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Modeling and Numerical Analysis of Kangaroo Lower Body based on Constrained Dynamics of Hybrid Serial-Parallel Floating-Base Systems

Enrico Mingo Hoffman, Andrea Curti, Narcis Miguel, Sai Kishor Kothakota, Adria Roig, and Luca Marchionni

Abstract—This paper presents the modeling and numerical analysis of the Kangaroo lower body prototype, a novel bipedal humanoid robot developed and manufactured by PAL Robotics. Kangaroo features high-power linear electric actuators combined with unique serial-parallel hybrid chains, which allow for the positioning of all the leg actuators near the base of the robot in order to improve the overall mass distribution. To model and analyze such complex nonlinear mechanisms, we employ a constrained formulation that is extended to account for floatingbase systems in contact with the environment. A comparison is made to demonstrate the significant improvements achieved with TALOS, another humanoid bipedal robot designed by PAL Robotics, in terms of equivalent Cartesian inertia at the feet and centroidal angular momentum. Finally, the paper includes numerical experiments conducted through simulation and preliminary tests performed on the actual Kangaroo platform.

I. INTRODUCTION

In recent years, significant advancements in mechanics and control influenced the mechanical design of humanoid biped robots, especially the incorporation of lightweight and impact-resilient mechanics, coupled with powerful actuation systems in the lower body. As a result, advanced control techniques applied to locomotion have emerged, accompanied by the capability of traversing challenging terrains with the ultimate goal of bridging the gap between laboratorybased research and the practical applicability of humanoid biped robots in less structured real-world scenarios.

In the past, the design of biped humanoid robots has predominantly relied on well-established systems, such as the HRP series, [1], [2], or ASIMO, [3], characterized by stiff position control, serial kinematics, electrical rotary actuators, and high reductions gearboxes, (see also [4], [5]), with very few exceptions, e.g. LOLA [6] and TULIP [7]. While these robots demonstrated the ability to perform complex motions, their interaction with the environment and agility were considerably limited.

During the DARPA Robotics Challenge (DRC), humanoid torque-controlled robots with high-power capabilities were specifically designed for intervention in disaster scenarios [8], showcasing significant advancements, particularly in terms of interaction capabilities. However, these systems exhibited certain limitations, including slow movements and low-impact resilience [9], [10], [11], [12]. The sluggishness of these robots can be attributed to the heavy weight of the platforms and their high inertia, which stemmed from the utilization of actuators positioned near the moving joints.



Fig. 1: Humanoid bipedal robots designed and produced by PAL Robotics since 2004.

Furthermore, the incorporation of gearboxes with high reduction ratios, such as harmonic drives, contributed to their lowimpact resilience. Another common characteristic of these systems was the use of torsional elements or strain gauges for joint torque measurement, rather than relying on current measurements. This approach was primarily necessitated by the high-friction nature of the high-reduction gearboxes, which made it challenging to estimate and compensate.

After the DRC, novel humanoid platforms have been developed following different design paradigms, in particular: relocating actuators to improve mass distribution, especially on the legs, and employing different actuation units than classical electrical actuators with harmonic drivers, to enhance impact robustness and torque/force control. Remarkable examples of this new generation of biped robots are the new Atlas from Boston Dynamics, see [13], Cassie/Digit from Agility Robotics [14], and UCLA's Artemis [15]. While Atlas relies on hydraulic actuation to achieve robustness and highpower impulsive motions, Cassie/Digit and Artemis use hightorque motors paired with high-efficiency low-reduction-ratio gearboxes located near the main body of the robot, with the motion transmitted through a series of closed linkages to reduce foot inertia.

Following this new trend in humanoid bipedal robotics, we present the modeling and analysis, of the lower body prototype of the Kangaroo robot, a bipedal platform recently developed in PAL Robotics (see Fig. 1), The mechanical design of the Kangaroo's anthropomorphic legs is based on novel high-power and robust linear actuation units located near the pelvis area with the actuation transferred to joints through a complex system of serial-parallel hybrid chains. Robotics systems designed with serial-parallel hybrid chains entail multiple advantages such as robustness during impacts,

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low inertia at the end-effectors, and lightweight leg structure.

The main contribution of this research paper is twofold. Firstly, we present the analysis of the lower body of Kangaroo including all the Degrees of Freedom (DOFs) (*a.k.a. fullmodel*), introducing a comprehensive study on floating-base systems presenting serial-parallel hybrid chains in contact with the environment, based on constrained multi-body modeling. Secondly, we carry on a comparison with the TALOS robot, focusing on dynamic quantities that are relevant in humanoid bipedal platforms, i.e. equivalent Cartesian inertia at the feet and Centroidal Angular Momentum.

The paper is organized as follows: Section II presents related works on humanoid bipedal platforms using closed linkages, their analysis, and control methods. Section III introduces the background on kinematic and dynamic modeling of floating-base robots presenting serial-parallel hybrid linkages. Section IV presents the Kangaroo robot and its lower-body kinematic structure, together with an analysis of the closed sub-mechanisms present in the legs, including joint limits and non-linear transmission effects. In Section V, we carry out the comparison against the TALOS biped robot. The full-model validation and preliminary experiments with a prototype of the Kangaroo robot are reported in Section VI. Finally, Section VII closes the paper with final remarks and future development.

II. RELATED WORKS

The main reason to adopt serial mechanisms relies on their straightforward structure, augmented workspace, and simple manufacturing and maintenance processes, therefore representing most of the time the state-of-the-art in robotics systems. For these reasons, most bipedal robots' legs consist of 6 DOFs serial mechanisms with actuators distributed along the whole kinematic chain, from the hip to the ankle, see for example [1], [3], [4], [5]. However, despite the aforementioned advantages, these types of robot architectures generally present only limited precision, low structural stiffness, and poor dynamic characteristics, related to mass and inertia distribution. In particular, the latter plays a fundamental role in agile locomotion and impact handling and mitigation. On the other hand, a parallel robot can provide higher stiffness, speed, accuracy, and payload capacity, at the cost of a reduced workspace and complex geometry, requiring careful analysis and control. [16] defines a series-parallel hybrid robot as a robot constituted by a serial or tree-type combination of serial and parallel mechanisms, combining the advantages of both worlds, but also inheriting their kinematic complexities.

The last few decades of research in humanoid bipedal robotics have shown that achieving high dynamic performance requires a stiff structure and good mass distribution [17]. These characteristics can be achieved using seriesparallel hybrid structures. In fact, more recently, several research works started to introduce mechanical leg designs that are based on series-parallel hybrid kinematics. In some cases, sub-mechanisms are based on parallel kinematics to achieve the lightweight, modular, and compact design of sub-parts. A typical case is the ankle, where motors are relocated closer to the knee by employing a four-bar linkage mechanism with a reduction of inertia at the leg end-effector, as in [9], [10], [18]. A first example of a humanoid bipedal robot strongly based on series-parallel hybrid kinematics chains is LOLA, from [6], which legs were designed mixing rotational, linear actuators, and parallel/differential transmissions. Other examples of humanoid bipedal systems where series-parallel hybrid chains were extensively used for the upper and lower body are HYDRA, [19], SAFFIR, [20], CARL, [21], and Disney's HYBRID LEG, [22], to name a few.

From the software and control point of view, the parallel mechanism is often handled by relying on an abstraction level that separates the parallel-closed chain computation into specialized functions (e.g. *transmissions* in ros_control [23]), or approximates the system as a serial chain.

In [24], is presented a Quadratic Programming (QP)based framework for whole-body control taking into account closed kinematic loops. The authors show how approaches that are based on serial-chain abstraction and separated into specialized functions lead to theoretical and practical disadvantages, in particular:

- the real physical limits induced by the mechanism can not be modeled using a simple box constraint on the serial model,
- the whole-body control solution may be less accurate as it does not consider the correct dynamics of the system,

The proposed framework is applied to a full-size humanoid robot with multiple series-parallel hybrid chains, named RH5, see [25], [26].

Recently, agile and dynamic walking has been achieved with platforms based on parallel mechanism legs, e.g. ATRIAS, [27], and series-parallel hybrid mechanism, e.g. Cassie and Digit [28]. ATRIAS leg design is based on a fourbar linkage driven by 2 motors for leg extension/retraction in common mode, and swing in differential mode. The design of ATRIAS has evolved in Cassie/Digit, employing a seriesparallel hybrid mechanism modeled by cutting the loops and adding consistency constraints, see [29].

In [22] is proposed an IK approach to retarget motions computed from a serial model onto a hybrid one to control the Disney's HYBRID LEG, see also [30]. However, the proposed approach is only kinematics-based.

The work in [16] reports (in Table 1) a comparison among different series-parallel hybrid humanoid robots, considering the number of parallel sub-mechanism modules and the provided free DOFs w.r.t. the total number of free DOFs. The same table is reported in Table I, considering only humanoid robots and augmented with the data for the Kangaroo robot. It is worth noticing that among all the considered humanoid bipedal systems, Kangaroo is the only one relocating the ankle's actuators at the back of the top rear femur, hence not presenting any actuator nor electronics under the knee. This unique feature makes Kangaroo a platform particularly resilient against impacts and potentially ideal to perform agile and dynamic locomotion.

TABLE I: Overview of parallel sub-mechanism based modules with different complexities in serial-parallel hybrid robots

Robot	Number of Parallel mechanisms (Free DOFs)					Composi	Composition Free DOFs	
Name(year)	1-DOF	2-DOF	3-DOF	4-DOF	6-DOF	Total	Parallel	
LOLA, [6]	2	2	-	-	-	25	6	
AILA, [31]	2	2	-	-	-	20	6	
Valkyrie, [10]	_	5	_	_	_	35	10	
TORO, [32]	2	_	_	_	_	27	2	
THOR, [33]	4	4	-	-	_	30	12	
SAFFIR, [34]	4	4	-	-	_	30	12	
LARMBOT, [35]	-	-	2	1	-	22	10	
TALOS, [12]	2	_	_	_	_	27	2	
RH5, [26]	5	5	-	-	_	32	15	
Disney Biped, [22]	-	-	-	-	2	12	12	
Kangaroo (2022)	4	4	-	-	-	12	12	

III. BACKGROUND ON MODELING SERIAL-PARALLEL HYBRID CHAINS IN FLOATING-BASE SYSTEMS

Classical approaches in modeling parallel linkages are based on geometrical analysis, leveraging case-specific analytical expressions for the linkage kinematics, as in [36] where the closed-form solution of a 2SPRR-U humanoid ankle is presented. This solution permits embedding the closed loop constraints directly in the equation of motions eventually saving computational cost and improving numerical accuracy. However, such closed-form solutions are not always available and do not generalize to arbitrary serialparallel hybrid mechanisms.

According to [37], a more comprehensive and promising method entails modeling these linkages as constrained multibody systems. This approach allows for the consideration of intricate geometries and dynamic effects associated with multiple serial-parallel hybrid chains. While the direct map between the linkage parameters and the resulting motion is no longer expressed analytically, it can still be determined through numerical computation. In this section, we recall this approach, including floating-base systems incorporating generic series-parallel hybrid chains, with the aim of using it for the modeling and analysis of the Kangaroo robot.

From now on, as we did in the previous sections, we will call *actuated* DOFs the ones that can provide force/torque, and *passive* DOFs the ones that can not. We will consider mechanisms with the following assumptions:

- **Assumption 1** The motion of the passive DOFs is totally constrained by the actuated DOFs;
- Assumption 2 The number of *free* DOFs coincides with the number of actuated DOFs.

A. Modeling of Serial-Parallel Hybrid Chains

Let's consider a simple 1DOF RRPR closed mechanism as the one depicted in Fig. 2. To model this closed linkage as a constrained multi-body system, the mechanism is first *opened* at one of the passive DOFs, represented by a dashed line in Fig. 2, forming two open kinematic chains. In this case, we have:

$$\boldsymbol{\theta} = \begin{bmatrix} \boldsymbol{\theta}_u \\ \boldsymbol{\theta}_a \end{bmatrix} \in \mathbb{R}^n, \tag{1}$$

with $\theta_u \in \mathbb{R}^m$ the passive DOFs and $\theta_a \in \mathbb{R}^{n-m}$ the actuated DOF, m < n. For the 1DOF RRPR mechanism considered, we have in particular n = 3 and with m = 2.



Fig. 2: 1DOF RRPR closed mechanism.

We denote with \mathcal{F}_a and \mathcal{F}_u the frames associated with each open chain, placed at the removed DOF, as shown in Fig. 2. Considering Assumption 1 and Assumption 2, we can enforce a constraint in the form:

$$^{a}\mathbf{f}_{u}\left(\boldsymbol{\theta}\right) = \mathbf{0} \in \mathbb{R}^{m}.$$
 (2)

along the constrained directions to keep the two frames overlapped to close the linkage. With θ_a as known quantities, the equation (2) consists in a non-linear system of m equations in m unknowns θ_u . For example, for the considered closed linkage, the motion is constrained on the local plane yz of frame \mathcal{F}_a , as shown in Fig. 2, resulting in the relative position of frame \mathcal{F}_u w.r.t. \mathcal{F}_a , along the local yz directions, being **0**. We can compute the constraint Jacobian from (2):

$$\mathbf{J}_{l}\left(\boldsymbol{\theta}\right)\dot{\boldsymbol{\theta}}=\mathbf{0},\tag{3}$$

with $\mathbf{J}_{l}(\boldsymbol{\theta}) \in \mathbb{R}^{m \times n}$ being the *relative Jacobian* between the frames \mathcal{F}_{u} and \mathcal{F}_{a} expressed in \mathcal{F}_{a} , such that:

$$\mathbf{J}_{l}\left(\boldsymbol{\theta}\right)\dot{\boldsymbol{\theta}} = {}^{a}\mathbf{J}_{a,u}\left(\boldsymbol{\theta}\right)\dot{\boldsymbol{\theta}} = {}^{a}\mathbf{v}_{a,u},\tag{4}$$

where ${}^{a}\mathbf{v}_{a,u}$ is the velocity along the constrained directions of the frame \mathcal{F}_{u} w.r.t. the frame \mathcal{F}_{a} expressed in frame \mathcal{F}_{a} (please refer to Appendix I on how to compute relative kinematics quantities). It is possible to divide the constraint Jacobian into its actuated and passive parts:

$$\mathbf{J}_{l}(\boldsymbol{\theta})\,\dot{\boldsymbol{\theta}} = \begin{bmatrix} \mathbf{J}_{l,u}\left(\boldsymbol{\theta}\right) & \mathbf{J}_{l,a}\left(\boldsymbol{\theta}\right) \end{bmatrix} \begin{bmatrix} \boldsymbol{\theta}_{u} \\ \dot{\boldsymbol{\theta}}_{a} \end{bmatrix} = \mathbf{0}, \qquad (5)$$

with $\mathbf{J}_{l,u}(\boldsymbol{\theta}) \in \mathbb{R}^{m \times m}$ the passive part, and $\mathbf{J}_{l,a}(\boldsymbol{\theta}) \in \mathbb{R}^{m \times n-m}$ the actuated part of the constraint Jacobian $\mathbf{J}_{l}(\boldsymbol{\theta})$. The constraint (3) can be written as:

$$\mathbf{J}_{l,u}\left(\boldsymbol{\theta}\right)\dot{\boldsymbol{\theta}}_{u}+\mathbf{J}_{l,a}\left(\boldsymbol{\theta}\right)\dot{\boldsymbol{\theta}}_{a}=\mathbf{0},\tag{6}$$

that permits to compute the passive velocities from the actuated ones, a.k.a. Differential Forward Kinematics (DFK):

$$\dot{\boldsymbol{\theta}}_{u} = -\mathbf{J}_{l,u}\left(\boldsymbol{\theta}\right)^{-1}\mathbf{J}_{l,a}\left(\boldsymbol{\theta}\right)\dot{\boldsymbol{\theta}}_{a} = \mathbf{J}_{m}\left(\boldsymbol{\theta}\right)\dot{\boldsymbol{\theta}}_{a},\qquad(7)$$

with $\mathbf{J}_{m}(\boldsymbol{\theta}) \in \mathbb{R}^{m \times n-m}$ named the *mapping Jacobian*, and considering $\mathbf{J}_{l,u}(\boldsymbol{\theta})$ invertible, thus not in singularity.

The passive accelerations $\ddot{\theta}_u$ can be computed from the (desired/measured) actuated accelerations $\ddot{\theta}_a$, positions θ and velocities $\dot{\theta}$. In fact, the constraint (2) can be easily expressed at the acceleration level:

$$\mathbf{J}_{l}\left(\boldsymbol{\theta}\right)\ddot{\boldsymbol{\theta}}+\dot{\mathbf{J}}_{l}\left(\boldsymbol{\theta},\dot{\boldsymbol{\theta}}\right)\dot{\boldsymbol{\theta}}=\mathbf{0}, \tag{8}$$

please refer to Appendix I-B for the computation of the relative bias term $\dot{\mathbf{J}}_l(\boldsymbol{\theta}, \dot{\boldsymbol{\theta}})\dot{\boldsymbol{\theta}}$. Splitting (8) into its actuated and passive parts leads to:

$$\begin{bmatrix} \mathbf{J}_{l,u} \left(\boldsymbol{\theta} \right) & \mathbf{J}_{l,a} \left(\boldsymbol{\theta} \right) \end{bmatrix} \begin{bmatrix} \ddot{\boldsymbol{\theta}}_{u} \\ \ddot{\boldsymbol{\theta}}_{a} \end{bmatrix} + \dot{\mathbf{J}}_{l} \left(\boldsymbol{\theta}, \dot{\boldsymbol{\theta}} \right) \dot{\boldsymbol{\theta}} = \mathbf{0}, \qquad (9)$$

therefore:

$$\ddot{\boldsymbol{\theta}}_{u} = -\mathbf{J}_{l,u} \left(\boldsymbol{\theta}\right)^{-1} \left(\mathbf{J}_{l,a} \left(\boldsymbol{\theta}\right) \ddot{\boldsymbol{\theta}}_{a} + \dot{\mathbf{J}}_{l} \left(\boldsymbol{\theta}, \dot{\boldsymbol{\theta}}\right) \dot{\boldsymbol{\theta}} \right) = \\ = \mathbf{J}_{m} \left(\boldsymbol{\theta}\right) \ddot{\boldsymbol{\theta}}_{a} - \mathbf{J}_{l,u} \left(\boldsymbol{\theta}\right)^{-1} \dot{\mathbf{J}}_{l} \left(\boldsymbol{\theta}, \dot{\boldsymbol{\theta}}\right) \dot{\boldsymbol{\theta}}.$$
(10)

The equations of motions consist of a constrained dynamic, see [38], and it can be written as:

$$\mathbf{M}(\boldsymbol{\theta})\ddot{\boldsymbol{\theta}} + \mathbf{h}(\boldsymbol{\theta}, \dot{\boldsymbol{\theta}}) = \mathbf{S}\boldsymbol{\tau} + \mathbf{J}_{l}\left(\boldsymbol{\theta}\right)^{T}\boldsymbol{\lambda}, \qquad (11)$$

with $\mathbf{M}(\boldsymbol{\theta}) \in \mathbb{R}^{n \times n}$ the inertia matrix, $\mathbf{h}(\boldsymbol{\theta}, \dot{\boldsymbol{\theta}}) \in \mathbb{R}^n$ non-linear terms consisting in Coriolis/Centrifugal and gravitational forces, $\boldsymbol{\tau} \in \mathbb{R}^{n-m}$ the actuated torques/-forces, $\boldsymbol{\lambda} \in \mathbb{R}^m$ the constrained forces, and $\mathbf{S} = [\mathbf{0}_{m \times n-m} \mathbf{I}_{n-m \times n-m}]^T$ the passive selection matrix such that $\mathbf{S}\boldsymbol{\tau} \in \mathbb{R}^n$. The dynamics in (11) can be as well divided into its actuated and passive parts:

$$\mathbf{M}_{u}\left(\boldsymbol{\theta}\right)\boldsymbol{\dot{\theta}} + \mathbf{h}_{u}(\boldsymbol{\theta},\boldsymbol{\dot{\theta}}) = \mathbf{J}_{l,u}\left(\boldsymbol{\theta}\right)^{T}\boldsymbol{\lambda},$$
(12a)

$$\mathbf{M}_{a}\left(\boldsymbol{\theta}\right)\ddot{\boldsymbol{\theta}} + \mathbf{h}_{a}\left(\boldsymbol{\theta},\dot{\boldsymbol{\theta}}\right) = \boldsymbol{\tau} + \mathbf{J}_{l,a}\left(\boldsymbol{\theta}\right)^{T}\boldsymbol{\lambda}, \quad (12b)$$

with $\mathbf{M}_{u}(\boldsymbol{\theta}) \in \mathbb{R}^{m \times n}$, $\mathbf{h}_{u}(\boldsymbol{\theta}, \dot{\boldsymbol{\theta}}) \in \mathbb{R}^{m}$, $\mathbf{M}_{a}(\boldsymbol{\theta}) \in \mathbb{R}^{n-m \times n}$ and $\mathbf{h}_{a}(\boldsymbol{\theta}, \dot{\boldsymbol{\theta}}) \in \mathbb{R}^{n-m}$. The constrained forces $\boldsymbol{\lambda}$ can be removed by using equation (12a):

$$\boldsymbol{\lambda} = \mathbf{J}_{l,u}(\boldsymbol{\theta})^{-T} \left(\mathbf{M}_u(\boldsymbol{\theta}) \boldsymbol{\ddot{\theta}} + \mathbf{h}_u(\boldsymbol{\theta}, \boldsymbol{\dot{\theta}}) \right).$$
(13)

Let's now define the quantity:

•••

$$\boldsymbol{\tau}_{a}(\boldsymbol{\theta}, \dot{\boldsymbol{\theta}}, \ddot{\boldsymbol{\theta}}, \boldsymbol{\tau}) = \boldsymbol{\tau} - \left(\mathbf{M}_{a}\left(\boldsymbol{\theta}\right) \ddot{\boldsymbol{\theta}} + \mathbf{h}_{a}(\boldsymbol{\theta}, \dot{\boldsymbol{\theta}}) \right), \quad (14)$$

the actuated torques, and

$$\boldsymbol{\tau}_{u}(\boldsymbol{\theta}, \dot{\boldsymbol{\theta}}, \ddot{\boldsymbol{\theta}}) = \mathbf{M}_{u}\left(\boldsymbol{\theta}\right)\ddot{\boldsymbol{\theta}} + \mathbf{h}_{u}(\boldsymbol{\theta}, \dot{\boldsymbol{\theta}})$$
 (15)

the *passive* torques. Equation (12b) can be rewritten using τ_a^{1} :

$$\boldsymbol{\tau}_{a} = -\mathbf{J}_{l,a} \left(\boldsymbol{\theta}\right)^{T} \boldsymbol{\lambda}$$
(16)

and substituting λ :

$$\boldsymbol{\tau}_{a} = -\mathbf{J}_{l,a} \left(\boldsymbol{\theta}\right)^{T} \mathbf{J}_{l,u} \left(\boldsymbol{\theta}\right)^{-T} \left(\mathbf{M}_{u} \left(\boldsymbol{\theta}\right) \boldsymbol{\ddot{\theta}} + \mathbf{h}_{u} (\boldsymbol{\theta}, \boldsymbol{\dot{\theta}})\right) = \\ = -\mathbf{J}_{l,a} \left(\boldsymbol{\theta}\right)^{T} \mathbf{J}_{l,u} \left(\boldsymbol{\theta}\right)^{-T} \boldsymbol{\tau}_{u} = \mathbf{J}_{m} \left(\boldsymbol{\theta}\right)^{T} \boldsymbol{\tau}_{u}$$
(17)

with:

$$\mathbf{J}_{m} \left(\boldsymbol{\theta}\right)^{T} = -\left(\mathbf{J}_{l,u} \left(\boldsymbol{\theta}\right)^{-1} \mathbf{J}_{l,a} \left(\boldsymbol{\theta}\right)\right)^{T} = \\ = -\mathbf{J}_{l,a} \left(\boldsymbol{\theta}\right)^{T} \mathbf{J}_{l,u} \left(\boldsymbol{\theta}\right)^{-T},$$
(18)

m

where equation (17) permits to map generic torques on the passive DOFs into the actuated ones. The inverse dynamics can finally be computed from (12b), plugging (13):

.

$$\tau = \mathbf{M}_{a}(\boldsymbol{\theta})\boldsymbol{\theta} + \mathbf{h}_{a}(\boldsymbol{\theta},\boldsymbol{\dot{\theta}}) - \mathbf{J}_{l,a}(\boldsymbol{\theta})^{T} \boldsymbol{\lambda} =$$

$$= \mathbf{M}_{a}(\boldsymbol{\theta})\boldsymbol{\ddot{\theta}} + \mathbf{h}_{a}(\boldsymbol{\theta},\boldsymbol{\dot{\theta}}) -$$

$$- \mathbf{J}_{l,a}(\boldsymbol{\theta})^{T} \mathbf{J}_{l,u}(\boldsymbol{\theta})^{-T} \left(\mathbf{M}_{u}(\boldsymbol{\theta})\boldsymbol{\ddot{\theta}} + \mathbf{h}_{u}(\boldsymbol{\theta},\boldsymbol{\dot{\theta}})\right) =$$

$$(19)$$

$$= \mathbf{M}_{a}(\boldsymbol{\theta})\boldsymbol{\ddot{\theta}} + \mathbf{h}_{a}(\boldsymbol{\theta},\boldsymbol{\dot{\theta}}) + \mathbf{J}_{m}(\boldsymbol{\theta})^{T} \left(\mathbf{M}_{u}(\boldsymbol{\theta})\boldsymbol{\ddot{\theta}} + \mathbf{h}_{u}(\boldsymbol{\theta},\boldsymbol{\dot{\theta}})\right)$$

where we have used the (18).

••

The constrained approach can be extended to complex mechanisms formed by multiple closed sub-mechanisms by stacking constraint Jacobians associated with each closed sub-mechanism, such that the total number of constraints will be equal to the number of passive DOFs. For example, the 2DOFs U-2RRPU differential mechanism depicted in Fig. 3a is modeled introducing a total of m = 8 constraints, split in 2 closed sub-mechanism, each one introducing 4 constraints:

$$\begin{cases} {}^{a}\mathbf{p}_{u}^{1}\left(\boldsymbol{\theta}\right) = \mathbf{0} \in \mathbb{R}^{3} & \text{for the relative position,} \\ {}^{a}\vartheta_{u}^{1}\left(\boldsymbol{\theta}\right) = c \in \mathbb{R} & \text{for the relative orientation,} \end{cases}$$
(20)

between the frames $\mathcal{F}_{a,1}$ and $\mathcal{F}_{u,1}$ depicted in Fig. 3b, with $\boldsymbol{\theta} \in \mathbb{R}^{10}$, modeling relative rotations using Euler angles $(\phi \ \vartheta \ \varphi)$ for simplicity².

Finally, given a joint-space selection matrix $\mathbf{P} \in \mathbb{R}^{n-m \times m}$ which selects a subset of the passive DOFs such that:

$$\dot{\mathbf{q}} = \mathbf{P}\dot{\boldsymbol{\theta}}_u, \qquad \boldsymbol{\tau}_j = \mathbf{P}\boldsymbol{\tau}_u,$$
 (21)

it is possible to compute the DIK problem w.r.t. (7):

$$\dot{\boldsymbol{\theta}}_a = (\mathbf{P} \mathbf{J}_m(\boldsymbol{\theta}))^{-1} \, \dot{\mathbf{q}}_d \tag{22}$$

and the inverse torque mapping problem w.r.t. (17):

$$\boldsymbol{\tau}_j = \left(\mathbf{J}_m(\boldsymbol{\theta})^T \mathbf{P}^T \right)^{-1} \boldsymbol{\tau}.$$
 (23)

Equation (22) can be used to impose a desired behavior to the passive DOFs, while equation (23) can be used to compute the equivalent *virtual* torque acting on these passive DOFs.

¹Here we drop dependencies of τ_a and τ_u

²In our implementation and in the experimental results we use a quaternion-based parameterization



(a) Full mechanism.



(b) Right brace of the mechanism.

Fig. 3: 2DOFs U-2RRPU differential mechanism.

B. Floating-Base Dynamics including Serial-Parallel Hybrid Chains

To model a floating-base system presenting hybrid serialparallel kinematic chains, we augment the generalized coordinates in (1) with a parameterization in SE(3) of the pose of the floating-base, and we denote with $\nu \in \mathbb{R}^{n+6}$ the generalized velocities:

$$\mathbf{q} = \begin{bmatrix} \mathbf{p} \\ \boldsymbol{\rho} \\ \boldsymbol{\theta} \end{bmatrix}, \qquad \boldsymbol{\nu} = \begin{bmatrix} \dot{\mathbf{p}} \\ \boldsymbol{\omega} \\ \dot{\boldsymbol{\theta}} \end{bmatrix}, \qquad (24)$$

with $\mathbf{p} \in \mathbb{R}^3$ and $\boldsymbol{\rho} \in \mathbf{S}^3 = \{\boldsymbol{\rho} \in \mathbb{R}^4 : \|\boldsymbol{\rho}\| = 1\}$, using quaternions for the orientation.

The robot can only exert forces from the contact points c, we denote with $\mathbf{F}_i \in \mathbb{R}^3$ the contact force vectors, with i = 1, ..., c, c = 8 in the case of double support when both feet are on the ground. The floating-base inverse dynamics model of this system consists of a set of equations:

$$\mathbf{M}(\mathbf{q})\dot{\boldsymbol{\nu}} + \mathbf{h}(\mathbf{q},\boldsymbol{\nu}) = \mathbf{S}\boldsymbol{\tau} + \mathbf{J}_c(\mathbf{q})^T \mathbf{F} + \mathbf{J}_l(\mathbf{q})^T \boldsymbol{\lambda}, \quad (25a)$$

$$\mathbf{J}_c(\mathbf{q})\dot{\boldsymbol{\nu}} + \dot{\mathbf{J}}_c(\mathbf{q}, \boldsymbol{\nu})\boldsymbol{\nu} = \mathbf{0}, \tag{25b}$$

$$\mathbf{J}_l(\mathbf{q})\dot{\boldsymbol{\nu}} + \dot{\mathbf{J}}_l(\mathbf{q}, \boldsymbol{\nu})\boldsymbol{\nu} = \mathbf{0}, \tag{25c}$$

with $\mathbf{M}(\mathbf{q}) \in \mathbb{R}^{n+6 \times n+6}$ the floating-base inertia matrix, $\mathbf{h}(\mathbf{q}, \boldsymbol{\nu}) \in \mathbb{R}^{n+6}$ the floating-base non-linear terms,

 $\mathbf{J}_{c}(\mathbf{q}) \in \mathbb{R}^{3c \times n+6}$ contacts Jacobians, $\mathbf{F} = \begin{bmatrix} \mathbf{F}_{1}^{T}, \dots, \mathbf{F}_{c}^{T} \end{bmatrix}^{T}$, $\mathbf{J}_{l}(\mathbf{q}) \in \mathbb{R}^{m \times n+6}$ closed linkages constraint Jacobians and $\boldsymbol{\lambda} \in \mathbb{R}^{m}$ the associated closed linkages constraint forces, $\mathbf{J}_{c}(\mathbf{q}, \boldsymbol{\nu})\boldsymbol{\nu} \in \mathbb{R}^{3c}$ and $\mathbf{J}_{l}(\mathbf{q}, \boldsymbol{\nu})\boldsymbol{\nu} \in \mathbb{R}^{m}$ the acceleration bias terms for contacts and closed linkages kinematics constraint, respectively, and finally $\mathbf{S} \in \mathbb{R}^{n+6 \times a}$ the selection matrix for the actuated torques.

We can divide the dynamics in (25a) into its actuated, passive, and underactuated, i.e. the floating-base, components:

$$\mathbf{M}_b(\mathbf{q})\dot{\boldsymbol{\nu}} + \mathbf{h}_b(\mathbf{q}, \boldsymbol{\nu}) = \mathbf{J}_{c,b}(\mathbf{q})^T \mathbf{F},$$
(26a)

$$\mathbf{M}_{u}(\mathbf{q})\dot{\boldsymbol{\nu}} + \mathbf{h}_{u}(\mathbf{q},\boldsymbol{\nu}) = \mathbf{J}_{c,u}(\mathbf{q})^{T}\mathbf{F} + \mathbf{J}_{l,u}(\mathbf{q})^{T}\boldsymbol{\lambda}, \quad (26b)$$
$$\mathbf{M}_{a}(\mathbf{q})\dot{\boldsymbol{\nu}} + \mathbf{h}_{a}(\mathbf{q},\boldsymbol{\nu}) = \boldsymbol{\tau} + \mathbf{J}_{c,a}(\mathbf{q})^{T}\mathbf{F} + \mathbf{J}_{l,a}(\mathbf{q})^{T}\boldsymbol{\lambda},$$

$$(26c)$$

with $\mathbf{M}_b(\mathbf{q}) \in \mathbb{R}^{6 \times n+6}$ and $\mathbf{h}_b(\mathbf{q}, \boldsymbol{\nu}) \in \mathbb{R}^6$, $\mathbf{M}_u(\mathbf{q}) \in \mathbb{R}^{m \times n+6}$ and $\mathbf{h}_u(\mathbf{q}, \boldsymbol{\nu}) \in \mathbb{R}^m$, $\mathbf{M}_a(\mathbf{q}) \in \mathbb{R}^{n-m \times n+6}$ and $\mathbf{h}_a(\mathbf{q}, \boldsymbol{\nu}) \in \mathbb{R}^{n-m}$, the inertia matrices and non-linear terms associated to the (underactuated) floating base, the passive part of the closed linkages and the remaining actuated part, respectively; $\mathbf{J}_{c,b}(\mathbf{q}) \in \mathbb{R}^{3c \times 6}$ and $\mathbf{J}_{c,u}(\mathbf{q}) \in \mathbb{R}^{3c \times m}$ the underactuated and passive parts of the closed kinematic chains respectively and, $\mathbf{J}_{c,a}(\mathbf{q}) \in \mathbb{R}^{3c \times n-m}$ its actuated part; $\mathbf{J}_{l,u}(\mathbf{q}) \in \mathbb{R}^{m \times m}$ the passive part of the constraint Jacobian and $\mathbf{J}_{l,a}(\mathbf{q}) \in \mathbb{R}^{3c \times n}$ its actuated part.

Taking inspiration from [39], a possible inverse dynamics (ID) scheme can be composed of the following three steps:

- 1) computation of the generalized accelerations $\dot{\nu}$ and contact forces F using (26a) and the constraints (25b) and (25c),
- 2) computation of Lagrange multipliers λ using (26b),
- 3) computation of actuation torques τ using (26c).

The first step can be addressed using a QP, constraining the contact forces inside linearized friction cones³:

$$\min_{\dot{\boldsymbol{\nu}},\mathbf{F}} \left\| \mathbf{J}_{b}(\mathbf{q})\dot{\boldsymbol{\nu}} - \mathbf{a}_{b,d} + \dot{\mathbf{J}}_{b}(\mathbf{q},\boldsymbol{\nu})\boldsymbol{\nu} \right\| + \epsilon \|\dot{\boldsymbol{\nu}}\| + \gamma \|\mathbf{F}\|$$
s.t. (26a), (25b), (25c) (27)

$$\bar{\mathbf{F}}^{t} \ge \mathbf{0}, \quad |\bar{\mathbf{F}}^{t}| \le \mu \bar{\mathbf{F}}^{n},$$

with $\mathbf{J}_b(\mathbf{q}) \in \mathbb{R}^{6 \times 6+n}$ the *task* Jacobian to move the floating base and $\mathbf{a}_{b,d} \in \mathbb{R}^6$ a proper Cartesian acceleration reference for the base, $\mathbf{\bar{F}}^t \in \mathbb{R}^c$ and $\mathbf{\bar{F}}^n \in \mathbb{R}^{2c}$ the tangential and normal components, respectively, of the contact forces \mathbf{F} rotated in the contact local frames, μ the friction cone coefficient, ϵ and γ scalar parameters for regularisation. Once optimal $\dot{\boldsymbol{\nu}}$ and \mathbf{F} are computed, the second step permits to compute the Lagrange multipliers:

$$\boldsymbol{\lambda} = \mathbf{J}_{l,u}(\mathbf{q})^{-T} \left(\mathbf{M}_u(\mathbf{q}) \boldsymbol{\dot{\nu}} + \mathbf{h}_u(\mathbf{q}, \boldsymbol{\nu}) - \mathbf{J}_{c,u}(\mathbf{q})^T \mathbf{F} \right).$$
(28)

Finally, the third step returns the actuated torques:

$$\boldsymbol{\tau} = \mathbf{M}_{a}(\mathbf{q})\dot{\boldsymbol{\nu}} + \mathbf{h}_{a}(\mathbf{q},\boldsymbol{\nu}) - \mathbf{J}_{c,a}(\mathbf{q})^{T}\mathbf{F} - \mathbf{J}_{l,a}(\mathbf{q})^{T}\boldsymbol{\lambda}.$$
 (29)

 3 Here we do not report further constraints which may include torque limits (by including equations (28) and (29)), joint limits, [40], and self-collision avoidance, [41], to name a few

Notice that the QP in (27) can be rewritten *projecting* the passive part into the actuated one and therefore considering only in the actuated quantities since the passive ones are dependent variables (for more details see Appendix II).

IV. ANALYSIS OF KANGAROO LOWER BODY

Fig. 4 shows the prototype of the Kangaroo robot highlighting the location of the linear actuators, which are all placed near the pelvis area. The lower body of the Kangaroo



Fig. 4: Prototype of the lower body of the Kangaroo robot.

robot consists of two legs, each one modeled with 32 passive rotational DOFs and 6 actuated linear DOFs. Therefore, the total number of DOFs in the Kangaroo model is 76, comprising of 64 passive DOFs and 12 active DOFs.⁴. The lower body prototype weighs $\approx 42 \ [kg]$ with the protection cages for the torso, not present in Fig. 4, and $\approx 36 \ [kg]$ without them. All the linear actuators, except for those located at the thighs, are equipped with relative linear encoders, while the thigh actuators are equipped with absolute linear encoders. To calibrate the relative linear encoders, an initial calibration is performed using absolute encoders mounted at certain passive DOFs.

Each leg of Kangaroo is a hybrid serial-parallel kinematic chain formed by 4 closed sub-mechanisms:

- 1DOF RRPR Hip Yaw,
- 2DOFs U-2RRPU Hip Pitch/Roll,
- 1DOF RRRR-RRRP-R Knee,
- 2DOFs U-2(RRRP-RRR-UU) Ankle Pitch/Roll.

We will now discuss each sub-mechanism separately, introducing a brief analysis of the kinematics singularities and the countermeasures taken to avoid them in the design of the platform. Table II reports an overview of the closed submechanisms with the number of active and passive DOFs (with the number of total passive DOFs), the number of constraints to model the linkage, the actuators limits, the DOFs limits, and the average human limits at the same DOF (see [42], [43]). In Figure 5, we present the normalized manipulability index for the left leg. This evaluation is conducted through a uniform sampling of the workspace, achieved by varying the linear actuators within the Hip Pitch/Roll and Knee sub-mechanisms.



Fig. 5: Normalized manipulability of the left leg. Position shown at the top, orientation at the bottom. View from the front on the left, and rear view on the right.

1DOF Hip Yaw: The Hip Yaw closed sub-mechanism 3D model, whose schematic is depicted in Fig. 2, is shown in the left portion of Fig. 6. This sub-mechanism exhibits two



Fig. 6: On the left is the Hip Yaw sub-mechanism, and on the right is the Hip Pitch/Roll sub-mechanism. The green arrows show the linear actuators, in blue the passive DOFs, and in red the main passive DOF(s) of the sub-mechanisms. The yellow arrow shows the position of the absolute rotary encoders used for initial calibration.

configuration singularities, occurring when the passive DOF $\theta_{u,1}$ (see Fig. 2) is at 0 [rad] and $-\pi$ [rad]. These singularities correspond to the linear actuator being approximately driven at 0.05 [m] and -0.05 [m] respectively, resulting in the sub-mechanism being reduced to a linear configuration.

⁴Here, we are not considering the floating-base DOFs

Closed sub-mechanism	#Active DOFs	#Passive DOFs	#Constraints	Actuator Lims [m]		DOFs Lims [deg]		Average Human Lims [deg]	
				min	max	min	max	min	max
Hip Yaw	1	2(3)	2	-0.02	0.02	-30	30	-30	60
Hip Roll	2	8(10)	0	0.04	0.04	-27	27	-30	45
Hip Pitch	2	8(10)	0	-0.04	0.04	-40	42	-135	20
Knee	1	6(8)	6	0	0.15	0	120	0	130
Ankle Roll	2	16(24)	16	0.02	0.02	-27	27	-45	15
Ankle Pitch	2	10(24)	10	-0.02	0.02	-36	34	-12	23

TABLE II: Overview of the number of parallel sub-mechanism based modules in Kangaroo lower body

Nonetheless, due to the limitations of the linear actuator of (-0.02, 0.02) [m], the sub-mechanism is unable to attain the singular configurations.

2DOFs Hip Pitch/Roll: The Hip Pitch/Roll, whose 3D model is shown in the right portion of Fig. 6, consists of a hybrid serial-parallel sub-mechanism equivalent to the schematic in Fig. 3a. This sub-mechanism introduces convex non-linear joint limits that can be computed knowing both the position limits in the linear actuator and passive DOFs, if any. In particular, it is possible to cast a QP-based IK, constrained by those limits, to explore the workspace of the sub-mechanism:

$$\min_{\boldsymbol{\dot{\theta}}} \left\| \dot{\boldsymbol{\theta}}_{a,d} - \dot{\boldsymbol{\theta}}_{a} \right\| \\
\text{s.t.} \quad \mathbf{J}_{l}(\boldsymbol{\theta}) \, \dot{\boldsymbol{\theta}} = \mathbf{0}, \qquad (30) \\
\frac{\boldsymbol{\theta} - \boldsymbol{\theta}}{dt} \leq \dot{\boldsymbol{\theta}} \leq \frac{\overline{\boldsymbol{\theta}} - \boldsymbol{\theta}}{dt},$$

with $[\underline{\theta}, \overline{\theta}]$ the minimum and maximum DOFs limits, respectively. By driving the linear actuators is possible to reconstruct the shape of the interested joints, as shown in Fig. 7. Recalling Fig. 3, a first set of singularities appear when the constraint ${}^{a}\vartheta_{u}^{1}$ and ${}^{a}\vartheta_{u}^{2}$ reach the 0 [rad] and $-\pi$ [rad]configurations, respectively when the two linear actuators are simultaneously driven to approximately -0.07 [m] and 0.05 [m]. Nevertheless, both singularities lie outside the linear actuator limits, (-0.04, 0.04) [m]. Unfortunately, a singularity arises when the two linear actuators are driven differentially to their extremes, such as -0.04 [m] and 0.04 [m], causing $\theta_{u,8}$ to approach $-\frac{\pi}{2}$ [rad] or $\frac{\pi}{2}$ [rad]. To address this issue, a physical constraint has been implemented for the Hip Roll, which can be enforced at the actuator level via the QP formulation in (30).

1DOF Knee: The Knee sub-mechanism is formed by the three closed planar linkages depicted in Fig. 8. This is the only mechanism that presents an absolute linear encoder mounted directly on the linear actuator, thus not requiring any initial calibration. The 3D model of the Knee is presented in the left part of Fig. 10, providing an insight into the thigh. The knee exhibits two primary singularities when the passive DOF $\theta_{u,2}$ (see Fig. 8) is in configurations of 0 [rad] and $-\pi$ [rad], respectively. In these instances, the linear actuator situated within the thigh is approximately positioned at 0.095 [m] and -0.095 [m], and the sub-

Hip Pitch-Roll Joint Limits



Fig. 7: Joint limits in 2DOFs U-2RRPU Hip Pitch/Roll. In blue (continuous line) are the real joint limits projected into the differential mechanism, while in red (dashed line) are the joint limits, as box constraints, when not projected.

mechanism is reduced to a line⁵. Nonetheless, the linear actuator is confined to a stroke of 0.15 m, bounded by limits of (-0.075, 0.075) m, preventing the Knee from attaining any singular configuration; furthermore the passive Knee motion is additionally mechanically restricted by plastic shells installed in front of the joint, serving the purpose of safeguarding the robot in the event of a fall, as depicted in Fig. 4.

2DOFs Ankle Pitch/Roll: The Ankle consists of a hybrid serial-parallel sub-mechanism formed by the three submechanisms depicted in Fig. 9. The right model in Fig. 10 shows the 3D visualization of the sub-mechanism. Joint limits are computed using the same methodology as for the 2DOFs Hip Pitch/Roll and reported in Fig. 11. The first singularity encountered in the Ankle occurs within sub-mechanism *b*. as depicted in Fig. 9 when the passive DOFs $\theta_{u,5}$ and $\theta_{u,15}$ reach 0 rad, corresponding to drive independently the linear actuators to approximately -0.035 m. A second singularity arises within sub-mechanism *a*., where

⁵For this analysis we are considering the 0 of the linear actuator centered at the $-\frac{\pi}{2}$ configuration of the $\theta_{u,2}$ DOF, see Fig. 8



Fig. 8: 1DOF RRRR-RRRP-R Knee simplified submechanisms: a. RRRR introducing 2 constraints; b. RRRP, introducing 2 constraints; c. RRRRRR introducing 2 constraints. Some passive DOFs are shared between the linkages.



Fig. 9: 2DOFs U-2(RRRP-RRR-UU) Ankle Pitch/Roll simplified sub-mechanisms: a. 2RRRP placed at the hip, introducing 4 constraints; b. 2RRRR transferring the motion from the hip to the knee introducing 4 constraints; c. U-2UR introducing 8 constraints. Some passive DOFs are shared between the linkages.

the passive DOFs $\theta_{u,2}$ and $\theta_{u,12}$ also assume values of 0 rad, coinciding with the separate actuation of linear actuators to approximately 0.027 m. Nevertheless, the linear actuators of the Ankle are limited to a stroke of 0.04 [m], with (-0.02, 0.02) [m] limits, preventing the aforementioned singular configurations and any singularity in common mode. The differential mode remains free of singularities within the actuator limits. However, these singularities are linked to the passive degree of freedom (DOF) $\theta_{u,22}$ reaching $\frac{\pi}{2}$ [rad] and



Fig. 10: On the left is the Knee sub-mechanism, and on the right is the Ankle Pitch/Roll sub-mechanism. The green arrows show the linear actuators, in blue the passive DOFs, and in red the main passive DOF(s) of the sub-mechanisms. The yellow arrow shows the position of the linear absolute encoder for the Knee and the absolute rotary encoders for the Ankle Pitch/Roll used for initial calibration.

 $-\frac{\pi}{2}$ [rad] (see Fig. 9), arises when the linear actuators are driven differentially, just slightly surpassing their limitations.

A. Non-Linear Transmission Analysis

The Ankle Pitch/Roll together with the Knee, form a nonlinear transmission (see Fig. 12) that permits keeping the orientation of the foot constant w.r.t. the Hip when the Knee linear actuator retracts/extends. In contrast with classical leg kinematics, this is achieved without moving the Ankle Pitch/Roll actuators, permitting, for example, in-place jumps by moving only the linear actuator located at the thigh, as shown in Fig. 13. On the contrary, to keep constant the value of the Ankle Pitch joint when moving retracting/extending the leg, it is necessary to actuate the Ankles linear actuators, as shown in Fig. 14.

Finally, Fig. 15 reports the non-linear transformation between Leg Length actuator force to foot sole normal force, while performing a squat motion and applying the maximum force at the actuator (5000 N). It is worth noticing that at the nominal configuration, the foot's sole normal force is between its minimum and maximum, when completely compressed it reaches its minimum, while at half compression, it reaches its maximum. From these plots, we can see that the non-linear transmission has been designed to be able to exert its maximum force when half-squatting, for example during a jump preparation. Being torque and speed inversely related, the non-linear transmission design was optimized to achieve high force when the robot is still standing at a normal height and high speed when the leg is crouched to accelerate for jumping efficiently.



Fig. 11: Joint limits in 2DOFs U-2(RRRP-RRR-UU) Ankle Pitch/Roll. In blue (continuous line) are the real joint limits projected into the differential mechanism, while in red (dashed line) are the joint limits, as box constraints, when not projected.



Fig. 12: In blue the (non-linear) relation between actuated Leg Length linear actuator and rotational knee pitch joint, retracting and extending the leg. The dashed line shows the line fitting the curve.

V. COMPARISON WITH TALOS

In this section, we conduct a comparison with another fullsize humanoid bipedal system, the TALOS robot, developed by PAL Robotics [12]. TALOS features a more traditional serial kinematics design for its legs, encompassing 6 actuated DOFs per leg, with motors distributed along the entire chain. Fig. 16 illustrates a comparison of the lower bodies between TALOS and Kangaroo. Notably, Kangaroo exhibits a lower



Fig. 13: Passive and active DOFs come into play during both the retraction and extension of the leg. The orange line represents the variation in the Ankle pitch joint which preserves the foot's orientation w.r.t. the hip. Notice that this adjustment is achieved with no intervention of the Ankle actuators.



Fig. 14: Passive and active DOFs during retraction and extension of the leg while keeping constant the Knee pitch joint. The violet and yellow lines show that to preserve the joint's position (orange line) throughout the motion, adjustments to the actuated Ankle DOFs are also required.

total mass compared to TALOS, and its CoM is positioned close to the pelvis. In contrast, TALOS has its CoM situated at knee height.

Let's now proceed to compare the equivalent Cartesian inertia matrix at the foott of both the Kangaroo and the TALOS robot. Considering an open-kinematic chain, the equivalent Cartesian inertia matrix *seen* at a certain frame



Fig. 15: Force Transmission from Knee Linear Actuator to Foot Sole. The topmost plot illustrates the normal force at the sole while performing a squat and applying a maximum linear force (5000 N) through the actuator. The maximum output force is depicted in yellow, while the minimum is shown in red. The middle plot displays the transformation ratio. The bottom plot depicts the actuator's position during the squat. Observing the graphs, the highest normal foot force is attained when the legs are half-compressed, whereas the lowest force is experienced when fully compressed.



Fig. 16: TALOS and Kangaroo lower-body comparison.

 \mathcal{F}_l w.r.t. the base frame \mathcal{F}_b can be computed as:

$$\mathbf{\Lambda} = \left(\mathbf{J}\mathbf{M}^{-1}\mathbf{J}^T + \epsilon \mathbf{I}\right)^{-1},\tag{31}$$

with $\Lambda \in \mathbb{R}^{6\times 6}$, J the Jacobian of \mathcal{F}_l in \mathcal{F}_b , M the joint space inertia matrix, $\mathbf{I} \in \mathbb{R}^{6\times 6}$ identity matrix and ϵ a regularisation term used when the Jacobian is near singularities. For serial-parallel hybrid linkages, it is possible to consider the same equation (31) using the projected versions of the Jacobian and inertia matrices (see Appendix II). The equivalent Cartesian inertia matrix at the *left_sole* frame, situated at the sole of the left foot, is calculated with respect to the *base_link* in a standard nominal configuration typically adopted for walking. In this configuration, the legs are nearly fully extended, as depicted in Fig. 17. The comparison



Fig. 17: On the left is the TALOS robot. The parts in evidence are the ones considered for the comparison with the Kangaroo robot, on the right.

involves evaluating the norm of the columns of the matrix Λ . Let Λ_T and Λ_K represent the equivalent Cartesian inertia matrices of TALOS and Kangaroo, respectively, computed at the foot. We denote the *j*-th columns of these matrices as $\lambda_{T,j}$ and $\lambda_{K,j}$. The *index* χ_j is computed as:

$$\chi_j = \frac{\|\boldsymbol{\lambda}_{T,j}\|}{\|\boldsymbol{\lambda}_{K,j}\|},\tag{32}$$

with $\chi_j > 1$ representing an improvement (reduced) equivalent Cartesian inertia along a certain direction.

Taking inspiration from the study conducted in [44], we extend our comparison to encompass the *Centroidal Angular Momentum Matrices* (CAMM) of Kangaroo and TALOS. The CAMM can be computed from the *System Momentum Matrix* (SMM), defined in [45] as the product between the *System Inertia* and *System Jacobian*:

$$\mathbf{A}_S = \mathbf{B}_S \mathbf{J}_S,\tag{33}$$

with $\mathbf{A}_S \in \mathbb{R}^{6N \times n+6}$ the System Momentum, N + 1 is the number of the links of the system, $\mathbf{B}_S \in \mathbb{R}^{6n \times 6n}$ the System Inertia matrix and $\mathbf{J}_S \in \mathbb{R}^{6n \times n+6}$ the System Jacobian (see [45] for details). The SMM in (33) can be projected at the *centroid* of the system such that $\mathbf{A}_G = \mathbf{X}_G^T \mathbf{A}_S$, with $\mathbf{X}_G \in \mathbb{R}^{6N \times 6}$ the stacking of adjoint matrices transforming motions from centroidal coordinates to link coordinates, and $\mathbf{A}_G \in \mathbb{R}^{6 \times n+6}$ defined as the Centroidal Momentum Matrix (CMM), that maps joint velocities into centroidal momentum $\mathbf{h} = \mathbf{A}_G \boldsymbol{\nu} \in \mathbb{R}^6$, with the CAMM the angular part of the CMM. As for the equivalent Cartesian Inertia matrix, it is possible to compute an equivalent CMM projected onto the closed linkage constraints.

We define an index similar to (32) computed at the same nominal configuration in Fig. 17:

$$\gamma_j = \frac{\|\boldsymbol{\alpha}_{T,j}\|}{\|\boldsymbol{\alpha}_{K,j}\|},\tag{34}$$

where $\alpha_{.,j}$ is the j - th column of the CAMM with an improvement (if > 1) in the mass distribution.

TABLE III: Comparison among Kangaroo and Talos

	χ_x	3.6
$\Lambda_{Linear} _{\epsilon=1e^{-5}}$	χ_y	4.1
16-16	χ_z	4.6
	χ_x	11.4
$\Lambda_{Angular} _{\epsilon=1e^{-5}}$	χ_y	4.9
5 16-16	χ_z	3.6
	γ_x	2.6
CAMM	γ_{y}	2.7
	γ_z	2.0

A succinct overview of the findings from this comparison is presented in Table III. The results highlight a substantial enhancement in the Kangaroo platform in comparison to Talos. This improvement pertains to both the equivalent Cartesian inertia observed at the feet and the Centroidal Angular Momentum Matrix (CAMM).

VI. PRELIMINARY EXPERIMENTS WITH FULL-MODEL

This section reports preliminary experiments on the real platform employing the full-model and applying algorithms derived from the constrained modeling presented in Section III. The Kangaroo full-model is constructed following the Unified Robotics Description Format (URDF) guidelines. However, the default URDF parser⁶ disallows joints with identical parent or child links, preventing the direct modeling of closed linkages. In contrast, the Simulation Description Format (SDF)⁷ permits such joints, enabling the simulation of closed linkages via the Gazebo Simulator [46]. For this reason, we designed a configuration YAML file and a novel parser that permits the use of the URDF of Kangaroo with control libraries (e.g. the CartesI/O framework, by [47]) where all the necessary closed-linkage constraints are listed. Finally, we developed a C++ library for the kinematics and dynamics modeling of series-parallel hybrid chains named Closed Linkage Library (CLL) based on RBDL [48] and Eigen [49]. The following experiments have been carried out using the CLL library on an Intel Core i7 CPU 2.30GHz \times 16 cores with 32 GiB RAM, and are presented in the video accompanying this paper.

A. Computation of Passive Kinematics Quantities and Initial Calibration

In general, when modeling serial-parallel hybrid linkages as constrained multi-body systems, it is necessary to have knowledge of the complete state vector denoted by θ and $\dot{\theta}$. However, this information may not be entirely accessible due to some passive quantities being either fully or partially unmeasurable. On the other hand, the actuated quantities typically remain observable. Furthermore, the linear actuators in Kangaroo are based on ball-screws, equipped with absolute encoders at the motor side, permitting a relative measurement of the linear displacement of the screw, hence working as

⁷http://sdformat.org/

linear relative encoders. For this reason, it is necessary to properly offset the initial value of the linear actuators to have a linear absolute reading, at every robot initialization. This calibration is made according to the arrangement of the closed kinematics and the measurements from the absolute encoders mounted on some of the passive DOFs.

In this section, we formulate an algorithm to estimate the passive state, which is fundamental to reconstructing the full state θ and $\dot{\theta}$. This algorithm can serve as well as a systematic procedure for the initial calibration. We consider having an initial estimation of the closed linkages DOF positions θ_k and velocities $\dot{\theta}_k$, and measured actuated positions $\bar{\theta}_a$ and velocities $\dot{\theta}_a$, at instant k. A new estimation for the passive velocities $\dot{\theta}_{u,k+1}$ can be obtained using:

$$\dot{\boldsymbol{\theta}}_{u,k+1} = \mathbf{J}_m\left(\boldsymbol{\theta}_k\right) \dot{\boldsymbol{\theta}}_a,\tag{35}$$

that leads to the estimation of the new DOF velocities:

$$\dot{\boldsymbol{\theta}}_{k+1} = \begin{bmatrix} \dot{\boldsymbol{\theta}}_{u,k+1} \\ \dot{\boldsymbol{\theta}}_a \end{bmatrix}.$$
 (36)

Considering equations (2) and (3), is possible to consider as well a properly computed error $\mathbf{e}_l(\boldsymbol{\theta}) \in \mathbb{R}^m$ associated to the closed linkage:

$$\mathbf{J}_{l}\left(\boldsymbol{\theta}\right)\dot{\boldsymbol{\theta}} = \alpha \mathbf{e}_{l}(\boldsymbol{\theta}),\tag{37}$$

which define the error dynamics $\dot{\mathbf{e}}_l + \alpha \mathbf{e}_l(\boldsymbol{\theta}) = \mathbf{0}$ converging to zero exponentially. Using (6) in (37) leads to the estimator:

$$\dot{\boldsymbol{\theta}}_{u,k+1}^{*} = \mathbf{J}_{m} \left(\boldsymbol{\theta}_{k}\right) \dot{\boldsymbol{\theta}}_{a,k} + \alpha \mathbf{J}_{l,u} \left(\boldsymbol{\theta}_{k}\right)^{-1} \mathbf{e}_{l}(\boldsymbol{\theta}_{k}), \qquad (38)$$

with $\dot{\theta}_{a,k} = \dot{\theta}_a + \beta (\bar{\theta}_a - \theta_{a,k})$, and α and β two scalars, positive, tunable gains.

A new estimation for the positions θ_{k+1} can be obtained using one-step Euler integration:

$$\boldsymbol{\theta}_{k+1} = \boldsymbol{\theta}_k + dt \dot{\boldsymbol{\theta}}_{k+1}^*, \tag{39}$$

given the time step dt and

$$\dot{\boldsymbol{\theta}}_{k+1}^* = \begin{bmatrix} \dot{\boldsymbol{\theta}}_{u,k+1}^* \\ \dot{\boldsymbol{\theta}}_{a,k} \end{bmatrix}.$$
 (40)

The two terms in (38) consist of a correction for the measurement and a correction for the constraint error. Both these terms are used to mitigate the drift phenomena induced by the discrete integration in (39), therefore the values of α and β are bounded by the rate of the estimation loop.

It is possible to define a simple calibration procedure based on the previous algorithm. We consider augmenting the constraint Jacobian with the measurement matrix E:

$$\mathbf{J}\left(\boldsymbol{\theta}\right) = \begin{bmatrix} \mathbf{J}_{l}\left(\boldsymbol{\theta}\right) \\ \mathbf{E} \end{bmatrix},\tag{41}$$

and the constraint error with the measurement error \mathbf{e}_m :

$$\mathbf{e}\left(\boldsymbol{\theta}\right) = \begin{bmatrix} \mathbf{e}_{l}(\boldsymbol{\theta}) \\ \mathbf{e}_{m} \end{bmatrix}.$$
 (42)

In the case of non-redundant measurements, as for the Kangaroo robot, we will have $\mathbf{J}(\boldsymbol{\theta}) \in \mathbb{R}^{n \times n}$ with $\mathbf{E} \in \mathbb{R}^{l \times n}$,

⁶https://github.com/ros/urdfdom

such as l + m = n, and $\mathbf{e}(\boldsymbol{\theta}) \in \mathbb{R}^n$. Each row in the measurement matrix \mathbf{E} is a zero row with a single "1" placed at the measured (passive) DOF. The measured error consists of the error associated with each measurement $e_i = \theta_{j,m} - \theta_j$, with $\mathbf{e}_m = [e_0, e_1, \dots, e_l]^T$. Finally, it is worth noticing that the reference for the closed linkage error $\mathbf{e}_l(\boldsymbol{\theta})$ is computed at a configuration where all the open linkages are properly closed. The calibration procedure is based on the iterative resolution of the linear system $\mathbf{J}(\boldsymbol{\theta}) \dot{\boldsymbol{\theta}} = \alpha \mathbf{e}(\boldsymbol{\theta})$, equivalent to (37), until the norm of the error is above a certain threshold ϵ : $\|\mathbf{e}(\boldsymbol{\theta})\| < \epsilon$. In Fig. 18 is reported the norm of the closed-linkage error, for 6 different initial measurements. It is possible to see that the initialization procedure converges below $1e^{-4}$ global error norm in a few iterations.





Fig. 18: Calibration procedure validation. Notice that the constraint error contains both linear and angular parts.

Concerning the computation of passive kinematics quantities, it is worth noticing that it can be seen as a tracking problem of the measured actuated positions, constrained by the closed linkages. The validation is therefore performed using sinusoidal position and velocity reference (measured) trajectories for the actuated DOFs, at 1 [kHz], that is tracked by the estimator algorithm, running as well at 1 [kHz], with parameters $\alpha = 1$ and $\beta = 1000$. In Fig. 19 are reported the position and velocity tracking between the estimated and measured actuated quantities, and the norm of the error of the closed linkage constraint. The average computation time for the algorithm, together with the update of the full-model, and computation of inverse dynamics torques using equations (28) and (29) without contact forces, is $\approx 0.38 \ [ms]$, using the partialPivLu⁸ method from the Eigen software library, and dense matrices representation. The motion performed in this numerical validation, together with results achieved on the real Kangaroo prototype is



Fig. 19: The top plot reports the tracking of measured (dashed) versus estimated (continuous) position and velocity, respectively, of the actuated DOFs. In the bottom plot the norm of the closed linkage constraint error (linear and angular).

reported in the accompanying video. Fig. 20 shows the kinematic estimation algorithm running on the real robot.



Fig. 20: The kinematic estimation running on the real Kangaroo platform.

B. Quasi-Static Contact Wrench Estimation

As of the current writing of this paper, the Kangaroo robot prototype does not incorporate any direct force/torque sensor measurements at its feet. By not housing electronics from the feet to the thigh, the robot becomes less vulnerable to impacts. However, in light of this design approach, there arises a need to estimate contact wrenches based on the force measurements of the actuators, all while considering the serial-parallel hybrid kinematics.

Assuming $\nu = 0$ and $\dot{\nu} = 0$, equations (28) and (29) become respectively:

$$\boldsymbol{\lambda} = \mathbf{J}_{l,u}(\mathbf{q})^{-T} \left(\mathbf{g}_u(\mathbf{q}) - \mathbf{J}_{c,u}(\mathbf{q})^T \mathbf{F} \right), \qquad (43a)$$

$$\boldsymbol{\tau} = \mathbf{g}_a(\mathbf{q}) - \mathbf{J}_{c,a}(\mathbf{q})^T \mathbf{F} - \mathbf{J}_{l,a}(\mathbf{q})^T \boldsymbol{\lambda}.$$
(43b)

⁸https://eigen.tuxfamily.org/dox/classEigen_1_ 1PartialPivLU.html

Substituting (43a) in (43b), leads to:

$$\boldsymbol{\tau} = \mathbf{g}_{a}(\mathbf{q}) - \mathbf{J}_{c,a}(\mathbf{q})^{T}\mathbf{F} + \mathbf{J}_{m}(\mathbf{q})^{T}\left(\mathbf{g}_{u}(\mathbf{q}) - \mathbf{J}_{c,u}(\mathbf{q})^{T}\mathbf{F}\right),$$
(44)

where we have used (18). Reorganizing leads to:

$$\boldsymbol{\tau} - \mathbf{g}_{a}(\mathbf{q}) - \mathbf{J}_{m}(\mathbf{q})^{T} \mathbf{g}_{u}(\mathbf{q}) =$$

$$= -\mathbf{J}_{c,a}(\mathbf{q})^{T} \mathbf{F} - \mathbf{J}_{m}(\mathbf{q})^{T} \mathbf{J}_{c,u}(\mathbf{q})^{T} \mathbf{F} =$$

$$= -\left(\mathbf{J}_{c,a}(\mathbf{q})^{T} + \mathbf{J}_{m}(\mathbf{q})^{T} \mathbf{J}_{c,u}(\mathbf{q})^{T}\right) \mathbf{F} =$$

$$= \mathbf{J}_{c,m}(\mathbf{q})^{T} \mathbf{F}, \quad (45)$$

that can be used to statically estimate contact forces given measured force/torques $\bar{\tau}$:

$$\left(\mathbf{J}_{c,m}(\mathbf{q})^{T}\right)^{\dagger}\left(\bar{\boldsymbol{\tau}}-\mathbf{g}_{a}(\mathbf{q})-\mathbf{J}_{m}\left(\mathbf{q}\right)^{T}\mathbf{g}_{u}(\mathbf{q})\right)=\mathbf{F},\quad(46)$$

with $\mathbf{J}_{c,m}(\mathbf{q}) \in \mathbb{R}^{3c \times a}$ and $(\cdot)^{\dagger}$ a properly computed pseudo-inverse. Notice that velocity measurements could be included to consider the effect of Coriolis/centrifugal terms. From the wrench estimation, we compute the Zero Moment Point (ZMP) being a fundamental quantity used in bipedal locomotion for stabilization, see [50], [51].

We conclude by showcasing an experiment involving the robot executing lateral swinging motions (see Fig. 21). In this experiment, although the motion of each sub-mechanism is individually controlled through geometric calculations, we utilize the full model for estimating the contact wrenches. The estimation of the axial force $\bar{\tau}$ in each linear actuator is computed from the measured motor torque τ_m as:

$$\bar{\tau} = \frac{\tau_m \cdot 2\pi \cdot \eta}{L},\tag{47}$$

with $\eta = 0.95$ the efficiency of the ball screw provided by the constructor, and L the screw lead, that for the leg length actuator is 0.1 [m] and for the hip and ankle actuators is 0.05 [m]. Fig. 22a reports the reconstructed contact wrench filtered using a low-pass second-order Butterworth filter with cuttoff frequency 5 [Hz]. It is possible to see that the estimated tangential forces correctly sum up at the weight of the robot without the cage, approximately 36 [kg]. In Fig. 22b are reported the computed ZMP and feet Centers of Pressure (COPs) using the estimated contact wrenches. The contact wrench algorithms estimation time is ≈ 0.03 [ms], using the fullPivLu⁹ method from the Eigen library, and dense matrices representation, allowing for fast real-time implementation.

VII. CONCLUSION AND FUTURE WORKS

In this paper, we presented the kinematic and dynamic modeling, and study of the lower body of Kangaroo, a novel humanoid bipedal robot designed and manufactured by PAL Robotics. Distinguished by its incorporation of seriesparallel hybrid chains, Kangaroo's unique design prompted us to delve into constraint-based modeling techniques for serial-parallel hybrid systems We conducted an in-depth



Fig. 21: In the top picture, the Kangaroo robot demonstrates lateral swing motions. In the bottom picture, RVIZ visualizes the estimated full configuration and contact wrenches.

analysis of Kangaroo's kinematics, such as the non-linear transmission mechanisms within the knee and differential parallel components governing hip and ankle movements. Furthermore, we computed the equivalent Cartesian inertia at the end-effector and the Centroidal Momentum matrix in the context of serial-parallel hybrid chains. This involved a comparative analysis with the Talos robot, a preceding humanoid creation from PAL Robotics. Our examination underscored the advancements achieved in Kangaroo's design. To facilitate our research, we introduced a suite of software tools designed to model, analyze, and control both fixed and floating-base robots housing serial-parallel hybrid chains as constrained multi-body systems. An example of our software contributions is the Closed Linkage Library, that facilitated the analyses and preliminary real-world experiments undertaken with the Kangaroo platform.

Future works will regard the deployment of Whole-Body Inverse Dynamics on the Kangaroo platform together with model predictive control algorithms to perform highly agile and dynamic motions, such as running and jumping. In particular, we envision the employment of a simplified, serial model of the robot for the model predictive control, i.e. considering only significative passive DOFs, and mapping the control input onto the full model using the IK and ID presented in this paper. We are also investigating the extension of passivity-based approaches, see [52], for Operational

⁹https://eigen.tuxfamily.org/dox/classEigen_1_ 1FullPivLU.html



Fig. 22: Results of the estimated contact wrenches using data from lateral swing experiment with the Kangaroo hardware.

Space Control to series-parallel hybrid chains. Finally, the tools developed in this work will be the basis for the design and development of the upper body for the Kangaroo robot, to evaluate different kinematic structures for the torso and the arms.

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APPENDIX I

COMPUTATION OF RELATIVE KINEMATICS

Considering three frames, denoted as w, a, and b as depicted in Fig. 23, our interest is in calculating relative kinematic values associated with frame b w.r.t. frame a, expressed in frame a, and using kinematics quantities of a and b expressed in frame w.



Fig. 23: Frames and notation.

According to Fig. 23, we denote with ${}^{w}\mathbf{p}_{a}$ and ${}^{w}\mathbf{p}_{b}$ the position of the frames a and b, respectively, w.r.t. the frame w. The position of frame a w.r.t. frame b expressed in frame w is denoted with ${}^{w}\mathbf{p}_{a,b} = {}^{w}\mathbf{p}_{b} - {}^{w}\mathbf{p}_{a}$. As previously stated, we consider known the following kinematics quantities relative to the frames a and b: ${}^{w}\mathbf{v}., {}^{w}\boldsymbol{\omega}., {}^{w}\mathbf{J}., {}^{w}\mathbf{a}., {}^{w}\boldsymbol{\omega}., \text{ and } {}^{w}\mathbf{J}.\mathbf{\dot{q}}$, respectively, the linear and angular velocities, Jacobian, linear and angular accelerations, and Cartesian acceleration bias, expressed in frame w. In the Jacobians, we denote with the superscript " \rightarrow " the *linear* part, while with " \angle " the *angular* part.

A. Relative Velocity and Jacobian

We first compute the relative linear and angular velocities of the frame b w.r.t. the frame a expressed in frame a, denoted respectively with ${}^{a}\mathbf{v}_{a,b}$ and ${}^{a}\boldsymbol{\omega}_{a,b}$:

$$\begin{bmatrix} {}^{a}\mathbf{v}_{a,b} \\ {}^{a}\boldsymbol{\omega}_{a,b} \end{bmatrix} = \begin{bmatrix} {}^{a}\mathbf{R}_{w} & \mathbf{0} \\ \mathbf{0} & {}^{a}\mathbf{R}_{w} \end{bmatrix} \begin{bmatrix} {}^{w}\mathbf{v}_{b} \\ {}^{w}\boldsymbol{\omega}_{b} \end{bmatrix} - \mathbf{A} \begin{bmatrix} {}^{w}\mathbf{v}_{a} \\ {}^{w}\boldsymbol{\omega}_{a} \end{bmatrix}, \quad (48)$$

with ${}^{a}\mathbf{R}_{w}$ the rotation matrix from a to w such that ${}^{a}\mathbf{R}_{w}{}^{w}\mathbf{v}_{a} = {}^{a}\mathbf{v}_{a}$ and ${}^{w}\mathbf{p}_{a,b_{[\times]}}$ the skew-symmetric matrix from ${}^{w}\mathbf{p}_{a,b}$ and:

$$\mathbf{A} = \begin{bmatrix} {}^{a}\mathbf{R}_{w} & -{}^{a}\mathbf{R}_{w}{}^{w}\mathbf{p}_{a,b_{[\times]}} \\ \mathbf{0} & {}^{a}\mathbf{R}_{w} \end{bmatrix}.$$
 (49)

We can further expand (48) in its linear and angular parts:

$${}^{a}\mathbf{v}_{a,b} = {}^{a}\mathbf{R}_{w}{}^{w}\mathbf{v}_{b} - {}^{a}\mathbf{R}_{w}{}^{w}\mathbf{v}_{a} + {}^{a}\mathbf{R}_{w}\left({}^{w}\mathbf{p}_{a,b} \times {}^{w}\boldsymbol{\omega}_{a}\right) =$$
$$= {}^{a}\mathbf{R}_{w}\left({}^{w}\mathbf{v}_{b} - {}^{w}\mathbf{v}_{a} - {}^{w}\boldsymbol{\omega}_{a} \times {}^{w}\mathbf{p}_{a,b}\right),$$
(50a)

$${}^{a}\boldsymbol{\omega}_{a,b} = {}^{a}\mathbf{R}_{w}{}^{w}\boldsymbol{\omega}_{b} - {}^{a}\mathbf{R}_{w}{}^{w}\boldsymbol{\omega}_{a} = {}^{a}\mathbf{R}_{w}\left({}^{w}\boldsymbol{\omega}_{b} - {}^{w}\boldsymbol{\omega}_{a}\right) =$$
$$= {}^{a}\mathbf{R}_{w}{}^{w}\boldsymbol{\omega}_{a,b}.$$
(50b)

From equation (48) is finally possible to extract the relative Jacobian ${}^{a}\mathbf{J}_{a,b}$:

$${}^{a}\mathbf{J}_{a,b} = \begin{bmatrix} {}^{a}\mathbf{R}_{w} & \mathbf{0} \\ \mathbf{0} & {}^{a}\mathbf{R}_{w} \end{bmatrix} {}^{w}\mathbf{J}_{b} - \begin{bmatrix} {}^{a}\mathbf{R}_{w} & {}^{-a}\mathbf{R}_{w} {}^{w}\mathbf{p}_{a,b_{[\times]}} \\ \mathbf{0} & {}^{a}\mathbf{R}_{w} \end{bmatrix} {}^{w}\mathbf{J}_{a}.$$
(51)

B. Relative Acceleration and Acceleration Bias

We want now to compute the relative linear and angular accelerations of the frame *b* w.r.t. the frame *a* expressed in frame *a*, denoted respectively with ${}^{a}\mathbf{a}_{a,b}$ and ${}^{a}\dot{\boldsymbol{\omega}}_{a,b}$.

Let begin with the relative angular acceleration obtained deriving equation (50b) w.r.t. time:

$${}^{a}\dot{\boldsymbol{\omega}}_{a,b} = {}^{a}\dot{\mathbf{R}}_{w}{}^{w}\boldsymbol{\omega}_{a,b} + {}^{a}\mathbf{R}_{w}{}^{w}\dot{\boldsymbol{\omega}}_{a,b}.$$
(52)

Notice that the derivative of the rotation matrix can be written as:

$${}^{a}\dot{\mathbf{R}}_{w} = -{}^{a}\mathbf{R}_{w}{}^{w}\boldsymbol{\omega}_{a[\times]},\tag{53}$$

that substituted in (52) gives the expression of the relative angular acceleration ${}^{a}\dot{\omega}_{a,b}$:

$${}^{a}\dot{\boldsymbol{\omega}}_{a,b} = {}^{a}\dot{\mathbf{R}}_{w}{}^{w}\boldsymbol{\omega}_{a,b} + {}^{a}\mathbf{R}_{w}{}^{w}\dot{\boldsymbol{\omega}}_{a,b} =$$

$$= -{}^{a}\mathbf{R}_{w}{}^{w}\boldsymbol{\omega}_{a} \times {}^{w}\boldsymbol{\omega}_{a,b} + {}^{a}\mathbf{R}_{w}{}^{w}\dot{\boldsymbol{\omega}}_{a,b} =$$

$$= {}^{a}\mathbf{R}_{w}\left({}^{w}\dot{\boldsymbol{\omega}}_{a,b} - {}^{w}\boldsymbol{\omega}_{a} \times {}^{w}\boldsymbol{\omega}_{a,b}\right) =$$

$$= {}^{a}\mathbf{R}_{w}\left({}^{w}\dot{\boldsymbol{\omega}}_{a,b} - {}^{w}\boldsymbol{\omega}_{a} \times {}^{w}\boldsymbol{\omega}_{b} + \underline{{}^{w}\boldsymbol{\omega}_{a}} \times {}^{w}\boldsymbol{\omega}_{a}\right) =$$

$$= {}^{a}\mathbf{R}_{w}\left({}^{w}\dot{\boldsymbol{\omega}}_{b,b} - {}^{w}\dot{\boldsymbol{\omega}}_{a} - {}^{w}\boldsymbol{\omega}_{a} \times {}^{w}\boldsymbol{\omega}_{b}\right). \tag{54}$$

From the expression (54) we can compute the angular part of the relative Cartesian acceleration bias ${}^{a}\dot{\mathbf{J}}_{a,b}^{\angle}\dot{\mathbf{q}}$, reported in equation (55).

Concerning the relative linear acceleration, its expression is reported in equation (56), obtained deriving equation (50a) w.r.t. time and using ${}^{w}\hat{\mathbf{a}}_{a,b} = {}^{w}\mathbf{a}_{b} - {}^{w}\mathbf{a}_{a}$ and ${}^{w}\hat{\mathbf{v}}_{a,b} = {}^{w}\mathbf{v}_{b} - {}^{w}\mathbf{v}_{a}$. Finally, from expression (56) we can compute the linear part of the relative Cartesian acceleration bias ${}^{a}\dot{\mathbf{j}}_{a,b}\dot{\mathbf{q}}$, reported in equation (57) with $\mathbf{b} = -2{}^{w}\omega_{a} \times {}^{w}\hat{\mathbf{v}}_{a,b} + {}^{w}\omega_{a} \times {}^{w}\omega_{a} \times {}^{w}\omega_{a}$.

APPENDIX II PROJECTED FLOATING-BASE DYNAMICS

This appendix presents the derivation of the floating-base dynamics projected into the closed linkage constraints. In particular, the final expression will linearly depend only on the actuated accelerations θ_a , base accelerations $[\mathbf{p}^T, \boldsymbol{\rho}^T]^T$, and contact forces **F**.

We first introduce the quantity $\mathbf{q}_a = [\mathbf{p}^T, \boldsymbol{\rho}^T, \boldsymbol{\theta}_a^T]^T$ and corresponding derivatives computed as for (24). Furthermore, we consider the quantities \mathbf{q} and $\boldsymbol{\nu}$ as known. We want to compute the equations of motions in the actuated accelerations $\dot{\boldsymbol{\nu}}_a$. With these concerns, we can rewrite the floating-base part of the floating-base dynamics (26a) as:

$$\mathbf{M}_{b,b}(\mathbf{q}) \begin{bmatrix} \ddot{\mathbf{p}} \\ \dot{\boldsymbol{\omega}} \end{bmatrix} + \mathbf{M}_{b,u}(\mathbf{q}) \ddot{\boldsymbol{\theta}}_u + \mathbf{M}_{b,a}(\mathbf{q}) \ddot{\boldsymbol{\theta}}_a + \mathbf{h}_b(\mathbf{q}, \boldsymbol{\nu}) = \mathbf{J}_{c,b}(\mathbf{q})^T \mathbf{F}$$
(58)

with $\mathbf{M}_{b,b}(\mathbf{q}) \in \mathbb{R}^{6 \times 6}$, $\mathbf{M}_{b,u}(\mathbf{q}) \in \mathbb{R}^{6 \times m}$ and $\mathbf{M}_{b,a}(\mathbf{q}) \in \mathbb{R}^{6 \times n-m}$. Substituting equation (10) and reorganizing leads to the floating-base part of the projected dynamics:

$$\mathbf{M}_{b,m}(\mathbf{q})\dot{\boldsymbol{\nu}}_a + \mathbf{h}_{b,m}(\mathbf{q},\boldsymbol{\nu}) = \mathbf{J}_{c,b}(\mathbf{q})^T \mathbf{F}, \qquad (59)$$

with:

$$\mathbf{M}_{b,m}(\mathbf{q}) = \begin{bmatrix} \mathbf{M}_{b,b}(\mathbf{q}) & \mathbf{M}_{b,u}(\mathbf{q})\mathbf{J}_m(\mathbf{q}) + \mathbf{M}_{b,a}(\mathbf{q}) \end{bmatrix},$$
(60a)
$$\mathbf{h}_{b,m}(\mathbf{q},\boldsymbol{\nu}) = \mathbf{h}_b(\mathbf{q},\boldsymbol{\nu}) - \mathbf{M}_{b,u}(\mathbf{q})\mathbf{J}_{l,u}(\mathbf{q})^{-1}\dot{\mathbf{J}}_l(\mathbf{q},\boldsymbol{\nu})\boldsymbol{\nu},$$
(60b)

where $\mathbf{M}_{b,m}(\mathbf{q}) \in \mathbb{R}^{6 \times n - m + 6}$ and $\mathbf{h}_{b,m}(\mathbf{q}, \boldsymbol{\nu}) \in \mathbb{R}^{6}$.

Given a generic task in operational space at the acceleration level:

$$\mathbf{J}_t(\mathbf{q})\dot{\boldsymbol{\nu}} + \dot{\mathbf{J}}_t(\mathbf{q}, \boldsymbol{\nu})\boldsymbol{\nu} = \mathbf{a}_d, \tag{61}$$

also, this can be expressed only using the actuated acceleration:

$$\mathbf{J}_{t,m}(\mathbf{q})\dot{\boldsymbol{\nu}}_a + \dot{\mathbf{J}}_{t,m}(\mathbf{q},\boldsymbol{\nu})\boldsymbol{\nu} = \mathbf{a}_d, \tag{62}$$

with:

$$\mathbf{J}_{t,m}(\mathbf{q}) = \begin{bmatrix} \mathbf{J}_{t,b}(\mathbf{q}) & \mathbf{J}_{t,a}(\mathbf{q}) + \mathbf{J}_{t,u}(\mathbf{q})\mathbf{J}_m(\mathbf{q}) \end{bmatrix},$$
(63a)

$$\dot{\mathbf{J}}_{t,m}(\mathbf{q},\boldsymbol{\nu})\boldsymbol{\nu} = \dot{\mathbf{J}}_t(\mathbf{q},\boldsymbol{\nu})\boldsymbol{\nu} - \mathbf{J}_{t,u}\mathbf{J}_{l,u}(\mathbf{q})^{-1}\dot{\mathbf{J}}_l(\mathbf{q},\boldsymbol{\nu})\boldsymbol{\nu}$$
(63b)

Equations (59) and (62) permits to rewrite the QP problem in (27) only in the actuated and base accelerations, and contact forces.

In the same way, it is possible to compute the Lagrange multipliers λ only in the actuated accelerations:

$$\boldsymbol{\lambda} = \mathbf{J}_{l,u}(\mathbf{q})^{-T} \left(\mathbf{M}_{u,m}(\mathbf{q}) \dot{\boldsymbol{\nu}}_a + \mathbf{h}_{u,m}(\mathbf{q}, \boldsymbol{\nu}) - \mathbf{J}_{c,u}(\mathbf{q})^T \mathbf{F} \right),$$
(64)

with:

$$\mathbf{M}_{u,m}(\mathbf{q}) = \begin{bmatrix} \mathbf{M}_{u,b}(\mathbf{q}) & \mathbf{M}_{u,u}(\mathbf{q})\mathbf{J}_m(\mathbf{q}) + \mathbf{M}_{u,a}(\mathbf{q}) \end{bmatrix},$$
(65a)

$$\mathbf{h}_{u,m}(\mathbf{q},\boldsymbol{\nu}) = \mathbf{h}_u(\mathbf{q},\boldsymbol{\nu}) - \mathbf{M}_{u,u}(\mathbf{q})\mathbf{J}_{l,u}(\mathbf{q})^{-1}\dot{\mathbf{J}}_l(\mathbf{q},\boldsymbol{\nu})\boldsymbol{\nu},$$
(65b)

$${}^{a}\dot{\boldsymbol{\omega}}_{a,b} = {}^{a}\mathbf{R}_{w}\left({}^{w}\dot{\boldsymbol{\omega}}_{b} - {}^{w}\dot{\boldsymbol{\omega}}_{a} - {}^{w}\boldsymbol{\omega}_{a} \times {}^{w}\boldsymbol{\omega}_{b}\right) =$$

$$= {}^{a}\mathbf{R}_{w}\left({}^{w}\mathbf{J}_{b}^{\angle}\ddot{\mathbf{q}} + {}^{w}\dot{\mathbf{J}}_{b}^{\angle}\dot{\mathbf{q}} - {}^{w}\mathbf{J}_{a}^{\angle}\ddot{\mathbf{q}} - {}^{w}\boldsymbol{\omega}_{a} \times {}^{w}\boldsymbol{\omega}_{b}\right) =$$

$$= {}^{a}\mathbf{R}_{w}\left({}^{w}\mathbf{J}_{b}^{\angle} - {}^{w}\mathbf{J}_{a}^{\angle}\right)\ddot{\mathbf{q}} + {}^{a}\mathbf{R}_{w}\left({}^{w}\dot{\mathbf{J}}_{b}^{\angle}\dot{\mathbf{q}} - {}^{w}\dot{\mathbf{\omega}}_{a} \times {}^{w}\boldsymbol{\omega}_{b}\right) =$$

$$= {}^{a}\mathbf{J}_{a,b}^{\angle}\ddot{\mathbf{q}} + {}^{a}\dot{\mathbf{J}}_{a,b}^{\angle}\dot{\mathbf{q}},$$
(55)

$${}^{a}\mathbf{a}_{a,b} = {}^{a}\dot{\mathbf{R}}_{w}{}^{w}\mathbf{v}_{a,b} + {}^{a}\mathbf{R}_{w}{}^{w}\mathbf{a}_{a,b} - {}^{a}\dot{\mathbf{R}}_{w}\left({}^{w}\boldsymbol{\omega}_{a} \times {}^{w}\mathbf{p}_{a,b}\right) - {}^{a}\mathbf{R}_{w}\left({}^{w}\dot{\boldsymbol{\omega}}_{a} \times {}^{w}\mathbf{p}_{a,b}\right) - {}^{a}\mathbf{R}_{w}\left({}^{w}\boldsymbol{\omega}_{a} \times {}^{w}\mathbf{v}_{a,b}\right) =$$

$$= -{}^{a}\mathbf{R}_{w}{}^{w}\boldsymbol{\omega}_{a} \times {}^{w}\mathbf{v}_{a,b} + {}^{a}\mathbf{R}_{w}{}^{w}\mathbf{a}_{a,b} + {}^{a}\mathbf{R}_{w}{}^{w}\boldsymbol{\omega}_{a} \times \left({}^{w}\boldsymbol{\omega}_{a} \times {}^{w}\mathbf{p}_{a,b}\right) - {}^{a}\mathbf{R}_{w}\left({}^{w}\dot{\boldsymbol{\omega}}_{a} \times {}^{w}\mathbf{p}_{a,b}\right) - {}^{a}\mathbf{R}_{w}\left({}^{w}\dot{\boldsymbol{\omega}}_{a} \times {}^{w}\mathbf{p}_{a,b}\right) - {}^{a}\mathbf{R}_{w}\left({}^{w}\boldsymbol{\omega}_{a} \times {}^{w}\mathbf{p}_{a,b}\right) - {}$$

$${}^{a}\mathbf{a}_{a,b} = {}^{a}\mathbf{R}_{w}\left({}^{w}\mathbf{a}_{b} - {}^{w}\mathbf{a}_{a} - {}^{w}\dot{\boldsymbol{\omega}}_{a} \times {}^{w}\mathbf{p}_{a,b} + \mathbf{b}\right) =$$

$$= {}^{a}\mathbf{R}_{w}\left[{}^{w}\mathbf{J}_{b}^{\rightarrow}\ddot{\mathbf{q}} + {}^{w}\dot{\mathbf{J}}_{b}^{\rightarrow}\dot{\mathbf{q}} - {}^{w}\mathbf{J}_{a}^{\rightarrow}\ddot{\mathbf{q}} - {}^{w}\dot{\mathbf{J}}_{a}^{\rightarrow}\dot{\mathbf{q}} - \left({}^{w}\mathbf{J}_{a}^{\measuredangle}\ddot{\mathbf{q}} + {}^{w}\dot{\mathbf{J}}_{a}^{\measuredangle}\dot{\mathbf{q}}\right) \times {}^{w}\mathbf{p}_{a,b} + \mathbf{b}\right] =$$

$$= {}^{a}\mathbf{R}_{w}\left({}^{w}\mathbf{J}_{b}^{\rightarrow} - {}^{w}\mathbf{J}_{a}^{\rightarrow} + {}^{w}\mathbf{p}_{a,b} \times {}^{w}\mathbf{J}_{a}^{\measuredangle}\right)\ddot{\mathbf{q}} + {}^{a}\mathbf{R}_{w}\left({}^{w}\dot{\mathbf{J}}_{b}^{\rightarrow}\dot{\mathbf{q}} - {}^{w}\dot{\mathbf{J}}_{a}^{\rightarrow}\dot{\mathbf{q}} - {}^{w}\dot{\mathbf{J}}_{a}^{\rightarrow}\dot{\mathbf{q}} \times {}^{w}\mathbf{p}_{a,b} + \mathbf{b}\right) =$$

$$= {}^{a}\mathbf{J}_{a,b}^{\rightarrow}\ddot{\mathbf{q}} + {}^{a}\dot{\mathbf{J}}_{a,b}^{\rightarrow}\dot{\mathbf{q}}.$$
(57)

where $\mathbf{M}_{u,b}(\mathbf{q}) \in \mathbb{R}^{m \times 6}$, $\mathbf{M}_{u,u}(\mathbf{q}) \in \mathbb{R}^{m \times m}$, $\mathbf{M}_{u,a}(\mathbf{q}) \in \mathbb{R}^{m \times n-m}$, and finally the actuated torques $\boldsymbol{\tau}$:

$$\boldsymbol{\tau} = \mathbf{M}_{a,m}(\mathbf{q})\dot{\boldsymbol{\nu}}_a + \mathbf{h}_{a,m}(\mathbf{q},\boldsymbol{\nu}) - \mathbf{J}_{c,a}(\mathbf{q})^T \mathbf{F} - \mathbf{J}_{l,a}(\mathbf{q})^T \boldsymbol{\lambda}.$$
(66)

with:

$$\begin{split} \mathbf{M}_{a,m}(\mathbf{q}) &= \begin{bmatrix} \mathbf{M}_{a,b}(\mathbf{q}) & \mathbf{M}_{a,u}(\mathbf{q})\mathbf{J}_m(\mathbf{q}) + \mathbf{M}_{a,a}(\mathbf{q}) \end{bmatrix}, \\ \mathbf{h}_{a,m}(\mathbf{q},\boldsymbol{\nu}) &= \mathbf{h}_a(\mathbf{q},\boldsymbol{\nu}) - \mathbf{M}_{a,u}(\mathbf{q})\mathbf{J}_{l,u}(\mathbf{q})^{-1}\dot{\mathbf{J}}_l(\mathbf{q},\boldsymbol{\nu})\boldsymbol{\nu}, \\ (67b) \end{split}$$

where $\mathbf{M}_{a,b}(\mathbf{q}) \in \mathbb{R}^{n-m \times 6}$, $\mathbf{M}_{a,a}(\mathbf{q}) \in \mathbb{R}^{n-m \times n-m}$, $\mathbf{M}_{a,u}(\mathbf{q}) = \mathbf{M}_{u,a}(\mathbf{q})^T \in \mathbb{R}^{n-m \times m}$.

The joint space part of the projected dynamics can be rewritten as:

$$\mathbf{M}_{j,m}(\mathbf{q})\dot{\boldsymbol{\nu}}_a + \mathbf{h}_{j,m}(\mathbf{q},\boldsymbol{\nu}) = \boldsymbol{\tau} + \mathbf{J}_{c,m,j}(\mathbf{q})^T \mathbf{F}, \quad (68)$$

with $\mathbf{M}_{j,m}(\mathbf{q}) \in \mathbb{R}^{n-m \times n-m}$: $\mathbf{M}_{j,m}(\mathbf{q}) = [\mathbf{M}_{j,m}, \mathbf{h}(\mathbf{q})]$

$$\mathbf{M}_{j,m}(\mathbf{q}) = \begin{bmatrix} \mathbf{M}_{j,m,b}(\mathbf{q}) & \mathbf{M}_{j,m,j}(\mathbf{q}) \end{bmatrix},$$
(69a)

$$\mathbf{M}_{j,m,b}(\mathbf{q}) = \mathbf{M}_{a,b}(\mathbf{q}) + \mathbf{J}_m^T(\mathbf{q})\mathbf{M}_{u,b}(\mathbf{q}), \tag{69b}$$

$$\mathbf{M}_{j,m,j}(\mathbf{q}) = \mathbf{M}_{a,a}(\mathbf{q}) + \mathbf{J}_m^T(\mathbf{q})\mathbf{M}_{u,u}(\mathbf{q})\mathbf{J}_m(\mathbf{q}) + \mathbf{M}_{a,u}(\mathbf{q})\mathbf{J}_m(\mathbf{q}) + \mathbf{J}_m(\mathbf{q})^T\mathbf{M}_{u,a}(\mathbf{q}), \quad (69c)$$

and $\mathbf{h}_{j,m}(\mathbf{q}, \boldsymbol{\nu}) \in \mathbb{R}^{n-m}$:

$$\mathbf{h}_{j,m}(\mathbf{q},\boldsymbol{\nu}) = \mathbf{h}_{a}(\mathbf{q},\boldsymbol{\nu}) + \mathbf{J}_{m}(\mathbf{q})^{T}\mathbf{h}_{u}(\mathbf{q},\boldsymbol{\nu}) - \\ - \mathbf{M}_{a,u}(\mathbf{q})\mathbf{J}_{l,u}(\mathbf{q})^{-1}\dot{\mathbf{J}}_{l}(\mathbf{q},\boldsymbol{\nu})\boldsymbol{\nu} - \\ - \mathbf{J}_{m}(\mathbf{q})^{T}\mathbf{M}_{u,u}(\mathbf{q})\mathbf{J}_{l,u}(\mathbf{q})^{-1}\dot{\mathbf{J}}_{l}(\mathbf{q},\boldsymbol{\nu})\boldsymbol{\nu}, \quad (70)$$

and $\mathbf{J}_{c,m,j}(\mathbf{q}) = \mathbf{J}_{c,a}(\mathbf{q}) + \mathbf{J}_{c,u}(\mathbf{q})\mathbf{J}_m(\mathbf{q})$, the joint part of the contact Jacobian projected as in (63a).