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HYPERELLIPTIC SURFACES ARE LOEWNER

MIKHAIL G. KATZ* AND STÉPHANE SABOURAU

ABSTRACT. We prove that C. Loewner's inequality for the torus is satisfied by all hyperelliptic surfaces X , as well. We first construct the Loewner loops on the (mildly singular) companion tori, locally isometric to X away from Weierstrass points. The loops are then transplanted to X , and surgered to obtain a Loewner loop on X .

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1. INTRODUCTION

The systole, $\text{sys}\pi_1(g)$, of a compact non simply connected Riemannian manifold (X, g) is the least length of a noncontractible loop $\gamma \subset X$:

$$\text{sys}\pi_1(g) = \min_{[\gamma] \neq 0 \in \pi_1(X)} \text{length}(\gamma). \quad (1.1)$$

This notion of systole is apparently unrelated to the systolic arrays of [Ku78]. We will be concerned with comparing this Riemannian invariant to the total area of the metric, as in Loewner's inequality (2.2). Higher dimensional optimal generalisations of Loewner's inequality are studied in [BK03, BK04, IK04, BCIK2]. The defining text for this material is [Gr99], with more details in [Gr83, Gr96]. See also the recent survey [CK03], as well as [KR04].

We will review the relevant literature in Section 2, state the main theorem in Section 3, and prove it in Sections 3 and 4.

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2. INEQUALITIES OF LOEWNER AND PU

The Hermite constant, denoted γ_n , can be defined as the optimal constant in the inequality

$$\text{sys}\pi_1(\mathbb{T}^n)^2 \leq \gamma_n \text{vol}(\mathbb{T}^n)^{2/n}, \quad (2.1)$$

over the class of all *flat* tori \mathbb{T}^n . Here γ_n is asymptotically linear in n , *cf.* [LLS90, pp. 334, 337]. The precise value is known for small n , *e.g.* $\gamma_2 = \frac{2}{\sqrt{3}}$, $\gamma_3 = 2^{\frac{1}{3}}$, \dots . An inequality of type (2.1) remains valid in the class of *all* metrics, but with a nonsharp constant on the order of n^{2n} [Gr83].

Around 1949, Charles Loewner proved the first systolic inequality, *cf.* [Pu52]. He showed that every Riemannian metric g on the torus \mathbb{T}^2 satisfies the inequality

$$\text{sys}\pi_1(g)^2 \leq \gamma_2 \text{area}(g), \quad (2.2)$$

while a metric satisfying the boundary case of equality in (2.2) is necessarily flat, and is homothetic to the quotient of \mathbb{C} by the lattice spanned by the cube roots of unity.

M. Gromov [Gr83, p. 50] (*cf.* [Ko87, Theorem 4, part (1)]) proved a general estimate which implies that if Σ_s is a compact orientable surface of genus s with a Riemannian metric, then

$$\frac{\text{sys}\pi_1(\Sigma_s)^2}{\text{area}(\Sigma_s)} < \frac{64}{4\sqrt{s} + 27}. \quad (2.3)$$

Thus the optimal systolic ratio tends to 0 as the genus increases without bound.

Remark 2.1. It was shown in [Gr83] (see also [KS04]) that asymptotically the optimal systolic ratio behaves as $C \frac{(\log s)^2}{s}$.

Another helpful estimate is found in [Gr83, Corollary 5.2.B]. Namely, every aspherical compact surface (Σ, g) admits a metric ball $B = B_p(\frac{1}{2} \text{sys}\pi_1(g)) \subset \Sigma$ of radius $\frac{1}{2} \text{sys}\pi_1(g)$, which satisfies

$$\text{sys}\pi_1(g)^2 \leq \frac{4}{3} \text{area}(B). \quad (2.4)$$

Furthermore, whenever a point $x \in \Sigma$ lies on a two-sided loop which is minimizing in its free homotopy class, the metric ball $B_x(r) \subset \Sigma$ of radius $r \leq \frac{1}{2} \text{sys}\pi_1(g)$ satisfies the estimate

$$2r^2 < \text{area}(B_x(r)). \quad (2.5)$$

Question 2.2. It follows from Gromov's estimate (2.3) that orientable surfaces Σ_s satisfy Loewner's inequality (2.2) if $s > 50$. This is improved in [KS04] to $s \geq 20$. Can the genus assumption be removed altogether?

A similar question for Pu's inequality [Pu52] has an affirmative answer. The generalisation is immediate from Gromov's inequality (2.4). Namely, every surface (X, g) which is not a 2-sphere satisfies

$$\text{sys}\pi_1(g)^2 \leq \frac{\pi}{2} \text{area}(g), \quad (2.6)$$

where the boundary case of equality in (2.6) is attained precisely when, on the one hand, the surface X is a real projective plane, and on the other, the metric g is of constant Gaussian curvature.

3. HYPERELLIPTIC SURFACES AND LOEWNER SURFACES

Recall that a Riemann surface X is called hyperelliptic if it admits a degree 2 meromorphic function, *cf.* [Mi95, p. 60-61] as well as [Mi95, Proposition 4.11, p. 92]. The associated ramified double cover

$$Q : X \rightarrow S^2$$

over the sphere S^2 is conformal away from the $2s+2$ ramification points, where s is the genus. Its deck transformation $J : X \rightarrow X$ is called the *hyperelliptic involution*. Such a holomorphic involution, if it exists, is uniquely characterized by the property of having precisely $2s+2$ fixed points. The fixed points of J are called Weierstrass points. Their images under Q will be referred to as *ramification points*.

We provide the following partial answer in the direction of Question 2.2. We will say that a surface is *Loewner* if it satisfies inequality (2.2). We prove that every hyperelliptic surface is Loewner. More precisely, we prove the following.

Theorem 3.1. *Let (X, g) be an orientable surface, where the metric g belongs to a hyperelliptic conformal class. Then (X, g) is Loewner.*

Since every genus 2 surface is hyperelliptic [FK92, Proposition III.7.2, page 100], we obtain the following corollary.

Corollary 3.2. *Every metric on the genus 2 surface is Loewner.*

Note that this is the first improvement, known to the authors, on Gromov's $3/4$ bound (2.4) in over 20 years, for surfaces of genus below 50, *cf.* Question 2.2. No extremal metric has as yet been conjectured in this genus, but it cannot be flat with conical singularities [Sa04]. The best available lower bound for the optimal systolic

ratio in genus 2 can be found in [CK03, section 2.2]. For genus $s \geq 3$, the theorem follows from the following proposition, *cf.* Remark 2.1 and [Kon03].

Proposition 3.3. *Every hyperelliptic surface Σ_s of genus s satisfies the estimate*

$$\frac{\text{sys}\pi_1(\Sigma_s)^2}{\text{area}(\Sigma_s)} \leq \frac{4}{s+1}.$$

Proof. Averaging the metric by the hyperelliptic involution $J : X \rightarrow X$ improves the systolic ratio, *cf.* [BCIK1]. Thus we may assume that the metric g is invariant under J . The distance between any pair of Weierstrass points is then at least $\frac{1}{2} \text{sys}\pi_1(\Sigma_s)$. Thus, the disks of radius $R = \frac{1}{4} \text{sys}\pi_1(\Sigma_s)$ centered at the Weierstrass points are disjoint. M. Gromov (and J. Hebda before him) proved that if the metric is extremal for the systolic inequality, the area of such a disk is at least

$$2R^2 = \frac{1}{8} \text{sys}\pi_1(\Sigma_s)^2,$$

cf. (2.5). The existence of an extremal metric was proved in [Gr83]. The latter result is still true in the class of hyperelliptic surfaces, proving the proposition. \square

4. PROOF OF THEOREM 3.1 IN GENUS 2

Let X be a genus 2 surface. Recall that X has a hyperelliptic involution J with 6 Weierstrass points.

The idea of the proof of Theorem 3.1 in genus 2 is to apply Loewner's inequality to certain *companion tori* of X , and to surger the resulting loops so as to obtain a Loewner loop on X . We may need the following lemma.

Lemma 4.1. *Let \mathbb{T}^2 be a torus endowed with a metric invariant under its hyperelliptic involution $J_{\mathbb{T}^2}$, with conical singularities with total angle less than 2π around each. Then the image of a systolic loop of \mathbb{T}^2 in S^2 under the hyperelliptic projection is a simple loop.*

Proof. Let $\gamma \subset \mathbb{T}^2$ be a systolic loop. Since $J_{\mathbb{T}^2}$ induces minus the identity homomorphism on $\pi_1(\mathbb{T}^2)$, the loops γ and $-J_{\mathbb{T}^2}(\gamma)$ are homotopic. In the hypotheses of our lemma, two homotopic systolic loops are necessarily disjoint. Hence the image of γ on S^2 is simple. \square

Definition 4.2. A *companion torus* $\mathbb{T}(a, b, c, d)$ of X is a torus whose ramification locus $\{a, b, c, d\} \subset S^2$ is a subset of the ramification locus of X .

As in the proof of Proposition 3.3, we can assume that the metric on X is invariant under J (see [BCIK1]). Therefore g descends to a metric g_0 , of half the area, on S^2 . Let's choose four of the 6 ramification points, say $a, b, c, d \in S^2$. Choose a double cover with ramification locus $\{a, b, c, d\}$, denoted

$$\mathbb{T}^2(a, b, c, d) \rightarrow S^2.$$

Pulling back the metric g_0 to the torus $\mathbb{T}^2(a, b, c, d)$, we obtain a metric of the same area as the surface X itself. This metric on the torus is smooth away from the two remaining points, where it has a conical singularity with total angle π around each. Consider a Loewner loop

$$LL \subset \mathbb{T}^2(a, b, c, d)$$

on this torus, *e.g.* a systolic loop realizing (2.2). Let L be the projection of LL to S^2 . The simple loop $L \subset S^2$ separates the four points a, b, c, d into two pairs, say a, b on one side and c, d , on the other. If the lift of L to X closes up, we obtain a Loewner loop on X and the theorem is proved. Thus, we may assume that the following three equivalent conditions are satisfied:

- (1) the lift of L to X does not close up;
- (2) the inverse image $Q^{-1}(L) \subset X$ is connected;
- (3) the loop L surrounds precisely 3 ramification points of Q .

Definition 4.3. The simple loop L partitions the sphere into two hemispheres, H_+ and H_- , with $a, b, e \in H_+$ and $c, d, f \in H_-$ where a, b, c, d, e, f are the 6 ramification points of Q .

Using a pair of companion tori, we will construct two loops on the sphere, defining two distinct partitions of the ramification locus into a pair of triples. The basic example to think of is the case of a centrally symmetric 6-tuple of points, *e.g.*, corresponding to the curve

$$y^2 = x^5 - x,$$

and a pair of generic great circles, such that each of the four digons contains at least one ramification point. We now construct a companion torus $\mathbb{T}(a, b, e, f)$.

Consider a Loewner loop $LL' \subset \mathbb{T}^2(a, b, e, f)$, and its projection $L' \subset S^2$. If its lift to X closes up, the theorem is proved. Therefore assume that the lift of L' to X does not close up, *i.e.* L' surrounds exactly 3 ramification points. Now L' separates the four points a, b, e, f into two pairs. Hence it defines a different splitting of the six points into two triples. The connected components of $L' \cap H_+$ form a nonempty finite collection of disjoint nonselfintersecting arcs α .

Each arc α divides H_+ into a pair of regions homeomorphic to disks. Such regions are partially ordered by inclusion. A minimal element for the partial order is necessarily a digon. Such a digon must contain at least one ramification point of Q (otherwise exchange the two sides of the digon between the loops L and L' , so as to decrease the total number of intersections, or else argue as in Lemma 4.1). It is clear that there are at least two such digons in H_+ .

Hence one of them, denoted $D \subset H_+$, must contain precisely one of the 3 ramification points of H_+ . We now exchange the two sides of D between the loops L, L' , obtaining two new loops M, M' . Each of the new loops surrounds a nonzero even number of ramification points. Since

$$\text{length}(M) + \text{length}(M') = \text{length}(L) + \text{length}(L'),$$

one of the loops M or M' is no longer than Loewner. Moreover, its lift to X closes up, producing a Loewner loop on X , as required.

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