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# Different possible behaviors of wavelet leaders of the Brownian motion

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## Abstract

The aim of this paper is to prove that wavelet leaders allow to get very fine properties of the trajectories of the Brownian motion: more precisely, we show that there exist (at least) three different behaviors for the size of wavelet leaders of the Brownian motion. Furthermore, they correspond to the three well-known behaviors of its oscillations, namely to be ordinary, rapid and slow. Some links between the oscillations and the size of wavelet leaders at a given point are also given.

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## 1 Introduction

The oscillation of a function is a widespread way of quantifying its regularity around a point. This classical notion has been widely used in analysis, for example in the frame of Hölder spaces, and in probability, as for instance in the study of sample path properties of stochastic processes. It is based on increments. More precisely, let  $f : \mathbb{R} \rightarrow \mathbb{R}$  be a continuous function; for each compact interval  $K$  of  $\mathbb{R}$ , we define the *oscillation* of  $f$  on  $K$  by

$$\text{Osc}(f, K) = \sup_{t, t' \in K} |f(t) - f(t')|. \quad (1)$$

In order to precisely know how smooth or how rough is  $f$  around some fixed point  $t \in \mathbb{R}$ , it is natural to try to determine as sharply as possible the asymptotic behavior of the oscillation  $\text{Osc}(f, I(t, \rho))$  when  $\rho \rightarrow 0^+$ . Here  $I(t, \rho)$  denotes the closed interval of center  $t$  and radius  $\rho > 0$ ; that is

$$I(t, \rho) = [t - \rho, t + \rho].$$

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On another hand, wavelet methods have become in the last decades a very powerful tool to finely study regularity properties of functions [5, 24, 22, 10]. In this context, wavelet coefficients can be viewed as generalized increments. In order to be more precise, one denotes by  $\psi : \mathbb{R} \rightarrow \mathbb{R}$  a compactly supported function of  $L^1(\mathbb{R})$  whose first moment vanishes, that is:

$$\int_{\mathbb{R}} \psi(x) dx = 0. \quad (2)$$

Therefore,  $\psi$  is called a *compactly supported wavelet*, and there exists a positive integer  $N$  such that the support of  $\psi$  is included in  $[-N, N]$ . The *wavelet coefficients*  $c_{J,K}$ ,  $(J, K) \in \mathbb{N} \times \mathbb{Z}$ , of  $f$  are defined by

$$c_{J,K} = 2^J \int_{\mathbb{R}} f(t) \psi(2^J t - K) dt = \int_{\mathbb{R}} f\left(\frac{x+K}{2^J}\right) \psi(x) dx = \int_{-N}^N f\left(\frac{x+K}{2^J}\right) \psi(x) dx. \quad (3)$$

Observe that (2) and (3) imply that

$$c_{J,K} = \int_{\mathbb{R}} \left( f\left(\frac{x+K}{2^J}\right) - f\left(\frac{K}{2^J}\right) \right) \psi(x) dx \quad (4)$$

$$= \int_{-N}^N \left( f\left(\frac{x+K}{2^J}\right) - f\left(\frac{K}{2^J}\right) \right) \psi(x) dx. \quad (5)$$

In what follows, we will use the notation  $c_\lambda$  to denote the wavelet coefficient  $c_{J,K}$ , where  $\lambda$  is the dyadic interval

$$\lambda = \lambda_{J,K} = \left[ \frac{K}{2^J}, \frac{K+1}{2^J} \right).$$

The interval  $\lambda_{J,K}$  provides the location of the wavelet  $\psi(2^J \cdot - K)$ . We denote by  $\Lambda$  the set of all dyadic intervals of  $\mathbb{R}$ . Moreover, at an arbitrary given scale  $J \in \mathbb{N}$ , we denote by  $\Lambda_J$  the set of all dyadic intervals of size  $2^{-J}$ . Amplitudes of wavelet coefficients  $c_\lambda$  located in some fixed interval  $\lambda$  (that is  $\lambda' \subseteq \lambda$ ) can be very fluctuating from one scale to another. In order to avoid such a drawback, the so-called *wavelet leaders*, which among other things offer the advantage of stability, have been introduced in [12]. They can be viewed as local suprema of the amplitudes of wavelet coefficients. Thus, a small value of the supremum associated with some interval  $\lambda$  means then that all the wavelet coefficients of the function  $f$  located in  $\lambda$  have small amplitudes. Roughly speaking this means that  $f$  is smooth on  $\lambda$  (see Section 2).

Let us now define the wavelet leaders of  $f$  in a precise way. For every  $(J, K) \in \mathbb{N} \times \mathbb{Z}$ , one denotes by  $N(\lambda_{J,K})$  the set

$$N(\lambda_{J,K}) = \{ \lambda_{J,K-1}, \lambda_{J,K}, \lambda_{J,K+1} \}.$$

Then, the *wavelet leader*  $d_{J,K}$  is defined by

$$d_{J,K} = \max_{\lambda \in N(\lambda_{J,K})} \sup_{\lambda' \subseteq \lambda} |c_{\lambda'}|. \quad (6)$$

Again, we use the notation  $d_\lambda$  if  $\lambda = \lambda_{J,K}$ .

**Remark 1.1.** In a probabilistic framework, the supremum on  $N(\lambda_{J,K})$  appearing in (6) can create correlation between wavelet leaders, even if it does not exist between wavelet

coefficients. It might seem more natural to consider only the dyadic intervals  $\lambda' \subseteq \lambda_{J,K}$  in the definition of the wavelet leaders so that the supremum at a given scale is taken on non-overlapping intervals. These coefficients are called the *restricted wavelet leaders* [11]. Let us point out that the methodology of our article can easily be adapted to their setting. In fact, we prefer to work with the wavelet leaders defined as in (6) since, in contrast with the restricted wavelet leaders, they provide a characterisation of pointwise Hölder spaces (see Section 2).

As wavelet coefficients can be viewed as generalized increments, wavelet leaders can be seen as generalized oscillations. It is therefore natural to wonder whether there are some links between the two notions: oscillation (see (1)) and wavelet leader (see (6)).

First, observe that for each fixed  $t \in \mathbb{R}$  and  $J \in \mathbb{N}$ , there exists a unique dyadic interval  $\lambda \in \Lambda_J$  such that  $t \in \lambda$ ; this dyadic interval is denoted by  $\lambda_J(t)$  and the corresponding wavelet leader by  $d_J(t)$ . If  $\lambda_{j,k} \subseteq \lambda$  for some  $\lambda \in N(\lambda_J(t))$ , one has

$$\left| t - \frac{k}{2^j} \right| \leq 2 \cdot 2^{-J},$$

hence, if  $x \in [-N, N]$ ,

$$\left| t - \frac{x+k}{2^j} \right| \leq (2+N)2^{-J}.$$

Combining these two inequalities with (5), (1) and (2), one gets that

$$\begin{aligned} |c_{j,k}| &\leq \int_{-N}^N \left| f\left(\frac{x+k}{2^j}\right) - f\left(\frac{k}{2^j}\right) \right| |\psi(x)| dx \\ &\leq c_0 \text{Osc}\left(f, I(t, (2+N)2^{-J})\right), \end{aligned}$$

where  $c_0 = \|\psi\|_{L^1(\mathbb{R})}$ . As a consequence, one has that

$$d_J(t) \leq c_0 \text{Osc}\left(f, I(t, (2+N)2^{-J})\right). \quad (7)$$

Conversely, Jaffard [9] has shown that generally speaking, under the assumption that  $f$  belongs to some Hölder space (see Section 2), the reverse inequality holds up to a logarithmic factor:

$$\text{Osc}\left(f, \lambda_J(t)\right) \leq C d_J(t) \log(d_J(t)) \quad (8)$$

for some constant  $C > 0$ . It can be derived from (7) and (8) that there is a priori a loss of information if one considers wavelet leaders instead of oscillations.

The problem we deal with in this article in the following: is this estimation (8) as good as possible, or can one expect an estimation without any logarithmic correction? This question appeared in a general setting in [8] where it is proved that this logarithmic correction is needed. In the present paper, we treat the particular case of Brownian motion. Despite the general inequality (8) which leads to believe that there is a loss of information if one works with wavelet leaders instead of oscillations, we show that wavelet leaders are precise enough to reflect very fine properties of the trajectories of Brownian motion, namely the coexistence in them of slow, fast and ordinary points.

The rest of this article is organized in the following way. In Section 2, we make some recalls on Brownian motion and gives some useful properties of its wavelet coefficients, also we present the notions of slow point, fast point and ordinary point. At the end of this section, we state the main result of the article (Theorem 2.5) which shows that the coexistence of slow, fast and ordinary points on Brownian trajectories can be revealed by wavelet leaders. Section 3 is devoted to the proof of Theorem 2.5. Finally, Section 4 gives some links between the behavior of oscillations at a point  $t$  and the size of the associated wavelet leaders  $d_J(t)$ .

## 2 Brownian motion and statement of the main result

The Brownian motion is an important Gaussian process that models many phenomena (see e.g. [18, 13, 14] and references therein). It is the unique real-valued centered Gaussian process  $B = \{B(t)\}_{t \in \mathbb{R}}$  defined on a probability space  $(\Omega, \mathcal{A}, \mathbb{P})$  such that

- almost surely,  $B(0) = 0$  and  $t \mapsto B(t)$  is a continuous function on the real line,
- it has stationary increments, more precisely for every  $t, s \in \mathbb{R}$ , one has  $B(t) - B(s) \sim \mathcal{N}(0, |t - s|)$ ,
- for any  $t_1 < t_2 < \dots < t_n$ ,  $n \geq 1$ , the increments  $B(t_2) - B(t_1), \dots, B(t_n) - B(t_{n-1})$  are independent.

From now on, if the value of  $B(t)$  has to be explicitly associated to an elementary event  $\omega \in \Omega$ , the notation  $B(t, \omega)$  will be used. It is well-known (see e.g. [13]) that there exists an event  $\Omega^* \subseteq \Omega$  of probability one satisfying the following property: for each  $\omega \in \Omega^*$  and for all  $h \in [0, 1/2)$ , there is a constant  $C(\omega) > 0$  such that for every  $t \in \mathbb{R}$ , the inequality

$$\text{Osc}(B(\cdot, \omega), I(t, \rho)) \leq C(\omega)\rho^h \quad (9)$$

holds for  $\rho > 0$  small enough. However, this inequality is not valid for  $h = 1/2$ . In terms of classical Hölder spaces (see e.g. [16]), it means that  $B$  belongs to the space  $\mathcal{C}^h(\mathbb{R})$  for every  $h < 1/2$  but fails to belong to  $\mathcal{C}^{1/2}(\mathbb{R})$ . This last property can be refined using the notion of generalized Hölder spaces  $\mathcal{C}^\tau(\mathbb{R})$ . These spaces are defined by replacing the value  $\rho^h$  in the inequality (9) by  $\tau(\rho)$ , for some modulus of continuity  $\tau$  (see [4]). The Brownian motion belongs to the Hölder space  $\mathcal{C}^\tau(\mathbb{R})$ , where the modulus of continuity  $\tau$  is given by

$$\tau(\rho) = \rho^{1/2} \sqrt{\log \rho^{-1}}$$

(see [13]). This means that there exists a constant  $C'(\omega) > 0$  such that for every  $t \in \mathbb{R}$ , one has

$$\text{Osc}(B(\cdot, \omega), I(t, \rho)) \leq C'(\omega)\rho^{1/2} \sqrt{\log \rho^{-1}} \quad (10)$$

for  $\rho > 0$  small enough.

These previous properties of the Brownian motion are uniform, but its pointwise regularities have also been studied. In this context, the Khintchin law of the iterated logarithm (see e.g. [13]) asserts that for almost every  $t \in \mathbb{R}$ ,  $B$  satisfies

$$\limsup_{\rho \rightarrow 0} \frac{|B(t + \rho, \omega) - B(t, \omega)|}{\rho^{1/2} \sqrt{\log \log \rho^{-1}}} = \sqrt{2}. \quad (11)$$

In particular, for almost every  $t \in \mathbb{R}$ , there exists a constant  $C''(t, \omega) > 0$  such that

$$\text{Osc}(B(\cdot, \omega), I(t, \rho)) \leq C''(t, \omega) \rho^{1/2} \sqrt{\log \log \rho^{-1}}$$

for every  $\rho > 0$  small enough. This is equivalent to the fact that  $B$  belongs to the generalized pointwise Hölder spaces  $\mathcal{C}^\tau(t)$  introduced recently in [4, 19], with  $\tau(\rho) = \rho^{1/2} \sqrt{\log \log \rho^{-1}}$ . One can go further and study the regularity properties at the points at which (11) is not true. The following theorem [13] summarizes some local properties of the Brownian motion: it shows the existence of three different possible behaviors of the oscillations of the Brownian motion. Note that the last point of this theorem follows from the Khintchin law of the iterated logarithm.

**Theorem 2.1.** *There exists an event  $\Omega^* \subseteq \Omega$  of probability 1 such that for every  $\omega \in \Omega^*$  and every non-empty open interval  $A$  of  $\mathbb{R}$ , there are  $t_o(\omega), t_r(\omega), t_s(\omega) \in A$  such that*

1.  $t_o(\omega)$  is an ordinary point of  $B(\cdot, \omega)$ , i.e.

$$0 < \limsup_{\rho \rightarrow 0^+} \left\{ \frac{\text{Osc}(B(\cdot, \omega), I(t_o(\omega), \rho))}{\rho^{1/2} \sqrt{\log \log(\rho^{-1})}} \right\} < +\infty;$$

2.  $t_r(\omega)$  is a fast or rapid point of  $B(\cdot, \omega)$ , i.e.

$$0 < \limsup_{\rho \rightarrow 0^+} \left\{ \frac{\text{Osc}(B(\cdot, \omega), I(t_r(\omega), \rho))}{\rho^{1/2} \sqrt{\log(\rho^{-1})}} \right\} < +\infty;$$

3.  $t_s(\omega)$  is a slow point of  $B(\cdot, \omega)$ , i.e.

$$0 < \limsup_{\rho \rightarrow 0^+} \left\{ \frac{\text{Osc}(B(\cdot, \omega), I(t_s(\omega), \rho))}{\rho^{1/2}} \right\} < +\infty.$$

Moreover, for every  $\omega \in \Omega^*$ , almost every  $t \in \mathbb{R}$  is an ordinary point of  $B(\cdot, \omega)$ .

As we have already mentioned in the introduction, the aim of this article is to study the behavior of the wavelet leaders of the Brownian motion. Wavelets methods appeared to be very useful to study global and pointwise regularities of a given function. Especially, they allow to characterize some functional spaces such as the Hölder spaces [10] and gives some numerical methods to study real-life signals (see among others [23, 25, 3, 10, 12, 17, 27, 2, 7, 6]). More precisely, let  $c_{J,K}$ ,  $(J, K) \in \mathbb{N} \times \mathbb{Z}$ , denote the sequence of wavelet coefficients of a locally bounded function  $f$ . Then, under some regularity properties on the considered wavelet (see [24]),  $f$  belongs to the Hölder space  $\mathcal{C}^h(\mathbb{R})$  if and only if there exists a constant  $C > 0$  such that

$$|c_{J,K}| \leq C 2^{-hJ} \quad \forall J \in \mathbb{N}, K \in \mathbb{Z}.$$

Similarly, the wavelet coefficients give a characterization of the generalized Hölder space  $\mathcal{C}^\tau(\mathbb{R})$  (see [21]). Namely, a function  $f$  belongs to  $\mathcal{C}^\tau(\mathbb{R})$  if and only if there exists a constant  $C > 0$  such that

$$|c_{J,K}| \leq C \tau(2^{-J}) \quad \forall J \in \mathbb{N}, K \in \mathbb{Z}.$$

In order to obtain equivalent results in the context of pointwise regularity, one has to work with the wavelet leaders of the function instead of its wavelet coefficients [4, 12, 10, 19]: if  $f$  belongs to the generalized pointwise Hölder space  $\mathcal{C}^\tau(t)$ , then there exists  $C' > 0$  such that

$$d_J(t) \leq C' \tau(2^{-J}) \quad \forall J \in \mathbb{N}. \quad (12)$$

Conversely, if  $f$  belongs to some Hölder space and if the inequality (12) holds, then

$$\text{Osc}(f, I(t, \rho)) \leq C' \tau(\rho) |\log \tau(\rho)|, \quad (13)$$

i.e.  $f$  belongs to the generalized pointwise Hölder space  $\mathcal{C}^{\tilde{\tau}}(t)$ , where

$$\tilde{\tau}(\rho) = \tau(\rho) |\log \tau(\rho)|.$$

From now on and in the rest of this paper, we assume that a wavelet whose support is included in  $[-N, N]$  is fixed, and we denote by  $c_{J,K}$  (resp.  $d_{J,K}$ ),  $(J, K) \in \mathbb{N} \times \mathbb{Z}$ , the wavelet coefficients (resp. the wavelet leaders) of the Brownian motion  $B$ . Let us now present some of the properties of the wavelet coefficients of the Brownian motion, arising directly from the properties of this stochastic process. We refer the reader to [1] for more details.

**Remark 2.2.** The equivalent definitions (3) and (4) of the wavelet coefficients gives that

- since  $B$  is a centered Gaussian process,  $\{c_{J,K} : (J, K) \in \mathbb{N} \times \mathbb{Z}\}$  is a centered Gaussian process as well;
- the stationarity of the increments of  $B$  gives that for every fixed  $J \in \mathbb{N}$ , the sequence  $\{c_{J,K} : K \in \mathbb{Z}\}$  is stationary in  $K$ ;
- the self-similarity of order 1/2 of  $B$  gives that for every  $J \in \mathbb{N}$ , the sequence  $\{c_{J,K} : K \in \mathbb{Z}\}$  has the same distribution as the sequence  $\{2^{-J/2} c_{0,K} : K \in \mathbb{Z}\}$ .

Thus, for every  $(J, K) \in \mathbb{N} \times \mathbb{Z}$ , one has

$$\mathbb{E}(c_{J,K}^2) = \mathbb{E}(c_{0,0}^2) 2^{-J} \quad \text{and so,} \quad c_{J,K} \sim \mathcal{N}(0, 2^{-J} \mathbb{E}(c_{0,0}^2)).$$

As a first result concerning the behavior of the wavelet leaders of the Brownian motion, Theorem 2.1 together with the characterization (12) directly give the following proposition.

**Proposition 2.3.** *Let  $\Omega^*$  be the event of probability 1 given in Theorem 2.1. For every  $\omega \in \Omega^*$  and every  $t \in \mathbb{R}$ ,*

1. *if  $t$  is an ordinary point of  $B(\cdot, \omega)$ , then*

$$\limsup_{J \rightarrow +\infty} \left\{ \frac{d_J(t, \omega)}{2^{-J/2} \sqrt{\log(J)}} \right\} < +\infty;$$

2. *if  $t$  is a rapid point  $B(\cdot, \omega)$ , then*

$$\limsup_{J \rightarrow +\infty} \left\{ \frac{d_J(t, \omega)}{2^{-J/2} \sqrt{J}} \right\} < +\infty;$$

3. if  $t$  is a slow point of  $B(\cdot, \omega)$ , then

$$\limsup_{J \rightarrow +\infty} \left\{ \frac{d_J(t, \omega)}{2^{-J/2}} \right\} < +\infty.$$

However, the logarithmic correction appearing in the characterization (13) does not allow to get the exact modulus of continuity for the three kinds of behavior of wavelet leaders appearing in Proposition 2.3. A first numerical work [15] has led to the idea that the behavior  $\rho^{1/2} \sqrt{\log \log(\rho^{-1})}$  of the ordinary points seems to be present across the scales of the wavelet coefficients and thus should be transposed in the framework of wavelet leaders. Namely, it leads to the idea that almost surely, for almost every point  $t$ , one has

$$0 < \limsup_{J \rightarrow +\infty} \left\{ \frac{d_J(t, \omega)}{2^{-J/2} \sqrt{\log(J)}} \right\} < +\infty.$$

This motivates us to consider the following definition.

**Definition 2.4.** Let  $\omega \in \Omega$  and  $t \in \mathbb{R}$ . We say that

1.  $t$  is a *leader-ordinary point* of  $B(\cdot, \omega)$  if

$$0 < \limsup_{J \rightarrow +\infty} \left\{ \frac{d_J(t, \omega)}{2^{-J/2} \sqrt{\log(J)}} \right\} < +\infty;$$

2.  $t$  is a *leader-rapid point* of  $B(\cdot, \omega)$  if

$$0 < \limsup_{J \rightarrow +\infty} \left\{ \frac{d_J(t, \omega)}{2^{-J/2} \sqrt{J}} \right\} < +\infty;$$

3.  $t$  is a *leader-slow point* of  $B(\cdot, \omega)$  if

$$0 < \limsup_{J \rightarrow +\infty} \left\{ \frac{d_J(t, \omega)}{2^{-J/2}} \right\} < +\infty.$$

The aim of the present paper is to prove the following result, which gives an equivalent of Theorem 2.1 in the context of wavelet leaders.

**Theorem 2.5.** *There exists an event  $\Omega_0^* \subseteq \Omega$  of probability 1 such that for every  $\omega \in \Omega_0^*$  and every non-empty open interval  $A$  of  $\mathbb{R}$ , there are  $t_o(\omega), t_r(\omega), t_s(\omega) \in A$  such that  $t_o(\omega)$  is a leader-ordinary point,  $t_r(\omega)$  is a leader-rapid point and  $t_s(\omega)$  is a leader-slow point of  $B(\cdot, \omega)$ . Moreover, for every  $\omega \in \Omega_0^*$ , almost every  $t \in \mathbb{R}$  is a leader-ordinary point of  $B(\cdot, \omega)$ .*



### 3 Proof of Theorem 2.5

This section is devoted to the proof of Theorem 2.5. It will be based on several successive lemma and propositions. Let us start with a straightforward remark.

**Remark 3.1.** Since the increments of  $B$  are independent and that the support of  $\psi$  is included in  $[-N, N]$ , the equality (5) shows that the wavelet coefficients  $c_{J_1, K_1}, \dots, c_{J_n, K_n}$  are independent as soon as

$$\left( \frac{K_i - N}{2^{J_i}}, \frac{K_i + N}{2^{J_i}} \right) \cap \left( \frac{K_l - N}{2^{J_l}}, \frac{K_l + N}{2^{J_l}} \right) = \emptyset \quad (14)$$

for every  $i \neq l$  in  $\{1, \dots, n\}$ . In particular, the coefficients  $c_{J, K}$  and  $c_{J, K'}$  are independent if  $|K' - K| \geq 2N$ .

This remark leads us to define the following condition.

**Definition 3.2.** Let  $n \geq 2$  and  $N \in \mathbb{N}$ . We say that the dyadic intervals  $\lambda_{J_1, K_1}, \dots, \lambda_{J_n, K_n}$  satisfy the condition  $(\mathcal{C}_N)$  if (14) is satisfied for every  $i \neq l$  in  $\{1, \dots, n\}$ .

Let us also introduce some notations. Fix  $(J, K) \in \mathbb{N} \times \mathbb{Z}$  and  $\lambda = \lambda_{J, K}$ . For any  $m \in \mathbb{N}$ , we denote by  $\mathcal{S}_{J, K, m}$  or  $\mathcal{S}_{\lambda, m}$  the finite set of cardinality  $2^m$  whose elements are the dyadic intervals of scale  $J + m$  included in  $\lambda_{J, K}$ ; roughly speaking, “ $\mathcal{S}_{J, K, m}$  is the set of descendants of  $\lambda_{J, K}$  at the  $m$ th generation”.

The following lemma allows to obtain a general lower bound for the size of the wavelet leaders of the Brownian motion.

**Lemma 3.3.** *Let us denote by  $\{\varepsilon_\lambda : \lambda \in \Lambda\}$  an arbitrary sequence of real-valued  $\mathcal{N}(0, 1)$  Gaussian random variables defined on a probability space  $\Omega$  and let us fix  $N \in \mathbb{N}$ . Let us assume that for every  $n \geq 2$  and every dyadic intervals  $\lambda_1, \dots, \lambda_n$  satisfying the condition  $(\mathcal{C}_N)$ , the variables  $\varepsilon_{\lambda_1}, \dots, \varepsilon_{\lambda_n}$  are independents. Then, there exists an event  $\Omega_1^* \subseteq \Omega$  of probability 1 such that, for every  $\omega \in \Omega_1^*$  and every  $t \in \mathbb{R}$ , one has*

$$\limsup_{j \rightarrow +\infty} \left\{ \max_{\substack{\lambda' \in \mathcal{S}_{\lambda, [\log_2(N)]+2} \\ \lambda \in N(\lambda_j(t))}} |\varepsilon_{\lambda'}(\omega)| \right\} > 0. \quad (15)$$

*Proof.* Let us fix  $(J, K) \in \mathbb{N} \times \mathbb{Z}$ . For any  $m \in \mathbb{N}$  and any  $S \in \mathcal{S}_{J, K, m}$ , there is a unique finite sequence  $(I_n)_{0 \leq n \leq m}$  of dyadic intervals which is decreasing in the sense of the inclusion and satisfies  $I_0 = \lambda_{J, K}$ ,  $I_m = S$  and  $I_n \in \mathcal{S}_{J, K, n}$  for all  $n \in \{1, \dots, m\}$ . Next, we consider the sequence  $(T_n)_{1 \leq n \leq m}$  of dyadic intervals constructed as follows: for every  $n \in \{1, \dots, m\}$ ,  $T_n$  is the unique dyadic interval of  $\mathcal{S}_{J, K, n}$  such that  $I_{n-1} = T_n \cup I_n$ . Note that, since the sequence  $(I_n)_{0 \leq n \leq m}$  is decreasing, this construction ensures that the intervals  $(T_n)_{1 \leq n \leq m}$  are pairwise disjoint. Moreover, let us also note that for every  $n \in \{1, \dots, m\}$ , one has  $T_n \in N(I_n)$ . For every  $n \in \{1, \dots, m\}$ , there is a dyadic interval  $T'_n \in \mathcal{S}_{T_n, [\log_2(N)]+2}$  such that

$$\left( \frac{k_n - N}{2^{j_n}}, \frac{k_n + N}{2^{j_n}} \right) \subseteq T'_n$$

where  $T'_n = \lambda_{j_n, k_n}$ . Consequently, by assumption, the corresponding Gaussian random variables  $(\varepsilon_{T'_n})_{1 \leq n \leq m}$  are independent. In the sequel, the set  $\{T'_n : 1 \leq n \leq m\}$  is denoted by  $\mathcal{T}'_{J,K,m}(S)$ .

Let  $c_0 = 2^{-3/2} \sqrt{\pi}$  and denote by  $p_0$  the probability that an arbitrary real-valued  $\mathcal{N}(0, 1)$  Gaussian random variable belongs to the interval  $(-c_0, c_0)$ . Elementary calculations allows to obtain that

$$0 < p_0 < \frac{1}{2}. \quad (16)$$

For all  $S \in \mathcal{S}_{J,K,m}$ , we denote by  $B_{J,K,m}(S)$  the Bernoulli random variable defined as

$$B_{J,K,m}(S) = \prod_{T' \in \mathcal{T}'_{J,K,m}(S)} \mathbf{1}_{\{|\varepsilon_{T'}| < c_0\}}. \quad (17)$$

Notice that, using the definition of  $p_0$  and the independence property of the random variables  $\varepsilon_{T'}$  for  $T' \in \mathcal{T}'_{J,K,m}(S)$ , one has

$$\mathbb{E}(B_{J,K,m}(S)) = p_0^m. \quad (18)$$

Next, let  $G_{J,K,m}$  be the random variable with values in  $\{0, \dots, 2^m\}$  defined as

$$G_{J,K,m} = \sum_{S \in \mathcal{S}_{J,K,m}} B_{J,K,m}(S).$$

Since the cardinality of  $\mathcal{S}_{J,K,m}$  equals  $2^m$ , using (18), one gets that  $\mathbb{E}(G_{J,K,m}) = (2p_0)^m$ . It follows from Fatou Lemma and (16) that

$$0 \leq \mathbb{E} \left( \liminf_{m \rightarrow +\infty} G_{J,K,m} \right) \leq \lim_{m \rightarrow +\infty} \mathbb{E}(G_{J,K,m}) = 0$$

Hence, the event

$$\Omega_{1,J,K}^* = \left\{ \omega \in \Omega : \liminf_{m \rightarrow +\infty} G_{J,K,m}(\omega) = 0 \right\} \quad (19)$$

has a probability equal to 1. Since  $\mathbb{N} \times \mathbb{Z}$  is a countable set, the probability of the event

$$\Omega_1^* = \bigcap_{(J,K) \in \mathbb{N} \times \mathbb{Z}} \Omega_{1,J,K}^* \quad (20)$$

is also equal to 1.

Let us now consider  $\omega \in \Omega_1^*$  and  $t \in \mathbb{R}$ , and let us prove that (15) is satisfied. We fix  $J \in \mathbb{N}$  and  $K = \lfloor 2^J t \rfloor$ , so that  $\lambda_{J,K} = \lambda_J(t)$ . Since for every  $m \in \mathbb{N}$ ,  $G_{J,K,m}$  takes values in  $\{0, \dots, 2^m\}$ , (19) and (20) imply that there are infinitely many  $m$  such that

$$B_{J,K,m}(S) = 0$$

for every  $S \in \mathcal{S}_{J,K,m}$ , i.e. using (17), there exists  $T' \in \mathcal{T}'_{J,K,m}(S)$  such that

$$|\varepsilon_{T'}| \geq c_0.$$

In particular, we have this result for  $S = \lambda_{J+m}(t)$ . In this case,  $T' \in \mathcal{S}_{\lambda, \lfloor \log_2(N) \rfloor + 2}$  with  $\lambda \in N(\lambda_{J+m}(t))$ . The conclusion follows.  $\square$

**Proposition 3.4.** *There exists an event  $\Omega_1^*$  of probability 1 such that for every  $\omega \in \Omega_1^*$  and every  $t \in \mathbb{R}$ , one has*

$$\limsup_{J \rightarrow +\infty} \left\{ \frac{d_J(t, \omega)}{2^{-J/2}} \right\} > 0.$$

*Proof.* For every  $J \in \mathbb{N}$  and every dyadic interval  $\lambda \in \Lambda_J$ , let us set

$$\varepsilon_\lambda = \frac{1}{\sqrt{2^{-J} \mathbb{E}(c_{0,0}^2)}} c_\lambda. \quad (21)$$

Observe that, using the general assumption that the support of the wavelet is included in  $[-N, N]$ , Remark 2.2 and Remark 3.1 imply that the assumptions of Lemma 3.3 are fulfilled. Therefore, there exists an event  $\Omega_1^* \subseteq \Omega$  of probability 1 such that, for every  $\omega \in \Omega_1^*$  and every  $t \in \mathbb{R}$ , one has

$$\limsup_{J \rightarrow +\infty} \left\{ \max_{\substack{\lambda' \in \mathcal{S}_{\lambda, \lfloor \log_2(N) \rfloor + 2} \\ \lambda \in N(\lambda_J(t))}} |\varepsilon_{\lambda'}(\omega)| \right\} > 0. \quad (22)$$

Note that using the definition (6) of the wavelet leaders and (21), one has

$$d_J(t, \omega) \geq \max_{\substack{\lambda' \in \mathcal{S}_{\lambda, \lfloor \log_2(N) \rfloor + 2} \\ \lambda \in N(\lambda_J(t))}} |c_{\lambda'}(\omega)| = \sqrt{2^{-(J + \lfloor \log_2(N) \rfloor + 2)} \mathbb{E}(c_{0,0}^2)} \max_{\substack{\lambda' \in \mathcal{S}_{\lambda, \lfloor \log_2(N) \rfloor + 2} \\ \lambda \in N(\lambda_J(t))}} |\varepsilon_{\lambda'}(\omega)|,$$

and together with (22), it implies that

$$\limsup_{J \rightarrow +\infty} \left\{ \frac{d_J(t, \omega)}{2^{-J/2}} \right\} > 0.$$

This gives the conclusion.  $\square$

As we will see in the proof of Theorem 2.5, this result will allow to get the existence of leader-slow points. Let us now focus on leader-ordinary points. First, let us recall the following classical lemma which provides asymptotic estimates on the tail behavior of a standard Gaussian distribution.

**Lemma 3.5.** *Let  $\varepsilon$  be an arbitrary real-valued  $\mathcal{N}(0, 1)$  Gaussian random variable. One has*

$$\lim_{x \rightarrow +\infty} \frac{\mathbb{P}(|\varepsilon| > x)}{(2\pi^{-1})^{1/2} x^{-1} e^{-x^2/2}} = 1.$$

**Lemma 3.6.** *Let us denote by  $\{\varepsilon_\lambda : \lambda \in \Lambda\}$  an arbitrary sequence of real-valued  $\mathcal{N}(0, 1)$  Gaussian random variables defined on a probability space  $\Omega$  and let us fix  $N \in \mathbb{N}$ . Let us assume that for every  $n \geq 2$  and every dyadic intervals  $\lambda_1, \dots, \lambda_n$  satisfying the condition  $(\mathcal{C}_N)$ , the variables  $\varepsilon_{\lambda_1}, \dots, \varepsilon_{\lambda_n}$  are independents. Then, there exists an event  $\Omega_2^* \subseteq \Omega$  of probability 1 such that, for every  $\omega \in \Omega_2^*$  and almost every  $t \in \mathbb{R}$ , one has*

$$\limsup_{j \rightarrow +\infty} \left\{ \frac{1}{\sqrt{\log(j)}} \max_{\substack{\lambda' \in \mathcal{S}_{\lambda, \lfloor \log_2(N) \rfloor + 2} \\ \lambda \in N(\lambda_j(t))}} |\varepsilon_{\lambda'}| \right\} > 0.$$

*Proof.* Within this proof, we will use the same notations as in the proof of Lemma 3.3. Let us fix  $t \in \mathbb{R}$ . If  $J \in \mathbb{N}$ , we set  $K = \lfloor 2^J t \rfloor$ . For every  $m \in \mathbb{N}$ , we consider the dyadic interval  $S = \lambda_{J+m}(t) \in \mathcal{S}_{J,K,m}$  and the associated sequence  $(T'_n)_{1 \leq n \leq m}$  of dyadic intervals. Next, we set

$$\mathcal{E}_{J,m}(t) := \left\{ \omega \in \Omega : \max_{1 \leq n \leq m} |\varepsilon_{T'_n}| \geq \sqrt{\log(2m)} \right\}$$

By construction, the Gaussian random variables  $\varepsilon_{T'_n}$ ,  $1 \leq n \leq m$ , are independent. Therefore, one has

$$\begin{aligned} \mathbb{P}(\mathcal{E}_{J,m}(t)) &= 1 - \prod_{1 \leq n \leq m} \mathbb{P}\left(|\varepsilon_{T'_n}| < \sqrt{\log(2m)}\right) \\ &= 1 - \left(1 - \mathbb{P}\left(|\varepsilon| > \sqrt{\log(2m)}\right)\right)^m \end{aligned}$$

where  $\varepsilon \sim \mathcal{N}(0,1)$ . Let us set  $C = 1/2 (2\pi^{-1})^{1/2} > 0$ . Using Lemma 3.5 and the fact that  $\log(1-x) \leq -x$  if  $x \in (0,1)$ , there exists  $M \in \mathbb{N}$  such that for any  $m > M$ , we have

$$\begin{aligned} \mathbb{P}(\mathcal{E}_{J,m}(t)) &\geq 1 - \left(1 - C \frac{e^{-\frac{1}{2} \log(2m)}}{\sqrt{\log(2m)}}\right)^m \\ &\geq 1 - \exp\left(-Cm \frac{e^{-\frac{1}{2} \log(2m)}}{\sqrt{\log(2m)}}\right) \\ &\geq 1 - \exp\left(-C \sqrt{\frac{m}{2 \log(2m)}}\right) \\ &\geq 1 - \exp(-m^\gamma) \end{aligned}$$

for  $\gamma \in (0, 1/2)$ . Consequently, one has in particular

$$\sum_{M \in \mathbb{N}} \mathbb{P}(\mathcal{E}_{2^M, 2^M}(t)) = +\infty.$$

In view of the fact that the events  $\mathcal{E}_{2^M, 2^M}(t)$ ,  $M \in \mathbb{N}$ , are independent, it follows from the Borel-Cantelli lemma that

$$\mathbb{P}\left(\bigcap_{M \in \mathbb{N}} \bigcup_{m \geq M} \mathcal{E}_{2^m, 2^m}(t)\right) = 1.$$

Therefore, for a fixed  $t \in \mathbb{R}$ , almost surely, there are infinitely many scales  $j \in \mathbb{N}$  such that

$$\max_{\substack{\lambda' \in \mathcal{S}_{\lambda, \lfloor \log_2(N) \rfloor + 2} \\ \lambda \in N(\lambda_j(t))}} |\varepsilon_{\lambda'}| \geq \sqrt{\log j}.$$

Fubini's theorem applied to the function  $(t, \omega) \in \mathbb{R} \times \Omega \mapsto \chi_{S(t)}(\omega)$ , where

$$S(t) = \left\{ \omega \in \Omega : \limsup_{j \rightarrow +\infty} \left\{ \frac{1}{\sqrt{\log j}} \max_{\substack{\lambda' \in \mathcal{S}_{\lambda, \lfloor \log_2(N) \rfloor + 2} \\ \lambda \in N(\lambda_j(t))}} |\varepsilon_{\lambda'}| \right\} < 1 \right\},$$

implies then that there is an event  $\Omega_2^* \subseteq \Omega$  of probability 1 on which for almost every  $t \in \mathbb{R}$ ,

$$\limsup_{j \rightarrow +\infty} \left\{ \frac{1}{\sqrt{\log j}} \max_{\substack{\lambda' \in \mathcal{S}_{\lambda, \lfloor \log_2(N) \rfloor + 2} \\ \lambda \in N(\lambda_j(t))}} |\varepsilon_{\lambda'}| \right\} > 0.$$

□

**Proposition 3.7.** *There exists an event  $\Omega_2^*$  of probability 1 such that for every  $\omega \in \Omega_2^*$  and almost every  $t \in \mathbb{R}$ , one has*

$$\limsup_{J \rightarrow +\infty} \left\{ \frac{d_J(t, \omega)}{2^{-J/2} \sqrt{\log(J)}} \right\} > 0.$$

*Proof.* We proceed as in the proof of Proposition 3.4. Using Lemma 3.6, the general assumption that the support of the wavelet is included in  $[-N, N]$ , Remark 2.2 and Remark 3.1, we know that there exists an event  $\Omega_2^* \subseteq \Omega$  of probability 1 such that, for every  $\omega \in \Omega_2^*$  and almost every  $t \in \mathbb{R}$ , one has

$$\limsup_{J \rightarrow +\infty} \left\{ \frac{1}{\sqrt{\log(J)}} \max_{\substack{\lambda' \in \mathcal{S}_{\lambda, \lfloor \log_2(N) \rfloor + 2} \\ \lambda \in N(\lambda_J(t))}} |c_{\lambda'}(\omega)| \right\} > 0,$$

where

$$\varepsilon_{\lambda} = \frac{1}{\sqrt{2^{-J} \mathbb{E}(c_{0,0}^2)}} c_{\lambda}.$$

In particular,

$$\max_{\substack{\lambda' \in \mathcal{S}_{\lambda, \lfloor \log_2(N) \rfloor + 2} \\ \lambda \in N(\lambda_J(t))}} |c_{\lambda'}(\omega)| = \sqrt{2^{-(J + \lfloor \log_2(N) \rfloor + 2)} \mathbb{E}(c_{0,0}^2)} \max_{\substack{\lambda' \in \mathcal{S}_{\lambda, \lfloor \log_2(N) \rfloor + 2} \\ \lambda \in N(\lambda_J(t))}} |\varepsilon_{\lambda'}(\omega)|.$$

Consequently, if  $\omega \in \Omega_2^*$ , for almost every  $t \in \mathbb{R}$ , one has

$$\limsup_{J \rightarrow +\infty} \left\{ \frac{d_J(t, \omega)}{2^{-J/2} \sqrt{\log(J)}} \right\} \geq \limsup_{J \rightarrow +\infty} \left\{ \frac{1}{2^{-J/2} \sqrt{\log(J)}} \max_{\substack{\lambda' \in \mathcal{S}_{\lambda, \lfloor \log_2(N) \rfloor + 2} \\ \lambda \in N(\lambda_J(t))}} |c_{\lambda'}(\omega)| \right\} > 0,$$

and the conclusion follows. □

Let us end with a result which will be useful for rapid points.

**Lemma 3.8.** *Let us denote by  $\{\varepsilon_{\lambda} : \lambda \in \Lambda\}$  an arbitrary sequence of real-valued  $\mathcal{N}(0, 1)$  Gaussian random variables defined on a probability space  $\Omega$  and let us fix  $N \in \mathbb{N}$ . Let us assume that for every  $n \geq 2$  and every dyadic intervals  $\lambda_1, \dots, \lambda_n$  satisfying the condition  $(\mathcal{C}_N)$ , the variables  $\varepsilon_{\lambda_1}, \dots, \varepsilon_{\lambda_n}$  are independents. Then, there exists an event  $\Omega_3^* \subseteq \Omega$  of probability 1 such that, for every  $\omega \in \Omega_3^*$  and every non-empty open interval  $A$  of  $\mathbb{R}$ , there is  $t \in A$  such that*

$$\limsup_{j \rightarrow +\infty} \left\{ \frac{|\varepsilon_{\lambda_j(t)}|}{\sqrt{j}} \right\} > 0.$$

*Proof.* To avoid making the notations heavier, we suppose that  $A = (0, 1)$ . The proof can be easily adapted in the general case. The conclusion follows then by covering  $\mathbb{R}$  with all open intervals with rational endpoints.

Let us fix  $a \in (0, 1)$  and  $C > 0$  such that  $C^2 < 2a \log 2$ . Let us also consider for every  $(j, l) \in \mathbb{N} \times \{0, \dots, \lfloor 2^{j(1-a)} \rfloor - 1\}$ , the event

$$\mathcal{F}_{j,l} := \left\{ \omega \in \Omega : \max_{k \in \{l \lfloor 2^{aj}/(2N) \rfloor, \dots, (l+1) \lfloor 2^{aj}/(2N) \rfloor - 1\}} |\varepsilon_{j,2kN}(\omega)| \geq C\sqrt{j} \right\}.$$

Besides, let  $j_0$  be the smallest  $j$  such that  $\lfloor 2^{aj}/(2N) \rfloor \geq 1$ .

Assume for a while that

$$\mathbb{P} \left( \mathbb{C} \left( \bigcap_{l \in \{0, \dots, \lfloor 2^{j(1-a)} \rfloor - 1\}} \mathcal{F}_{j,l} \right) \right) \quad (23)$$

is the general term of a convergent series; then the Borel-Cantelli lemma implies that

$$\mathbb{P} \left( \bigcup_{J \geq j_0} \bigcap_{j \geq J} \bigcap_{l \in \{0, \dots, \lfloor 2^{j(1-a)} \rfloor - 1\}} \mathcal{F}_{j,l} \right) = 1.$$

Now, let us set

$$\Omega_3^* := \bigcup_{J \geq j_0} \bigcap_{j \geq J} \bigcap_{l \in \{0, \dots, \lfloor 2^{j(1-a)} \rfloor - 1\}} \mathcal{F}_{j,l} \quad (24)$$

and let us consider  $\omega \in \Omega_3^*$ . For every  $j \geq j_0$ , let us set

$$G_j(\omega) := \left\{ k \in \{0, \dots, 2^j - 1\} : |\varepsilon_{j,k}(\omega)| \geq C\sqrt{j} \right\} \quad (25)$$

and

$$U_j(\omega) := \bigcup_{k \in G_j(\omega)} \left( \frac{k}{2^j}, \frac{k+1}{2^j} \right). \quad (26)$$

Finally, for every  $n \geq j_0$ , one considers

$$O_n(\omega) := \bigcup_{j \geq n} U_j(\omega).$$

This last open subset is dense in  $(0, 1)$ . Indeed, let us consider  $t \in (0, 1)$ ,  $j \geq j_0$  and  $k$  such that  $\lambda_j(t) = \lambda_{j,k}$ . Then, either there is  $l \in \{0, \dots, \lfloor 2^{j(1-a)} \rfloor - 1\}$  such that

$$k \in \{l \lfloor 2^{ja} \rfloor, \dots, (l+1) \lfloor 2^{ja} \rfloor - 1\},$$

or

$$k \in \{\lfloor 2^{j(1-a)} \rfloor \lfloor 2^{ja} \rfloor, \dots, 2^j - 1\}.$$

In the first case, using (24) and (25), there is  $k' \in \{l \lfloor 2^{aj}/(2N) \rfloor, \dots, (l+1) \lfloor 2^{aj}/(2N) \rfloor - 1\}$  such that  $2k'N \in G_j(\omega)$ . From (26), we get that  $t$  is at a distance at most  $2 \cdot 2^{j(a-1)}$  of  $U_j(\omega)$ . In the second case, there is  $k' \in \{(\lfloor 2^{j(1-a)} \rfloor - 1) \lfloor 2^{aj}/(2N) \rfloor, \dots, \lfloor 2^{j(1-a)} \rfloor \lfloor 2^{aj}/(2N) \rfloor - 1\}$  such that  $2k'N \in G_j(\omega)$ , and similarly, we get that  $t$  is at a distance at most  $c \cdot 2^{j(a-1)}$

of  $U_j(\omega)$ , for some constant  $c > 0$  depending only on  $N$  and  $a$ . The density follows. Hence, Baire's theorem gives that the set

$$\bigcap_{n \geq j_0} O_n(\omega)$$

is not empty. If  $t \in \bigcap_{n \geq j_0} O_n(\omega)$ , for every  $n \geq j_0$ , there is  $j \geq n$  such that  $|\varepsilon_{\lambda_j(t)}| \geq C\sqrt{j}$ , and it leads to the conclusion.

It remains then to show that (23) is the general term of a convergent series. Let us remark that the variables  $\varepsilon_{j,2kN}$  for  $k \in \{l\lfloor 2^{aj}/(2N) \rfloor, \dots, (l+1)\lfloor 2^{aj}/(2N) \rfloor - 1\}$  and  $l \in \{0, \dots, \lfloor 2^{j(1-a)} \rfloor - 1\}$  are independent. Consequently, one has

$$\begin{aligned} & \mathbb{P} \left( \mathfrak{C} \left( \bigcap_{l \in \{0, \dots, \lfloor 2^{j(1-a)} \rfloor - 1\}} \mathcal{F}_{j,l} \right) \right) \\ &= 1 - \prod_{l \in \{0, \dots, \lfloor 2^{j(1-a)} \rfloor - 1\}} \left( 1 - \prod_{k \in \{l\lfloor 2^{aj}/(2N) \rfloor, \dots, (l+1)\lfloor 2^{aj}/(2N) \rfloor - 1\}} \mathbb{P} \left( |\varepsilon_{j,2kN}| < C\sqrt{j} \right) \right) \\ &= 1 - \left( 1 - \mathbb{P} \left( |\varepsilon| < C\sqrt{j} \right)^{\lfloor 2^{aj}/(2N) \rfloor} \right)^{\lfloor 2^{j(1-a)} \rfloor} \\ &= 1 - \left( 1 - \left( 1 - \mathbb{P} \left( |\varepsilon| \geq C\sqrt{j} \right) \right)^{\lfloor 2^{aj}/(2N) \rfloor} \right)^{\lfloor 2^{j(1-a)} \rfloor} \\ &\leq 1 - \exp \left( 2^{j(1-a)} \log(1 - x_j) \right) \end{aligned} \tag{27}$$

where  $\varepsilon \sim \mathcal{N}(0, 1)$  and

$$x_j = \left( 1 - \mathbb{P} \left( |\varepsilon| \geq C\sqrt{j} \right) \right)^{\lfloor 2^{aj}/(2N) \rfloor}.$$

Let us remark that  $x_j$  is always positive and tends to 0 as  $j \rightarrow +\infty$ . Indeed, let us set  $C' = (1/2)(2\pi^{-1})^{1/2}$ . Using Lemma 3.5 and the fact that  $\log(1 - x) \leq -x$  if  $x \in (0, 1)$ , there exists  $J \in \mathbb{N}$  such that for any  $j \geq J$ ,

$$\begin{aligned} 0 \leq x_j &\leq (1 - C' \sqrt{j} \exp(-C^2 j/2))^{\lfloor 2^{aj}/(2N) \rfloor} \\ &\leq \exp \left( - \left\lfloor \frac{2^{aj}}{2N} \right\rfloor C' \sqrt{j} \exp(-C^2 j/2) \right) \\ &\leq \exp \left( -C'' \sqrt{j} \exp(j(a \log 2 - C^2/2)) \right), \end{aligned} \tag{28}$$

where  $C''$  is a strictly positive constant depending only on  $a$ ,  $N$  and  $C$ . The expression (28) tends to 0 since  $C^2 < 2a \log 2$ . Moreover, the same argument shows that  $2^{j(1-a)} x_j$  tends to 0. Using the fact that  $\log(1 - x) = -x + o(x)$  and  $\exp(x) = 1 + x + o(x)$  as  $x \rightarrow 0$ , we obtain that, for any  $\epsilon > 0$ , the expression (27) is upper bounded by

$$2^{j(1-a)} (\epsilon(x_j + \epsilon x_j) + x_j + \epsilon x_j) \tag{29}$$

for  $j$  large enough. Therefore the expression (27) is the general term of a convergent series using the inequality (29) and the inequality (28).  $\square$

**Proposition 3.9.** *There exists an event  $\Omega_3^*$  of probability 1 such that for every  $\omega \in \Omega_3^*$  and every non-empty open interval  $A$  of  $\mathbb{R}$ , there is  $t(\omega) \in A$  such that*

$$\limsup_{J \rightarrow +\infty} \left\{ \frac{d_J(t(\omega), \omega)}{2^{-J/2} \sqrt{J}} \right\} > 0.$$

*Proof.* We proceed as in the proof of Propositions 3.4 and 3.7, using Lemma 3.8, the assumption that the support of the wavelet is included in  $[-N, N]$ , Remark 2.2 and Remark 3.1.  $\square$

We are now able to prove Theorem 2.5.

*Proof of Theorem 2.5.* Let us recall (see (10)) that there exists an event  $\Omega_4^* \subseteq \Omega$  of probability 1 such that for every  $\omega \in \Omega_4^*$ ,

$$\limsup_{\rho \rightarrow 0^+} \left\{ \frac{\text{Osc} \left( B(\cdot, \omega), I(t, \rho) \right)}{\rho^{1/2} \sqrt{\log(\rho^{-1})}} \right\} < +\infty$$

for every  $t \in \mathbb{R}$ . In view of the inequality (7),

$$\limsup_{J \rightarrow +\infty} \left\{ \frac{d_J(t, \omega)}{2^{-J/2} \sqrt{J}} \right\} < +\infty \quad (30)$$

holds for every  $t \in \mathbb{R}$ .

Let us consider the event

$$\Omega_0^* := \Omega^* \cap \Omega_1^* \cap \Omega_2^* \cap \Omega_3^* \cap \Omega_4^*$$

of probability 1, where the event  $\Omega^*$  (resp.  $\Omega_1^*$ ,  $\Omega_2^*$  and  $\Omega_3^*$ ) is the event of Theorem 2.1 (resp. Proposition 3.4, Proposition 3.7 and Proposition 3.9). Let us fix  $\omega \in \Omega_0^*$  and let us consider a non-empty open interval  $A$  of  $\mathbb{R}$ .

Let us first show that almost every  $t \in \mathbb{R}$  is a leader-ordinary point of  $B(\cdot, \omega)$ . Using Theorem 2.1, we know that almost every  $t \in \mathbb{R}$  is an ordinary point of  $B(\cdot, \omega)$ . Together with Proposition 2.3 and Proposition 3.7, this implies that for almost every  $t \in \mathbb{R}$ ,

$$0 < \limsup_{J \rightarrow +\infty} \left\{ \frac{d_J(t, \omega)}{2^{-J/2} \sqrt{\log(J)}} \right\} < +\infty.$$

In particular, there exist leader-ordinary points of  $B(\cdot, \omega)$  in  $A$ .

Secondly, Proposition 3.9 shows that there exists  $t_r(\omega) \in A$  such that

$$\limsup_{J \rightarrow +\infty} \left\{ \frac{d_J(t_r(\omega), \omega)}{2^{-J/2} \sqrt{J}} \right\} > 0.$$

This result combined with the equation (30) implies that the point  $t_r(\omega)$  is a leader-rapid point of  $B(\cdot, \omega)$ .

Finally, Theorem 2.1 and Proposition 2.3 show that there exists  $t_s(\omega) \in I$  such that

$$\limsup_{J \rightarrow +\infty} \left\{ \frac{d_J(t_s(\omega), \omega)}{2^{-J/2}} \right\} < +\infty.$$

Using Proposition 3.4, we see that the point  $t_s(\omega)$  is a leader-slow point of  $B(\cdot, \omega)$ .  $\square$



## 4 Some links between the behaviors of the oscillation and the wavelet leaders

Theorem 2.5 shows that there are (at least) three different behaviors for the size of the wavelet leaders of the Brownian motion and furthermore, that they correspond to those of the oscillations. A natural question is to determine, for  $t \in \mathbb{R}$  fixed, whether the behavior of  $\text{Osc} \left( B, I(t, \rho) \right)$  as  $\rho \rightarrow 0$  is the same as  $d_J(t)$  as  $J \rightarrow +\infty$ , and conversely. This section gives a partial answer to this question.

First, let us note that the inequality (7) directly leads to the following proposition, which is an equivalent of Proposition 2.3.

**Proposition 4.1.** *Let  $\Omega_0^*$  be the event of probability 1 given in Theorem 2.5. For every  $\omega \in \Omega^*$  and every  $t \in \mathbb{R}$ ,*

1. *if  $t$  is a leader-ordinary point of  $B(\cdot, \omega)$ , then*

$$\limsup_{\rho \rightarrow 0^+} \left\{ \frac{\text{Osc} \left( B(\cdot, \omega), I(t, \rho) \right)}{\rho^{1/2} \sqrt{\log \log(\rho^{-1})}} \right\} > 0;$$

2. *if  $t$  is a leader-rapid point of  $B(\cdot, \omega)$ , then*

$$\limsup_{\rho \rightarrow 0^+} \left\{ \frac{\text{Osc} \left( B(\cdot, \omega), I(t, \rho) \right)}{\rho^{1/2} \sqrt{\log(\rho^{-1})}} \right\} > 0;$$

3. *if  $t$  is a leader-slow point of  $B(\cdot, \omega)$ , then*

$$\limsup_{\rho \rightarrow 0^+} \left\{ \frac{\text{Osc} \left( B(\cdot, \omega), I(t, \rho) \right)}{\rho^{1/2}} \right\} > 0.$$

We can now give some links between the behaviors of the oscillations and the size of the wavelet leaders.

**Theorem 4.2.** *Let  $\Omega_0^*$  be the event of probability 1 given in Theorem 2.5 and let us fix  $\omega \in \Omega_0^*$ .*

1. *For every  $t \in \mathbb{R}$ ,*

- *if  $t$  is a leader-rapid point of  $B(\cdot, \omega)$ , then  $t$  is a rapid point of  $B(\cdot, \omega)$ ,*
- *if  $t$  is a slow point of  $B(\cdot, \omega)$ , then  $t$  is a leader-slow point of  $B(\cdot, \omega)$ .*

2. *For almost every  $t \in \mathbb{R}$ ,  $t$  is an ordinary point and a leader-ordinary point of  $B(\cdot, \omega)$ .*

*Proof.* If  $t$  is a leader-rapid point, Proposition 4.1 and the equation (10) directly imply that  $t$  is a rapid point. If  $t$  is a slow point, Proposition 2.3 and Proposition 3.4 give that  $t$  is a leader-slow point. Finally, the result for the ordinary points can be deduced from the last parts of Theorem 2.1 and 2.5.  $\square$

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