t_{k+1}) \\ v = \alpha(Z) \\ \hat{p}(D, t) = 0 \\ \hat{p}_x(0, t) = 0 \\ \hat{p}_t = \hat{p}_{xx} + g(\hat{p}(x, t) + Z_1) - Z_2 & x \in [0, D] \end{cases} \quad (4)$$

with

$$\hat{u}(x, t) = \hat{p}(x, t) + Z_1$$

if we consider the errors  $\tilde{p}(x, t) = \hat{p}(x, t) - p(x, t)$  and  $\tilde{u}(x, t) = \hat{u}(x, t) - u(x, t)$ , then

Let us consider the dynamical error system  $e = Z - X$  and  $\tilde{u} = \hat{u} - u$ , then for any  $x \in [0, D]$ , we obtain

$$\begin{cases} \dot{e}_i = e_{i+1} - l_i \theta^i e_1 - l_i \theta^i \tilde{p}(0, t) & i = 1, \dots, n-1 \\ \dot{e}_n = f(Z, \alpha(Z(t_k))) - f(X, \alpha(Z(t_k))) - l_n \theta^n e_1 - l_n \theta^n \tilde{p}(0, t) \\ t \in [t_k, t_{k+1}) \\ \dot{X}_i = X_{i+1}, & i = 1, \dots, n-1 \\ \dot{X}_n = f(X, \alpha(Z(t_k))) & t \in [t_k, t_{k+1}) \\ \tilde{p}(D, t) = 0 \\ \tilde{p}_x(0, t) = 0 \\ \tilde{p}_t = \tilde{p}_{xx} + g(\tilde{p}(x, t) + Z_1) - g(p(x, t) + X_1) - e_2, & x \in [0, D] \\ \tilde{u}(x, t) = \tilde{p}(x, t) - e_1 \end{cases} \quad (5)$$

By using the classical change of coordinates  $\xi_i = \theta^{1-i} e_i$ , we derive

$$\begin{cases} \dot{\xi}_i = \theta \xi_{i+1} - \theta l_i \xi_1 - l_i \theta \tilde{p}(0, t), & i = 1, \dots, n-1 \\ \dot{\xi}_n = \theta^{1-n} [f(Z, \alpha(Z(t_k))) - f(X, \alpha(Z(t_k)))] - \theta l_n \xi_1 - \theta l_n \tilde{p}(0, t) \\ t \in [t_k, t_{k+1}) \\ \dot{X}_i = X_{i+1}, & i = 1, \dots, n-1 \\ \dot{X}_n = f(X, \alpha(Z(t_k))) & t \in [t_k, t_{k+1}) \\ \tilde{p}(D, t) = 0 \\ \tilde{p}_x(0, t) = 0 \\ \tilde{p}_t = \tilde{p}_{xx} + g(\tilde{p}(x, t) + Z_1) - g(p(x, t) + X_1) - \theta \xi_2 \\ \tilde{u}(x, t) = \tilde{p}(x, t) - \xi_1 \end{cases}$$

Let us now introduce the following augmented vector state

$$\eta = [\xi, X]^T, \quad \text{where } \xi = (\xi_1, \dots, \xi_n)^T.$$

Then the above system can be written as

$$\begin{cases} \dot{\eta} = F_0(\eta, \tilde{p}(0, t)) \\ \tilde{p}(D, t) = 0 \\ \tilde{p}_x(0, t) = 0 \\ \tilde{p}_t = \tilde{p}_{xx} + g(\tilde{p}(x, t) + Z_1) - g(p(x, t) + X_1) - \theta \xi_2 \\ \tilde{u}(x, t) = \tilde{p}(x, t) - \xi_1 \end{cases} \quad (6)$$

where  $F_0(\eta, \tilde{p}(0, t))$  is given by

$$\begin{pmatrix} \theta \xi_{i+1} - \theta l_i \xi_1 - l_i \theta \tilde{p}(0, t), & i = 1, \dots, n-1 \\ \theta^{1-n} [f(\Delta \xi + X, \alpha(Z(t_k))) - f(X, \alpha(Z(t_k)))] \\ -\theta l_n \xi_1 - \theta l_n \tilde{p}(0, t) \\ X_{i+1}, & i = 1, \dots, n-1 \\ f(X, \alpha(\Delta \xi(t_k) + X)(t_k)) \end{pmatrix}. \quad (7)$$

where  $\Delta = \text{diag}(1, \dots, \theta^{n-1})$ .

*Remark 1.* The well-posedness problem of the system (6) can be proven by using the work of Pazy (1983) and by using similar arguments to those used in Katz and Fridman. (2020) and Katz and Fridman. (2022) for each interval  $[t_k, t_{k+1})$ . For instance, it is not difficult to see that the infinite dimensional part of the system (6) can be written in the Hilbert space  $L^2(0, D)$  as an ordinary differential equation:  $\dot{\tilde{p}}(t) = \mathcal{A}_0 \tilde{p}(t) + \tilde{g}(\tilde{p}(t), \xi(t)) - \theta \xi_2$  with  $\tilde{g}(\tilde{p}(t), \xi(t)) = g(\tilde{p}(t) + Z_1) - g(p(t) + X_1)$  and  $\mathcal{A}_0 = \frac{\partial^2}{\partial x^2}$  which is defined on the dense domain  $\mathcal{D}(\mathcal{A}_0)$

$$\mathcal{D}(\mathcal{A}_0) = \{\tilde{p} \in \mathcal{H}^2(0, D) : \tilde{p}_x(0) = \tilde{p}(D) = 0\}$$

The domain  $\mathcal{D}(\mathcal{A}_0) \subseteq \mathcal{H}^2(0, D) \subseteq L^2(0, D)$ . Furthermore as in Katz and Fridman. (2022) let us consider the operator

$$\tilde{\mathcal{A}} = \begin{pmatrix} \mathcal{A}_0 & 0 & 0 \\ 0 & \theta(A - LC) & 0 \\ 0 & 0 & A \end{pmatrix}.$$

Then following the same arguments than in Ahmed-Ali et al. (2023), we can conclude that, the Cauchy-problem

(6) admits a unique classical solution  $(\tilde{p}, \eta) \in (\mathcal{C}^0[0, \infty); \mathcal{H}) \cap (\mathcal{C}^1(0, \infty); \mathcal{H})$  such that  $(\tilde{p}(t), \eta(t)) \in \mathcal{D}(\mathcal{A}_0) \times \mathbb{R}^n \times \mathbb{R}^n \forall t > t_0$ . We can also do similar analysis for observer (4).

**Theorem 2.** Consider system (1). Then, there exist  $\theta_1, D^*(\theta), \bar{h}(\theta)$  such that  $\forall \theta > \theta_1, \forall D \in (0, D^*(\theta))$  and  $\forall h \in (0, \bar{h}(\theta))$ , the observation error system (6) converges exponentially to zero in the sense of the norm :  $(|\eta(t)|^2 + \int_0^D \tilde{u}^2(x, t)dx)^{\frac{1}{2}}$

**Proof.** In order to prove the exponential stability of the system (6), we will divide the proof into three parts: for the infinite dimensional sub-system, for the finite dimensional one and finally for the overall error system.

*Infinite dimensional sub-system*

In this first part we will analyse the sub-system

$$\begin{cases} \tilde{p}_t = \tilde{p}_{xx} + g(\hat{p}(x, t) + Z_1) - g(p(x, t) + X_1) - \theta \xi_2 \\ \tilde{p}(D, t) = 0 \\ \tilde{p}_x(0, t) = 0 \end{cases} \quad (8)$$

In order to do this, we consider the following Lyapunov functional :

$$W = \frac{1}{2} \int_0^D \tilde{p}^2(x, t)dx + \frac{1}{2} \int_0^D \tilde{p}_x^2(x, t)dx$$

Then, the time derivative of W(t) along the trajectory of the system (5) is

$$\begin{aligned} \dot{W}(t) &= \int_0^D \tilde{p}(x, t)\tilde{p}_t(x, t)dx + \int_0^D \tilde{p}_x(x, t)\tilde{p}_{xt}(x, t)dx \\ &= \int_0^D \tilde{p}(x, t)(\tilde{p}_{xx}(x, t) + \tilde{g} - \theta \xi_2)dx \\ &\quad + \int_0^D \tilde{p}_x(x, t)\tilde{p}_{tx}(x, t)dx \end{aligned}$$

where

$$\tilde{g} = g(\hat{p}(x, t) + Z_1) - g(p(x, t) + X_1).$$

Notice that under Hypotheses (H3), we have

$$|\tilde{g}|^2 \leq 2K_g (|\tilde{p}(x, t)|^2 + |\eta|^2)$$

From the fact that  $\tilde{p}(D, t) = 0$ , then we have  $\tilde{p}_t(D, t) = 0$

and by using the integration by parts on  $[0, D]$ , we can easily derive that

$$\begin{aligned} \dot{W} &= - \int_0^D \tilde{p}_x^2(x, t)dx - \theta \int_0^D \tilde{p}(x, t)\xi_2 dx - \int_0^D \tilde{p}_{xx}^2(x, t)dx \\ &\quad + \int_0^D p(x, t)\tilde{g}dx - \int_0^D \tilde{p}_{xx}\tilde{g}dx + \theta \int_0^D \tilde{p}_{xx}\xi_2 dx \end{aligned}$$

If we use Young's inequality, and under Hypotheses (H3) then,

$$\begin{aligned} \dot{W} &\leq - \int_0^D \tilde{p}_x^2(x, t)dx - \frac{1}{2} \int_0^D \tilde{p}_{xx}^2(x, t)dx \\ &\quad + \left(\frac{1}{4} + 24K_g^2\right) \int_0^D \tilde{p}^2(x, t)dx \\ &\quad + D(24K_g^2 + 12\theta^2)|\xi|^2 \end{aligned}$$

Now by By using Wirtinger's inequality Fridman and Blichovsky (2012), we can also derive

$$\begin{aligned} \dot{W} + 2\delta W &\leq -\left(\frac{\pi^2}{4D^2} - \frac{1}{4} - 24K_g^2 - \delta\right) \int_0^D \tilde{p}^2(x, t)dx \\ &\quad - \left(\frac{\pi^2}{8D^2} - \delta\right) \int_0^D \tilde{p}_x^2(x, t)dx \\ &\quad + D(24K_g^2 + 12\theta^2)|\xi|^2 \end{aligned}$$

*Finite-dimensional system*

Let us now analyse the unperturbed sub-system

$$\dot{\eta} = F_0(\eta, 0).$$

By considering the following Lyapunov function,

$$V(\eta) = \frac{1}{\theta^{2(n-1)}} V_0(X) + \xi^T P \xi \quad (9)$$

where P is a positive definite symmetric matrix which satisfies

$$P(A - LC) + (A - LC)^T P = -I_n$$

Now let us compute the derivative of V along the solution of the unperturbed system, we obtain

$$\dot{V} = \frac{\partial V}{\partial \eta} F_0(\eta, 0).$$

After some computations as in Khalil and Praly (2014), we can easily derive the following inequality:

$$\begin{aligned} \dot{V} &\leq -\frac{c_4}{2\theta^{2(n-1)}} |X|^2 - \left[\theta - 2\lambda_{max}(P)K_0 - \frac{4c_3^2}{c_4} K_0^2\right] |\xi|^2 \\ &\quad + 8\frac{c_3^2}{c_4} K_0^2 |\xi(t) - \xi(t_k)|^2 \\ &\quad + \frac{8}{\theta^{2(n-1)}} \frac{c_3^2}{c_4} K_0^2 |X(t) - X(t_k)|^2 \end{aligned} \quad (10)$$

for some positive constant  $K_0$  independent of  $\theta$ , and  $c_4$  defined in (2).

Choosing  $\theta > \theta_0$  such that :

$$\theta_0 = \max \left\{ 2\lambda_{max}(P)K_0 + \frac{2c_3^2}{c_4} K_0^2 + \frac{c_4}{2}, 1 \right\} \quad (11)$$

we deduce that

$$\begin{aligned} \dot{V} &\leq -\frac{c_4}{2\theta^{2(n-1)}} |\eta|^2 + 8\frac{c_3^2}{c_4} K_0^2 |\xi(t) - \xi(t_k)|^2 \\ &\quad + \frac{8}{\theta^{2(n-1)}} \frac{c_3^2}{c_4} K_0^2 |X(t) - X(t_k)|^2 \end{aligned}$$

Now let us consider the following Lyapunov functional

$$\begin{aligned} W_1(\eta) &= \frac{1}{\theta^{2(n-1)}} V_0(X) + \xi^T P \xi \\ &\quad + V_1 + V_2 \end{aligned} \quad (12)$$

where

$$V_1 = \frac{\alpha_1}{\theta^{2(n-1)}} h^2 e^{2\delta h} \int_{t_k}^t e^{2\delta(s-t)} |\dot{X}(s)|^2 ds - \frac{\pi^2}{4} \frac{\alpha_1}{\theta^{2(n-1)}} \int_{t_k}^t |X(s) - X(t_k)|^2 ds \quad (13)$$

and

$$V_2 = \alpha_2 h^2 e^{2\delta h} \int_{t_k}^t e^{2\delta(s-t)} |\dot{\xi}(s)|^2 ds - \frac{\pi^2}{4} \alpha_2 \int_{t_k}^t |\xi(s) - \xi(t_k)|^2 ds. \quad (14)$$

where  $\alpha_1$  and  $\alpha_2$  are two positive constants which will be determined later. By using the result of Selivanov and Fridman (2016), which uses generalized Wirtinger's inequality, we easily deduce that both  $V_1$  and  $V_2$  are nonnegative and does not grow at the jumps occurring at instants  $t_k$ . The time-derivative of  $W_1$ , along the sub-system

$$\dot{\eta} = F(\eta, \tilde{p}(0, t))$$

, satisfies :

$$\begin{aligned} \dot{W}_1 \leq & -\frac{c_4}{2\theta^{2(n-1)}} |X|^2 - \left[ \theta - 2\lambda_{max}(P)K_0 - \frac{4c_3^2}{c_4} K_0^2 \right] |\xi|^2 \\ & + 2\theta \xi^T PL\tilde{p}(0, t) + 8\frac{c_3^2}{c_4} K_0^2 |\xi(t) - \xi(t_k)|^2 \\ & + \frac{8}{\theta^{2(n-1)}} \frac{c_3^2}{c_4} K_0^2 |X(t) - X(t_k)|^2 \\ & - 2\delta V_1 + \frac{\alpha_1}{\theta^{2(n-1)}} h^2 e^{2\delta h} |\dot{X}(t)|^2 - \frac{\pi^2}{4} \frac{\alpha_1}{\theta^{2(n-1)}} |X(t) - X(t_k)|^2 \\ & - 2\delta V_2 + \alpha_2 h^2 e^{2\delta h} |\dot{\xi}(t)|^2 - \frac{\pi^2}{4} \alpha_2 |\xi(t) - \xi(t_k)|^2 \end{aligned} \quad (15)$$

Using Hypotheses **(H1-H2-H3)** and from the definition of the system (6) and (7) , we can deduce the following inequalities :

$$|\dot{X}(t)| \leq |A| \cdot |X(t)| + |b|K_0|Z(t_k)| \quad (16)$$

and

$$|\dot{X}(t)| \leq |A| \cdot |X(t)| + |b|K_0 \left( \theta^{n-1} |\xi(t_k) - \xi(t)| \right) + |b|K_0 \left( \theta^{n-1} |\xi(t)| + |X(t)| + |X(t_k) - X(t)| \right). \quad (17)$$

By using Young's inequality, and since  $|A| = |b| = 1$ , then we derive

$$|\dot{X}(t)|^2 \leq 8(1 + K_0^2) |X(t)|^2 + 8K_0^2 \left( \theta^{2(n-1)} |\xi(t_k) - \xi(t)|^2 \right) + 8K_0^2 \left( \theta^{2(n-1)} |\xi(t)|^2 + |X(t_k) - X(t)|^2 \right) \quad (18)$$

On the the hand, we have also the following inequality :

$$|\dot{\xi}(t)| \leq (\theta|A - LC| + K_0) |\xi(t)| + \theta L\tilde{p}(0, t) \quad (19)$$

Combining the above inequalities, we easily deduce :

$$\begin{aligned} \dot{W}_1 + 2\delta W_1 \leq & -\frac{1}{2\theta^{2(n-1)}} [c_4 - 16\alpha_1 h^2 e^{2\delta h} (1 + K_0^2) - 4\delta c_2] |X|^2 \\ & - \left[ \theta - 2\lambda_{max}(P)(K_0 + \delta) - \frac{4c_3^2}{c_4} K_0^2 \right] |\xi|^2 \\ & + [8K_0^2 \alpha_1 h^2 e^{2\delta h} + \alpha_2 h^2 e^{2\delta h} (\theta|A - LC| + K_0)^2 + \beta] |\xi|^2 \\ & - \frac{1}{\theta^{2(n-1)}} \left[ \alpha_1 \left( \frac{\pi^2}{4} - 8h^2 e^{2\delta h} K_0^2 \right) - 8\frac{c_3^2}{c_4} K_0^2 \right] |X(t) - X(t_k)|^2 \\ & - \left[ \alpha_2 \left( \frac{\pi^2}{4} - 2h^2 e^{2\delta h} (\theta|A - LC| + K_0)^2 \right) \right] |\xi(t) - \xi(t_k)|^2 \\ & + \left[ 8\alpha_1 h^2 e^{2\delta h} K_0^2 + 8\frac{c_3^2}{c_4} K_0^2 \right] |\xi(t) - \xi(t_k)|^2 \\ & + (2\theta^2 |L|^2 \alpha_2 h^2 e^{2\delta h} + \frac{1}{\beta} \theta^2 |PL|^2) D \int_0^D \tilde{p}_x^2(x, t) dx \end{aligned} \quad (20)$$

for positive constants  $\delta > 0$  and  $\beta > 0$ .

### Stability of the overall error system

At this stage, we consider the Lyapunov functional  $W_2 = W + W_1$ .

If we choose  $\alpha_1$  and  $\alpha_2$  such that

$$\begin{aligned} \alpha_1 &= \frac{8\frac{c_3^2}{c_4} K_0^2}{\left( \frac{\pi^2}{4} - 8h^2 e^{2\delta h} K_0^2 \right)} \\ \alpha_2 &= \frac{8\alpha_1 h^2 e^{2\delta h} K_0^2 + 8\frac{c_3^2}{c_4} K_0^2}{\frac{\pi^2}{4} - 2h^2 e^{2\delta h} (\theta|A - LC| + K_0)^2} \end{aligned} \quad (21)$$

then to guaranty that  $\alpha_1 > 0$  and  $\alpha_2 > 0$ , we have to choose  $h$  sufficiently small such that

$$\begin{aligned} \frac{\pi^2}{4} - 8h^2 e^{2\delta h} K_0^2 &> 0 \\ \frac{\pi^2}{4} - 2h^2 e^{2\delta h} (\theta|A - LC| + K_0)^2 &> 0 \end{aligned} \quad (22)$$

Then from the two previous parts, we obtain the following inequality

$$\begin{aligned} \dot{W}_2 + 2\delta W_2 \leq & -\frac{1}{2\theta^{2(n-1)}} [c_4 - 16\alpha_1 h^2 e^{2\delta h} (1 + K_0^2) - 4\delta c_2] |X|^2 \\ & - \left[ \theta - 2\lambda_{max}(P)(K_0 + \delta) - \frac{4c_3^2}{c_4} K_0^2 - D(24K_g^2 + 12\theta^2) \right] |\xi|^2 \\ & + [8K_0^2 \alpha_1 h^2 e^{2\delta h} + \alpha_2 h^2 e^{2\delta h} (\theta|A - LC| + K_0)^2 + \beta] |\xi|^2 \\ & - \left( \frac{\pi^2}{4D^2} - \frac{1}{4} - 24K_g^2 - \delta \right) \int_0^D \tilde{p}^2(x, t) dx \\ & - \left( \frac{\pi^2}{8D^2} - \delta - (2\theta^2 |L|^2 \alpha_2 h^2 e^{2\delta h}) \right) \int_0^D \tilde{p}_x^2(x, t) dx \\ & - \frac{1}{\beta} \theta^2 |PL|^2 D \int_0^D \tilde{p}_x^2(x, t) dx \end{aligned} \quad (23)$$

In order to ensure the exponential stability, it is sufficient to guaranty the following inequalities

$$\begin{aligned}
c_4 - 16\alpha_1 h^2 e^{2\delta h} (1 + K_0^2) &> 0 \\
\theta - 2\lambda_{max}(P)K_0 - \frac{4c_3^2}{c_4}K_0^2 - D(24K_g^2 + 12\theta^2) \\
- 8K_0^2\alpha_1 h^2 e^{2\delta h} - \alpha_2 h^2 e^{2\delta h} (\theta|A - LC| + K_0)^2 - \beta &> 0 \\
\frac{\pi^2}{4D^2} - \frac{1}{4} - 24K_g^2 - \delta &> 0 \\
\frac{\pi^2}{8D^2} - (2\theta^2|L|^2\alpha_2 h^2 e^{2\delta h} + \frac{1}{\beta}\theta^2|PL|^2)D &> 0. \quad (24)
\end{aligned}$$

for a positive fixed constant  $\delta > 0$  which represents the speed of convergence. Then in order to satisfy the above inequalities, we can choose  $h$  sufficiently small such that the inequalities  $\alpha_2 h^2 e^{2\delta h} (\theta|A - LC| + K_0)^2 < \frac{1}{2}$  and  $c_4 - 16\alpha_1 h^2 e^{2\delta h} (1 + K_0^2) > 0$  hold. Please note that it's not difficult to see that there exists  $\bar{h}$  such that for all  $h \in [0, \bar{h}(\theta))$  the above inequalities and the inequalities (22) hold since the terms involving  $h$  are increasing when  $h$  increases. We can also choose  $D$  sufficiently small such that the two last inequalities of (24) hold and the term  $D(24K_g^2 + 12\theta^2)$  involved in the second inequality (24) remains less than  $\frac{1}{2}$ . Indeed we can easily deduce that

$$D < D^*(\theta) = \min \{D_1^*(\theta), D_2^*(\theta)\} \quad (25)$$

where

$$D_1^*(\theta) = \min \left\{ \frac{\pi}{2} \sqrt{\frac{1}{\frac{1}{4} + 24K_g^2 + \delta}}, \frac{1}{2(24K_g^2 + 12\theta^2)} \right\}$$

and

$$D_2^*(\theta) = \left( \frac{\pi^2}{8(2\theta^2|L|^2\alpha_2 h^2 e^{2\delta h} + \frac{1}{\beta}\theta^2|PL|^2 + \delta)} \right)^{1/3}$$

After satisfying the above inequalities, we have to satisfy the second inequality of (24) by choosing  $\theta > \max \{\theta_1, \theta_0\}$  where  $\theta_0$  is defined in (11) and

$$\theta_1 = 1 + 2\lambda_{max}(P)(K_0 + \delta) + \frac{4c_3^2}{c_4}K_0^2 + 8K_0^2\alpha_1 h^2 e^{2\delta h} + \beta \quad (26)$$

and  $\beta$  is a tuning positive constant. This ends the proof.

*Remark 3.* The expression for  $D^*(\theta)$  shows that there is a trade off between  $\theta$  and  $D$ . Indeed as  $\theta$  increases,  $D^*(\theta)$  decreases. This also true for  $\theta$  and  $D$ : the larger the observer gain  $\theta$ , the smaller the length of the PDE. Notice that the gain of the observer does not depend on  $D$ . Then the condition (25) means only a restriction on the class of systems. Please note also that the tuning constant  $\beta$  can be used to realize the above trade off and  $\delta$  is used to fix the speed of convergence.

#### 4. CONCLUSION

A sampled-data case of the design of high-gain observer-based output feedback control for a nonlinear systems with sensors described by parabolic PDEs has been proposed by bringing into light explicitly the expected trade-off between the gain of the observer and the length of the PDE. Further works will be undertaken in the same vein by considering actuators PDEs and sampling in both input and output. On the other hand, we are working on the relaxation of the globally Lipschitz condition.

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