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ONE-CYCLE SWEEPOUT ESTIMATES OF ESSENTIAL SURFACES IN CLOSED RIEMANNIAN MANIFOLDS

STÉPHANE SABOURAU

ABSTRACT. We present new-curvature one-cycle sweepout estimates in Riemannian geometry, both on surfaces and in higher dimension. More precisely, we derive upper bounds on the length of one-parameter families of one-cycles sweeping out essential surfaces in closed Riemannian manifolds. In particular, we show that there exists a homotopically substantial one-cycle sweepout of the essential sphere in the complex projective space, endowed with an arbitrary Riemannian metric, whose one-cycle length is bounded in terms of the volume (or diameter) of the manifold. This is the first estimate on sweepout volume in higher dimension without curvature assumption. We also give a detailed account of the situation for compact Riemannian surfaces with or without boundary, in relation with questions raised by P. Buser and L. Guth.

1. INTRODUCTION

At the end of his essay on systolic geometry [Gr96, §4.A.7], M. Gromov suggested to consider min-max volumes over families of cycles (or sweepouts) satisfying nontrivial topological conditions as generalizations of the notion of systole. These min-max volumes – known under different names, *e.g.*, waist, width, diastole, slicing – have recently been studied in numerous works, *cf.* [Gr03], [Gu09], [BS10], [L16], [S17], [GL17], [LZ18], [LMN] for a selection of references focusing on volume comparison inequalities, and have led to several applications including major breakthroughs in Riemannian geometry, *cf.* [MN14], [MN17]. Keeping in mind the analogy with systolic geometry, a natural question to ask is whether the different widths of a closed Riemannian manifold provide universal lower bounds on its volume (see [Gu10] for a general discussion). Occurrences of such lower bounds are referred to as width inequalities or sweepout estimates.

A prototype of a sweepout estimate is given by the following curvature-free geometric inequality, *cf.* [BS10].

Theorem 1.1 ([BS10]). *Every closed Riemannian surface M of genus g satisfies*

$$\inf_f \sup_{t \in \mathbb{R}} \text{length } f^{-1}(t) \leq C \sqrt{g+1} \sqrt{\text{area}(M)} \quad (1.1)$$

where $f : M \rightarrow \mathbb{R}$ runs over all continuous (Morse) function and C is an explicit universal constant.

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In this inequality, the fibers $f^{-1}(t)$ define a homotopically nontrivial one-cycle sweepout of the surface and the min-max length of the left-hand side of (1.1) represents the 1-width of M . Note that the rough dependence on the genus in this inequality, *i.e.*, $\sqrt{g+1}$, is optimal and differs from the optimal one, equal to $\frac{\log(g)}{\sqrt{g}}$, appearing in systolic inequalities, *cf.* [Gr83],[Gr96].

If we insist on the fibers to be connected, we have to consider continuous map $f : M \rightarrow \Gamma$ to graphs. Of course, the inequality (1.1) still holds, but it might be possible to improve the multiplicative constant. For instance, L. Guth asked in his ICM article [Gu10] if every closed Riemannian surface M satisfies

$$\inf_f \sup_{t \in \Gamma} \text{length } f^{-1}(t) \leq C \sqrt{\text{area}(M)} \quad (1.2)$$

where $f : M \rightarrow \Gamma$ runs over all continuous function to some graph Γ and C is a universal constant. This is related to an old question of P. Buser about the sharp value of the Bers constant for pants decomposition of hyperbolic surfaces.

In higher dimension, one-cycle sweepout estimates have been obtained for closed 3-manifolds with positive Ricci curvature, *cf.* [LZ18]. But the question remains open without curvature assumption (see [Gu10] for specific questions about one-cycle sweepout estimates on 3-manifolds). Similar “hypersurface” sweepout estimates have also been established for closed manifolds with nonnegative Ricci curvature, *cf.* [S17], [GL17]. Contrarily to the surface case, “hypersurface” sweepout estimates do not exist without restriction on the Riemannian metrics, even on n -spheres with $n \geq 3$, *cf.* [PS].

In this article, we present new curvature-free one-cycle sweepout estimates both on surfaces and in higher dimensions. More precisely, we derive upper bounds on the length of one-parameter families of one-cycles sweeping out essential surfaces in closed manifolds. A motivating example is given by the complex projective space $\mathbb{C}P^n$, endowed with an arbitrary Riemannian metric, along with its essential sphere $\mathbb{C}P^1$. By analogy with Theorem 1.1, we can ask whether there exists a homotopically substantial one-cycle sweepout of $\mathbb{C}P^n$ (more precisely, a one-parameter family of one-cycles of $\mathbb{C}P^n$ sweeping out $\mathbb{C}P^1$) whose one-cycle length is at most $\sim \text{vol}(\mathbb{C}P^n)^{\frac{1}{2n}}$.

Before stating our main results, we need to be more precise and define the k -width of a closed n -manifold M as there are many notions of width depending on the kind of sweepouts we look at. A first natural choice would be to consider the k -dimensional fibers of the continuous maps from M to the $(n-k)$ -dimensional Euclidean space \mathbb{R}^{n-k} or to an $(n-k)$ -dimensional simplicial complex (already here, the choice of the target space matters). However, this choice might be too rigid and presents a couple of drawbacks in some situations. Indeed, the sweepouts are made of pairwise disjoint cycles, which might appear too restrictive for some applications. Moreover, the family of k -cycles thus-obtained sweep out the whole manifold M with an $(n-k)$ -dimensional parameter space. This prevents us from working with one-parameter families of one-cycles sweeping out the essential sphere $\mathbb{C}P^1$

in $\mathbb{C}P^n$ for instance. As a second choice, one could consider maps $P \rightarrow \mathcal{Z}_k(M)$ from a p -dimensional simplicial complex P to Almgren's k -cycle space on M , cf. [A62]. But this raises continuity issues as we would like to include in our definition sweepouts of the classical “three-legged starfish” made of single loops and parameterized by the tripod tree T , see Figure 5 in the introduction of [P81]. Indeed, such a map $T \rightarrow \mathcal{Z}_1(S^2)$ is not continuous for any reasonable topology on the one-cycle space, including the flat norm topology.

Actually, the notion of width we will use – at least for surfaces (we will relax it a bit in higher dimension) – has been introduced by M. Gromov, see [Gr, §6].

Definition 1.2. Let M be a closed surface. A *one-cycle sweepout* of M is a family of one-cycles

$$\varphi[h^{-1}(t)]$$

in M , with $t \in \Gamma$, where $h : \Sigma \rightarrow \Gamma$ is a continuous map from a closed surface Σ to a finite graph Γ and $\varphi : \Sigma \rightarrow M$ is a continuous map. Such a one-cycle sweepout is *homologically substantial* if

$$\varphi_*([\Sigma]) \neq 0 \in H_2(M; \mathbb{k}).$$

Here, the homology coefficients are in $\mathbb{k} = \mathbb{Z}$, if M is orientable, and in $\mathbb{k} = \mathbb{Z}_2$, otherwise.

In this definition, we will further assume that the vertices of Γ are of degree one or three, and that the preimages $h^{-1}(t)$ are

- simple closed curves if t is not a vertex;
- point curves if t is a degree one vertex;
- figure-eight curves if t is a degree three vertex.

Define the *width* of a closed Riemannian surface M as

$$W(M) = \inf_{\varphi, h} \sup_{t \in \Gamma} \text{length } \varphi[h^{-1}(t)]$$

where the infimum is taken over all maps φ, h defining a homologically substantial one-cycle sweepout of M .

With this definition, we can prove the following curvature-free geometric inequalities.

Theorem 1.3. *Every closed Riemannian surface M of genus g satisfies*

$$W(M) \leq 144\sqrt{6} \sqrt{\text{area}(M)} \tag{1.3}$$

$$W(M) \leq 4 \text{diam}(M). \tag{1.4}$$

Observe that the first inequality (1.3) is similar to the inequality (1.1), except for the dependence on the genus. Note also that the second inequality (1.4) has no counterpart if one replaces $h : \Sigma \rightarrow \Gamma$ with a continuous function $h : \Sigma \rightarrow \mathbb{R}$ in Definition 1.2, see [L14] for counterexamples.

Actually, we prove further width inequalities on surfaces, similar to (1.3) and (1.4). In particular, we establish sweepout estimates for compact Riemannian surfaces with boundary, cf. Section 4. We also bound the topological complexity of the homologically substantial one-cycle sweepouts required to have a width-diameter inequality, cf. Theorem 5.9.

The first inequality (1.3) is closely related to Buser-Guth’s question, see (1.2). The difference is two-fold: first, the cycles in the one-cycle sweepouts we consider may intersect with each other, but more importantly, these cycles arise from a slicing of a surface Σ mapped to M with no control on the genus of Σ , and so on the “complexity” of the sweepout (at least for the area estimate (1.3)). We do not know whether this construction can be adapted to obtain one-cycle sweepouts satisfying an area estimate similar to (1.3) from a surface Σ with the same genus as M , which would solve Buser’s problem.

In higher dimension, homotopically substantial one-cycle sweepouts of length at most $\sim \text{vol}(M)^{\frac{1}{n}}$ do not exist in general. This can already be observed on the manifold $S^2 \times S^3$ endowed with the product metric of the standard unit sphere S^2 with a sphere S^3 of small volume. Still, under some topological conditions on the manifold, similar upper bounds to (1.3) and (1.4) hold true if one replaces homologically substantial one-cycle sweepouts with homotopically substantial ones in the definition of the width. (We refer to Definition 2.1 for an extended definition of the width.)

More specifically, we can state the following result, where the Φ -width $W_\Phi(M)$ is defined in Definition 2.1. Note that this is the first curvature-free geometric inequality involving sweepout volume in higher dimension.

Theorem 1.4. *Let M be a closed n -manifold and $\Phi : M \rightarrow K$ be a continuous map to a CW-complex K with $\pi_i(K) = 0$ for every $i \geq 3$. Suppose that $\Phi_*[M] \neq 0 \in H_n(K; \mathbb{k})$. Then every Riemannian metric on M satisfies*

$$W_\Phi(M) \leq c_n \text{vol}(M)^{\frac{1}{n}}$$

$$W_\Phi(M) \leq c'_n \text{diam}(M)$$

where c_n and c'_n are explicit constants depending only on n .

Theorem 1.4 applies for instance to the torus \mathbb{T}^n (more generally, to closed manifolds admitting a nonpositively curved metric such as closed hyperbolic manifolds) and to the complex projective space $\mathbb{C}P^n$ (more generally, to closed symplectic manifolds). A manifold satisfying the topological condition of Theorem 1.4 is sometimes called a two-essential manifold, at least when the manifold is simply connected.

There is no universal upper bound on the 2-systole – defined as the minimal area of a homologically nontrivial surface – of $\mathbb{C}P^n$, endowed with an arbitrary metric, in terms of the volume of the manifold, *cf.* [KS01]. On the other hand, a (sharp) systolic inequality relating the *stable* 2-systole of $\mathbb{C}P^n$ to its volume can be found in [Gr99, Theorem 4.36]. However, even though the stable 2-systole is related to the area of some minimal surfaces, it is not clear how to combine this result with Theorem 1.1 to derive a sweepout estimate as in Theorem 1.4.

The proofs of Theorem 1.3 and Theorem 1.4 follow the same strategy and will therefore be carried out simultaneously in the first part of the paper. They both rely on the notion of filling radius, a Riemannian invariant introduced by M. Gromov, *cf.* Definition 3.1, and one of the main tool to establish curvature-free geometric inequalities. More specifically, we will show that

the filling radius of a two-essential Riemannian manifold is bounded from below in terms of its width, *cf.* Theorem 3.3. The main results will immediately follow from general filling estimates. In the second part of the paper, we will focus on the surface case: proving sweepout estimates for compact Riemannian surfaces with nonempty boundary, *cf.* Section 4, and bounding the topological complexity of the one-cycle sweepouts in the width-diameter inequality, *cf.* Section 5.

2. HOMOTOPICALLY SUBSTANTIAL ONE-CYCLE SWEEPOUTS

In this section, we introduce the notion of homotopically substantial one-cycle sweepouts (relaxing the definition of homologically substantial one-cycle sweepouts) and define the width in higher dimension. We also presents topological conditions ensuring the existence of homotopically substantial one-cycle sweepouts and generalize classical Morse-theoretical comparison results to our setting.

Definition 2.1. Let M be a closed manifold and $\Phi : M \rightarrow K$ be a continuous map to a CW-complex K . A one-cycle sweepout of M is a family of one-cycles

$$\varphi[h^{-1}(t)]$$

in M , with $t \in \Gamma$, where $h : \Sigma \rightarrow \Gamma$ is a continuous map from a compact “surface” (more precisely, a compact two-dimensional complex) Σ to a finite graph Γ and $\varphi : \Sigma \rightarrow M$ is a continuous map. Such a one-cycle sweepout is *homotopically Φ -substantial* if the composite map

$$\Phi \circ \varphi : \Sigma \rightarrow K$$

is not homotopic to any map

$$\Sigma \xrightarrow{h} \Gamma \rightarrow K$$

which factors through $h : \Sigma \rightarrow \Gamma$.

In this definition, we will further assume that the vertices of Γ are of degree one or three, and that the preimages $h^{-1}(t)$ are

- simple closed curves if t is not a vertex;
- point curves if t is a degree one vertex;
- simple closed curves or figure-eight curves if t is a degree three vertex.

As previously, define the Φ -width of a closed Riemannian manifold M as

$$W_{\Phi}(M) = \inf_{\varphi, h} \sup_{t \in \Gamma} \text{length } \varphi[h^{-1}(t)]$$

where the infimum is taken over all maps φ, h defining a homotopically Φ -substantial one-cycle sweepout of M .

If homotopically Φ -substantial one-cycle sweepouts do not exist, we let $W_{\Phi}(M) = 0$, but this case is not interesting for us.

Remark 2.2. Suppose that K is the Eilenberg-MacLane space $K(\mathbb{Z}, 2) = \mathbb{C}P^{\infty}$. Then a one-cycle sweepout of M given by $\varphi : \Sigma \rightarrow M$ and $h : \Sigma \rightarrow \Gamma$ is homotopically Φ -substantial if and only if the homotopy class of $\Phi \circ \varphi$ is

nontrivial in $[\Sigma, K(\mathbb{Z}; 2)]$, since K is simply connected. Following the fundamental relationship between cohomology and Eilenberg-MacLane spaces, *cf.* [Ha02, Theorem 4.57], this is equivalent to

$$(\Phi \circ \varphi)^*(\alpha) \neq 0$$

where $\alpha \in H^2(K; \mathbb{Z})$ is the fundamental cohomology class of $K = K(\mathbb{Z}; 2)$ in $H^2(K; \mathbb{Z}) = \text{Hom}(H_2(K; \mathbb{Z}); \mathbb{Z})$ given by the inverse of the Hurewicz isomorphism $\pi_2(K) = \mathbb{Z} \rightarrow H_2(K; \mathbb{Z})$.

Remark 2.3. Working with homotopically substantial one-cycle sweepouts instead of merely homologically substantial ones allows us to consider more general manifolds. For instance, homologically substantial one-cycle sweepouts do not exist on hyperbolic integer homology 3-spheres since their second homology group is trivial. However, homotopically substantial one-cycle sweepouts exist on these manifolds and, more generally, on closed n -manifolds admitting a nonpositively curved metric since such manifolds have nonfree fundamental groups (indeed, their fundamental groups have one end, while nontrivial free groups have two or infinitely many ends depending whether they are isomorphic to the infinite cyclic group or not), *cf.* Proposition 2.4. In particular, the Φ -width of every closed Riemannian n -manifold admitting a nonpositively curved metric is nontrivial when Φ is the identity map, *cf.* Proposition 2.5, and Theorem 1.4 applies to these manifolds.

The following immediate proposition gives a couple of criteria ensuring the existence of homotopically substantial one-cycle sweepouts.

Proposition 2.4. *Let M be a closed n -manifold and $\Phi : M \rightarrow M$ be the identity map.*

- (1) *If $\pi_1(M)$ is trivial, then M admits a homotopically Φ -substantial one-cycle sweepout if and only if $H_2(M; \mathbb{Z}) = \pi_2(M)$ is nontrivial.*
- (2) *If $\pi_1(M)$ is nonfree, then M admits a homotopically Φ -substantial one-cycle sweepout.*

Proof. For the point (1). If $H_2(M; \mathbb{Z})$ (or $\pi_2(M)$) is nontrivial, take a closed surface Σ representing a nontrivial homology (or homotopy) class along with a Morse function $f : \Sigma \rightarrow \mathbb{R}$ on it. Then the induced map $h : \Sigma \rightarrow \Gamma$ to the Reeb graph Γ of f defines a homotopically Φ -substantial one-cycle sweepout. Conversely, let $\varphi : \Sigma \rightarrow M$ be a continuous map defined on a compact “surface” Σ . Since M is simply connected, there exists a deformation of $\varphi : \Sigma \rightarrow M$ taking the 1-skeleton of Σ to a single point of M . This deformation factors as $\Sigma \rightarrow \Sigma/\Sigma^{(1)} \rightarrow M$, where the first map is the quotient map obtained by collapsing the 1-skeleton $\Sigma^{(1)}$ of Σ to a point. Now, since the quotient $\Sigma/\Sigma^{(1)}$ is a bouquet of two-spheres and $\pi_2(M)$ is trivial, the second map $\Sigma/\Sigma^{(1)} \rightarrow M$ is homotopically trivial. Therefore, no map $\varphi : \Sigma \rightarrow M$ gives rise to a homotopically Φ -substantial one-cycle sweepout of M .

For the point (2). Consider the 2-skeleton $\Sigma = M^{(2)}$ of M for a given triangulation of the manifold. The inclusion map $\Sigma \hookrightarrow M$, which induces a π_1 -isomorphism, is not homotopic to any map $\Sigma \rightarrow \Gamma \rightarrow M$ factoring through a graph Γ . Otherwise, the isomorphism $\pi_1(\Sigma) \rightarrow \pi_1(M)$ induced by the inclusion map would split as $\pi_1(\Sigma) \rightarrow \pi_1(\Gamma) \rightarrow \pi_1(M)$ through $\pi_1(\Gamma)$.

Thus, $\pi_1(M) \simeq \pi_1(\Sigma)$ would be a subgroup of the free group $\pi_1(\Gamma)$ and so would be a free group. \square

The min-max process over homotopically Φ -substantial one-cycle sweepouts shares similar properties with the classical min-max process over the loop space as illustrated by the following straightforward proposition.

Proposition 2.5. *Let M be a closed Riemannian n -manifold and $\Phi : M \rightarrow K$ be a continuous map to a CW-complex K . Suppose that homotopically Φ -substantial one-cycle sweepouts exist on M . Then*

$$W_\Phi(M) > 0.$$

Moreover, when M is a closed Riemannian surface, we have $W(M) \geq \text{scg}(M)$, where $\text{scg}(M)$ is the length of the shortest closed geodesic on M .

Proof. Let $\varphi : \Sigma \rightarrow M$ and $h : \Sigma \rightarrow \Gamma$ be the maps involved in the definition of a one-cycle sweepout of M , cf. Definition 2.1. By definition, the map h is a trivial circle bundle over the interiors of the edges of Γ . Furthermore, the fibers over the vertices of Γ are either point curves or figure-eight curves. It follows that Γ admits a lift Γ' on Σ under h . Thus, every fiber $h^{-1}(t)$ has a distinguished basepoint x_t given by its intersection with Γ' . If the length of every one-cycle $\varphi[h^{-1}(t)]$ is less than the length $\text{sgl}(M)$ of the shortest pointed geodesic loop of M (for instance, less than twice the injectivity radius), then applying Birkhoff's curve shortening flow to this one-parameter family of one-cycles with basepoints $\varphi(x_t)$ gives rise to a continuous retraction of the one-cycles $\varphi[h^{-1}(t)]$ to x_t . Therefore, the map $\varphi : \Sigma \rightarrow M$ is homotopic to the composite map

$$\Sigma \xrightarrow{h} \Gamma \simeq \Gamma' \xrightarrow{\varphi} M.$$

Consequently, the one-cycle sweepout $\varphi[h^{-1}(t)]$ is not homotopically Φ -substantial for any map $\Phi : M \rightarrow K$. Hence, $W_\Phi(M) \geq \text{sgl}(M) > 0$.

Suppose that M is a closed Riemannian surface. Let us argue by contradiction. As mentioned above, the map $h : \Sigma \rightarrow \Gamma$ is a trivial circle bundle over the interiors of the edges of Γ . Thus, the surface Σ is made of cylinders \mathcal{C} foliated by the fibers of h over the non-vertex points of Γ . In this construction, the boundary components of the cylinders \mathcal{C} collapse to point curves over the degree one vertices of Γ , while they agree either with the simple loops forming the figure-eight curves $h^{-1}(\tau)$ or with the whole figure-eight curves $h^{-1}(\tau)$ (seen as limits of simple closed curves) over the degree three vertices τ of Γ .

Now, we want to deform the images under φ of the boundary components of these cylinders to point curves in a way coherent with the gluing of the cylinders of Σ . To do so, we apply the disk flow [HS94] (which does not require a parametrization of the curves and does not increase their number of self-intersection points) to the images of the figure-eight curves $h^{-1}(\tau)$ corresponding to the degree three vertices τ of Γ . These curves converge to point curves in finite time since they are shorter than any closed geodesic on M . The disk flow also preserves their decomposition into double loops as images of figure-eight curves. (Actually, one of the two loops forming the image of such figure-eight curve might shrink to a point before the end

of the flow convergence, but we can still consider this point as a loop and follow it along the flow.) Thus, the map $\varphi : \Sigma \rightarrow M$ is homotopic to a map $\Sigma \rightarrow \Sigma' \rightarrow M$, where the first map $\Sigma \rightarrow \Sigma'$ is the quotient map obtained by collapsing each fiber of h over a vertex point of Γ to a point.

By construction, the quotient space Σ' is formed of spheres \mathcal{C}' attached to each other, where the spheres \mathcal{C}' are obtained by collapsing the boundary components of the cylinders \mathcal{C} of Σ to points (the two poles of the spheres). Observe that the spheres \mathcal{C}' are foliated by latitude circles whose images under $\Sigma' \rightarrow M$ are given by the images under φ of the fibers of h foliating the cylinders \mathcal{C} and the homotopies contracting the images of the boundary components of \mathcal{C} to points. By construction, the length of the images of these latitude circles is at most $W(M)$.

Now, if $W(M) < \text{scg}(M)$, we deduce from the standard Birkhoff min-max principle over the loop space that the map $\mathcal{C}' \rightarrow M$ is contractible. Putting together these maps, we deduce that the map $\Sigma' \rightarrow M$, and so $\Sigma \rightarrow \Sigma' \rightarrow M$, induces a trivial homomorphism between the second homology groups. Therefore, the one-cycle sweepout $\varphi[h^{-1}(t)]$ is not homologically substantial. Hence, $W(M) \geq \text{scg}(M)$. \square

Remark 2.6. In higher dimension, the Φ -width of M might be smaller than the length of the shortest closed geodesic on M . Indeed, the critical one-cycles giving the Φ -width might include unions of two geodesic loops based at the same point with starting and ending unit tangent vectors which do not lie in the same tangent two-plane.

3. CURVATURE-FREE ONE-CYCLE SWEEPOUT ESTIMATES

In this section, we introduce the notion of filling radius and present a couple of filling inequalities. Then we prove a new filling inequality involving the width of two-essential Riemannian manifolds, which implies the sweepout estimates of Theorem 1.3 and Theorem 1.4.

Let us recall the notion of filling radius introduced by M. Gromov in [Gr83] to established systolic inequalities on essential manifolds.

Definition 3.1. Let (M, g) be a closed Riemannian n -manifold. Denote by d_g the distance on M induced by the Riemannian metric g . The map

$$i : (M, d_g) \hookrightarrow (L^\infty(M), \|\cdot\|)$$

defined by $i(x)(\cdot) = d_g(x, \cdot)$ is an embedding from the metric space (M, d_g) into the Banach space $L^\infty(M)$ of bounded functions on M endowed with the sup-norm $\|\cdot\|$. This natural embedding, also called the Kuratowski embedding, is an isometry between metric spaces. We will consider M isometrically embedded into $L^\infty(M)$.

The *filling radius* of (M, g) , denoted by $\text{FillRad}(M, g)$, is the infimum of the positive reals ρ such that

$$(i_\rho)_*([M]) = 0 \in H_n(U_\rho(M); \mathbb{k})$$

where $i_\rho : M \hookrightarrow U_\rho(M)$ is the inclusion into the ρ -tubular neighborhood of M in $L^\infty(M)$, and $[M] \in H_n(M; \mathbb{k})$ is the fundamental class of M . Here, the homology coefficients are in $\mathbb{k} = \mathbb{Z}$, if M is orientable, and in $\mathbb{k} = \mathbb{Z}_2$, otherwise.

The filling radius satisfies the following fundamental bounds respectively obtained by M. Gromov [Gr83] and M. Katz [K83].

Theorem 3.2 ([Gr83], [K83]). *Let M be a closed Riemannian n -manifold. Then*

$$\begin{aligned}\text{FillRad}(M) &\leq c_n \text{vol}(M)^{\frac{1}{n}} \\ \text{FillRad}(M) &\leq \frac{1}{3} \text{diam}(M)\end{aligned}$$

where c_n is an explicit constant depending only on n .

When $n = 2$, one can take $c_2 = 12\sqrt{6}$.

Combining Theorem 3.2 with the filling estimate given by Theorem 3.3 below immediately lead to Theorem 1.3 and Theorem 1.4.

Theorem 3.3. *Let M be a closed n -manifold and $\Phi : M \rightarrow K$ be a continuous map to a CW-complex K with $\pi_i(K) = 0$ for every $i \geq 3$. Suppose that $\Phi_*[M] \neq 0 \in H_n(K; \mathbb{k})$. Then every Riemannian metric on M satisfies*

$$\text{FillRad}(M) \geq \frac{1}{12} W_\Phi(M).$$

When M is a closed surface, the same inequality also holds with $W(M)$ instead of $W_\Phi(M)$.

We can now proceed to the proof of Theorem 3.3 indicating the changes to make along the way to cover the surface case.

I - Set-up. By definition of the filling radius, the fundamental class $[M]$ of M vanishes in the ρ -neighborhood $U_\rho(M)$ of M in $L^\infty(M)$, where $\rho > \text{FillRad}(M)$. More precisely,

$$(i_\rho)_*([M]) = 0 \in H_n(U_\rho(M); \mathbb{k})$$

Therefore, there exists a continuous map $\sigma : P \rightarrow U_\rho(M)$ defined on an $(n+1)$ -pseudomanifold P with boundary such that the restriction $\sigma : \partial P \rightarrow M$ represents $[M]$ in $H_n(M; \mathbb{k})$, that is,

$$\sigma_*[\partial P] = [M] \in H_n(M; \mathbb{k}). \quad (3.1)$$

Recall that an $(n+1)$ -pseudomanifold with boundary is an $(n+1)$ -dimensional simplicial complex P such that

- every simplex of P is a face of some $(n+1)$ -simplex of P ;
- every n -simplex of P is the face of at most two $(n+1)$ -simplices of P .

The boundary ∂P of an $(n+1)$ -pseudomanifold P with boundary is the simplicial n -subcomplex of P formed of the n -simplices of P which are the faces of exactly one $(n+1)$ -simplex of P .

Suppose that $\rho < \frac{1}{12} W_\Phi(M)$. The usual argument to obtain a contradiction and derive a lower bound on the filling radius of M consists in constructing a continuous map $P \rightarrow K$ which agrees with $\Phi \circ \sigma$ on ∂P , contradicting the assumption $\Phi_*[M] \neq 0$. We will show how to carry out this argument in the higher dimensional case. However, this might not be possible in the surface case. In this special setting, we will construct a continuous map $F : Q \rightarrow M$ from a different 3-pseudomanifold Q with the same

boundary as P which agrees with σ on $\partial Q = \partial P$. Combined with (3.1), this will yield the relation

$$[M] = \sigma_*[\partial P] = F_*[\partial Q] = 0$$

leading to a contradiction.

II - Extension to the 1-skeleton. Denote by P^k the k -skeleton of P . Subdividing P if necessary, we can assume that the diameter of the images by σ of the simplices of P is less than $\varepsilon > 0$, with $\varepsilon < \min\{W_\Phi(M) - 2\rho, \frac{1}{10}\text{inj}(M)\}$. We first define a map

$$f : P^0 \cup \partial P \rightarrow M$$

with $f|_{\partial P} = \sigma$ by sending each vertex $p_i \in P^0$ to a nearest point from $\sigma(p_i)$ in M , as we wish. Since the inclusion $i : M \hookrightarrow U_\rho(M)$ is distance-preserving, every pair p_i, p_j of adjacent vertices of P satisfies

$$\begin{aligned} d_M(f(p_i), f(p_j)) &\leq d_{L^\infty}(f(p_i), \sigma(p_j)) + d_{L^\infty}(\sigma(p_i), \sigma(p_j)) + d_{L^\infty}(\sigma(p_j), f(p_j)) \\ &< \delta \end{aligned}$$

with $\delta = 2\rho + \varepsilon < \frac{1}{6}W_\Phi(M)$. We extend the map f to P^1 by taking the edges of $P \setminus \partial P$ to minimizing segments joining the images of their endpoints, as we wish. By construction, the boundary of every 2-simplex of P is sent by f to a loop of length less than 3δ .

III - Local extension to part of the 2-skeleton. Let us examine the obstruction to extend $f : P^1 \cup \partial P \rightarrow M$ to the 2-skeleton of P . Given a 2-simplex Δ of P , we apply the Birkhoff curve shortening flow, to the (simple) closed curve $f(\partial\Delta^2)$. Without loss of generality, we can always assume that the metric is generic, and that the Birkhoff curve shortening flow converges either to a point curve or to a (nontrivial) closed geodesic. Actually, we could use any convergent curve shortening flow (leaving simple loops simple in the surface case such as the disk flow, *cf.* [HS94]). Two cases may occur:

- (F₁) If the flow converges to a point curve, then the map f extends to Δ^2 .
- (F₂) If the flow converges to a (nontrivial) closed geodesic, then the map f extends to the punctured face Δ_ϵ^2 obtained by removing a small ϵ -disk from Δ^2 around its center.

The punctured face Δ_ϵ^2 is a topological annulus with *outer boundary component* the boundary of Δ^2 and *inner boundary component* the boundary of the small ϵ -disk removed from Δ^2 . By construction, the face Δ^2 – in the case (F₁) – or the annulus Δ_ϵ^2 – in the case (F₂) – is foliated by concentric closed curves whose images by f correspond to the homotopy given by the curve shortening flow. Note that the images of these concentric closed curves are simple in the surface case since, in this case, simple loops remain simple through the Birkhoff curve shortening flow (except when reduced to a point). Anyway, in any case, the images of these concentric closed curves are of length at most 3δ .

The union of P^1 with the faces Δ^2 corresponding to the case (F₁) and the punctured faces Δ_ϵ^2 corresponding to the case (F₂) forms a two-dimensional

space denoted by P_0^2 . By construction, the map $f : P^1 \cup \partial P \rightarrow M$ extends to

$$f : P_0^2 \cup \partial P \rightarrow M.$$

Consider the graph Γ lying in the dual 1-skeleton of P^1 in P^3 defined as follows:

- The vertices of Γ are formed of
 - (i) the centers of the 3-simplices of P^3 ;
 - (ii) the centers of the faces of P^3 corresponding to the case (F_2) ;
 - (iii) the midpoints of the segments joining the centers of the 3-simplices of P^3 to the centers of the faces of the same 3-simplex corresponding to the case (F_1) .
- The edges of Γ are the segments joining the centers of the 3-simplices of P^3 to the other vertices of Γ lying in the same 3-simplex.

Thus defined, the graph Γ inherits a natural bipartite graph structure with the vertices corresponding to the centers of the 3-simplices of P^3 on one side, *cf.* (i), and the other vertices on the other side, *cf.* (ii)-(iii). Observe that the vertices of Γ given by (i) are of degree 4, the vertices of Γ given by (ii) are of degree at least 2 and the vertices of Γ given by (iii) are of degree 1. Actually, in the surface case, the vertices of Γ given by (ii) are of degree exactly 2 (and, therefore, can be considered as non-vertex points of Γ). Indeed, in this case, the faces of $P = P^3$ corresponding to the case (F_2) do not lie in ∂P and each of them is the face of exactly two 3-simplices of P by definition of a 3-pseudomanifold.

IV - Global extension to the whole 2-skeleton and the 3-skeleton.

In the absence of obstructions, the map $f : P_0^2 \cup \partial P \rightarrow M$ clearly extends to P^3 away from a small ϵ -tubular neighborhood $U_\epsilon(\Gamma)$ of Γ in P^3 . This yields a map

$$f : (P^3 \setminus U_\epsilon(\Gamma)) \cup \partial P \rightarrow M.$$

However, the map f may not extend to the whole 3-skeleton P^3 . In particular, its restriction

$$\varphi : \Sigma \rightarrow M$$

to the compact “surface” (more precisely, the compact 2-complex) $\Sigma = \partial U_\epsilon(\Gamma)$ may not extend to $U_\epsilon(\Gamma)$.

In the surface case, it follows from the discussion above that $U_\epsilon(\Gamma)$ is a compact 3-manifold with boundary and that $\Sigma = \partial U_\epsilon(\Gamma)$ is a genuine closed surface. In this case, we will show that the map $\varphi : \Sigma \rightarrow M$, which does not necessarily extend to $P = P^3$, extends to a different 3-complex with the same boundary. More precisely, we will show that the map $F = \varphi : \Sigma \rightarrow M$ induces a trivial homomorphism between the two-dimensional homology groups.

In the higher dimensional case, we consider the composite map

$$F = \Phi \circ f : (P^3 \setminus U_\epsilon(\Gamma)) \cup \partial P \rightarrow K$$

and extend it to $P^3 \cup \partial P$ as follows.

For every 3-simplex Δ^3 of P^3 , consider the radial projection

$$\pi_{\Delta^3} : \Delta^3 \setminus \{*\} \rightarrow P^2$$

which takes every point x distinct from the center $*$ of Δ^3 to the point of $\partial\Delta^3$ lying in the ray $[*, x)$ arising from $*$ and passing through x . The restriction of the radial projection π_{Δ^3} to Σ gives rise to a homeomorphism

$$\Sigma \cap \Delta^3 \xrightarrow{\sim} \partial\Delta^3 \setminus U_\epsilon(\Gamma). \quad (3.2)$$

This homeomorphism allows us to decompose $\Sigma \cap \Delta^3$ into four pieces corresponding to the preimages of the faces of $\partial\Delta^3$. Topologically, these pieces are either disks or annuli depending whether the matching faces correspond to the case (F_1) or (F_2) .

Now, we can define a map

$$h : \Sigma \rightarrow \Gamma$$

by giving its restriction

$$h_{\Delta^3} : \Sigma \cap \Delta^3 \rightarrow \Gamma \cap \Delta^3$$

to the intersection of Σ with each 3-simplex Δ^3 of P^3 . Actually, identifying $\Sigma \cap \Delta^3$ with $\partial\Delta^3 \setminus U_\epsilon(\Gamma)$ through the homeomorphism (3.2) given by π_{Δ^3} , we define h_{Δ^3} on each face Δ^2 or punctured face Δ_ϵ^2 of $\partial\Delta^3 \setminus U_\epsilon(\Gamma)$ corresponding to the case (F_1) or (F_2) as follows.

In this construction, the map h_{Δ^3} maps Δ^2 to the edge \mathcal{E} of $\Gamma \cap \Delta^3$ arising from the center of Δ^3 and pointing to the center of Δ^2 in such a way that

- it takes $\partial\Delta^2$ to the vertex of Γ corresponding to the center of Δ^3 ;
- the concentric closed curves foliating Δ^2 previously described agree with the level curves of the restriction $h_{\Delta^3} : \Delta^2 \rightarrow \mathcal{E}$.

In particular, the map h_{Δ^3} sends the center of Δ^2 to the endpoint of \mathcal{E} opposite to the center of Δ^3 , *cf.* (iii).

Similarly, the map h_{Δ^3} maps Δ_ϵ^2 to the edge \mathcal{E} of $\Gamma \cap \Delta^3$ joining the center of Δ^3 to the center of the face of Δ^3 containing Δ_ϵ^2 in such a way that

- it takes the outer boundary component of Δ_ϵ^2 to the vertex of Γ corresponding to the center of Δ^3 ;
- it takes the inner boundary component of Δ_ϵ^2 to the vertex of Γ corresponding to the face of Δ^3 containing Δ_ϵ^2 ;
- the concentric closed curves foliating Δ_ϵ^2 previously described agree with the level curves of the restriction $h_{\Delta^3} : \Delta_\epsilon^2 \rightarrow \mathcal{E}$.

By construction, the preimages of the degree 1 vertices of Γ under $h : \Sigma \rightarrow \Gamma$ are points. The preimages of the (degree 4) vertices corresponding to the centers of the 3-simplices of P agree with the 1-skeleta of these 3-simplices under the homeomorphisms (3.2). Their images by $\varphi : \Sigma \rightarrow M$ are of length less than 6δ . Meanwhile, the preimages of all the other points of Γ are simple closed curves of Σ and their images by $\varphi : \Sigma \rightarrow M$ are of length less than 3δ . In any case, for every $t \in \Gamma$, we have

$$\text{length } \varphi[h^{-1}(t)] < 6\delta.$$

Actually, we can do slightly better altering this construction (we only sketch some features of the new construction leaving the details to the reader). Inserting a new edge $[v, w]$ at each degree 4 vertex u of Γ corresponding to the centers of the 3-simplices of P as in Figure 1 gives rise to

a new graph Γ' .

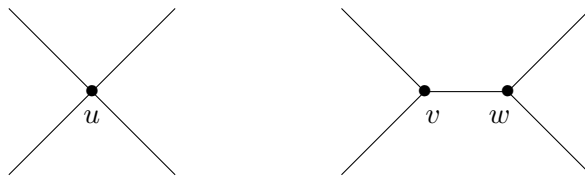


FIGURE 1. Inserting an edge

Similarly, we define a new compact “surface” Σ' by inserting cylinders in Σ in such a way that the map h extends to a map $h' : \Sigma' \rightarrow \Gamma'$ where the circles foliating the added cylinders agree with the level curves over the new edges in Γ' . While $h^{-1}(u)$ corresponds to the graph T given by the 1-skeleton of a tetrahedron, the level curves $h'^{-1}(v)$ and $h'^{-1}(w)$ correspond to the graph T with one edge removed, where the edges removed in $h'^{-1}(v)$ and $h'^{-1}(w)$ are in opposite position in T . Moreover, this construction can be performed so that the level curves over the edge $[v, w]$ cover at most five of the six edges of T . Therefore, for every $t \in \Gamma$, we have

$$\text{length } \varphi'[h'^{-1}(t)] < 5\delta$$

where $\varphi' : \Sigma' \rightarrow \Sigma \xrightarrow{\varphi} M$ is the map obtained by collapsing the cylinders inserted in Σ and applying φ . Similarly, we can replace every degree $k \geq 4$ vertex with $k - 2$ trivalent vertices by inserting $k - 3$ new edges at each of these vertices. Thus, the new graph so-obtained has only vertices of degree one or three. We can accordingly modify our sweepouts so that they are made of point curves, simple closed curves or figure-eight curves depending on the degree of the point t in the parameterizing graph. This can be performed without increasing the length of the image of the sweepout curves over 6δ . Further minor technical adjustments need to be made in the rest of the argument to be consistent, but we will skip them and work with the original sweepout $\varphi[h^{-1}(t)]$.

Now, by our choice of δ , every curve in our sweepout is of length less than $W_\Phi(M)$. By definition of the Φ -width, it follows that the composite map $F = \Phi \circ \varphi : \Sigma \rightarrow K$ is homotopic to a map

$$\Sigma \xrightarrow{h} \Gamma \rightarrow K$$

which factors through h . That is, the map $F : \Sigma \rightarrow K$ extends to $U_\epsilon(\Gamma)$ since the ϵ -tubular neighborhood $U_\epsilon(\Gamma)$ of Γ in P^3 is homeomorphic to the quotient $\Sigma \times [0, 1] / \sim$, where $(x, 1) \sim (y, 1)$ if and only if $h(x) = h(y)$. This yields a map

$$F : P^3 \cup \partial P \rightarrow K.$$

In the surface case, the argument is similar. By definition of the width, since every curve in our sweepout is of length less than $W(M)$, the map $\varphi : \Sigma \rightarrow M$ takes the fundamental class of the genuine closed surface Σ to zero. Therefore, this map extends to a 3-complex with boundary Σ . This yields a map $F : Q \rightarrow M$ defined on some 3-complex Q with the same

boundary as P which agrees with σ on $\partial Q = \partial P$. Hence a contradiction in this case.

V - Extension to higher dimensional skeleta. Back to the higher dimension case, since $\pi_3(K) = 0$, there is no obstruction to extend any map $S^3 \rightarrow K$ from the 3-sphere to the 4-ball. In particular, the restriction of F to the boundary $\partial\Delta^4$ of every 4-simplex Δ^4 of P not lying in ∂P extends to Δ^4 . This yields a map

$$F : P^4 \cup \partial P \rightarrow K.$$

Applying this argument to higher-dimensional skeleta one at a time, we obtain the desired extension

$$F : P \rightarrow K$$

of $\Phi \circ \sigma : \partial P \rightarrow K$, which finishes the proof of Theorem 3.3.

4. ONE-CYCLE SWEEPOUTS OF COMPACT SURFACES WITH BOUNDARY

In this section, we establish width inequalities for compact Riemannian surfaces with nonempty boundary. We will first consider the simple case of the disk before moving to more general surfaces.

Theorem 4.1. *Let M be a Riemannian disk. Then there exists a continuous map $h : M \rightarrow \Gamma$ to a finite tree Γ , sending the boundary of M to a terminal vertex of Γ , such that the fibers of h are simple closed curves, point curves or figure-eight curves with*

$$\text{length } h^{-1}(t) \leq \max\{2 \text{length}(\partial M), 16R\}$$

for every $t \in \Gamma$, where $R = \max_{x \in M} d(x, \partial M)$.

Proof. Slightly perturbing the metric if necessary, we can assume that M has finitely many closed geodesics of length at most $L_M := \max\{2 \text{length}(\partial M), 16R\}$.

In the absence of such geodesic, the curve shortening flow applied to ∂M converges to a point and gives rise to a foliation of M . The curves of this foliation, of length at most $\text{length}(\partial M)$, are the level curves of a continuous map $h : M \rightarrow [0, 1]$ satisfying the conclusion of Theorem 4.1. This concludes the proof of the theorem in this case.

Let us argue by induction on the number n_M of closed geodesics of length at most L_M . Applying the curve shortening flow to ∂M as previously, we can assume that M has a geodesic boundary. We will consider two cases.

Case I - Suppose that ∂M is not a length-minimizing closed geodesic. That is, there exist two antipodal points x_1 and x_2 on ∂M , separating ∂M into two arcs α_1 and α_2 of the same length, such that

$$d(x_1, x_2) < \frac{1}{2} \text{length}(\partial M).$$

Let α be the length-minimizing arc of M joining x_1 to x_2 . Note that α is shorter than α_1 and α_2 . The loops $\alpha \cup \alpha_1$ and $\alpha \cup \alpha_2$ bound two convex domains of M and converge to two closed geodesics γ_1 and γ_2 (possibly reduced to point curves) in each of these domains under the curve shortening flow. Denote by P the region lying between ∂M and the two curves

γ_1 and γ_2 . Since the curve shortening flow preserves the curve convexity, we obtain – after a slight perturbation – a foliation of P by simple closed curves, possibly point curves, and figure-eight curves, where the curves of this foliation are of length at most

$$\text{length}(\alpha \cup \alpha_1) + \text{length}(\alpha \cup \alpha_2) < 2 \text{length}(\partial M).$$

The construction is rather obvious (first deforming ∂M along α back and forth) and will be omitted. This gives rise to a continuous map $h_P : P \rightarrow T$ to the tripod tree T , sending the three boundary components ∂M , γ_1 and γ_2 of P to the three terminal vertices of T , where the fibers of h_P are the curves of the foliation. In particular, $\text{length } h^{-1}(t) \leq 2 \text{length}(\partial M)$.

By construction, the disk D_i bounded by γ_i satisfies

$$\begin{aligned} \text{length}(\partial D_i) &\leq \text{length}(\partial M) \\ R_i &\leq R \end{aligned}$$

where $R_i = \max_{x \in D_i} d(x, \partial D_i)$. Moreover, we also have $n_{D_i} < n_M$.

By induction, we obtain two maps $h_i : D_i \rightarrow \Gamma_i$ to finite trees Γ_i satisfying the conclusion of Theorem 4.1. Putting together $h_P : P \rightarrow T$ and the maps $h_i : D_i \rightarrow \Gamma_i$, we define the desired map $h : M \rightarrow \Gamma$, where Γ is the tree obtained by gluing Γ_1 and Γ_2 to the tripod tree T at the terminal vertices corresponding to γ_1 and γ_2 .

Case II - Suppose that ∂M is a length-minimizing closed geodesic. In this case, the isometric embedding $i : \partial M \hookrightarrow L^\infty(\partial M)$ given by the Kuratowski embedding, *cf.* Definition 3.1, isometrically extends to M , *cf.* [Gr83, §1.1]. In particular,

$$\text{FillRad}(\partial M) \leq R.$$

Since $\text{length}(\partial M) \leq 6 \text{FillRad}(\partial M)$ from [Gr83, Lemma 1.2.B], we derive the upper bound

$$\text{length}(\partial M) \leq 6R.$$

Let $x_0 \in M$ such that $d(x_0, \partial M) = R$. From [CE75, Lemma 8.15] (see also [CE75, Lemma 6.2]), whose proof relies on the first variation formula, we can show that there exist three geodesic arcs α_1 , α_2 and α_3 joining x_0 to ∂M of length R dividing M into three convex domains Δ_1 , Δ_2 and Δ_3 . (Actually, two geodesic arcs might be enough, but in this case, one of the two convex domains will be empty.) Each boundary component $\partial \Delta_i$ converges to a simple closed geodesic γ_i (possibly reduced to a point curve) in Δ_i under the curve shortening flow. As previously, we obtain a foliation of the region Q bounded by ∂M and the three curves γ_1 , γ_2 and γ_3 by simple closed curves and figure-eight curves (deforming ∂M along the three arcs α_i back and forth) of length at most

$$\text{length}(\partial M) + 6R \leq 8R.$$

This gives rise to a continuous map $h_Q : Q \rightarrow \Gamma_Q$ to the union Γ_Q of two tripod trees sending the four boundary components ∂M , γ_1 , γ_2 and γ_3 of Q to the four terminal vertices of Γ_Q , where the fibers of h_Q are the curves of the foliation. In particular, $\text{length } h^{-1}(t) \leq 8R$.

By construction, the disk D_i bounded by γ_i satisfies

$$\begin{aligned} \text{length}(\partial D_i) &\leq 8R \\ R_i &\leq R \end{aligned}$$

and $n_{D_i} < n_M$.

By induction, we obtain three maps $h_i : D_i \rightarrow \Gamma_i$ to finite trees Γ_i , sending the boundary of D_i to a terminal vertex of Γ_i , such that the fibers of h_i are simple closed curves, point curves or figure-eight curves of length at most $16R$. Combining the map $h_Q : Q \rightarrow \Gamma_Q$ with the maps $h_i : D_i \rightarrow \Gamma_i$ as previously, we obtain a map $h : M \rightarrow \Gamma$ satisfying the conclusion of Theorem 4.1, where Γ is the finite tree obtained by gluing together Γ_1 , Γ_2 and Γ_3 to Γ at the terminal vertices corresponding to γ_1 , γ_2 and γ_3 . \square

Now, we consider the case of general surfaces with nonempty boundary and extend the notion of width to this setting.

Definition 4.2. Consider a compact Riemannian surface M with nonempty boundary. Let

$$L(\partial M) = \max_C \text{length}(C)$$

where C runs over the components of ∂M . Note that $L(\partial M)$ does not represent the total length of ∂M , but the length of its longest boundary component.

Define the *relative width* of M as

$$W(M, \partial M) = \inf_{\varphi, h} \sup_{t \in \Gamma} \text{length} \varphi[h^{-1}(t)] \quad (4.1)$$

where the infimum is taken over

- the (relative) degree ± 1 maps $\varphi : (\Sigma, \partial \Sigma) \rightarrow (M, \partial M)$ defined on a compact surface Σ with boundary;
- the continuous maps $h : \Sigma \rightarrow \Gamma$ to a finite graph where every boundary component of Σ is sent to a terminal vertex of Γ and the preimage of every terminal vertex of Γ is either a point or a boundary component of Σ .

As previously, the homology coefficients are in \mathbb{Z} or \mathbb{Z}_2 depending whether M is orientable or not. Note that $W(M, \partial M) \geq L(\partial M)$.

Replacing Σ with another compact surface with boundary filling some of the boundary components of Σ , we can further assume that the images of the boundary components of Σ are noncontractible in M (but we will not make use of this feature).

Regarding the inequalities (1.3) and (1.4), two natural questions to ask are whether

$$W(M, \partial M) \leq C_1 \sqrt{\text{area}(M)} + C_2 L(\partial M) \quad (4.2)$$

and if

$$W(M, \partial M) \leq C'_1 \text{diam}(M) + C'_2 L(\partial M) \quad (4.3)$$

where C_1, C_2, C'_1 and C'_2 are universal constants.

In the following, we give a negative answer to the first inequality and a positive answer to the second. We also show that the first inequality holds true if one replaces $L(\partial M)$ with the total length of ∂M .

More precisely, we first prove

Proposition 4.3. *There exists a compact Riemannian surface M with boundary such that*

- *the area of M and the length of each component of ∂M are arbitrarily small;*
- *the relative width $W(M, \partial M)$ of M is arbitrarily large.*

Proof. Consider a flat torus T of systole at least ℓ , where $\ell > 0$ is large. Fix $\varepsilon > 0$ small. Remove sufficiently many disjoint disks D_i of perimeter at most ε from T so that the area of the resulting surface $M = T \setminus \cup_i D_i$ is at most ε . Thus, $\text{area}(M) \leq \varepsilon$ and $L(\partial M) \leq \varepsilon$.

Let $\varphi : (\Sigma, \partial\Sigma) \rightarrow (M, \partial M)$ and $h : \Sigma \rightarrow \Gamma$ be two maps involved in the definition of the relative width of M for which the min-max value (4.1) is attained up to ε . That is,

$$\sup_{t \in \Gamma} \text{length } \varphi[h^{-1}(t)] < W(M, \partial M) + \varepsilon. \quad (4.4)$$

Every component C of $\partial\Sigma$ is sent to some circle ∂D_i in T and can be contracted to a point inside D_i through a length-nonincreasing homotopy of loops C_t in D_i . This gives rise to a continuous map

$$\hat{\varphi} : \hat{\Sigma} \rightarrow T$$

from the closed surface $\hat{\Sigma}$ obtained by collapsing every boundary component of Σ to a point with

$$\hat{\varphi}([\Sigma]) \neq 0 \in H_2(T; \mathbb{Z}).$$

This also gives rise to a continuous map

$$\hat{h} : \hat{\Sigma} \rightarrow \hat{\Gamma}$$

extending h , where $\hat{\Gamma}$ is a graph obtained from Γ by adding an interval to each terminal vertex of Γ corresponding to a component of $\partial\Sigma$, such that the image by $\hat{\varphi}$ of the fibers over these intervals are given by the homotopies C_t . Thus,

$$\sup_{t \in \hat{\Gamma}} \text{length } \hat{\varphi}[\hat{h}^{-1}(t)] = \sup_{t \in \Gamma} \text{length } \varphi[h^{-1}(t)]. \quad (4.5)$$

Observe that the family $\hat{\varphi}[\hat{h}^{-1}(t)]$ defines a homologically substantial one-cycle sweepout of T , cf. Definition 1.2. Combining the relation (4.5) with the bound (4.4) and Proposition 2.5, we immediately deduce that

$$\ell \leq W(T) \leq W(M, \partial M) + \varepsilon$$

for every $\varepsilon > 0$. Hence, $W(M, \partial M) \geq \ell$. □

Remark 4.4. Working directly on M without introducing the width $W(T)$ of the flat torus T is more technical. At least, we do not have a short argument to conclude in this case.

Now, we show that the inequality (4.3) holds. More precisely, we have

Theorem 4.5. *Let M be a compact Riemannian surface with nonempty boundary. Then*

$$W(M, \partial M) \leq \frac{8}{3} \text{diam}(M) + \frac{10}{3} L(\partial M).$$

Proof. Applying the curve shortening flow, we can assume that the boundary components of M are closed geodesics. Let \hat{M} be the closed surface obtained by isometrically attaching round hemispheres H_i along the boundary components of M .

Since every arc of a round hemisphere with endpoint lying in its boundary can be deformed into an arc of the equator through a length-nonincreasing homotopy keeping its endpoints fixed, we derive that every length-minimizing arc of \hat{M} joining two points of M entirely lies in M . As every point on a round hemisphere of equatorial length L is at distance at most $\frac{L}{4}$ from its boundary, we deduce that

$$\text{diam}(\hat{M}) \leq \text{diam}(M) + \frac{1}{2}L(\partial M). \quad (4.6)$$

Let us show that

$$\text{FillRad}(\hat{M}) \geq \frac{1}{8}W(M, \partial M) - \frac{1}{4}L(\partial M). \quad (4.7)$$

Consider an ε -fine geodesic triangulation of M for $\varepsilon > 0$ small enough, and extend it into a geodesic triangulation of \hat{M} with extra vertices the poles x_i of the hemispheres H_i attached to M . Thus, all triangles of the triangulation of \hat{M} are ε -small except for the triangles lying in the round hemispheres, which are of diameter less than $\frac{1}{4}L(\partial M) + \varepsilon$.

Fix $\rho > \text{FillRad}(\hat{M})$. Consider \hat{M} as embedded in $L^\infty(\hat{M})$ and denote by $U_\rho(\hat{M})$ the ρ -neighborhood of \hat{M} in $L^\infty(\hat{M})$, cf. Definition 3.1. By definition of the filling radius, there exists a continuous map $\sigma : \hat{P} \rightarrow U_\rho(\hat{M})$ defined on a 3-pseudomanifold \hat{P} with boundary such that the restriction $\sigma|_{\partial\hat{P}} : \partial\hat{P} \rightarrow \hat{M}$ of σ to $\partial\hat{P}$ is a simplicial homeomorphism. In particular,

$$\sigma_*[\partial\hat{P}] = [\hat{M}].$$

Subdividing the simplices of \hat{P} (but not by barycentric subdivision in order to preserve the simplicial structure on $\partial\hat{P}$), we can assume that the images by σ of the simplices of \hat{P} are of diameter less than $\frac{1}{4}L(\partial M) + \varepsilon$.

The vertices of \hat{P} decompose into three parts, namely the isolated vertices, the peripheral vertices and the regular vertices, defined as follows:

- The *isolated* vertices of \hat{P} are the vertices of $\partial\hat{P}$ which are sent to the poles of the hemispheres of \hat{M} by the simplicial homeomorphism $\sigma|_{\partial\hat{P}}$.

Without loss of generality, we can assume that every isolated vertex of the pseudomanifold \hat{P} is adjacent to a single vertex of $\hat{P} \setminus \partial\hat{P}$.

- The *peripheral* vertices of \hat{P} are the vertices of $\hat{P} \setminus \partial\hat{P}$ adjacent to the isolated vertices of \hat{P} .
- The *regular* vertices of \hat{P} are formed of all the other vertices of \hat{P} .

Let P be the 3-pseudomanifold with boundary formed of all the simplices of \hat{P} with vertices the peripheral and regular vertices of \hat{P} . There is a natural deformation retract of \hat{P} onto P sending every isolated vertex p_i of \hat{P} to its adjacent peripheral vertex q_i . Denote by D_i the union of the 2-simplices

of ∂P around the peripheral vertex q_i . It is a cone with principal vertex q_i over the preimage of ∂H_i under $\sigma|_{\partial \hat{P}}$.

Suppose that $\rho < \frac{1}{8}W(M, \partial M) - \frac{1}{4}L(\partial M) - \frac{\varepsilon}{4}$. We define a continuous map $f : \hat{P}^1 \cup \partial \hat{P} \rightarrow \hat{M}$ extending $\sigma|_{\partial \hat{P}}$ as follows (compare with the proof of Theorem 3.3, Part II).

As the map f is given by $\sigma|_{\partial \hat{P}}$ on $\partial \hat{P}$, it maps every isolated vertex p_i of \hat{P} to the pole x_i of some hemisphere H_i of \hat{M} . We extend f to the peripheral vertex q_i of \hat{P} adjacent to p_i by sending q_i to some (generic) point in the connected component ∂H_i of ∂M . The only requirement on this generic point of ∂H_i is that it should not be antipodal to any vertex of the triangulation of M (for technical reasons). Thus, the image $f(q_i)$ lies in ∂H_i and

$$\begin{aligned} d_{L^\infty}(f(q_i), \sigma(q_i)) &\leq d_{L^\infty}(f(q_i), \sigma(p_i)) + d_{L^\infty}(\sigma(p_i), \sigma(q_i)) \\ &< \frac{1}{4}L(\partial M) + \frac{1}{4}L(\partial M) + \varepsilon \end{aligned} \quad (4.8)$$

as $\sigma(p_i) = x_i$ and $\sigma(q_i) \in \partial H_i$.

By definition, the map f also takes every regular vertex v of \hat{P} to a nearest point from $\sigma(v)$ in $M \subset L^\infty(\hat{M})$ away from the hemispheres. The point $\sigma(v)$ is at distance at most ρ from \hat{M} , hence at distance at most $\rho + \frac{1}{4}L(\partial M)$ from M since every point of an hemisphere H_i is at distance at most $\frac{1}{4}L(\partial M)$ from its boundary. Thus,

$$d_{L^\infty}(f(v), \sigma(v)) < \rho + \frac{1}{4}L(\partial M). \quad (4.9)$$

Note that if v is a regular vertex of $\partial \hat{P}$ then $f(v) = \sigma(v)$ as required.

Now, we need to define the image of every edge $[v_1, v_2]$ of \hat{P} under f . If both $f(v_1)$ and $f(v_2)$ lie in the same connected component ∂H_i of ∂M , the map f sends $[v_1, v_2]$ to the shortest arc of ∂H_i . Otherwise, the map f sends $[v_1, v_2]$ to a minimizing segment of \hat{M} between $f(v_1)$ and $f(v_2)$. With this definition, the boundary of every 2-simplex of D_i is sent to a contractible loop of ∂H_i , except for one whose image by f agrees with the equatorial loop ∂H_i of length at most $L(\partial M)$. In general, arguing as in the proof of Theorem 3.3, Part II, we deduce from the bounds (4.8) and (4.9) that the image by f of every edge of \hat{P} is a geodesic arc of \hat{M} of length at most

$$\ell := 2\rho + \frac{1}{2}L(\partial M) + \varepsilon.$$

Thus, the map f sends the boundary of every 2-simplex of \hat{P} to a simple loop of length at most 3ℓ .

We now want to extend $f : \hat{P}^1 \cup \partial \hat{P} \rightarrow \hat{M}$ to \hat{P} . Observe that the edges of \hat{P} between two non-isolated vertices have their images in M , while the edges of \hat{P} joining an isolated vertex to a peripheral or regular vertex have their images in the hemispheres of \hat{M} . This implies that the restriction of f to P^1 defines a map $f : P^1 \rightarrow M$ with values in M .

Arguing as in the proof of Theorem 3.3, Parts III and IV, we define an extension

$$f : P \setminus U_\epsilon(\Gamma) \rightarrow M$$

of $f : P^1 \rightarrow M$ to P away from a small ϵ -tubular neighborhood $U_\epsilon(\Gamma)$ of some finite graph Γ in the dual 1-skeleton of P^1 in P . The restriction of f to the compact surface with boundary $\Sigma = \overline{\partial U_\epsilon(\Gamma)} \setminus \partial P$, defined as the closure of $\partial U_\epsilon(\Gamma) \setminus \partial P$, is denoted by

$$\varphi : (\Sigma, \partial\Sigma) \rightarrow (M, \partial M).$$

Observe that the boundary components of Σ correspond to the 2-simplices of the D_i 's (one for each i) whose images are noncontractible in the ∂H_i 's. By construction, the map φ gives rise to a homeomorphism between $\partial\Sigma$ and ∂M . This implies that the relative degree of φ is ± 1 . Furthermore, the map φ comes with a map

$$h : \Sigma \rightarrow \Gamma$$

taking every boundary component of Σ to a terminal vertex of Γ , such that the preimages of the terminal vertices of Γ are either points or boundary components of Σ . In addition, the images by φ of the fibers of h are of length at most 4ℓ . Thus,

$$\text{length } \varphi[h^{-1}(t)] < W(M, \partial M).$$

This contradicts the definition of the relative width, *cf.* (4.1). Therefore, the filling radius estimate (4.7) holds true.

Combining the bounds (4.7) and (4.6) with the general estimate given by the second inequality of Theorem 3.2, we derive that

$$\frac{1}{8} W(M, \partial M) - \frac{1}{4} L(\partial M) \leq \text{FillRad}(\hat{M}) \leq \frac{1}{3} \text{diam}(\hat{M}) \leq \frac{1}{3} \text{diam}(M) + \frac{1}{6} L(\partial M).$$

Hence,

$$W(M, \partial M) \leq \frac{8}{3} \text{diam}(M) + \frac{10}{3} L(\partial M).$$

□

From Proposition 4.3, there does not exist any relative width inequality similar to (4.2). However, such an inequality holds true if one replaces $L(\partial M)$ with the total length of ∂M .

Theorem 4.6. *Let M be a compact Riemannian surface with boundary. Then*

$$W(M, \partial M) \leq 96\sqrt{6} \sqrt{\text{area}(M)} + \left(96\sqrt{\frac{3}{\pi}} + 2\right) \text{length}(\partial M).$$

Proof. We consider the closed surface \hat{M} obtained from M by gluing round hemispheres as in the proof of Theorem 4.5. The filling radius estimate (4.7) combined with Theorem 3.2 yields the bound

$$\frac{1}{8} W(M, \partial M) - \frac{1}{4} L(\partial M) \leq \text{FillRad}(\hat{M}) \leq 12\sqrt{6} \sqrt{\text{area}(\hat{M})}.$$

By construction,

$$\text{area}(\hat{M}) \leq \text{area}(M) + \frac{1}{2\pi} \text{length}(\partial M)^2.$$

Hence, from the classical relation $\sqrt{a+b} \leq \sqrt{a} + \sqrt{b}$, we derive the following bound

$$\frac{1}{8} W(M, \partial M) - \frac{1}{4} L(\partial M) \leq 12\sqrt{6} \sqrt{\text{area}(M)} + 12\sqrt{\frac{3}{\pi}} \text{length}(\partial M).$$

The desired inequality immediately follows. \square

5. TOPOLOGICAL COMPLEXITY OF THE ONE-CYCLE SWEEPOUTS

The surface Σ involved in the definition of the width of a closed Riemannian surface M , *cf.* Definition 1.2, is not necessarily homeomorphic to M . A natural question to ask is whether the width inequalities (1.3) and (1.4) of Theorem 1.3 hold when Σ is homeomorphic to M .

In this section, we show that a width-diameter inequality similar to (1.4) holds for homologically substantial one-cycle sweepouts obtained from a surface Σ of genus at most twice the genus of M , *cf.* Theorem 5.9. Note, however, that our approach only works for the width-diameter inequality (1.4). We do not know how to impose an *a priori* bound on the Euler characteristic of Σ for a width-area inequality as in (1.3).

Before stating and proving a width-diameter inequality for homologically substantial one-cycle sweepouts of bounded topological complexity, we establish an intermediate result of independent interest regarding quantitative triangulations of Riemannian surfaces.

Definition 5.1. Every closed surface M can be described by gluing together finitely many triangles along their edges. Such collection of triangles defines a “*triangulation*” of M . This amounts to saying that M is a Δ -complex, *cf.* [Ha02, §2.1].

With this definition, the 2-torus admits a “*triangulation*” with only two triangles, whose vertices are identified into a single point, while a genuine triangulation, whose triangles are uniquely determined by their vertices, has at least 14 triangles.

The following result, which is a direct consequence of Proposition 5.5 and Proposition 5.6, shows that we can control the number and the size of the triangles in the triangulations of Riemannian surfaces.

Theorem 5.2. *Let M be a closed Riemannian surface of genus g . Then there exists a “*triangulation*” of M with $4g - 2$ triangles of side length at most $3 \text{diam}(M)$.*

Remark 5.3. By taking a barycentric subdivision, we can deduce a similar result for genuine triangulations. Specifically, there exists a triangulation of M with $24g - 12$ triangles of side length at most $\frac{9}{2} \text{diam}(M)$.

Remark 5.4. It would be rather surprising if this result was not already known. However, we were unable to find a reference in the literature.

The proof we present is more involved than first expected. The starting point is Gromov’s classical result asserting that the fundamental group of M can be generated by the homotopy classes of simple loops of length at most $2 \text{diam}(M)$, *cf.* [Gr99, Proposition 3.22]. Using this result, we can easily

open up the surface into a polygon. But it is not clear how to obtain a triangulation of this polygon from the previous simple loops. Proposition 5.6 is then required at this point.

Another way to proceed would be to make use of the length-three relations in the presentation of the fundamental group of M by homotopy classes of loops of length at most $2 \operatorname{diam}(M)$, cf. [Gr99, Proposition 5.28], since these relations correspond to triangles in the surface. However, these triangles may overlap and it is not clear how to derive a “triangulation” from them.

Let us construct by induction a collection of geodesic loops $\gamma_1, \dots, \gamma_{2g}$ based at a fixed point x_0 in M as follows. The loop γ_1 is the shortest homologically nontrivial loop of M based at x_0 . Define by induction the loop γ_{i+1} as the shortest loop of M based at x_0 homologically independent with $\gamma_1, \dots, \gamma_i$. Cutting open the surface M along the loops $\gamma_1, \dots, \gamma_{2g}$ gives rise to a $4g$ -gon D satisfying the following geometric features.

Proposition 5.5.

- (1) *The sides of D are of length at most $2 \operatorname{diam}(M)$.*
- (2) *Every point of D is at distance at most $\operatorname{diam}(M)$ from one of its vertices.*

Proof. The fundamental group of M , and so its first integral homology group, is generated by the homotopy classes of the loops of length at most $2 \operatorname{diam}(M)$ based at x_0 , cf. [Gr99, Proposition 3.22]. This immediately implies that $\operatorname{length}(\gamma_i) \leq 2 \operatorname{diam}(M)$. As the sides of D correspond to the geodesic arcs γ_i , we immediately derive the point (1).

Observe that no minimizing arc with endpoint x_0 intersects any loop γ_i of M away from its endpoints. Otherwise, there would be a minimizing arc α intersecting γ_i at exactly two points x_0 and x , which can be taken as the endpoints of α . Denote by γ_i^\pm the two arcs of γ_i with the same endpoints as α . Since α is length-minimizing, the piecewise geodesic loops $\alpha \cup \gamma_i^\pm$ are of length at most $\operatorname{length}(\gamma_i)$. Moreover, as the arc α forms a nonzero nonflat angle with γ_i^\pm at x , the loops $\alpha \cup \gamma_i^\pm$ can be deformed into shorter arcs with the same basepoint x_0 . By construction of γ_i , this implies that both loops $\alpha \cup \gamma_i^\pm$ are homologous to some linear combinations of the loops $\gamma_1, \dots, \gamma_{i-1}$. As the loop γ_i itself is homologous to the sum of the loops $\alpha \cup \gamma_i^\pm$, it follows that γ_i is homologous to a linear combination of the loops $\gamma_1, \dots, \gamma_{i-1}$. Hence a contradiction.

Let x be a point of M not lying in any loop γ_i . From the previous observation, every minimizing arc c from x to the basepoint x_0 intersects the curves γ_i only at its endpoint x_0 . Thus, the arc c lies in D . As the vertices of D correspond to the basepoint x_0 , we deduce that the distance in D from x to one of the vertices of D equals the length of c , which is at most $\operatorname{diam}(M)$. Hence the point (2). \square

Proposition 5.6. *Let D be a Riemannian polygon with n vertices. Suppose that*

- (1) *the edges of D are of length at most 2δ ;*
- (2) *every point of D is at distance at most δ from the set of vertices of D .*

Then there exists a “triangulation” of D with $n-2$ triangles compatible with the polygonal structure of D whose triangle side length is at most 3δ .

Proof. We argue by induction on the number n of vertices of D . The result is obvious for $n = 3$. So we can assume $n > 3$.

Denote by v_1, \dots, v_n the vertices of D . Let V_i be the Voronoi cells centered at v_i

$$V_i = \{x \in D \mid |xv_i| \leq |xv_j| \text{ for every } j \neq i\}.$$

The cells V_i are topological disks since they are simply connected. Note that every point of V_i is at distance at most δ from the center v_i of the cell, cf. Item (2) of Proposition 5.6.

By definition, a *barrier* of D is an arc of the boundary ∂V_i of some Voronoi cell V_i , intersecting ∂D only at its endpoints.

Case I - Suppose that every pair of Voronoi cells with nonempty intersection are centered at adjacent vertices.

The following lemma describes the boundary components of the Voronoi cells and how these cells fit together.

Lemma 5.7.

- (1) The boundary of a Voronoi cell contains one or two barriers of D .
- (2) Two Voronoi cells meet along a barrier of D or have an empty intersection.
- (3) Three distinct Voronoi cells have an empty intersection.

Proof. Suppose there exists a Voronoi cell V whose boundary ∂V contains at least three barriers of D . Let α_{\pm} be two barriers of D lying in ∂V adjacent to the center v of V and let α be another barrier of D lying in ∂V different from α_{\pm} , cf. Figure 2. Note that α is non-adjacent to v .

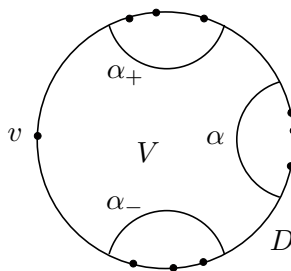


FIGURE 2. Voronoi cell with at least three barriers

The vertices of D across from α from the vertex v (that is, the vertices of D lying in the connected component of $D \setminus \alpha$ not containing v) are not adjacent to v . Indeed, they are separated from v by the vertices of D across from α_{\pm} from v . Thus, the Voronoi cells intersecting V along α are centered at vertices non-adjacent to v , which is impossible in Case I.

Suppose the point (2) does not hold. In this case, there exists a point lying at the intersection of three Voronoi cells. Under the assumption of Case I, the centers of these three Voronoi cells are pairwise adjacent, which is impossible if $n > 3$.

The point (3) is an immediate consequence of the previous point. \square

For $\varepsilon > 0$ small enough, the ε -neighborhoods V_i^ε of the Voronoi cells V_i deformation retract to V_i and form an open cover \mathcal{U} of D . Consider the nerve Γ of \mathcal{U} defined as a simplicial complex with vertices x_1, \dots, x_n (in bijection with the centers of the Voronoi cells) such that for every $k \geq 0$, the vertices x_{i_0}, \dots, x_{i_k} span a k -simplex if and only if the intersection $\bigcap_{p=0}^k V_{i_p}^\varepsilon$ is nonempty.

The following lemma describes the structure of the nerve Γ .

Lemma 5.8. *For $\varepsilon > 0$ small enough, the nerve Γ of \mathcal{U} is an interval.*

Proof. Observe that the nerve Γ is a graph since three distinct Voronoi cells (and so their ε -neighborhoods, for ε small enough) have an empty intersection from Lemma 5.7.(3). By Lemma 5.7.(2)-(3), every nonempty intersection of sets in the cover \mathcal{U} is contractible. From the nerve theorem, cf. [Ha02, Corollary 4G.3], we deduce that the nerve Γ of \mathcal{U} is homotopically equivalent to D . Therefore, the graph Γ is a tree. This tree does not have any vertex of degree three or more, otherwise a Voronoi cell would meet at least three other Voronoi cells and so would have at least three barriers lying in its boundary, which would contradict Lemma 5.7.(1). Thus, we conclude that the graph Γ is an interval. \square

We deduce from Lemma 5.8 and Lemma 5.7.(1)-(2) that the barriers of D form a sequence of nonselfintersecting arcs $\alpha_1, \dots, \alpha_{n-1}$ such that for every $i = 1, \dots, n-1$, the arc α_i separates $\alpha_1, \dots, \alpha_{i-1}$ from $\alpha_{i+1}, \dots, \alpha_{n-1}$ in D , cf. Figure 3. Up to permutation of the indices in the Voronoi cells, each cell V_i is bounded by α_{i-1} and α_i (where α_0 and α_n are empty sets).

The vertices v_2, \dots, v_{n-1} lie in the same arc of ∂D with endpoints v_1 and v_n . Otherwise, there would be two Voronoi cells V_i and V_{i+1} meeting along α_i with $2 \leq i \leq n-2$ whose centers, separated by v_1 and v_n , would not be adjacent, which is impossible in Case I. We deduce that every vertex v_i is adjacent to v_{i-1} and v_{i+1} , where the indices are taken modulo n . These vertices decompose the circle ∂D into n arcs $\overline{v_i v_{i+1}}$ of length at most 2δ with endpoints v_i and v_{i+1} .

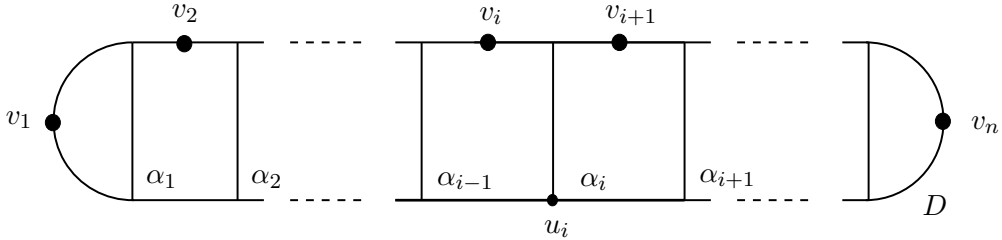


FIGURE 3. Sequence of ordered Voronoi cells in D

Denote by u_i the intersection point of α_i with $\overline{v_1 v_n}$. Clearly, the point u_i is at distance at most 2δ from v_1 . Since u_i lies in V_i , the distance between u_i and v_i is at most δ . Combining these estimates, we derive that the segment $[v_1 v_i]$ (choose one) is of length at most 3δ .

These segments, for i running between 3 and $n - 1$, decompose the polygon D into $n - 2$ triangles with vertices v_1, v_j, v_{j+1} , where $j = 2, \dots, n - 1$. By construction, the length of the edges of these triangles is at most 3δ . This finishes the proof in the Case I.

Case II - There exist two non-adjacent vertices v_i and v_j whose Voronoi cells intersect at some point p .

By construction, the point p is closer to v_i and v_j than to any other vertex v_k . In particular, $|pv_i| = |pv_j| \leq \delta$. Thus, the arc formed of the two minimizing segments $[p, v_i]$ and $[p, v_j]$ (choose any) divide D into two polygons D' and D'' , which satisfy the assumption (1) of Proposition 5.6. (In this construction, the point p is not a vertex of D' and D'' .)

We will denote by d_D the distance induced on D and by $d_{D'}$ and $d_{D''}$ the length distance induced on D' and D'' . Note that $d_D \leq d_{D'}$ and $d_D \leq d_{D''}$. Let x be a point in the interior of D' . Denote by v the closest vertex of D from x . If the (minimizing) segment $[x, v]$ lies in D' , then

$$d_{D'}(x, v) = d_D(x, v) \leq \delta \quad (5.1)$$

by the assumption (2) of Proposition 5.6. Otherwise, the segment $[x, v]$ intersects $[p, v_i]$ or $[p, v_j]$. Say it intersects $[p, v_i]$ at some point q . We have $|qv_i| \leq |qv|$, otherwise

$$\begin{aligned} |pv_i| &= |pq| + |qv_i| \\ &> |pq| + |qv| \\ &> |pv| \end{aligned}$$

which contradicts the definition of v . Thus,

$$\begin{aligned} |xv| &= |xq| + |qv| \\ &\geq |xq| + |qv_i| \\ &\geq |xv_i|. \end{aligned}$$

Actually, by smoothing out at q the piecewise geodesic arc formed of the two segments $[x, q]$ and $[q, v_i]$, we derive that x is closer from v_i than from v , that is, $|x, v| > |xv_i|$, which contradicts the definition of v .

Therefore, the distance with respect to $d_{D'}$ between x and the set of vertices of D' is at most δ , cf. (5.1). The same holds for points of D'' with the vertices of D'' . That is, both D' and D'' satisfy the assumption (2) of Proposition 5.6.

Now, since the vertices v_i and v_j are non-adjacent in D , the numbers of vertices of D' and D'' , denoted by n' and n'' , are less than n . By induction, the polygons D' and D'' can be "triangulated" into $n' - 2$ and $n'' - 2$ triangles of side length at most 3δ . Putting together these "triangulations", we obtain a "triangulation" of D compatible with its polygonal structure with $n' + n'' - 4$ triangles of side length at most 3δ . Now, as each of the vertices v_i and v_j are split into two vertices in the new configuration, we derive that

$$n' + n'' = n + 2$$

which shows that the “triangulation” of D previously obtained has $n - 2$ triangles as desired. \square

As previously mentioned, the combination of Proposition 5.5 and Proposition 5.6 immediately yields Theorem 5.2.

We can now prove the main result of this section about a topologically refined width-diameter inequality.

Theorem 5.9. *Let M be a closed Riemannian surface of genus g . Then there exists a homologically substantial one-cycle sweepout of M defined from some maps $\varphi : \Sigma \rightarrow M$ and $h : \Sigma \rightarrow \Gamma$ as in Definition 1.2, where Σ is a closed surface of genus at most $2g$, such that*

$$\text{length } \varphi[h^{-1}(t)] \leq 18 \text{ diam}(M)$$

for every $t \in \Gamma$.

Proof. For Riemannian two-spheres, *i.e.*, $g = 0$, the theorem immediately follows from Theorem 4.1. Thus, we can assume that $g \geq 1$.

Even though the proof of Theorem 5.9 does not rely on filling radius estimates as it was the case with Theorem 3.3, it is based on similar obstruction arguments.

Suppose the conclusion of Theorem 5.9 does not hold. We want to derive a contradiction by constructing a map $f : Q \rightarrow M$ from a 3-pseudomanifold Q with $\partial Q = M$ such that the restriction of f to ∂Q is the identity map.

Choose a “triangulation” of M as in Theorem 5.2. Consider an abstract cone P over M with the coned-off “triangulation” induced from M . We extend the identity map on M to the 1-skeleton of P as follows. Send the principal vertex of P to any point in M . Then send every edge of P arising from the principal vertex to a segment of M joining the images of its endpoints. The length of this segment is at most $\text{diam}(M)$. Thus, the images of the boundaries of the 2-simplices of P based at its principal vertex are loops of total length at most $5 \text{ diam}(M)$. Note that the perimeter of each triangle in the “triangulation” of M is at most $9 \text{ diam}(M)$.

Now, contract these loops to point curves – case (F_1) – or closed geodesics – case (F_2) – by applying a curve shortening flow as in the proof of Theorem 3.3, Part III. This gives rise to a map

$$f : P_0^2 \cup \partial M \rightarrow M$$

extending the identity map, where P_0^2 is the 2-skeleton of P with a small disk removed around the centers of the faces corresponding to the case (F_2) .

Consider the graph Γ lying in the dual 1-skeleton of P^1 in P defined as in the proof of Theorem 3.3, Part III. For every 3-simplex Δ^3 of P^3 , replace the edge of $\Gamma \cap \Delta^3$ pointing to the center of the face Δ^2 of $\partial \Delta^3$ lying in $M = \partial P$ with an edge of half its length. This ensures that the resulting graph Γ' does not have any terminal vertex lying in M (a property that may fail for Γ as the boundaries of the triangles of the “triangulation” of M may not converge to point curves under the curve shortening flow). For every 3-simplex Δ^3 of P^3 , we can also replace the edge of $\Gamma \cap \Delta^3$ pointing to the

center of the face Δ^2 of $\partial\Delta^3$ lying in $M = \partial P$ with the tree Γ_{Δ^2} given in Theorem 4.1 along with a map $h_{\Delta^2} : \Delta^2 \subset M \rightarrow \Gamma_{\Delta^2}$. The resulting graph is denoted by Γ'' .

As in the proof of Theorem 3.3, Part III, the map $f : P_0^2 \cup \partial M \rightarrow M$ extends to

$$f : (P \setminus U_\epsilon(\Gamma')) \cup \partial P \rightarrow M.$$

For $\epsilon > 0$ small enough, the ϵ -neighborhood $U_\epsilon(\Gamma')$ of Γ' is a compact 3-manifold. Its boundary $\Sigma = \partial U_\epsilon(\Gamma')$ is a closed surface obtained by assembling punctured spheres, one for each 3-simplex of P . These spheres have at most three holes. Thus, the genus of Σ is bounded by half the number of 3-simplices in P plus one. That is,

$$\text{genus}(\Sigma) \leq 2g.$$

Now, we want to define a map $h : \Sigma \rightarrow \Gamma''$ by giving its restriction

$$h_{\Delta^3} : \Sigma \cap \Delta^3 \rightarrow \Gamma''$$

to the intersection of Σ with each 3-simplex Δ^3 of P^3 by identifying $\Sigma \cap \Delta^3$ with $\partial\Delta^3 \cap U_\epsilon(\Gamma')$ as in the proof of Theorem 3.3. Specifically, we define h_{Δ^3} on each face Δ^2 or punctured face Δ_ϵ^2 of $\partial\Delta^3 \setminus U_\epsilon(\Gamma')$ not lying in M as in the proof of Theorem 3.3, Part IV. For the face Δ^2 of Δ^3 lying in M , the restriction of h_{Δ^3} to Δ^2 agrees with the map $h_{\Delta^2} : \Delta^2 \rightarrow \Gamma_{\Delta^2}$ whose fibers are of length at most $18 \text{diam}(M)$, *cf.* Theorem 4.1. Thus, the restriction $\varphi : \Sigma \rightarrow M$ of f to Σ comes along with a map

$$h : \Sigma \rightarrow \Gamma''$$

such that the images by φ of the fibers of h are of length less than $18 \text{diam}(M)$. (See the proof of Theorem 3.3, Part IV, for further details.) That is, for every $t \in \Gamma$, we have

$$\text{length } \varphi[h^{-1}(t)] < 18 \text{diam}(M).$$

By assumption, this implies that the map $\varphi : \Sigma \rightarrow M$ takes the fundamental class $[\Sigma]$ of Σ to zero. Therefore, the map φ extends to a 3-pseudomanifold with boundary Σ . This yields a map $F : Q \rightarrow M$ defined on some 3-pseudomanifold Q with the same boundary as P , which agrees with the identity map on $\partial Q = M$. Hence the desired contradiction. \square

Remark 5.10. Theorem 4.5 provides an upper bound on the relative width of a genus g surface M in terms of its diameter and the length of its longest boundary component. As in Theorem 5.9, we can impose on this estimate an *a priori* $2g$ bound on the genus of the surface Σ involved in the homologically substantial one-cycle sweepouts considered in the definition of the relative width, *cf.* Definition 4.2.

REFERENCES

- [A62] F. Almgren: The homotopy groups of the integral cycle groups. *Topology* 1 (1962) 257–299.
- [BS10] F. Balacheff, S. Sabourau: Diastolic and isoperimetric inequalities on surfaces. *Ann. Sci. Éc. Norm. Supér. (4)* 43 (2010), no. 4, 579–605.
- [CE75] J. Cheeger, D. Ebin: Comparison theorems in Riemannian geometry, North-Holland, Amsterdam, 1975.

- [GL17] P. Glynn-Adey, Y. Liokumovich: Width, Ricci curvature, and minimal hypersurfaces, *J. Diff. Geom.* 105 (2017), no. 1, 33–54.
- [Gr83] M. Gromov: Filling Riemannian manifolds. *J. Diff. Geom.* 18 (1983), no. 1, 1–147.
- [Gr96] M. Gromov: Systoles and intersystolic inequalities. In Actes de la Table Ronde de Géométrie Différentielle (Luminy, 1992), *Sémin. Congr.* 1, Soc. Math. France, (1996), 291–362.
- [Gr99] M. Gromov: Metric Structures for Riemannian and Non-Riemannian Spaces. *Progr. Math.* 152. Birkhäuser, 1999.
- [Gr03] M. Gromov: Isoperimetry of waists and concentration of maps. *Geom. Funct. Anal.* 13 (2003), no. 1, 178–215.
- [Gr] M. Gromov: Morse spectra, homology measures, spaces of cycles and parametric packing problems. See arXiv:1710.03616.
- [Gu07] L. Guth: The width-volume inequality. *Geom. Funct. Anal.* 17 (2007), no. 4, 1139–1179.
- [Gu09] L. Guth: Minimax problems related to cup powers and Steenrod squares. *Geom. Funct. Anal.* 18 (2009), no. 6, 1917–1987.
- [Gu10] L. Guth: Metaphors in systolic geometry. *Proc. Internat. Congr. Math.* (2010), vol. II, 745–768.
- [HS94] J. Hass, P. Scott: Shortening curves on surfaces. *Topology* 33 (1994), no. 1, 25–43.
- [Ha02] A. Hatcher: Algebraic topology. Cambridge University Press, 2002.
- [K83] M. Katz: The filling radius of two-point homogeneous spaces. *J. Diff. Geom.* 18 (1983), no. 3, 505–511.
- [KS01] M. Katz, A. Suci: Systolic freedom of loop space. *Geom. Funct. Anal.* 11 (2001), no. 1, 60–73.
- [L14] Y. Liokumovich: Surfaces of small diameter with large width. *J. Topol. Anal.* 6 (2014), no. 3, 383–396.
- [L16] Y. Liokumovich: Families of short cycles on Riemannian surfaces. *Duke Math. J.* 165 (2016), no. 7, 1363–1379.
- [LMN] Y. Liokumovich, F. Marques; A. Neves: Weyl law for the volume spectrum. *Ann. of Math.*, to appear. See arXiv:1607.08721.
- [LZ18] Y. Liokumovich, X. Zhou: Sweeping out 3-manifold of positive Ricci curvature by short 1-cycles via estimates of min-max surfaces. *Int. Math. Res. Not.* 4 (2018), 1129–1152. See arXiv:1510.02896.
- [MN14] F. Marques, A. Neves: Min-max theory and the Willmore conjecture. *Ann. of Math. (2)* 179 (2014), no. 2, 683–782.
- [MN17] F. Marques, A. Neves: Existence of infinitely many minimal hypersurfaces in positive Ricci curvature. *Invent. Math.* 209 (2017), no. 2, 577–616.
- [PS] P. Papasoglu, E. Swenson: A sphere hard to cut. See arXiv:1509.02307.
- [P81] J. Pitts: Existence and regularity of minimal surfaces on Riemannian manifolds. *Math. Notes*, vol. 27, Princeton University Press, 1981.
- [S17] S. Sabourau: Volume of minimal hypersurfaces in manifolds with nonnegative Ricci curvature. *J. Reine Angew. Math.* 731 (2017), 1–19.

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