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A SYSTOLIC-LIKE EXTREMAL GENUS TWO SURFACE

STÉPHANE SABOURAU AND ZEINA YASSINE

ABSTRACT. It is known that the genus two surface admits a piecewise flat metric with conical singularities which is extremal for the systolic inequality among all nonpositively curved metrics. We prove that this piecewise flat metric is also critical for slow metric variations, without curvature restrictions, for another type of systolic inequality involving the lengths of the shortest noncontractible loops in different free homotopy classes.

1. INTRODUCTION

We are interested in optimal geometric inequalities relating the area of a closed Riemannian surface Σ to the lengths of the shortest loops in certain homotopy classes. A typical example is given by the systolic inequalities, which relate the area of the surface to the systole, that is, the length of the shortest noncontractible loop in Σ . The first known systolic inequality is due to C. Loewner in 1949, who proved that every Riemannian two-torus (\mathbb{T}^2, g) satisfies

$$\text{area}(g) \geq \frac{\sqrt{3}}{2} \text{sys}^2(g),$$

where $\text{sys}(g)$ denotes the systole of the torus. Furthermore, the equality holds if and only if the torus is endowed with a flat hexagonal metric. C. Loewner did not publish his result, however it was mentioned by his student P. Pu, *cf.* [Pu52], who established a sharp systolic inequality on the projective plane \mathbb{RP}^2 . In this case, the equality is attained precisely by the round metrics. More than thirty years later, C. Bavard, *cf.* [Ba86] (see also [Sak88]) proved a sharp systolic inequality on the Klein bottle \mathbb{K} . Here, the extremal metrics are not smooth. These are the only manifolds for which an optimal systolic inequality is known. All these systolic inequalities can be written as follows. If

$$\text{sys}(g) \geq \text{sys}(g_0)$$

then

$$\text{area}(g) \geq \text{area}(g_0) \tag{1.1}$$

where g_0 is the extremal metric in each of these cases.

Optimal systolic-like inequalities, *i.e.*, inequalities that relate the area to the product of the lengths of the shortest loops or arcs in different relative homotopy classes are known only for the two-torus \mathbb{T}^2 , *cf.* [Ke67], the Klein

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bottle \mathbb{K} , *cf.* [Ba06, EY15], and the Mobius band \mathbb{M} , *cf.* [Pu52, Bl61, Ba06, EY15].

For instance, the following result follows as a particular case of L. Keen's optimal systolic-like inequality for the two-torus, *cf.* [Ke67]. Let (\mathbb{T}^2, g_0) be a flat torus. Denote by $\text{sys}_1(g)$ and $\text{sys}_2(g)$ the lengths of the shortest loops on \mathbb{T}^2 inducing a basis in homology. If

$$\begin{cases} \text{sys}_1(g) \geq \text{sys}_1(g_0) \\ \text{sys}_2(g) \geq \text{sys}_2(g_0) \end{cases}$$

then

$$\text{area}(g) \geq \text{area}(g_0).$$

About ten years ago, M. Katz and the first author, *cf.* [KS06], proved that there exists a piecewise flat metric g_0 on the genus two surface Σ_2 which is extremal among all nonpositively curved Riemannian metrics. This metric is composed of six identical flat regular octagons Ω_k with the identifications given in Figure 1. It admits regions where only one systolic loop, *i.e.*, a noncontractible loop of length the systole, passes through every point. Hence this metric cannot be extremal for the general systolic inequality, *i.e.*, without restriction on the curvature. This follows from a classical result in the field which says that at least two systolic loops pass through every point of a systolically extremal surface. In fact, the non-extremality of the metric g_0 for the systolic inequality can also be deduced from a result of [Sa04], which says that no flat metric with conical singularities is extremal for the systolic inequality in genus two.

We need to introduce a few definitions related to the structure of this piecewise flat surface.

Definition 1.1. Simple closed geodesics of (Σ_2, g_0) parallel to the edges of the octagons Ω_k fall into two categories of curves of the same length: curves passing through opposite sides of the octagons and curves not passing through opposite sides of the octagons. The simple closed geodesics of the former category of curves will be referred to as ℓ_1 -loops. Similarly, the simple closed geodesics of the latter category will be referred to as ℓ_2 -loops. Note that the ℓ_1 -loops are the systolic loops of g_0 . However the ℓ_2 -loops are not the second shortest noncontractible loops of g_0 since there exist isolated shorter noncontractible loops which are not systolic.

Definition 1.2. Denote by Λ_i the subset of the free homotopy classes generated by the ℓ_i -loops. For a Riemannian metric g on Σ_2 , define

$$\ell_i(g) = \inf_{\langle \gamma \rangle \in \Lambda_i} \ell_g(\gamma). \quad (1.2)$$

Note that $\ell_1(g_0) = \text{sys}(g_0)$.

We observe that the piecewise flat metric g_0 defined on the genus two surface Σ_2 may be a potential extremal metric for another systolic problem on Σ_2 involving ℓ_1 and ℓ_2 . The reason for such a claim follows from some geometric properties of the surface (Σ_2, g_0) :

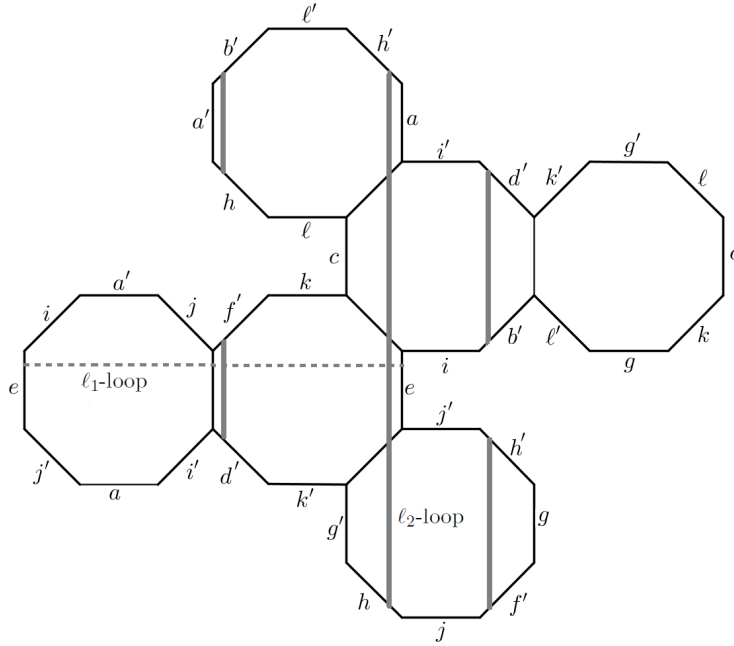


FIGURE 1. The critical surface (Σ_2, g_0) is isometric to the to the piecewise flat genus two surface composed of six identical regular octagons with the identifications given in the figure.

- First, as mentioned before, the systolic loops of a systolically extremal surface cover the surface. In our case, the ℓ_1 -loops and the ℓ_2 -loops cover the surface Σ_2 . Here, by an ℓ_i -loop we mean a loop of length $\ell_i(g_0)$ in (Σ_2, g_0) whose free homotopy class lies in Λ_i .
- Second, the unit tangent vectors of these ℓ_i -loops are well distributed on each tangent plane of the surface. More precisely, their convex hull forms a regular octagon on these tangent planes. In general, a similar pattern occurs for systolically extremal surfaces. Indeed, the convex hull of the unit tangent vectors of the systolic loops on all the known systolically extremal surfaces is symmetric.
- Finally, systolically extremal surfaces of genus at least two tend to have flat regions, *cf.* [Ca96, Br96], and the same applies for systolic-like inequalities.

In [Ca96], E. Calabi described two piecewise flat genus three surfaces and conjectured that one of them is the global minimum for the systolic inequality. Later, the first author, *cf.* [Sa11], proved that these two genus three surfaces are critical for the systolic inequality with respect to slow metric variations. Of course, the systolic volume functional is not necessarily differentiable and an adequate notion of criticality needs to be introduced. The notion of criticality used in [Sa11] was introduced in [Na96, EI00] to study the Riemannian surfaces that maximize the product of the area with the first nonzero eigenvalue of the Laplacian. Other notions of systolically critical metrics were used in [Bal06, Bal10, Sa10, AB14].

In this article we show that the metric g_0 on the genus two surface Σ_2 is critical in the sense of [Na96, EI00, Sa11] for slow metric variations for some systolic-like inequality. More precisely, we prove

Theorem 1.3. *Let (g_t) be a slow metric variation of g_0 defined on the genus two surface Σ_2 . If $\ell_1(g_t) \geq \ell_1(g_0)$ and $\ell_2(g_t) \geq \ell_2(g_0)$, then*

$$\text{area}(g_t) \geq \text{area}(g_0) + o(t).$$

The slow metric variations involved in Theorem 1.3 are analogous to those defined in [Sa11]. We refer to Definition 4.1 for a precise definition of these metric variations and to the last section of [Sa11] for examples. Observe for instance that deforming the regular octagons composing the extremal surface (Σ_2, g_0) into non-regular octagons gives rise to a slow metric variation.

With the exception of some recent proofs of systolic inequalities, *cf.* [Iv02, Sa10], and systolic-like inequalities, *cf.* [SY16], on the projective plane \mathbb{RP}^2 , the two-torus \mathbb{T}^2 and the Mobius band \mathbb{M} , all the other known proofs of the aforementioned optimal geometric inequalities require the uniformization theorem as a main tool. In our proof of Theorem 1.3, we do not make use of the uniformization theorem. Instead, we rely on recent calibrating methods, *cf.* [Iv02, Iv11, Sa11]. Briefly, we embed the universal cover $\tilde{\Sigma}_2$ of the genus two surface Σ_2 into an infinite-dimensional space \mathbb{R}^∞ as follows. Given that $\ell_i(g) \geq \ell_i(g_0)$, we construct a 1-Lipschitz equivariant map

$$\Psi^g : \tilde{\Sigma}_2 \rightarrow \mathbb{R}^\infty$$

using the Busemann functions induced by the lifts of the ℓ_i -loops of Σ_2 , *cf.* Section 3. Then, we introduce an appropriate infinitesimally calibrating two-form ω on \mathbb{R}^∞ whose pull-back passes to the quotient on the surface Σ_2 . This allows us to show that

$$\int_{\Sigma_2} (\Psi^g)^* \omega \leq c_0 \text{area}(\Sigma_2, g)$$

for every Riemannian metric g on Σ_2 , where c_0 is a sharp positive constant. Moreover, the equality holds if $g = g_0$. Finally, we prove that if $(g_t)_{t \geq 0}$ is a slow metric variation then

$$\int_{\Sigma_2} (\Psi^{g_t})^* \omega - \int_{\Sigma_2} (\Psi^{g_0})^* \omega = o(t).$$

This completes the proof, *cf.* Section 4.

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2. AN EXTREMAL PIECEWISE FLAT METRIC IN GENUS TWO

In this section we provide a description for the critical piecewise flat genus two surface and we introduce some notations and definitions.

The Bolza surface \mathfrak{B} is a genus two Riemann surface. It is the smooth completion of the smooth affine algebraic curve

$$y^2 = x^5 - x. \tag{2.1}$$

The set $\{-1, 0, 1, -i, i, \infty\}$ of roots of the polynomial (2.1) (including the point at infinity) corresponds to the set of vertices of the regular octahedral triangulation of the Riemann sphere S^2 under the conformal stereographic projection. Hence, these six points can be identified with the ramification points of the ramified conformal double cover $Q : \mathfrak{B} \rightarrow S^2$, or equivalently, with the Weierstrass points of \mathfrak{B} .

The conformal class of the Bolza surface \mathfrak{B} admits a piecewise flat non-positively curved Riemannian metric g_0 with 16 conical singularities whose orientation-preserving isometry group is isomorphic to the automorphism group of \mathfrak{B} . This metric g_0 has been introduced by M. Katz and the first author in [KS06], where they prove that it is extremal for the systolic inequality among all nonpositively curved metrics on the genus two surface Σ_2 . The metric g_0 , defined up to homothety, on Σ_2 is isometric to the piecewise flat genus two surface composed of six identical flat regular octagons Ω_k with the identifications given in Figure 1. The Weierstrass points of \mathfrak{B} correspond to the centers of the octagons and the 16 conical singularities are located at their vertices.

Denote by \hbar the length of an edge of a regular octagon Ω_k of g_0 . We have

$$\text{area}(g_0) = 12(1 + \sqrt{2})\hbar^2.$$

The systole $\ell_1(g_0)$ of the surface (Σ_2, g_0) is computed in [KS06, Lemma 3.2]. Namely,

$$\ell_1(g_0) = 2(1 + \sqrt{2})\hbar.$$

A direct computation shows that

$$\ell_2(g_0) = 4(2 + \sqrt{2})\hbar.$$

Recall that an ℓ_i -loop is a loop in (Σ_2, g_0) of length $\ell_i(g_0)$ and whose free homotopy class lies in $\Lambda_i = \Lambda_i(g_0)$, *cf.* Definition 1.1 and Definition 1.2. We define an ℓ_i -band as follows.

Definition 2.1. Let $\langle \alpha \rangle$ be a free homotopy class in Λ_i . The ℓ_i -loops in $\langle \alpha \rangle$ are parallel to each other and form a flat cylinder $\mathcal{B}_{\langle \alpha \rangle}$ of height \hbar if $i = 1$ and $\frac{\sqrt{2}}{2}\hbar$ if $i = 2$. Such a cylinder $\mathcal{B}_{\langle \alpha \rangle}$ will be called an ℓ_i -band of (Σ_2, g_0) . The *soul* of an ℓ_i -band $\mathcal{B}_{\langle \alpha \rangle}$ is the ℓ_i -loop of $\mathcal{B}_{\langle \alpha \rangle}$ equidistant from the boundary components of the ℓ_i -band.

The intersections of the ℓ_i -bands decompose the surface Σ_2 into 150 polygons, *cf.* Figure 2. These polygons define four regions $\mathcal{R}_1, \dots, \mathcal{R}_4$ depending on the number of ℓ_i -loops that pass through every point in their interior. More precisely, such a polygon Δ_k lies in a region \mathcal{R}_k if through every point in its interior pass exactly $2k$ oriented ℓ_1 -loops and $8 - 2k$ oriented ℓ_2 -loops, *cf.* Figure 2. Hence, every region \mathcal{R}_k with $k \neq 4$ is composed of exactly 48 identical polygons Δ_k (with 8 polygons in each octagon), while the region \mathcal{R}_4 is composed of only 6 identical polygons Δ_4 . Observe that the polygons $\Delta_1, \Delta_2, \Delta_3$ and Δ_4 have the shape of a right isosceles triangle, a kite, a right isosceles triangle and a (small) octagon. Furthermore, the right isosceles triangles Δ_1 are the only polygons which have an edge in common with the six regular flat octagons forming the critical surface (Σ_2, g_0) , namely their hypotenuses coincide with the edges of the regular octagons Ω_k .

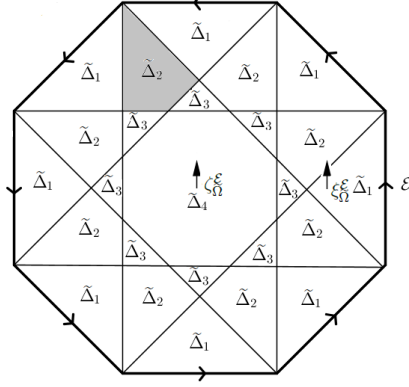


FIGURE 2. A flat regular octagon $\tilde{\Omega}$ of the metric \tilde{g}_0 .

Fix an orientation on Σ_2 . Let Ω be one of the six regular octagons composing the surface (Σ_2, g_0) . Denote by $(\tilde{\Sigma}_2, \tilde{g}_0)$ the universal cover of (Σ_2, g_0) . Let $\tilde{\Omega}$ be a lift of Ω in the universal cover $\tilde{\Sigma}_2$ and let \mathcal{E} be an (oriented) edge of the regular octagon $\tilde{\Omega}$. Moreover, let $\tilde{\Delta}_k$ be a lift of the polygon Δ_k in $\tilde{\Sigma}_2$. The souls of the ℓ_i -bands lift to $\tilde{\Sigma}_2$ as follows.

Definition 2.2. An ℓ_1 -direction of $(\tilde{\Sigma}_2, \tilde{g}_0)$ is a \tilde{g}_0 -unit vector $\zeta = \zeta_{\tilde{\Omega}}^{\xi}$ based at the center of $\tilde{\Omega}$ and pointing in the same direction as \mathcal{E} , cf. Figure 2. The unit vector ζ induces a geodesic line \tilde{c}_{ζ} on $(\tilde{\Sigma}_2, \tilde{g}_0)$ such that

$$\tilde{c}'_{\zeta}(0) = \zeta.$$

The projection c_{ζ} of \tilde{c}_{ζ} to (Σ_2, g_0) is a closed geodesic curve which coincides with the soul of the ℓ_1 -band $\mathcal{B}_{\langle c_{\zeta} \rangle}$.

Now, denote by $x_{\mathcal{E}}$ the point of $\tilde{\Omega}$ lying in the perpendicular bisector of \mathcal{E} and at distance $\frac{\sqrt{2}}{4}h$ from \mathcal{E} . (Recall that h is the side length of $\tilde{\Omega}$.) Note that the point $x_{\mathcal{E}}$ lies in the right isosceles triangle $\tilde{\Delta}_1$ of $\tilde{\Omega}$ with hypotenuse \mathcal{E} .

Definition 2.3. An ℓ_2 -direction of $(\tilde{\Sigma}_2, \tilde{g}_0)$ is a \tilde{g}_0 -unit vector $\xi = \xi_{\tilde{\Omega}}^{\xi}$ based at $x_{\mathcal{E}}$ and pointing in the same direction as the edge \mathcal{E} , cf. Figure 2. This unit vector ξ induces a geodesic line \tilde{c}_{ξ} on $(\tilde{\Sigma}_2, \tilde{g}_0)$ such that

$$\tilde{c}'_{\xi}(0) = \xi.$$

The projection c_{ξ} of \tilde{c}_{ξ} to (Σ_2, g_0) is a closed geodesic curve which coincides with the soul of the ℓ_2 -band $\mathcal{B}_{\langle c_{\xi} \rangle}$.

Remark 2.4. In what follows, we will replace the symbols ζ and ξ by ν when there is no need to distinguish between an ℓ_1 -direction and an ℓ_2 -direction. Moreover, the indices $\tilde{\Omega}$ and \mathcal{E} in $\nu_{\tilde{\Omega}}^{\xi}$ will be omitted.

3. BUSEMANN FUNCTIONS AND CALIBRATING FORMS

In this section, we introduce some notations and preliminary results that will be useful in our proof of Theorem 1.3.

Definition 3.1. Let ν be an ℓ_i -direction of $(\tilde{\Sigma}_2, \tilde{g}_0)$. Extend ν to a map, still denoted by

$$\nu : \mathbb{R} \rightarrow U_{\tilde{g}_0} \tilde{\Sigma}_2$$

defined as $\nu(s) = \tilde{c}'_\nu(s)$. In particular, $\nu(0) = \nu$.

Along with the metric g_0 , consider another Riemannian metric g defined on Σ_2 and denote by \tilde{g} its lift to $\tilde{\Sigma}_2$. Throughout the article, we will assume that

$$\ell_1(g) \geq \ell_1(g_0) \text{ and } \ell_2(g) \geq \ell_2(g_0).$$

Definition 3.2. Let ν be a \tilde{g}_0 -unit vector tangent to a geodesic line induced by an ℓ_i -direction of $(\tilde{\Sigma}_2, \tilde{g}_0)$. Define the *Busemann function* $b_\nu^g : \tilde{\Sigma}_2 \rightarrow \mathbb{R}$ as

$$b_\nu^g(x) = \limsup_{t \rightarrow \infty} d_{\tilde{g}}(x, \tilde{c}_\nu(t)) \ell_i(g_0) - t \ell_g(\langle c_\nu \rangle)$$

where \tilde{c}_ν is the \tilde{g}_0 -geodesic line induced by ν and c_ν is its projection to Σ_2 .

Here, $\ell_g(\langle c_\nu \rangle)$ denotes the smallest length of a noncontractible loop in the free homotopy class $\langle c_\nu \rangle$ of c_ν with respect to the Riemannian metric g .

Remark 3.3. Contrary to the original definition of a Busemann function, cf.[BH99], the line \tilde{c}_ν is geodesic for \tilde{g}_0 but not for \tilde{g} . Moreover, the function

$$h : \mathbb{R} \rightarrow \mathbb{R} \\ t \mapsto d_{\tilde{g}}(x, \tilde{c}_\nu(t)) \ell_i(g_0) - t \ell_g(\langle c_\nu \rangle)$$

which appears in Definition 3.2, is not necessarily non-increasing.

Proposition 3.4. *The real function h is bounded.*

Proof. By the triangle inequality, we obtain

$$h(t) \geq d_{\tilde{g}}(\tilde{c}_\nu(0), \tilde{c}_\nu(t)) \ell_i(g_0) - t \ell_g(\langle c_\nu \rangle) - d_{\tilde{g}}(x, \tilde{c}_\nu(0)) \ell_i(g_0). \quad (3.1)$$

Let β be a length-minimizing loop of (Σ_2, g) in the homotopy class of c_ν , that is, $\ell_g(\beta) = \ell_g(\langle c_\nu \rangle)$. Let $[t]_i$ be the integer part of $\frac{t}{\ell_i(g_0)}$, that is,

$$[t]_i = \max\{k \in \mathbb{N} \mid t \geq k \ell_i(g_0)\}.$$

We have

$$[t]_i \ell_i(g_0) - t \geq -\ell_i(g_0). \quad (3.2)$$

Since the genus two surface is orientable, the $[t]_i$ -iterate of β is length-minimizing in its homotopy class with respect to g , cf. [Gr99, Lemma 4.32]. Let $\tilde{\gamma}$ in $\tilde{\Sigma}_2$ be a length-minimizing arc connecting $\tilde{c}_\nu(0)$ to $\tilde{c}_\nu([t]_i)$. Its projection γ to Σ_2 is homotopic to $c_\nu^{[t]_i}$ and its length satisfies the two relations

$$\ell_g(\gamma) = d_{\tilde{g}}(\tilde{c}_\nu(0), \tilde{c}_\nu([t]_i))$$

and

$$\ell_g(\gamma) \geq [t]_i \ell_g(\langle c_\nu \rangle). \quad (3.3)$$

Combining (3.1), (3.2) and (3.3), we deduce that the map h is bounded from below.

Now, we show that the map h is bounded from above. Indeed, by the triangle inequality, we derive that

$$h(t) \leq \{d_{\tilde{g}}(x, \tilde{c}_\nu(0)) + d_{\tilde{g}}(\tilde{c}_\nu(0), \tilde{c}_\nu([t]_i)) + d_{\tilde{g}}(\tilde{c}_\nu([t]_i), \tilde{c}_\nu(t)) - [t]_i \ell_g(\langle c_\nu \rangle)\} \ell_i(g_0).$$

Moreover,

$$d_{\tilde{g}}(\tilde{c}_\nu(0), \tilde{c}_\nu([t]_i)) - [t]_i \ell_g(\langle c_\nu \rangle) \leq 2 \operatorname{diam}(\Sigma_2, g). \quad (3.4)$$

Indeed, let γ be a g -length minimizing loop in the free homotopy class $\langle c_\nu \rangle$, that is, $\ell_g(\gamma) = \ell_g(\langle c_\nu \rangle)$. Let α be a length-minimizing arc joining $c_\nu(0)$ to γ . We deduce that

$$d_{\tilde{g}}(\tilde{c}_\nu(0), \tilde{c}_\nu([t]_i)) \leq \ell_g(\alpha \cup \gamma^{[t]_i} \cup \alpha^{-1}) \leq \ell_g(\gamma^{[t]_i}) + \operatorname{diam}(\Sigma_2, g).$$

Then, the inequality (3.4) is satisfied. Furthermore,

$$d_{\tilde{g}}(\tilde{c}_\nu([t]_i), \tilde{c}_\nu(t)) \leq \operatorname{diam}(\Sigma_2, g).$$

This shows that h is bounded from above. \square

Definition 3.5. The map $\Psi_\nu^g : \tilde{\Sigma}_2 \rightarrow \mathbb{R}$ induced by an ℓ_i -direction ν , cf. Definitions 2.2 and 2.3, is defined as

$$\Psi_\nu^g(x) = \frac{1}{\ell_i(g_0) \ell_g(\langle c_\nu \rangle)} \int_{\mathbb{R}/\ell_i(g_0)\mathbb{Z}} \frac{b_{\nu(s)}^g(x) - b_{-\nu(s)}^g(x)}{2} ds,$$

where $\nu(s) = \tilde{c}_\nu(s)$, cf. Definition 3.1.

Example 3.6. If $g = g_0$, the g_0 -gradient $\nabla_x \Psi_\nu^{g_0}$ is a g_0 -unit tangent vector parallel to and pointing in the same direction as the ℓ_i -direction ν .

Proposition 3.7. *Let ν be an ℓ_i -direction of $(\tilde{\Sigma}_2, \tilde{g}_0)$. The map Ψ_ν^g satisfies the following properties:*

- (1) *It is 1-Lipschitz and differentiable almost everywhere.*
- (2) $\Psi_{-\nu}^g = -\Psi_\nu^g$.
- (3) *The differential of the map Ψ_ν^g depends only on the oriented ℓ_i -band $\tilde{\mathcal{B}}$ generated by the vector ν .*

Proof. For (1), the map Ψ_ν^g is 1-Lipschitz for the supremum norm as the Busemann function b_ν^g is $\ell_i(g_0)$ -Lipschitz and $\ell_i(g) \geq \ell_i(g_0)$. Hence, it is differentiable almost everywhere by the Rademacher theorem. Part (2) follows directly from Definition 3.5. For (3), let ν and η be two ℓ_i -directions generating the same oriented ℓ_i -band. That is, their basepoints lie in the same ℓ_i -band and the vectors point in the same direction. Recall that $\nu(s) = \tilde{c}_\nu(s)$, cf. Definition 3.1, and the same with η . The definition of ℓ_i -directions and the assumption on the vectors ν and η imply that

$$\eta(s) = \nu \left(s + \frac{k}{n} \ell_i(g_0) \right)$$

for some integer k with $n = 2$ if $i = 1$ and $n = 8$ if $i = 2$. We derive easily from Definition 3.5 that

$$d\Psi_\eta^g = d\Psi_\nu^g.$$

This completes the proof. \square

Let Γ be the deck transformation group of $\tilde{\Sigma}_2$. The \tilde{g}_0 -geodesic line \tilde{c}_ν induces a unique element γ_ν in Γ that leaves \tilde{c}_ν globally invariant and such that

$$\gamma_\nu \cdot \tilde{c}_\nu(s) = \tilde{c}_\nu(s + \ell_i(g_0)).$$

Moreover, we define an action \star of Γ on the set of unit tangent vectors to the geodesic line \tilde{c}_ν as

$$\gamma \star \nu(s) = (\gamma \cdot \tilde{c}_\nu)'(s)$$

where $\gamma \in \Gamma$.

We state now the following properties of the map Ψ_ν^g .

Proposition 3.8. *Let ν be an ℓ_i -direction of $(\tilde{\Sigma}_2, \tilde{g}_0)$. The map $\Psi_\nu^g : \tilde{\Sigma}_2 \rightarrow \mathbb{R}$ satisfies the following properties:*

(1) *It is equivariant, that is, for every γ in Γ ,*

$$\Psi_{\gamma \star \nu}^g(\gamma \cdot x) = \Psi_\nu^g(x).$$

(2) *It passes to the quotient by the cyclic subgroup $\langle \gamma_\nu \rangle$ and induces a map*

$$\bar{\Psi}_\nu^g : \tilde{\Sigma}_2 / \langle \gamma_\nu \rangle \rightarrow \mathbb{R} / \ell_i(g_0)\mathbb{Z}$$

defined as

$$\bar{\Psi}_\nu^g(\gamma_\nu \cdot x) \equiv \Psi_\nu^g(x) \pmod{\ell_i(g_0)}.$$

Proof. Let $x \in \tilde{\Sigma}_2$ and $\gamma \in \Gamma$. Since γ is an isometry and $\gamma \cdot \tilde{c}_\nu = \tilde{c}_{\gamma \star \nu}$, we derive

$$\ell_g(\langle c_{\gamma \star \nu} \rangle) = \ell_g(\langle c_\nu \rangle).$$

Moreover, since the distance $d_{\tilde{g}}$ induced by the Riemannian metric \tilde{g} is Γ -invariant, we derive that $b_{\gamma \star \nu}^g(\gamma \cdot x) = b_\nu^g(x)$. Hence, the desired result

$$\Psi_{\gamma \star \nu}^g(\gamma \cdot x) = \Psi_\nu^g(x).$$

Now, since $\gamma_\nu^{-1} \cdot \tilde{c}_\nu(t) = \tilde{c}_\nu(t - \ell_i(g_0))$, we derive from the Γ -invariance of $d_{\tilde{g}}$ that

$$\begin{aligned} d_{\tilde{g}}(\gamma_\nu \cdot x, \tilde{c}_\nu(t)) &= d_{\tilde{g}}(x, \gamma_\nu^{-1} \cdot \tilde{c}_\nu(t)) \\ &= d_{\tilde{g}}(x, \tilde{c}_\nu(t - \ell_i(g_0))). \end{aligned}$$

Hence,

$$\begin{aligned} b_\nu^g(\gamma_\nu \cdot x) &= \limsup_{t \rightarrow \infty} \ell_i(g_0) d_{\tilde{g}}(x, \tilde{c}_\nu(t - \ell_i(g_0))) - t \ell_g(\langle c_\nu \rangle) \\ &= b_\nu^g(x) - \ell_i(g_0) \ell_g(\langle c_\nu \rangle). \end{aligned}$$

We deduce that

$$\Psi_\nu^g(\gamma_\nu \cdot x) = \Psi_\nu^g(x) - \ell_i(g_0). \quad (3.5)$$

□

Remark 3.9. As a consequence of Proposition 3.8 and the relation (3.5), we deduce the following couple of points:

- (1) $\gamma^*(d\Psi_{\gamma \star \nu}^g) = d\Psi_\nu^g$ for every $\gamma \in \Gamma$,
- (2) The map $\Psi_\nu^g - \Psi_\nu^{g_0}$ passes to the quotient by the cyclic subgroup $\langle \gamma_\nu \rangle$ and induces a real function which will be (improperly) denoted by

$$\bar{\Psi}_\nu^g - \bar{\Psi}_\nu^{g_0} : \tilde{\Sigma}_2 / \langle \gamma_\nu \rangle \rightarrow \mathbb{R}.$$

By definition, this function satisfies

$$(\bar{\Psi}_\nu^g - \bar{\Psi}_\nu^{g_0})(x) = \Psi_\nu^g(x) - \Psi_\nu^{g_0}(x).$$

Actually, these two properties hold if we replace Ψ_ν^g with the Busemann function b_ν^g .

Let $\tilde{\Omega}$ be a regular octagon of the universal cover $(\tilde{\Sigma}_2, \tilde{g}_0)$ of the surface (Σ_2, g_0) . We order the (oriented) edges $(\mathcal{E}^j)_{1 \leq j \leq 8}$ of $\tilde{\Omega}$ with respect to the cyclic order induced by the orientation of $\tilde{\Omega}$ by fixing an initial edge \mathcal{E}^1 . Denote

$$\nu^j = \nu_{\tilde{\Omega}}^{\mathcal{E}^j}.$$

We state the following definition.

Definition 3.10. Let $\mathbb{R}^\infty = \prod_{\tilde{\Omega}} \mathbb{R}^8$ be the infinite product of \mathbb{R}^8 where $\tilde{\Omega}$ runs over all the regular octagons of $(\tilde{\Sigma}_2, \tilde{g}_0)$. Let

$$\Psi^g : \tilde{\Sigma}_2 \rightarrow \mathbb{R}^\infty$$

be the map defined as

$$\begin{aligned} \pi_{\tilde{\Omega}} \circ \Psi^g|_{\tilde{\Omega}} : \tilde{\Omega} &\rightarrow \mathbb{R}^8 \\ x &\mapsto (\Psi_{\nu^j}^g(x))_{j=1}^8 \end{aligned}$$

where

$$\pi_{\tilde{\Omega}} : \mathbb{R}^\infty \rightarrow \mathbb{R}^8$$

is the canonical projection to the factor \mathbb{R}^8 of \mathbb{R}^∞ corresponding to $\tilde{\Omega}$. The vector ν^j is the ℓ_i -direction that is parallel and point in the same direction as the \tilde{g}_0 -unit vector based at x and tangent to the lift of an oriented ℓ_i -loop.

Example 3.11. Consider the regular octagon $\tilde{\Omega}$ in Figure 2. Suppose that the numbering of the edges of $\partial\tilde{\Omega}$ follows a cyclic order starting at \mathcal{E} , then we have

$$\Psi_{\tilde{\Delta}_2}^g = (\Psi_{\zeta^1}^g, \Psi_{\zeta^2}^g, \Psi_{\xi^3}^g, \Psi_{\xi^4}^g, \Psi_{\zeta^5}^g, \Psi_{\zeta^6}^g, \Psi_{\xi^7}^g, \Psi_{\xi^8}^g)$$

where $\tilde{\Delta}_2$ is the gray polygon in Figure 2.

Remark 3.12. Note that the map Ψ^g is not well defined in the lift of the boundaries of the polygons Δ since there is some ambiguity in the choice of ν^j .

Definition 3.13. Consider the two-form

$$\omega_0 = \sum_{j=1}^8 dx_j \wedge dx_{j+2}$$

in \mathbb{R}^8 . We define a two-form $\tilde{\omega}$ on $\tilde{\Sigma}_2$ as the pull-back of ω_0 by the map

$$\pi_{\tilde{\Omega}} \circ \Psi^g|_{\tilde{\Omega}} : \tilde{\Omega} \rightarrow \mathbb{R}^8$$

on every regular octagon $\tilde{\Omega}$. It can be expressed on $\tilde{\Omega}$ as

$$\tilde{\omega} = \sum_{j=1}^8 d\Psi_{\nu^j}^g \wedge d\Psi_{\nu^{j+2}}^g$$

where $\nu^j = \nu_{\tilde{\Omega}}^j$ and all indices j are taken modulo 8. This defines a two-form on $\tilde{\Sigma}_2$ that we still denote by $\tilde{\omega}$.

Note that the two-form $\tilde{\omega}$ in Definition 3.13 is defined almost everywhere since the maps Ψ_ν^g are almost everywhere differentiable, *cf.* Proposition 3.7.(1).

Proposition 3.14. *The two-form $\tilde{\omega}$ passes to the quotient on Σ_2 .*

Proof. Let $\gamma \in \Gamma$. Observe that γ preserves the octagonal decomposition of the universal cover $(\tilde{\Sigma}_2, \tilde{g}_0)$ and its orientation. Therefore, for every regular octagon $\tilde{\Omega}$, there exists a permutation $\sigma = \sigma_\gamma$ of $\{1, \dots, 8\}$ such that

$$\gamma^{-1}(\mathcal{E}_\Omega^j) = \mathcal{E}_{\gamma^{-1}(\tilde{\Omega})}^{\sigma(j)}.$$

That is,

$$\gamma^{-1}(\nu_\Omega^j) = \nu_{\gamma^{-1}(\tilde{\Omega})}^{\sigma(j)}.$$

Hence,

$$d\Psi_{\gamma^{-1}\star\nu^j}^g = d\Psi_{\nu^{\sigma(j)}}^g.$$

By the equivariance property of the map Ψ_{ν^j} , *cf.* Remark 3.9, we derive

$$\begin{aligned} \gamma^*\tilde{\omega} &= \sum_{j=1}^8 d\Psi_{\gamma^{-1}\star\nu^j}^g \wedge d\Psi_{\gamma^{-1}\star\nu^{j+2}}^g \\ &= \sum_{j=1}^8 d\Psi_{\nu^{\sigma(j)}}^g \wedge d\Psi_{\nu^{\sigma(j+2)}}^g. \end{aligned}$$

Finally, since the permutation σ arises from an orientation-preserving isometry γ , the angle between $\nabla_x \Psi_{\nu^{\sigma(j)}}^g$ and $\nabla_x \Psi_{\nu^{\sigma(j+2)}}^g$ is equal to the angle between $\nabla_{\gamma(x)} \Psi_{\nu^j}^g$ and $\nabla_{\gamma(x)} \Psi_{\nu^{j+2}}^g$ for every $x \in \tilde{\Sigma}_2$ at which $\tilde{\omega}$ is defined. Therefore,

$$\sigma(j+2) = \sigma(j) + 2.$$

Reindexing the sum in the expression of the two-forms ω_0 and $\tilde{\omega}$, we derive that

$$\gamma^*\tilde{\omega} = \tilde{\omega}.$$

□

Remark 3.15. The induced two-form on Σ_2 will be denoted by ω .

4. PROOF OF THEOREM 1.3

We briefly restate the statement of Theorem 1.3. Let (g_t) be a slow metric variation of g_0 defined on Σ_2 . If $\ell_1(g_t) \geq \ell_1(g_0)$ and $\ell_2(g_t) \geq \ell_2(g_0)$, then

$$\text{area}(g_t) \geq \text{area}(g_0) + o(t).$$

Before proceeding to the proof, let us first define what we mean by a slow metric variation.

Definition 4.1. A *slow metric variation* $(g_t)_{t \geq 0}$ of g_0 is a smooth deformation of g_0 (more precisely, g_t is a Riemannian metric with conical singularities on M smoothly varying with t on $M \setminus \{\text{conical singularities of } g_0\}$) such that for every ℓ_i -direction ν , the following condition is satisfied

$$\|d(b_\nu^{g_t} - b_{-\nu}^{g_t}) - d(b_\nu^{g_0} - b_{-\nu}^{g_0})\|_{L^2(B_\nu)} = o(\sqrt{t}), \quad (4.1)$$

where B_ν is a fundamental domain in $(\tilde{\Sigma}_2, \tilde{g}_0)$ of the band $\mathcal{B}_\nu = \mathcal{B}_{\langle c_\nu \rangle}$.

Note that the map $b_\nu^{g_t} - b_\nu^{g_0}$ passes to the quotient by the cyclic subgroup $\langle c_\nu \rangle$, cf. Remark 3.9.(2).

Remark 4.2. A metric variation (g_t) should have conical singularities as the Riemannian metric g_0 has conical singularities, cf. Section 2.

Definition 4.3. The \mathcal{F} -functional of the map

$$\Psi^g : \tilde{\Sigma}_2 \rightarrow \mathbb{R}^\infty$$

introduced in Definition 3.10 is defined as

$$\mathcal{F}(\Psi^g) = \int_{\Sigma_2} \omega,$$

where ω is the quotient two-form on Σ_2 introduced in Section 3.

The proof of Theorem 1.3 follows directly from the next two propositions.

Proposition 4.4. *The two-form ω is a calibrating form for the map Ψ^g . That is,*

$$\mathcal{F}(\Psi^g) \leq 8 \text{ area}(g)$$

with equality for $g = g_0$.

Proof. Let $x \in \tilde{\Delta}$ be a point at which the maps $\Psi_{\nu^j}^g$ are differentiable. Let v be a \tilde{g} -unit vector in the tangent plane $T_x \tilde{\Sigma}_2$. Since the map $\Psi_{\nu^j}^g$ is 1-Lipschitz, cf. Proposition 3.7, we derive that

$$|d_x \Psi_{\nu^j}^g(v)| \leq 1.$$

Hence

$$\|\tilde{\omega}\|_{\tilde{g}} = \left\| \sum_{j=1}^8 d\Psi_{\nu^j}^g \wedge d\Psi_{\nu^{j+2}}^g \right\|_{\tilde{g}} \leq 8.$$

The desired inequality directly follows.

Let \mathcal{D} denote a fundamental domain of the surface (Σ_2, g_0) in its universal cover made of regular octagons $\tilde{\Omega}$. By Definition 4.3, the \mathcal{F} -functional of the almost everywhere differentiable map Ψ^g is defined as

$$\begin{aligned} \mathcal{F}(\Psi^g) &= \int_{\Sigma_2} \omega \\ &= \sum_{\tilde{\Omega} \in \mathcal{D}} \int_{\tilde{\Omega}} \sum_{j=1}^8 d\Psi_{\nu^j}^g \wedge d\Psi_{\nu^{j+2}}^g. \end{aligned}$$

When $g = g_0$, the vectors ν^j and ν^{j+2} form an oriented orthonormal basis. The same holds for the covectors $d_x \Psi_{\nu^j}^{g_0}$ and $d_x \Psi_{\nu^{j+2}}^{g_0}$ from Example 3.6. Hence,

$$d_x \Psi_{\nu^j}^{g_0} \wedge d_x \Psi_{\nu^{j+2}}^{g_0} = d\tilde{g}_0$$

where $d\tilde{g}_0$ represents the area form of the Riemannian metric \tilde{g}_0 . In particular, we derive

$$\mathcal{F}(\Psi^{g_0}) = 8 \text{ area}(g_0)$$

after passing to the quotient. \square

Remark 4.5. An analogous two-form ω along systolic directions could be defined on the genus three surface to recover the result of [Sa11] without using the Euclidean metrics.

Proposition 4.6. *Let (g_t) be a slow metric variation of g_0 . Then*

$$\mathcal{F}(\Psi^{g_t}) - \mathcal{F}(\Psi^{g_0}) = o(t).$$

Proof. Let $\nu = \nu_{\tilde{\Omega}}^{\mathcal{E}}$, where $\tilde{\Omega}$ is a regular octagon of $(\tilde{\Sigma}_2, \tilde{g}_0)$ of side length \hbar . By definition, an ℓ_i -band $\mathcal{B}_\nu = \mathcal{B}_{\langle c_\nu \rangle}$ is the quotient of

$$\tilde{\mathcal{B}}_\nu = \{x \in \tilde{\Sigma}_2 \mid d_{\tilde{g}_0}(x, \tilde{c}_\nu) \leq \lambda\}$$

by the cyclic subgroup $\langle c_\nu \rangle$, where λ is equal to $\frac{1}{2}\hbar$ if $i = 1$ and equal to $\frac{\sqrt{2}}{4}\hbar$ if $i = 2$. We still denote by ν the vector given by the systolic direction after passing to the quotient.

Recall that the differential $d\Psi_\nu^g$ of the map Ψ_ν^g depends only on the oriented ℓ_i -band $\tilde{\mathcal{B}}$ generated by the vector ν (or rather the oriented geodesic line \tilde{c}_ν), but not on ν , *cf.* Proposition 3.7.(3). Moreover, by passing to the quotient, we can rearrange the sum in Definition 4.3 to be taken over all the oriented ℓ_i -bands \mathcal{B} of (Σ_2, g_0) as follows

$$\mathcal{F}(\Psi^g) = \sum_{\mathcal{B}} \int_{\mathcal{B}} d\bar{\Psi}_{\nu_{\mathcal{B}}}^g \wedge d\bar{\Psi}_{(\nu_{\mathcal{B}})^\perp}^g$$

where $\nu_{\mathcal{B}}$ is an ℓ_i -direction that generates the ℓ_i -band \mathcal{B} and the vector $(\nu_{\mathcal{B}})^\perp$ is an ℓ_1 -direction or an ℓ_2 -direction orthogonal to the boundary $\partial\mathcal{B}$ and whose basepoint lies in the same regular octagon Ω as the vector $\nu_{\mathcal{B}}$. More precisely, if $\nu_{\mathcal{B}}$ points in the direction of an edge \mathcal{E}^j of Ω , then $(\nu_{\mathcal{B}})^\perp$ points in the direction of the edge \mathcal{E}^{j+2} , where the edges follow the cyclic order on $\partial\Omega$.

In particular, we have

$$\mathcal{F}(\Psi^{g_t}) - \mathcal{F}(\Psi^{g_0}) = I_1 + I_2,$$

where

$$I_1 = \sum_{\mathcal{B}} \int_{\mathcal{B}} d(\bar{\Psi}_{\nu_{\mathcal{B}}}^{g_t} - \bar{\Psi}_{\nu_{\mathcal{B}}}^{g_0}) \wedge d\bar{\Psi}_{(\nu_{\mathcal{B}})^\perp}^{g_0} - d(\bar{\Psi}_{(\nu_{\mathcal{B}})^\perp}^{g_t} - \bar{\Psi}_{(\nu_{\mathcal{B}})^\perp}^{g_0}) \wedge d\bar{\Psi}_{\nu_{\mathcal{B}}}^{g_0} \quad (4.2)$$

and

$$I_2 = \sum_{\mathcal{B}} \int_{\mathcal{B}} d(\bar{\Psi}_{\nu_{\mathcal{B}}}^{g_t} - \bar{\Psi}_{\nu_{\mathcal{B}}}^{g_0}) \wedge d(\bar{\Psi}_{(\nu_{\mathcal{B}})^\perp}^{g_t} - \bar{\Psi}_{(\nu_{\mathcal{B}})^\perp}^{g_0}). \quad (4.3)$$

First, we show that $I_1 = 0$. Recall that from Proposition 3.7.(2), we have

$$-\Psi_{(\nu_{\mathcal{B}})^\perp}^g = \Psi_{-(\nu_{\mathcal{B}})^\perp}^g$$

and the same holds for $\bar{\Psi}$. Moreover, since the oriented angle between the vectors $\nu_{\mathcal{B}}$ and $(\nu_{\mathcal{B}})^\perp$ is the same as the one between the vectors $-(\nu_{\mathcal{B}})^\perp$ and $\nu_{\mathcal{B}}$, we can write

$$I_1 = 2 \sum_{\mathcal{B}} \int_{\mathcal{B}} d(\bar{\Psi}_{\nu_{\mathcal{B}}}^{g_t} - \bar{\Psi}_{\nu_{\mathcal{B}}}^{g_0}) \wedge d\bar{\Psi}_{(\nu_{\mathcal{B}})^\perp}^{g_0}$$

after arranging the oriented bands \mathcal{B} in the sum (4.2). Moreover, the one-forms $d\bar{\Psi}_{(\nu_{\mathcal{B}})^\perp}^{g_0}$ are constant over the bands \mathcal{B} . Therefore, we derive

$$\sum_{\mathcal{B}} \int_{\mathcal{B}} d(\bar{\Psi}_{\nu_{\mathcal{B}}}^{g_t} - \bar{\Psi}_{\nu_{\mathcal{B}}}^{g_0}) \wedge d\bar{\Psi}_{(\nu_{\mathcal{B}})^\perp}^{g_0} = \sum_{\mathcal{B}} \int_{\mathcal{B}} d\left((\bar{\Psi}_{\nu_{\mathcal{B}}}^{g_t} - \bar{\Psi}_{\nu_{\mathcal{B}}}^{g_0})d\bar{\Psi}_{(\nu_{\mathcal{B}})^\perp}^{g_0}\right).$$

Now, let v be a tangent vector to the boundary $\partial\mathcal{B}$. The vector v is parallel to the ℓ_i -direction $\nu_{\mathcal{B}}$. Moreover, by definition, the unit vectors $\nu_{\mathcal{B}}$ and $(\nu_{\mathcal{B}})^\perp$ form an orthonormal basis. Since the g_0 -gradient of $\bar{\Psi}_{\nu_{\mathcal{B}}}^{g_0}$ satisfies

$$\nabla\bar{\Psi}_{\nu_{\mathcal{B}}}^{g_0} = \nu_{\mathcal{B}}$$

for every ℓ_i -direction $\nu_{\mathcal{B}}$, *cf.* Example 3.6, and the same holds with $(\nu_{\mathcal{B}})^\perp$, we have

$$d\bar{\Psi}_{(\nu_{\mathcal{B}})^\perp}^{g_0}(v) = g_0((\nu_{\mathcal{B}})^\perp, v) = 0.$$

Then by Stokes' formula, we derive that

$$\begin{aligned} \sum_{\mathcal{B}} \int_{\mathcal{B}} d\left((\bar{\Psi}_{\nu_{\mathcal{B}}}^{g_t} - \bar{\Psi}_{\nu_{\mathcal{B}}}^{g_0})d\bar{\Psi}_{(\nu_{\mathcal{B}})^\perp}^{g_0}\right) &= \sum_{\mathcal{B}} \int_{\partial\mathcal{B}} (\bar{\Psi}_{\nu_{\mathcal{B}}}^{g_t} - \bar{\Psi}_{\nu_{\mathcal{B}}}^{g_0})d\bar{\Psi}_{(\nu_{\mathcal{B}})^\perp}^{g_0} \\ &= 0. \end{aligned}$$

Note that $\bar{\Psi}_{\nu_{\mathcal{B}}}^{g_t} - \bar{\Psi}_{\nu_{\mathcal{B}}}^{g_0}$ is well defined as a real function and not merely as a map with values in the circle $\mathbb{R}/\ell_i(g_0)\mathbb{Z}$, *cf.* Remark 3.9.(2). Hence, the integral I_1 vanishes.

Now, we need to show that $I_2 = o(t)$. Note that this is the only place where we will make use of the slow metric variation assumption.

Let ν be an ℓ_i -direction ν . From the definition of the function $\Psi_\nu^{g_t}$, *cf.* Definition 3.5, and after differentiation, we obtain

$$\begin{aligned} d\Psi_\nu^{g_t}(x) &= \frac{1}{\ell_i(g_0)\ell_{g_t}(\langle c_\nu \rangle)} \int_{\mathbb{R}/\ell_i(g_0)\mathbb{Z}} \frac{1}{2} d(b_{\nu(s)}^{g_t} - b_{-\nu(s)}^{g_t})(x) ds \\ &= \frac{1}{\ell_i(g_0)\ell_{g_0}(\langle c_\nu \rangle)} \int_{\mathbb{R}/\ell_i(g_0)\mathbb{Z}} \frac{1}{2} d(b_{\nu(s)}^{g_t} - b_{-\nu(s)}^{g_t})(x) ds + O(t) \end{aligned}$$

where the error term $O(t)$ is uniformly bounded with respect to x .

Hence,

$$d(\Psi_\nu^{g_t} - \Psi_\nu^{g_0})(x) = \frac{1}{\ell_i(g_0)\ell_{g_0}(\langle c_\nu \rangle)} \int_{\mathbb{R}/\ell_i(g_0)\mathbb{Z}} \frac{1}{2} \Delta_{\nu(s)}^{g_t}(x) ds + O(t)$$

where

$$\Delta_{\nu(s)}^{g_t} = d(b_{\nu(s)}^{g_t} - b_{-\nu(s)}^{g_t}) - d(b_{\nu(s)}^{g_0} - b_{-\nu(s)}^{g_0}).$$

By definition of a slow metric variation, *cf.* Definition 4.1, we have

$$\|\Delta_{\nu(s)}^{g_t}\|_{L^2(B_\nu)} = o(\sqrt{t}).$$

Since $\bar{\Psi}_\nu^{g_t} - \bar{\Psi}_\nu^{g_0} = \Psi_\nu^{g_t} - \Psi_\nu^{g_0}$ after some quotient identifications, *cf.* Remark 3.9.(2), we deduce

$$\|d(\bar{\Psi}_{\nu_{\mathcal{B}}}^{g_t} - \bar{\Psi}_{\nu_{\mathcal{B}}}^{g_0})\|_{L^2(\mathcal{B})} = \|d(\Psi_{\nu_{\mathcal{B}}}^{g_t} - \Psi_{\nu_{\mathcal{B}}}^{g_0})\|_{L^2(\mathcal{B})} = o(\sqrt{t}). \quad (4.4)$$

By the Cauchy-Schwartz inequality and after rearranging the terms in the sum as previously, the term I_2 defined in (4.3) satisfies the following upper bound

$$\begin{aligned} I_2 &\leq \sqrt{\sum_{\mathcal{B}} \int_{\mathcal{B}} \|d(\bar{\Psi}_{\nu_{\mathcal{B}}}^{g_t} - \bar{\Psi}_{\nu_{\mathcal{B}}}^{g_0})\|^2} \times \sqrt{\sum_{\mathcal{B}} \int_{\mathcal{B}} \|d(\bar{\Psi}_{(\nu_{\mathcal{B}})^{\perp}}^{g_t} - \bar{\Psi}_{(\nu_{\mathcal{B}})^{\perp}}^{g_0})\|^2} \\ &\leq \sum_{\mathcal{B}} \|d(\bar{\Psi}_{\nu_{\mathcal{B}}}^{g_t} - \bar{\Psi}_{\nu_{\mathcal{B}}}^{g_0})\|_{L^2(\mathcal{B})}^2 \end{aligned}$$

Combined with the relation (4.4), this implies that $I_2 = o(t)$. \square

From Proposition 4.4 and Proposition 4.6, we can write

$$\text{area}(g_t) - \text{area}(g_0) \geq \frac{1}{8} (\mathcal{F}(\Psi^{g_t}) - \mathcal{F}(\Psi^{g_0})) = o(t)$$

which finishes the proof of Theorem 1.3.

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