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On the computation of controlled invariant and output invisible subspaces for parameter-dependent systems

Valessa V. Viana, Jérémie Kreiss, and Marc Jungers. *

Abstract

This paper proposes a new approach for computing controlled invariant and output invisible subspaces of parameter-dependent systems. Non-strictly proper systems are considered, and the system matrices can exhibit a polynomial dependence on the parameter. The proposed approach is applied in both contexts of constant and parameter-dependent inputs making the subspace invariant, referred to as the generalized or generalized adaptively controlled invariant and output invisible subspaces. A large discussion unifies and makes the link with several contributions to the literature. Moreover, a procedure to compute one of the parameter-dependent input trajectories is provided. A numerical example is presented to demonstrate the effectiveness of the strategy.

1 Introduction

The geometric control theory is of great importance for many problems related to control theory [1–3], where the notion of controlled invariance, or equivalently (A, B) -invariance, is a core concept. It corresponds to the fact of containing the trajectories of the system in a particular sub-region of the whole state space by applying an adequate input. If these trajectories also result in a null output one, we speak of output invisibility. Controlled invariant and output invisible subspaces are widely studied for linear time-invariant (LTI) systems because it has been a powerful tool applied among others (i) for system invertibility [4, 5], (ii) in the disturbance decoupling problem [6], (iii) in the characterization of input redundancy [7], or (iv) for the design of control allocation methods [8].

In the last decades, parameter-dependent systems have gained significant importance in control theory. This representation of a system considers parameters that account for deviations from the original model. This approach provides also a more realistic model compared to the LTI representation since it can capture unmodeled dynamics, non-deterministic parameters, or nonlinearities. When going into the context of controlled invariance for parameter-dependent systems, the following question naturally arises: Can we characterize controlled invariant and output invisible subspaces that would also be independent from the parameters? To answer this question, [9] introduced the concept

of generalized controlled invariant subspaces and [10] a similar one called robust controlled invariant subspaces. Although neither of these two subspaces depend on the parameters, their difference lies in the fact that the input trajectories used to make the subspaces invariant can depend or not on the parameters. Indeed, these input trajectories must be independent from the parameters for the generalized controlled invariant subspaces, whereas they are allowed to depend on them for the robust controlled invariant ones.

Obtaining a tractable characterization of generalized or robust controlled invariant subspaces in the general framework of parameter-dependent systems is a difficult problem that is still open. In this context, a characterization is obtained for the generalized controlled invariant subspaces of affine parameter-dependent systems in [9] under the assumptions that the system matrices cannot depend on the same parameters and that the system is strictly proper. [11] extends this result by removing these two assumptions. Regarding the so-called robust controlled invariant subspaces, [10] and [12] proposed characterizations with an infinite number of conditions to verify. Later, [13–15] present results with a finite number of conditions under some specific assumptions (strictly proper systems, linear dependence on the parameter, input matrix independent from the parameter, etc).

In this paper, we aim to propose a new and general approach able to handle both generalized and robust controlled invariant subspaces in the case of polynomial dependence on the parameter and non-strictly proper systems. It is worth mentioning that the main contributions of the paper in this context are: (i) an equivalent and tractable characterization for the generalized controlled invariant subspaces and (ii) a tractable characterization for the robust controlled invariant subspaces in the case where the input trajectories making the subspace invariant also depend polynomially on the parameter. In addition, this paper provides, as another contribution, a procedure to compute one of the parameter-dependent input trajectories making invariant the robust controlled invariant subspace.

The paper is organized as follows. In Section 2, we recall some tools from the geometric control theory for LTI systems. These tools are extended to parameter-dependent systems in Section 3. A technical result is presented in Section 4 which is the core of the main results of the paper. Section 5 relates the largest subspace computed in the previous section to controlled invariant and output invisible subspaces of parameter-dependent systems. Section 6 provides some numerical examples to highlight the applicability

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of our strategy. Finally, in Section 7, some concluding remarks are presented.

Notation. Let \mathbb{R} denote the set of real numbers, \mathbb{R}^+ denote the set of positive real numbers including the zero, \mathbb{N} the set of natural numbers including the zero, and \mathbb{N}^* the set of natural numbers not including the zero. For a matrix M , M^\top means its transpose. $0_{n \times m}$ stands for the zero matrix of n rows and m columns, and I_n stands for the identity matrix of dimension n . For a set \mathcal{V} , \mathcal{V}^N stands for the Cartesian product $\mathcal{V} \times \mathcal{V} \times \dots \times \mathcal{V}$, N times. The notation $\text{diag}_{i=0}^N M_i$ denotes a diagonal matrix composed of the matrices M_i in the matrix block position (i, i) . $\text{Im}\{V\}$ is the vector space spanned by the columns of matrix V . $|\cdot|$ stands for the cardinality of a set. The Kronecker product is denoted by \otimes . The inverse map of set \mathcal{V} by an application B is given by $B^{-1}\mathcal{V} = \{u(t) \in \mathbb{R}^m; Bu(t) \in \mathcal{V}\}$, where B is not necessarily invertible.

2 Background on geometric control for LTI systems

In this section we recall some concepts from the geometric control theory for LTI systems [1, 2], including the notion of controlled invariant subspaces, which is widely discussed in the literature, and the concept of output invisibility of subspaces. Then, consider an LTI system as follows,

$$\begin{cases} \dot{x}(t) = Ax(t) + Bu(t), \\ y(t) = Cx(t) + Du(t), \end{cases} \quad (1)$$

where $x(t) \in \mathbb{R}^n$ is the state, $x(0) := x_0 \in \mathbb{R}^n$, $u(t) \in \mathbb{R}^m$ is the input, $y(t) \in \mathbb{R}^p$ is the output of the system, and A, B, C, D are constant matrices of appropriate dimension.

Definition 1 ([2, Section 7.3]). *A subspace $\mathcal{V} \subseteq \mathbb{R}^n$ is (A, B) -invariant and output invisible if for any $x_0 \in \mathcal{V}$ there exists an input function u such that the state trajectories remain in \mathcal{V} and the output is zero for all times, i.e., $\forall x_0 \in \mathcal{V}, \exists u$ such that $x_u(t, x_0) \in \mathcal{V}$ and $y_u(t, x_0) = 0$ for all $t \geq 0$.*

From [2, Section 7.3], the controlled invariance and output invisibility of the subspace \mathcal{V} , for the LTI system (1), can be verified by the satisfaction of the following condition:

$$\begin{bmatrix} A \\ C \end{bmatrix} \mathcal{V} \subseteq \mathcal{V} \times \{0_{p \times 1}\} + \text{Im} \left\{ \begin{bmatrix} B \\ D \end{bmatrix} \right\}, \quad (2)$$

or equivalently there exists a matrix $F \in \mathbb{R}^{m \times n}$, that is called a *friend* of \mathcal{V} , such that

$$\left(\begin{bmatrix} A \\ C \end{bmatrix} + \begin{bmatrix} B \\ D \end{bmatrix} F \right) \mathcal{V} \subseteq \mathcal{V} \times \{0_{p \times 1}\}. \quad (3)$$

In general, such a friend F is not unique. A procedure to compute the set of friends F of \mathcal{V} was proposed in [16]. Note that every controlled invariant and output invisible subspace \mathcal{V} is included in the largest controlled invariant and output invisible subspace, denoted by \mathcal{V}^* , which is also sometimes called the weakly unobservable subspace. An algorithm with a finite number of steps to compute the largest controlled invariant subspace of an LTI system was proposed in [6].

3 Controlled invariant subspaces of parameter-dependent systems

Consider now the following linear parameter-dependent system,

$$\begin{cases} \dot{x}(t) = A(\theta(t))x(t) + B(\theta(t))u(t), \\ y(t) = C(\theta(t))x(t) + D(\theta(t))u(t), \end{cases} \quad (4)$$

where $x(t) \in \mathbb{R}^n$ is the state, $x(0) := x_0 \in \mathbb{R}^n$, $u(t) \in \mathbb{R}^m$ is the input, $y(t) \in \mathbb{R}^p$ is the output of the system, and $\theta(t) \in \mathbb{R}^l$ is a parameter vector subject to variations at time t . $A(\theta(t)), B(\theta(t)), C(\theta(t)), D(\theta(t))$ are matrices of appropriate dimensions. For simplicity, the time dependence of these matrices will be omitted.

As presented in the introduction of this paper, two main frameworks exist for dealing with parameter-independent controlled invariant and output invisible subspaces of parameter-dependent systems: the generalized and robust controlled invariance. In both cases, the subspace presents robustness concerning the parameter but for the so-called robust controlled invariant subspace, the input is parameter-dependent so that the term *robust* may be confusing. Then, to avoid confusion, this paper adopts a different terminology and refers to it as *generalized adaptively* controlled invariance, as it involves an input that *adapts* according to the parameters. It is important to highlight that the two terms, *robust* and *generalized adaptively* controlled invariance, denote the same concept. We present the definitions of the two subspaces, generalized and generalized adaptively, in Definitions 2 and 3, respectively.

Definition 2 ([11]). *A subspace $\mathcal{V} \subseteq \mathbb{R}^n$ is a generalized controlled invariant and output invisible subspace if for any $x_0 \in \mathcal{V}$, there exists an input function u such that $\forall \theta \in \mathbb{R}^l, x_u(t, \theta, x_0) \in \mathcal{V}$ and $y_u(t, \theta, x_0) = 0$ for all $t \geq 0$.*

Definition 3. *A subspace $\mathcal{V} \subseteq \mathbb{R}^n$ is a generalized adaptively controlled invariant and output invisible subspace if for any $x_0 \in \mathcal{V}$, there exists an input function $u(t, \theta)$, where $\theta: \mathbb{R}^+ \rightarrow \mathbb{R}^l$, such that $x_u(t, \theta, x_0) \in \mathcal{V}$ and $y_u(t, \theta, x_0) = 0$, for all $\theta \in \mathbb{R}^l$ and $t \geq 0$.*

In the sequel, we recall from [11] the characterization of the *generalized* controlled invariance and output invisibility of a subspace \mathcal{V} in Lemma 1, and we extend the characterization given by [13] of *generalized adaptively* controlled invariance and output invisibility of a subspace \mathcal{V} to non-strictly proper parameter-dependent systems in Lemma 2.

Lemma 1 ([11]). *The following statements are equivalent:*

- (i) $\mathcal{V} \subseteq \mathbb{R}^n$ is a generalized controlled invariant and output invisible subspace,
- (ii) There exists a matrix $F \in \mathbb{R}^{m \times n}$, such that, for all $\theta \in \mathbb{R}^l$

$$(A(\theta) + B(\theta)F)\mathcal{V} \subseteq \mathcal{V}, \quad (C(\theta) + D(\theta)F)\mathcal{V} = 0. \quad (5)$$

Proof. (i) \Rightarrow (ii) Consider that \mathcal{V} is a generalized controlled invariant and output invisible subspace. Then, for

every $x_0 \in \mathcal{V}$, there exists an input function u such that $x_u(t, \theta, x_0) \in \mathcal{V}$ for all $\theta \in \mathbb{R}^l$. It follows that $\dot{x}(0^+) = \lim_{t \rightarrow 0^+} \frac{1}{t}(x_u(t, \theta, x_0) - x_0) \in \mathcal{V}$, and there exists $u_0 \in \mathbb{R}^m$ such that

$$\begin{bmatrix} A(\theta) \\ C(\theta) \end{bmatrix} x_0 + \begin{bmatrix} B(\theta) \\ D(\theta) \end{bmatrix} u_0 \in \mathcal{V} \times \{0_{p \times 1}\}. \quad (6)$$

for all $\theta \in \mathbb{R}^l$. Then, consider that $V \in \mathbb{R}^{n \times q}$ be a basis of \mathcal{V} where $q \leq n$ corresponds to the dimension of \mathcal{V} . For each column v of V there exists a vector-valued function $\tilde{v} : \mathbb{R}^l \rightarrow \mathcal{V}$, and a vector $\tilde{u} \in \mathbb{R}^m$ such that

$$\begin{bmatrix} A(\theta) \\ C(\theta) \end{bmatrix} v = \begin{bmatrix} B(\theta) \\ D(\theta) \end{bmatrix} \tilde{u} + \begin{bmatrix} \tilde{v}(\theta) \\ 0_{p \times 1} \end{bmatrix}. \quad (7)$$

Since matrix V is a basis of \mathcal{V} , then $\tilde{v}(\theta)$ can be expressed as $\tilde{v}(\theta) = V\tilde{y}(\theta)$, where $\tilde{y} : \mathbb{R}^l \rightarrow \mathbb{R}^q$. Thus, it is clear that there exist a matrix $U = [\tilde{u}_1, \dots, \tilde{u}_q] \in \mathbb{R}^{m \times q}$ and a matrix $Y : \mathbb{R}^l \rightarrow \mathbb{R}^{q \times q}$ such that

$$\begin{bmatrix} A(\theta) \\ C(\theta) \end{bmatrix} V = \begin{bmatrix} B(\theta) \\ D(\theta) \end{bmatrix} U + \begin{bmatrix} V \\ 0_{p \times q} \end{bmatrix} Y(\theta). \quad (8)$$

Finally, since V is full column rank, we have that $V^\top V$ is invertible. Then, we can define the gain $F = -U(V^\top V)^{-1}V^\top$, which is a friend of \mathcal{V} , leading to the inclusion (5).

(ii) \Rightarrow (i) For any $x_0 \in \mathcal{V}$, consider $u(t) = Fx(t)$, with F verifying (5), then $x_u(t, \theta, x_0) \in \mathcal{V}$ for all $\theta \in \mathbb{R}^l$. \square

Lemma 2. *The following statements are equivalent:*

(i) $\mathcal{V} \subseteq \mathbb{R}^n$ is a generalized adaptively controlled invariant and output invisible subspace,

(ii) There exists a matrix $F : \mathbb{R}^l \rightarrow \mathbb{R}^{m \times n}$, such that, for all $\theta \in \mathbb{R}^l$

$$(A(\theta) + B(\theta)F(\theta))\mathcal{V} \subseteq \mathcal{V}, \quad (C(\theta) + D(\theta)F(\theta))\mathcal{V} = 0. \quad (9)$$

(iii) \mathcal{V} satisfies, for all $\theta \in \mathbb{R}^l$

$$\begin{bmatrix} A(\theta) \\ C(\theta) \end{bmatrix} \mathcal{V} \subseteq \mathcal{V} \times \{0_{p \times 1}\} + \text{Im} \left\{ \begin{bmatrix} B(\theta) \\ D(\theta) \end{bmatrix} \right\}. \quad (10)$$

Proof. (i) \Rightarrow (iii) Following the developments of the proof of Lemma 1, we obtain

$$\begin{bmatrix} A(\theta) \\ C(\theta) \end{bmatrix} x_0 + \begin{bmatrix} B(\theta) \\ D(\theta) \end{bmatrix} u_0(\theta) \in \mathcal{V} \times \{0_{p \times 1}\}, \quad (11)$$

where, in this case, u_0 depends on θ , since u may depend on it. Equation (11) leads directly to the inclusion (10).

(iii) \Rightarrow (ii) Let us assume that inclusion (10) holds. By following the same proof of item (i) \Rightarrow (ii) of Lemma 1, with a matrix $U(\theta)$ depending on θ in this case, we have

$$\begin{bmatrix} A(\theta) \\ C(\theta) \end{bmatrix} V = \begin{bmatrix} B(\theta) \\ D(\theta) \end{bmatrix} U(\theta) + \begin{bmatrix} V \\ 0_{p \times q} \end{bmatrix} Y(\theta). \quad (12)$$

Then, the gain $F(\theta) = -U(\theta)(V^\top V)^{-1}V^\top$ is a friend of \mathcal{V} and satisfies relations (9).

(ii) \Rightarrow (i) For any $x_0 \in \mathcal{V}$, applying the control $u(t) = F(\theta)x(t)$, with $F(\theta)$ satisfying (9), leads $x_u(t, \theta, x_0) \in \mathcal{V}$ for all $\theta \in \mathbb{R}^l$. \square

The conditions presented in Lemmas 1 and 2 are challenging to verify since we have to check them for all $\theta \in \mathbb{R}^l$, resulting in an infinite number of conditions to be satisfied. Therefore, it is essential to have a framework that can characterize controlled invariant and output invisible subspaces of parameter-dependent systems with tractable conditions. However, this requires making assumptions about the dependence on the parameters, so it is crucial to develop a strategy with a good balance between tractable conditions and assumptions imposed on the parameter-dependent system. It would also be interesting to characterize both subspaces, generalized and generalized adaptively, using a unified approach and derive a generic algorithm to compute the largest subspace \mathcal{V}^* for both cases. Additionally, there is no procedure in the literature for designing such a parameter-dependent input into the set of inputs that renders the subspace controlled invariant. For some systems, finding a parameter-dependent input of the subspace is straightforward, but this does not hold for all cases. Thus, developing a methodology for designing such a parameter-dependent input is another induced goal.

4 Technical results

In this section, let us introduce some preliminary results related to operations on subspaces. Consider the following given matrices $\mathbf{A} \in \mathbb{R}^{n \times n}$, $\mathbf{B} \in \mathbb{R}^{n \times r}$, $\mathbf{C} \in \mathbb{R}^{p \times n}$, $\mathbf{D} \in \mathbb{R}^{p \times r}$, where $n, p, r, Z \in \mathbb{N}^*$, and a subspace $\mathcal{V} \subseteq \mathbb{R}^n$.

Lemma 3. *The following statements are equivalent:*

(i) $\mathcal{V} \subseteq \mathbb{R}^n$ satisfies

$$\begin{bmatrix} \mathbf{A} \\ \mathbf{C} \end{bmatrix} \mathcal{V} \subseteq (\mathcal{V}^Z \times \{0_{p \times 1}\}^Z) + \text{Im} \left\{ \begin{bmatrix} \mathbf{B} \\ \mathbf{D} \end{bmatrix} \right\}, \quad (13)$$

(ii) There exists a matrix $\mathbf{F} \in \mathbb{R}^{r \times n}$, such that

$$\left(\begin{bmatrix} \mathbf{A} \\ \mathbf{C} \end{bmatrix} + \begin{bmatrix} \mathbf{B} \\ \mathbf{D} \end{bmatrix} \mathbf{F} \right) \mathcal{V} \subseteq \mathcal{V}^Z \times \{0_{p \times 1}\}^Z. \quad (14)$$

Proof. (i) \Rightarrow (ii) Let us assume that inclusion (13) holds, and let $V \in \mathbb{R}^{n \times q}$ be a basis¹ of \mathcal{V} where $q \leq n$ corresponds to the dimension of \mathcal{V} . Then, for each column v of V there exist Z vectors $\tilde{v}_i \in \mathcal{V}$, $i = 1, \dots, Z$, and a vector $\tilde{u} \in \mathbb{R}^r$ such that

$$\begin{bmatrix} \mathbf{A} \\ \mathbf{C} \end{bmatrix} v = \begin{bmatrix} \mathbf{B} \\ \mathbf{D} \end{bmatrix} \tilde{u} + (\tilde{v}_1^\top \cdots \tilde{v}_Z^\top \ 0_{1 \times p} \cdots 0_{1 \times p})^\top. \quad (15)$$

$\tilde{v}_i \in \mathcal{V}$ can be expressed as $\tilde{v}_i = V\tilde{y}_i$, where $\tilde{y}_i \in \mathbb{R}^q$. Thus, by selecting each column of V for v in (15), it is clear that there exist a matrix $U \in \mathbb{R}^{r \times q}$ and a matrix $Y \in \mathbb{R}^{q \times q}$ such that

$$\begin{bmatrix} \mathbf{A} \\ \mathbf{C} \end{bmatrix} V = \begin{bmatrix} \mathbf{B} \\ \mathbf{D} \end{bmatrix} U + \begin{bmatrix} \text{diag}_{i=1}^Z V \\ 0_{p \times q} \end{bmatrix} Y. \quad (16)$$

¹Note that when $\mathcal{V} = \{0\}$, its dimension q is zero such that the basis V is empty. In this case, $\mathbf{F} = 0$ holds by definition. This trivial situation is omitted in our algorithm.

Finally, since $V^\top V$ being invertible, we can define the following gain satisfying inclusion (14)

$$\mathbf{F} = -U(V^\top V)^{-1}V^\top. \quad (17)$$

(ii) \Rightarrow (i) Let us assume that there exists \mathbf{F} such that inclusion (14) holds. Then, it is straightforward that inclusion (13) also holds. \square

Denote $\mathcal{V}_{(\mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{D})} = \mathcal{V}_S$ the set of subspaces \mathcal{V} that satisfy (13). Then, we present the following lemma.

Lemma 4. *The set \mathcal{V}_S admits a unique maximal element, which is denoted by \mathcal{V}^* . It means that if $\mathcal{V} \in \mathcal{V}_S$, then $\mathcal{V} \subseteq \mathcal{V}^*$.*

Proof. The proof is inspired by the developments in [6] and [11]. By definition, the singleton vector is an element of \mathcal{V}_S , i.e., $\{0_{n \times 1}\} \in \mathcal{V}_S$. Moreover, \mathcal{V}_S is closed by addition, i.e. if $\mathcal{V}_1 \in \mathcal{V}_S$ and $\mathcal{V}_2 \in \mathcal{V}_S$, $\mathcal{V}_1 + \mathcal{V}_2 \in \mathcal{V}_S$. From [1, Lemma 4.4], since \mathcal{V}_S is a nonempty subspace and closed under addition, then \mathcal{V}_S contains a supremal element denoted by \mathcal{V}^* . Thus, if $\mathcal{V} \in \mathcal{V}_S$, $\dim(\mathcal{V} + \mathcal{V}^*) \leq \dim(\mathcal{V}^*)$, which implies $\mathcal{V} \subseteq \mathcal{V}^*$. On the other hand, assume that there exist two distinct maximal elements, these subspaces are included in each other by the definition of maximality. Then, they are identical, proving that the maximal element $\mathcal{V}^* \in \mathcal{V}_S$ is unique. \square

An iterative algorithm to compute the maximal subspace \mathcal{V}^* is presented in Algorithm 1. Furthermore, MATLAB tools to compute the geometric subspaces were developed in [17].

Algorithm 1: Computation of \mathcal{V}^* .

input : $\mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{D}$

output: \mathcal{V}^*

1 $\mathcal{V}_0 \leftarrow \mathbb{R}^n$, $k = 0$;

2 **repeat**

3 $\mathcal{V}_{k+1} \leftarrow \begin{bmatrix} \mathbf{A} \\ \mathbf{C} \end{bmatrix}^{-1} \left(\mathcal{V}_k^Z \times \{0_{p \times 1}\}^Z + \text{Im} \left\{ \begin{bmatrix} \mathbf{B} \\ \mathbf{D} \end{bmatrix} \right\} \right)$; (18)

4 $k = k + 1$;

5 **until** $\mathcal{V}_{k+1} = \mathcal{V}_k$;

6 **return** $\mathcal{V}^* = \mathcal{V}_k$;

Proposition 1. *Algorithm 1 converges in at most n steps and returns the largest controlled invariant and output invisible subspace \mathcal{V}^* .*

Proof. Let us prove, by recurrence, that the given algorithm returns the largest subspace such that (13) or (14) holds. First, we prove that the sequence $\{\mathcal{V}_k\}_{k \in \mathbb{N}}$ is not increasing in the sense of inclusion. We have $\mathcal{V}_0 = \mathbb{R}^n \supseteq \mathcal{V}_1$, then considering that $\mathcal{V}_{k-1} \supseteq \mathcal{V}_k$ and applying some manipulations, it yields

$$\mathcal{V}_{k-1}^Z \times \{0_{p \times 1}\}^Z + \text{Im} \left\{ \begin{bmatrix} \mathbf{B} \\ \mathbf{D} \end{bmatrix} \right\} \supseteq \mathcal{V}_k^Z \times \{0_{p \times 1}\}^Z + \text{Im} \left\{ \begin{bmatrix} \mathbf{B} \\ \mathbf{D} \end{bmatrix} \right\},$$

and finally

$$\begin{aligned} \mathcal{V}_k &= \begin{bmatrix} \mathbf{A} \\ \mathbf{C} \end{bmatrix}^{-1} \left(\mathcal{V}_{k-1}^Z \times \{0_{p \times 1}\}^Z + \text{Im} \left\{ \begin{bmatrix} \mathbf{B} \\ \mathbf{D} \end{bmatrix} \right\} \right) \supseteq \\ &\begin{bmatrix} \mathbf{A} \\ \mathbf{C} \end{bmatrix}^{-1} \left(\mathcal{V}_k^Z \times \{0_{p \times 1}\}^Z + \text{Im} \left\{ \begin{bmatrix} \mathbf{B} \\ \mathbf{D} \end{bmatrix} \right\} \right) = \mathcal{V}_{k+1}. \end{aligned}$$

Furthermore, we are also able to prove by recurrence that $\mathcal{V}_k \supseteq \mathcal{V}^*$. We also have $\mathcal{V}_0 = \mathbb{R}^n \supseteq \mathcal{V}^*$. Consider $\mathcal{V}_k \supseteq \mathcal{V}^*$, and let us show that $\mathcal{V}_{k+1} \supseteq \mathcal{V}^*$,

$$\begin{aligned} \mathcal{V}_{k+1} &= \begin{bmatrix} \mathbf{A} \\ \mathbf{C} \end{bmatrix}^{-1} \left(\mathcal{V}_k^Z \times \{0_{p \times 1}\}^Z + \text{Im} \left\{ \begin{bmatrix} \mathbf{B} \\ \mathbf{D} \end{bmatrix} \right\} \right) \supseteq \\ &\begin{bmatrix} \mathbf{A} \\ \mathbf{C} \end{bmatrix}^{-1} \left((\mathcal{V}^*)^Z \times \{0_{p \times 1}\}^Z + \text{Im} \left\{ \begin{bmatrix} \mathbf{B} \\ \mathbf{D} \end{bmatrix} \right\} \right) = \mathcal{V}^*. \end{aligned}$$

Thus, $\{\mathcal{V}_k\}_{k \in \mathbb{N}}$ is a non-increasing sequence, and there exists one integer $K \leq n - 1$ such that Algorithm 1 converges in finite time and $\mathcal{V}_{K+1} = \mathcal{V}_K$, then it follows that $\mathcal{V}_K \in \mathcal{V}_S$. Since there exists a maximal element \mathcal{V}^* , we have $\mathcal{V}_K \supseteq \mathcal{V}^* \supseteq \mathcal{V}_K$, which is equivalent to the equality $\mathcal{V}_K = \mathcal{V}^*$, completing the proof. \square

Regarding geometric control theory, Algorithm 1 is an extension of the one provided in [6], allowing to compute controlled invariant subspaces of LTI systems (1). It is recovered by setting $Z = 1$, $\mathbf{A} = A$, $\mathbf{B} = B$, $\mathbf{C} = C$, and $\mathbf{D} = D$. In the context of this paper, this algorithm can be used to compute the largest controlled invariant and output invisible subspace of parameter-dependent systems. It means that Lemma 3 and the associated Algorithm 1 can be applied in order to provide a method to compute \mathcal{V}^* and a related friend F or $F(\theta)$ depending on the context of Lemmas 1 or 2.

5 Main results

In this section, we formalize a strategy for characterizing controlled invariant and output invisible subspaces of a parameter-dependent system. We consider systems that exhibit polynomial dependence on the parameter over the unit simplex set, such that a homogeneous polynomial representation of the system matrices can be obtained. By representing the system matrices in a homogeneous polynomial form, we are able to make the characterization problem more tractable. Then, to obtain the conditions from our approach, we assume that the matrices of system (4) meet the following criteria.

Assumption 1. *The parameter $\theta(t)$ lies in a compact set $\Theta \subset \mathbb{R}^l$. There exist an integer $N \in \mathbb{N}^*$ and a vector-valued $\alpha : \Theta \rightarrow \Lambda_N$, with $\Lambda_N = \left\{ \alpha \in \mathbb{R}^N, \alpha_i \geq 0, \forall i = 1, \dots, N, \sum_{i=1}^N \alpha_i = 1 \right\}$, such that the matrices of system (4) can be represented as homogeneous polynomial matrices with degrees n_A, n_B, n_C , and n_D with respect to the parameter $\alpha(\theta)$ (denoted α is the sequel to enlighten the notation).*

Note that a generic polynomial dependent system where the compact set Θ is a polytope can always be rewritten in an equivalent manner that satisfies Assumption 1.

Furthermore, we establish a particular structure for the friend $F(\theta)$, which is also a homogeneous polynomial function of degree $n_F \geq 0$ ($n_F = 0$ when we want F independent from θ) with respect to $\alpha(\theta)$, enabling us to compute a parameter-dependent friend of the subspace.

Recalling some representations of homogeneous polynomials from [18] and [19] and related notations, a homogeneous polynomial matrix of arbitrary degree n_A is given in a generalized form by $A(\alpha) = \sum_{k \in \mathcal{K}_{n_A}} \alpha^k A_k$, where $\alpha^k = \alpha_1^{k_1} \alpha_2^{k_2} \dots \alpha_N^{k_N}$ and $k = k_1 k_2 \dots k_N$. For $i = 1, \dots, N$, k_i are nonnegative and integer elements, $\mathcal{K}_n := \{k \in \mathbb{N}^N : k_1 + k_2 + \dots + k_N = n\}$ is the set of N -tuples with all possible combinations of k such that the sum of all k_i elements is equal to n , $N_A = |\mathcal{K}_{n_A}|$ and $N_{A+B} = |\mathcal{K}_{n_A+n_B}|$ are, respectively, the cardinal of the sets \mathcal{K}_{n_A} and $\mathcal{K}_{n_A+n_B}$, $\alpha_1^{k_1} \alpha_2^{k_2} \dots \alpha_N^{k_N}$ are monomials with $\alpha \in \Lambda_N$, and $A_k \in \mathbb{R}^{n \times n}$ are matrices indexed by the elements of \mathcal{K}_{n_A} . Moreover, the vertical concatenation and the horizontal concatenation of matrices A_k for all k elements in the set \mathcal{K}_{n_A} are given by $\text{cat}_v \{A_k\}$, and $\text{cat}_h \{A_k\}$, respectively. Con-

sider also the indicator function $\delta : \mathcal{K}_n \times \mathcal{K}_n \rightarrow \{0, 1\}$, where $(k, k') \mapsto \delta(k, k')$ with $\delta(k, k') = 1$ if $k = k'$, or 0 otherwise, such that $\text{cat}_h \delta_{kj} = [\delta_{k_1 j} \quad \delta_{k_2 j} \quad \dots \quad \delta_{k_N j}]$, that is, for example, equal to $[1 \quad 0 \quad \dots \quad 0]$ if $j = k_1$. These notations will be useful later in this paper and are illustrated by the following example.

Example 1. Consider a system of the form (4), where

$$A(\theta) = \begin{bmatrix} 1 & 1 \\ \theta^2 & 1 \end{bmatrix}, B(\theta) = \begin{bmatrix} 1 & 0 \\ 0 & \theta \end{bmatrix}, C = [0 \quad 1], \quad (19)$$

with $0 \leq \theta \leq 1$. To deal with the polynomial dependence on θ , we can represent it as a homogeneous polynomial system, such that $\alpha_1 = \theta$, $\alpha_2 = 1 - \alpha_1$, $C = [0 \quad 1]$,

$$A(\alpha) = \begin{bmatrix} (\alpha_1 + \alpha_2)^2 & (\alpha_1 + \alpha_2)^2 \\ \alpha_1^2 & (\alpha_1 + \alpha_2)^2 \end{bmatrix}, \\ B(\alpha) = \begin{bmatrix} \alpha_1 + \alpha_2 & 0 \\ 0 & \alpha_1 \end{bmatrix}.$$

Here, we have $N = 2$, $n_A = 2$, and $\mathcal{K}_A = \{(2, 0), (1, 1), (0, 2)\}$, $N_A = 3$, $n_B = 1$, $\mathcal{K}_B = \{(1, 0), (0, 1)\}$, $N_B = 2$, leading us to $A(\alpha) = \alpha_1^2 A_{20} + \alpha_1 \alpha_2 A_{11} + \alpha_2^2 A_{02}$ and $B(\alpha) = \alpha_1 B_{10} + \alpha_2 B_{01}$, with $B_{10} = I_2$,

$$A_{20} = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}, A_{11} = \begin{bmatrix} 2 & 2 \\ 0 & 2 \end{bmatrix}, \\ B_{01} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, A_{02} = \frac{1}{2} A_{11}.$$

Moreover,

$$\text{cat}_v \{A_k\}_{k \in \mathcal{K}_2} = [A_{20}^\top \quad A_{11}^\top \quad A_{02}^\top]^\top, \\ \text{cat}_h \{A_k\}_{k \in \mathcal{K}_2} = [A_{20} \quad A_{11} \quad A_{02}].$$

Then, in the following, looking for a friend that is a homogeneous polynomial on the parameter, we establish the equivalence between conditions (5) and (9) from Lemmas 1 and 2 with condition (14) from Lemma 3 by specifying auxiliary matrices \mathbf{A} , \mathbf{B} , \mathbf{C} , and \mathbf{D} . By this equivalence, we can compute both generalized or generalized adaptively controlled invariant and output invisible subspaces of parameter-dependent systems with conditions from Lemma 3, allowing us to use Algorithm 1 computing generalized or generalized adaptively \mathcal{V}^* . In Theorem 1, we demonstrate the equivalence between condition (9) in Lemma 2 and condition (14) in Lemma 3.

Theorem 1. Under Assumption 1, define

$$\mathbf{A} = \text{cat}_v \left\{ (n_B + n_F - n_A)! \sum_{\substack{\tau \in \mathcal{K}_{(n_B+n_F-n_A)} \\ j-\tau \in \mathcal{K}_{n_A}}} \frac{A_{j-\tau}}{\pi(\tau)} \right\}, \\ \mathbf{B} = \text{cat}_v \left\{ \sum_{j \in \mathcal{K}_{n_B+n_F}} \left(\sum_{\substack{\ell \in \mathcal{K}_{n_F} \\ j-\ell \in \mathcal{K}_{n_B}}} \text{cat}_h \delta_{i\ell} \otimes B_{j-\ell} \right) \right\}, \\ \mathbf{C} = \text{cat}_v \left\{ (n_D + n_F - n_C)! \sum_{\substack{\tau \in \mathcal{K}_{(n_D+n_F-n_C)} \\ j-\tau \in \mathcal{K}_{n_C}}} \frac{C_{j-\tau}}{\pi(\tau)} \right\}, \\ \mathbf{D} = \text{cat}_v \left\{ \sum_{j \in \mathcal{K}_{n_D+n_F}} \left(\sum_{\substack{\ell \in \mathcal{K}_{n_F} \\ j-\ell \in \mathcal{K}_{n_D}}} \text{cat}_h \delta_{i\ell} \otimes D_{j-\ell} \right) \right\},$$

where $\pi(\tau) = (\tau_1!) (\tau_2!) \dots (\tau_N!)$ and, without loss of generality, that $n_B + n_F \geq n_A$ and $n_D + n_F \geq n_C$. Then the two following statements are equivalent.

(i) There exists a homogeneous polynomial $F : \Lambda_N \rightarrow \mathbb{R}^{m \times n}$ of degree n_F with respect to α such that, for all $\alpha \in \Lambda_N$,

$$(A(\alpha) + B(\alpha)F(\alpha))\mathcal{V} \subseteq \mathcal{V}, \quad (20)$$

$$(C(\alpha) + D(\alpha)F(\alpha))\mathcal{V} = 0. \quad (21)$$

(ii) There exists a matrix $\mathbf{F} \in \mathbb{R}^{N_F \times n}$, such that

$$\left(\begin{bmatrix} \mathbf{A} \\ \mathbf{C} \end{bmatrix} + \begin{bmatrix} \mathbf{B} \\ \mathbf{D} \end{bmatrix} \mathbf{F} \right) \mathcal{V} \subseteq \mathcal{V}^{Z_1} \times \{0_{p \times 1}\}^{Z_2}, \quad (22)$$

where $Z_1 = N_{B+F}$ and $Z_2 = N_{D+F}$.

Proof. (i) \Rightarrow (ii) There exists an $F : \Lambda_N \rightarrow \mathbb{R}^{m \times n}$ that is called a friend of \mathcal{V} , such that (20)-(21) hold. The multiplication of polynomials $B(\alpha)$ and $F(\alpha)$ is $B(\alpha)F(\alpha) = \left(\sum_{k \in \mathcal{K}_{n_B}} \alpha^k B_k \right) \left(\sum_{j \in \mathcal{K}_{n_F}} \alpha^j F_j \right)$. Thanks to $\alpha^k B_k \alpha^j F_j = \alpha^{k+j} B_k F_j$ and by applying a change of variables $k+j = \ell \in \mathcal{K}_{n_B+n_F}$, we can represent $B(\alpha)F(\alpha)$ in the following form

$$B(\alpha)F(\alpha) = \sum_{\substack{j \in \mathcal{K}_{n_B+n_F} \\ \ell \in \mathcal{K}_{n_F} \\ j-\ell \in \mathcal{K}_{n_B}}} \alpha^j B_{j-\ell} F_\ell. \quad (23)$$

Moreover, as $\sum_{i=1}^N \alpha_i = 1$ and² $n_B + n_F \geq n_A$, the polynomial matrix $A(\alpha)$ can also be rewritten as

²If $n_B + n_F < n_A$, we can use the same technique to increase the degree of $B(\alpha)F(\alpha)$ to obtain the same degree of $A(\alpha)$.

$(\sum_{i=1}^N \alpha_i)^{n_T} A(\alpha)$, where $n_T = (n_B + n_F - n_A)$, to have the same degree of the polynomial $B(\alpha)F(\alpha)$. By following the technique presented by [20] and [21, Section 2.24], we obtain the representation

$$A(\alpha) = n_T! \sum_{\substack{j \in \mathcal{K}_{n_B+n_F} \\ \tau \in \mathcal{K}_{n_T} \\ j-\tau \in \mathcal{K}_{n_A}}} \alpha^j \frac{A_{j-\tau}}{\pi(\tau)}. \quad (24)$$

Since the polynomials from (23) and (24) have the same degree $n_B + n_F$, the sum $A(\alpha) + B(\alpha)F(\alpha)$ can be rearranged by gathering together the monomials α^j for all $j \in \mathcal{K}_{n_B+n_F}$. That implies that condition (20) is equivalent to

$$\left[n_T! \sum_{\substack{\tau \in \mathcal{K}_{n_T} \\ j-\tau \in \mathcal{K}_{n_A}}} \frac{A_{j-\tau}}{\pi(\tau)} + \sum_{\substack{\ell \in \mathcal{K}_{n_F} \\ j-\ell \in \mathcal{K}_{n_B}}} B_{j-\ell} F_\ell \right] \mathcal{V} \subseteq \mathcal{V}, \quad (25)$$

for all $j \in \mathcal{K}_{n_B+n_F}$. The same procedure applies for $C(\alpha) + D(\alpha)F(\alpha)$, such that condition (21) can be represented as

$$\left[n_P! \sum_{\substack{\tau \in \mathcal{K}_{n_T} \\ j-\tau \in \mathcal{K}_{n_C}}} \frac{C_{j-\tau}}{\pi(\tau)} + \sum_{\substack{\ell \in \mathcal{K}_{n_F} \\ j-\ell \in \mathcal{K}_{n_D}}} D_{j-\ell} F_\ell \right] \mathcal{V} \subseteq \mathcal{V}, \quad (26)$$

for all $j \in \mathcal{K}_{n_D+n_F}$, where $n_P = n_D + n_F - n_C$. Moreover, we can rewrite (25) and (26) as condition (22), where $\mathbf{F} = \text{cat}_v \{F_\ell\}$ is the vertical concatenation of F_ℓ for all $\ell \in \mathcal{K}_{n_F}$.

(ii) \Rightarrow (i) For any $x_0 \in \mathcal{V}$, the control $u(t) = \sum_{\ell \in \mathcal{K}_{n_F}} \theta^\ell F_\ell x(t)$ ensures that $x_u(t, \theta, x_0) \in \mathcal{V}$. \square

Note that differently from (20) and (21), condition (22) does not depend anymore on α , then we do not have to check the conditions for all $\alpha \in \Lambda_N$, i.e., we have a finite number of conditions to be satisfied. It is also noteworthy that Theorem 1 and Lemma 3 depend on the degree n_F of the parameter-dependent friend $F(\alpha)$. Hence, this is also the case of the subset \mathcal{V}^* provided by Algorithm 1. Nevertheless, as stated by the following proposition, we can show that increasing the degree n_F leads to potential larger subspaces \mathcal{V}^* . That is to say, a hierarchy related to the degree n_F .

Proposition 2. *The sequence $\{\mathcal{V}_{n_F}^*\}_{n_F \in \mathbb{N}}$ is non-strictly increasing in the sense of inclusions, where $\mathcal{V}_{n_F}^*$ is the subset returned by Algorithm 1 for a given degree n_F .*

Proof. Choose $n_F \in \mathbb{N}$. Suppose that $F_{n_F}(\alpha)$ and $\mathcal{V}_{n_F}^*$ satisfy (20)–(21), where $F_{n_F}(\alpha)$ is a friend of degree n_F associated to the subspace $\mathcal{V}_{n_F}^*$. Then, we also have a friend $F_{n_F+1}(\alpha) = \left(\sum_{i=1}^N \alpha_i\right) F_{n_F}(\alpha)$ of degree $n_F + 1$ verifying $(A(\alpha) + B(\alpha)F_{n_F+1}(\alpha))\mathcal{V}_{n_F}^* \subseteq \mathcal{V}_{n_F}^*$ and $(C(\alpha) + D(\alpha)F_{n_F+1}(\alpha))\mathcal{V}_{n_F}^* = 0$. That yields, by definition of the largest subspace, that $\mathcal{V}_{n_F}^* \subseteq \mathcal{V}_{n_F+1}^*$. \square

From Theorem 1, we can derive conditions related to the *generalized* controlled invariant subspaces by setting $n_F = 0$, indicating that the friend cannot be parameter-dependent. The equivalence between condition (5) in

Lemma 1 and condition (14) in Lemma 3 is established in Corollary 1, such that Algorithm 1 can also be used to compute the largest generalized controlled invariant subspace.

Corollary 1. *Under Assumption 1, define*

$$\begin{aligned} \mathbf{A} &= \text{cat}_v \left\{ (n_B - n_A)! \sum_{\substack{\tau \in \mathcal{K}_{n_B-n_A} \\ j-\tau \in \mathcal{K}_{n_A}}} \frac{A_{j-\tau}}{\pi(\tau)} \right\}, \\ \mathbf{C} &= \text{cat}_v \left\{ (n_D - n_C)! \sum_{\substack{\tau \in \mathcal{K}_{n_D-n_C} \\ j-\tau \in \mathcal{K}_{n_C}}} \frac{C_{j-\tau}}{\pi(\tau)} \right\}, \\ \mathbf{B} &= \text{cat}_v \{B_k\}, \quad \mathbf{D} = \text{cat}_v \{D_k\}, \\ &\quad k \in \mathcal{K}_{n_B} \quad k \in \mathcal{K}_{n_D} \end{aligned}$$

and, without loss of generality, that $n_B \geq n_A$ and $n_D \geq n_C$. Then the two following statements are equivalent.

(i) *There exists a matrix $F \in \mathbb{R}^{m \times n}$ such that (20) and (21) hold.*

(ii) *There exists a matrix $\mathbf{F} \in \mathbb{R}^{m \times n}$, such that (22) holds.*

Proof. The proof comes from Theorem 1 with $n_F = 0$ and $\mathbf{F} = F$. \square

Remark 1. *The procedure to compute a friend comes from Lemma 3. Since we find \mathcal{V}^* applying Algorithm 1 we have the basis V of \mathcal{V}^* . By solving a system of linear equations (16), where $[U^\top \ Y^\top]^\top$ is the variable, we find the matrix U allowing us to compute \mathbf{F} by relation (17).*

Particular cases: The strategy presented in this paper is a generalized framework dealing both with polynomial systems considering polynomial or constant friends and with the cases already covered in the literature, that are presented now.

1. Systems with linear dependence and with constant friends [11]: The system matrices present a linear dependence on the parameter, then $n_A = n_B = n_C = n_D = 1$ and $(A, B, C, D)(\alpha) = \sum_{i=1}^N \alpha_i (A_i, B_i, C_i, D_i)$. The bold matrices are

$$\begin{aligned} \mathbf{A} &= [A_1^\top, \dots, A_N^\top]^\top, \quad \mathbf{B} = [B_1^\top, \dots, B_N^\top]^\top, \\ \mathbf{C} &= [C_1^\top, \dots, C_N^\top]^\top, \quad \mathbf{D} = [D_1^\top, \dots, D_N^\top]^\top, \end{aligned}$$

and $Z_1 = Z_2 = N$. We obtain here the generalized controlled invariant and output invisible subspace by applying the algorithm when the dependence is linear.

2. Systems with linear dependence and with friends depending linearly on the parameter [14]: The system matrices present a linear dependence on the parameter, then $n_A = n_B = n_C = n_D = 1$. Moreover, the friend also presents a linear dependence on the parameter, i.e., $F(\alpha) = \sum_{i=1}^N \alpha_i F_i$ leading to $n_F = 1$. Then, the expressions of the bold matrices are given by

$$\begin{aligned} \mathbf{A} &= f(\{A_i\}_{i=1}^N), \quad \mathbf{B} = g(\{B_i\}_{i=1}^N), \\ \mathbf{C} &= f(\{C_i\}_{i=1}^N), \quad \mathbf{D} = g(\{D_i\}_{i=1}^N), \end{aligned}$$

where $f(\{A_i\}_{i=1}^N)$ and $g(\{B_i\}_{i=1}^N)$ are given, respectively, by

$$\begin{bmatrix} A_1 \\ \vdots \\ A_N \\ \hline A_1 + A_2 \\ \vdots \\ A_1 + A_N \\ \hline A_2 + A_3 \\ \vdots \\ A_2 + A_N \\ \hline \vdots \\ A_{N-1} + A_N \end{bmatrix}, \text{ and } \begin{bmatrix} B_1 & \cdots & \cdots & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & \cdots & \cdots & B_N \\ \hline B_2 & B_1 & \cdots & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ B_N & 0 & \cdots & \cdots & B_1 \\ \hline 0 & B_3 & B_2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & B_N & 0 & \cdots & B_2 \\ \hline \vdots \\ 0 & 0 & \cdots & B_N & B_{N-1} \end{bmatrix},$$

$\mathbf{F} = [F_1^\top, \dots, F_N^\top]^\top$, and $Z_1 = Z_2 = \frac{N(N+1)}{2}$. We obtain here the generalized adaptively controlled invariance and output invisible subspace by the algorithm when the parameter dependence is linear.

Remark 2. Since we consider a specific form of homogeneous polynomial function for the friends, we do not have an exact equivalence between the largest generalized adaptively controlled invariant and output invisible subspace of Definition 3 and the largest subspace \mathcal{V}^* found with our strategy. It happens due to the fact that the friend may depend in a generic way on the parameter (Section 6 highlights this particularity). Meanwhile, we can have this equivalence for all cases where a friend can be represented as a homogeneous polynomial function.

6 Numerical example

In this section, we present a numerical example to illustrate the application of our strategy and highlight the benefits of considering polynomial dependence on the parameter.

Example 1. (continued) Recall the polynomial system (19). The simplest way to deal with such a system and to recover the results of the literature that mostly deals with affine dependence on the parameters is to take an affine representation of the system (19):

$$A(\rho) = \begin{bmatrix} 1 & 1 \\ \rho_2 & 1 \end{bmatrix}, B(\rho) = \begin{bmatrix} 1 & 0 \\ 0 & \rho_1 \end{bmatrix}, C = [0 \quad 1],$$

where $\rho_1 = \theta$, $\rho_2 = \theta^2$. However, this representation is not equivalent and could be conservative, since the link between ρ_1 and ρ_2 is omitted to obtain it. To compute the largest generalized adaptively subspace \mathcal{V}^* with the strategy from this paper, we choose a degree $n_F = 1$ for the friend, build matrices $\mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{D}$, accordingly to Theorem 1, and apply Algorithm 1. Then, considering the affine dependence, we find $\mathcal{V}^* = \{0\}$. Note that increasing n_F does not increase \mathcal{V}^* in this case.

On the other hand, we can also represent system (19) by a homogeneous polynomial representation, as presented in

Section 5. In this case, we have an equivalent representation of the original system. Since our strategy also deals with this type of system, we can apply the same approach to compute \mathcal{V}^* . Then, choosing a degree $n_F = 1$ for the friend F , we build matrices $\mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{D}$ and apply Algorithm 1. Now, we find a $\mathcal{V}^* = [1 \ 0]^\top \mathbb{R}$, which has a larger dimension than the \mathcal{V}^* found with the affine-dependent system. Moreover, our strategy also provides one of the friends which is

$$F(\alpha) = \begin{bmatrix} f_{11} & f_{12} \\ f_{21} & f_{22} \end{bmatrix}, \quad (27)$$

where $f_{11} = f_{12} = f_{22} = 0$ and $f_{21} = -\alpha_1 = -\theta$.

This difference between the largest subspaces obtained with each representation happens because, for the affine-dependent system, the set of friends for the subspace $\mathcal{V} = [1 \ 0]^\top \mathbb{R}$ is of the form (27) with $f_{11}, f_{12}, f_{22} : \Lambda_N \mapsto \mathbb{R}$ and $f_{21} = -\frac{\rho_2}{\rho_1}$. Since the link between the parameters ρ_2 and ρ_1 is not considered, the friends are rational functions, and our strategy, as any strategy in the literature, cannot handle it. Moreover, f_{21} is not defined when $\rho_1 = 0$, then $\mathcal{V} = [1 \ 0]^\top \mathbb{R}$ cannot be a generalized adaptively controlled invariant and output invisible subspace for all possible ρ . Then, the algorithm returns $\mathcal{V}^* = \{0\}$. By considering the homogeneous polynomial representation, we have the equivalence with the original system such that we directly obtain a friend of the subspace $\mathcal{V} = [1 \ 0]^\top \mathbb{R}$ without a rational dependence. Therefore, for system (19), we have an advantage in computing the largest generalized adaptively controlled invariant and output invisible subspace \mathcal{V}^* by considering the homogeneous polynomial representation instead of the affine one.

It is interesting to highlight that if we consider the homogeneous polynomial representation and a degree $n_F = 0$, the largest subspace is given by $\mathcal{V}^* = \{0\}$. It means that there is no F constant which is a friend of the subspace $\mathcal{V}^* = [1 \ 0]^\top \mathbb{R}$.

7 Conclusion

This paper presents a novel framework and practical tools for computing controlled invariant and output invisible subspaces of non-strictly proper parameter-dependent systems that depend polynomially on the parameters. Both contexts of controlled invariant subspaces such that the inputs that make the subspace invariant are independent of the parameters or dependent on them are considered, being called generalized and generalized adaptively controlled invariant subspaces, respectively. In the case of generalized adaptively controlled invariance, we assume that the inputs making the subspace invariant depend polynomially on the parameters. Then, the algorithm to compute the largest subspace requires setting a degree of dependence on the input, where increasing the degree of input dependence can result in a subspace of a larger dimension. Future work should explore if there is a maximum degree n_F for the input dependence such that the dimension of the subspace remains constant even increasing n_F . Moreover, the possibility of considering a more general dependence of the input on the parame-

ters than a polynomial one also seems interesting, since we have shown in the numerical example section that even for an affine parameter-dependent system, the inputs making a subspace invariant might be at least rational.

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