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A PU–BONNESEN INEQUALITY

MIKHAIL G. KATZ AND STÉPHANE SABOURAU

ABSTRACT. We prove an inequality of Bonnesen type for the real projective plane, generalizing Pu’s systolic inequality for positively-curved metrics. The remainder term in the inequality, analogous to that in Bonnesen’s inequality, is a function of $R - r$ (suitably normalized), where R and r are respectively the circumradius and the inradius of the Weyl–Lewy Euclidean embedding of the orientable double cover. We exploit John ellipsoids of a convex body and Pogorelov’s rigidity theorem.

1. PU, BONNESEN, AND WEYL–LEWY

Pu’s inequality asserts that $\text{area}(g) - \frac{2}{\pi} \text{sys}^2(g) \geq 0$ for every Riemannian metric g on \mathbb{RP}^2 where equality is satisfied if and only if g has constant Gaussian curvature ([19], 1952). The inequality has recently been strengthened [11] to

$$\text{area}(g) - \frac{2}{\pi} \text{sys}^2(g) \geq 2\pi \text{Var}_\mu(f) \tag{1.1}$$

where the variance Var_μ is with respect to the probability measure μ induced by the constant curvature unit area metric g_0 in the conformal class of g , so that $g = f^2 g_0$. A similar strengthened version exists for Loewner’s torus inequality; see [8]. However, the remainder term $\text{Var}_\mu(f)$ in these inequalities does not exhibit any explicit relation to the geometry of the metric. In this paper we seek a strengthening of Pu’s inequality where the relation to the metric is more explicit.

Bonnesen’s inequality and its analogs involve a strengthening of the isoperimetric inequality of the following type:

$$L^2 - 4\pi A \geq f(R, r), \tag{1.2}$$

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where L is the length of a Jordan curve in \mathbb{R}^2 , A is the area of the region bounded by the curve, R is the circumradius and r is the inradius [4, p. 3]. In Bonnesen's inequality, one has $f(R, r) = \pi^2(R - r)^2$. Additional inequalities exist with f dependent on parameters A or L , namely $f(R, r) = A^2\left(\frac{1}{r} - \frac{1}{R}\right)^2$ and $f(R, r) = L^2\left(\frac{R-r}{R+r}\right)^2$ (ibid.).

Now suppose $(\mathbb{R}\mathbb{P}^2, g)$ has positive curvature. Then by the generalisation of the Weyl–Lewy theorem [26], [14], the orientable double cover S_g admits a unique (up to congruence) isometric embedding as a convex surface $S_g \subseteq \mathbb{R}^3$ (Nirenberg [16], Lu [15]). Here, we seek an analog of the inequality (1.1) with a remainder term exhibiting a more explicit relation to the geometry of S_g . The relation is expressed in terms of the difference $R - r$ between the circumscribed and inscribed radii of S_g , analogous to (1.2). Let R and r be the circumradius and inradius of $S_g \subseteq \mathbb{R}^3$, respectively.

Theorem 1.1. *There exists a monotone continuous function $\lambda(t) > 0$ for $t > 0$ such that if $(\mathbb{R}\mathbb{P}^2, g)$ has positive Gaussian curvature, then*

$$\frac{\text{area}(g)}{\text{sys}^2(g)} - \frac{2}{\pi} \geq \lambda\left(\frac{R - r}{\text{sys}}\right), \quad (1.3)$$

where $\lambda(t)$ is asymptotically linear as $t \rightarrow \infty$, and R and r are the circumscribed and inscribed radii of the Euclidean embedding of the orientable double cover $S_g \subseteq \mathbb{R}^3$.

Our argument produces asymptotically linear bounds for the remainder term for large $R - r$ (when the systole is normalized) that can easily be made effective. It would be interesting to develop effective lower bounds for the remainder term for small $R - r$ as in Volkov [25].

Generalisations of Pu's inequality are studied in [2], [7], [9], [12], [13], and elsewhere.

2. RIGIDITY IN PU'S INEQUALITY FOR SINGULAR METRICS

In this section, we review some aspects of Reshetnyak's work on Alexandrov surfaces (*i.e.*, singular surfaces with bounded integral curvature) focusing on nonnegatively curved metrics.

We refer to [21], [22], [24], [5] and [6] for a more detailed exposition.

Let (S^2, g_n) be a sequence of nonnegatively curved Riemannian metrics on S^2 such that the curvature measures $\omega_n = K_{g_n} dg_n$ weakly converge to ω . The measure difference $\mu = \omega - dg_0$ is a Radon measure of zero total mass on S^2 . Suppose that $\mu(\{x\}) < 2\pi$ for every $x \in S^2$. Then $d_{g_n} \rightarrow d$ for the uniform topology; see [24, Theorem 6.2] or [20]. Here d is the intrinsic distance induced by $g = e^u g_0$, where u is the potential of μ with respect to the canonical metric g_0 on S^2 (*i.e.*, $\mu = \Delta_{g_0} u$

in the distribution sense). Note that u is expressed via the Green function $G : S^2 \times S^2 \rightarrow \mathbb{R} \cup \{+\infty\}$ through the formula

$$u(x) = \int_{S^2} G(x, y) d\mu(y). \quad (2.1)$$

For general Alexandrov surfaces, the potential u is the difference of two subharmonic functions, and one has $u \in \cap_{p < 2} W^{1,p}(S^2)$; see [24] and references therein (this won't be needed though). Furthermore, $\text{area}(g_n) \rightarrow \text{area}(g)$ and

$$\text{area}(g) = \int_{S^2} e^u dg_0.$$

This result readily applies to a sequence of smooth convex spheres $S_n \subseteq \mathbb{R}^3$ converging to a convex sphere S (bounding a convex body) for the Hausdorff topology in \mathbb{R}^3 . Equivalently, it applies to a sequence of nonnegatively curved metrics g_n on S^2 Gromov-Hausdorff converging (without collapse) to a metric on S^2 with nonnegative curvature in Alexandrov's sense. Indeed, in this case, the distance on S_n uniformly converges to the distance on S ; see [1, §III.1] or [3, Lemma 10.2.7]. Furthermore, the condition $\mu_S(\{x\}) < 2\pi$, where $x \in S$, is always satisfied on the boundary S of a convex body in \mathbb{R}^3 ; see [1, §V.3]. Thus, the metric on S can be written as $g = e^u g_0$ for some weakly regular function u on S as above.

For such singular metrics, the characterization of the equality case in Pu's inequality still holds. More generally, we have

Theorem 2.1. *Let $\mathbb{R}P^n$ be the projective n -space with a singular Riemannian metric $g = \bar{f}^2 g_0$ with nonzero systole, where g_0 is the standard round metric and $\bar{f} \in L^n$ is a nonnegative function. Then*

$$\frac{\text{vol}(\mathbb{R}P^n, g)}{\text{sys}(\mathbb{R}P^n, g)^n} \geq \frac{\text{vol}(\mathbb{R}P^n, g_0)}{\text{sys}(\mathbb{R}P^n, g_0)^n}.$$

Furthermore, when $n = 2$ and $\bar{f} = e^u$, where u is the potential of a Radon measure μ of zero total mass given by (2.1), equality holds if and only if g is a metric of constant Gaussian curvature.

Proof. Denote by f the lift of \bar{f} to the double orientable cover S^n of $\mathbb{R}P^n$. Let us recall a simple version of Santaló's formula on the standard sphere S^n ; see [23, IV.19.4]. The space Γ of closed oriented geodesics on S^n is a $(2n - 2)$ -dimensional manifold admitting a natural symplectic structure whose corresponding natural volume measure is denoted by ν . Every integrable function $F : S^n \rightarrow \mathbb{R}$ satisfies

$$\int_{S^n} F dg_{S^n} = \frac{1}{\text{vol}(S^{n-1}, g_0)} \int_{\gamma \in \Gamma} \int_{S^1} F(\gamma(t)) dt d\nu(\gamma). \quad (2.2)$$

Taking $F \equiv 1$, we observe that $\nu(\Gamma) = \frac{1}{2\pi} \text{vol}(S^{n-1}, g_0) \text{vol}(S^n, g_0)$.
 Applied to $F = f^n$, Santaló's formula yields

$$\text{vol}(S^n, g) = \int_{S^n} f^n dg_0 = \frac{1}{\text{vol}(S^{n-1}, g_0)} \int_{\gamma \in \Gamma} \int_{S^1} f(\gamma(t))^n dt d\nu(\gamma).$$

By Hölder's inequality

$$(2\pi)^{\frac{n-1}{n}} \left(\int_{S^1} f(\gamma(t))^n dt \right)^{\frac{1}{n}} \geq \int_{S^1} f(\gamma(t)) dt = \text{length}_g(\gamma),$$

we deduce

$$\begin{aligned} \text{vol}(S^n, g) &\geq \frac{1}{(2\pi)^{n-1} \text{vol}(S^{n-1}, g_0)} \left(\int_{\gamma \in \Gamma} \text{length}_g(\gamma) d\nu(\gamma) \right)^n \\ &\geq \frac{2^n \nu(\Gamma)}{(2\pi)^{n-1} \text{vol}(S^{n-1}, g_0)} \text{sys}(\mathbb{RP}^n, g)^n \end{aligned}$$

since $\text{length}_g(\gamma) \geq 2 \text{sys}(\mathbb{RP}^n, g)$ for every $\gamma \in \Gamma$. Combined with the value of $\nu(\Gamma)$, we derive

$$\text{vol}(\mathbb{RP}^n, g) \geq \frac{\frac{1}{2} \text{vol}(S^n, g_0)}{\pi^n} \text{sys}(\mathbb{RP}^n, g)^n$$

as desired. Furthermore, equality holds if and only if equality holds in Hölder's inequality, that is, if f is constant almost everywhere.

Suppose that $n = 2$ and $f = e^u$, where u is given by (2.1). Since f , and so u , are constant almost everywhere, we have $\mu = \Delta_{g_0} u = 0$. By (2.1), this implies that u , and so f , are constant everywhere. Hence the extremal metric has constant curvature. \square

Specifically, we have

Corollary 2.2. *Let \mathbb{RP}^2 be the projective plane with a singular Riemannian metric $g = e^u g_0$, where u is the potential of a Radon measure μ of zero total mass given by (2.1). Then*

$$\text{area}(\mathbb{RP}^2, g) \geq \frac{2}{\pi} \text{sys}(\mathbb{RP}^2, g)^2$$

with equality if and only if g is a metric of constant Gaussian curvature.

3. PU'S INEQUALITY WITH AN $R - r$ REMAINDER

In this section, we present two proofs of Theorem 1.1. The first one follows an extrinsic geometry approach, while the second relies on intrinsic geometry arguments.

First proof of Theorem 1.1. We will exploit John ellipsoids as well as Pogorelov's rigidity theorem.

Step 1. Consider a pair of John ellipsoids $E \subseteq S_g \subseteq \sqrt{3}E$ as in [10]. Let $a \leq b \leq c$ be the principal axes of $E \subseteq \mathbb{R}^3$. Then up to a uniform multiplicative constant, each nonnegative curved metric g on $\mathbb{R}P^2$ has the following properties:

- (1) $\text{sys}(g) \sim b$;
- (2) the inradius r of S_g satisfies $r \sim a$;
- (3) the circumradius R of S_g satisfies $R \sim c$;
- (4) $\text{area}(g) \sim bc$.

Here, we write $\alpha \sim \beta$ if there exist two positive constants C and C' (which do not depend on g) such that $C\alpha \leq \beta \leq C'\alpha$.

This follows from the existence of distance-decreasing nearest-point projections from $\sqrt{3}E$ to S_g and from S_g to E , due to the convexity of S_g .

Since the relation (1.3) is scale-invariant, we can introduce a normalisation $b = 1$. Then the systole is uniformly bounded above and below by item (1). For a sequence of metrics with $c \rightarrow \infty$, we have $\text{area}(g) \sim c$ and therefore the inequality (1.3) follows from items (3) and (4), including the linear asymptotic behavior of the remainder term as $t \rightarrow \infty$.

Step 2. By the characterization of the equality case in Pu's inequality (see Corollary 2.2), it remains to show that if for a sequence of metrics (with $b = 1$) the difference $\text{area} - \frac{2}{\pi} \text{sys}^2$ tends to 0 then S_g converges to a round metric. For such a sequence, the major axis c is uniformly bounded in view of estimates (1) and (4). We need to show that the minor axis a stays away from zero in such a sequence of metrics. Suppose $a \rightarrow 0$. Then $\text{sys}(g)$ tends to the width W of the Jordan curve obtained as the boundary of the orthogonal projection of S_g to the plane spanned by the principal axes b and c . Similarly, $\text{area}(g)$ tends to the area A of the planar region $D \subseteq \mathbb{R}^2$ bounded by the curve. Since D is a centrally symmetric convex planar domain, we have $W^2 \leq \frac{4}{\pi}A$. Therefore, in the limit, we obtain

$$\text{area} - \frac{2}{\pi} \text{sys}^2 = A - \frac{2}{\pi} W^2 \geq A \left(1 - \frac{8}{\pi^2}\right) > 0$$

which contradicts the assumption that $\text{area} - \frac{2}{\pi} \text{sys}^2$ tends to 0. Hence, the minor axis a stays uniformly away from 0.

Thus, for a family of metrics with the difference $\text{area} - \frac{2}{\pi} \text{sys}^2$ tending to 0, the corresponding John ellipsoids have uniformly bounded eccentricity.

Step 3. Convex sets of uniformly bounded eccentricity form a compact family by the Blaschke selection theorem. Let $\lambda(t)$ be the minimum of $\frac{\text{area}}{\text{sys}^2} - \frac{2}{\pi}$ among nonnegatively curved metrics with $\frac{R-r}{\text{sys}} \geq t$. Every metric which is not internally isometric to the metric of constant curvature, satisfies $R - r > 0$.

Step 4. Now suppose the metric of the convex surface is internally isometric to the metric of constant curvature. It follows that the surface is congruent to the standard one by Pogorelov's rigidity theorem [17, p. 167] (see Prosanov [18] for a discussion of the status of the various proofs of this rigidity result), and therefore $R - r = 0$. Thus $\lambda(t) > 0$ for $t > 0$. \square

Our second proof of Theorem 1.1 relies on a more intrinsic argument.

Second proof of Theorem 1.1. Since the relation (1.3) is scale-invariant, we can assume that $\text{sys}(g) = 1$.

Consider the lift γ on $S_g \subseteq \mathbb{R}^3$ of a systolic loop of $(\mathbb{R}\mathbb{P}^2, g)$. Note that γ is a simple closed geodesic of S_g of length $L = 2 \text{sys}(g)$. Furthermore, $2\pi r \leq L$. Thus, $r \leq \frac{1}{\pi}$. Let p be a point of S_g at maximal intrinsic distance from γ . There exist at least two arcs of length $D = d_{S_g}(p, \gamma)$ starting at p and ending perpendicularly at γ . These two arcs along with γ decompose the hemisphere of S_g bounded by γ into two isosceles triangles Δ_i . Now, consider the comparison triangles T_i with the same side lengths in the Euclidean plane. By Toponogov's theorem, the area of Δ_i is greater or equal to the area of T_i . Also, one of the edges of one of the triangles Δ_i (and so of T_i) is of length at least $\frac{L}{2}$, and its other two edges are of length D . If D goes to infinity (while the systole of g is fixed), the area of T_i goes to infinity and so does the area of Δ_i (and $\mathbb{R}\mathbb{P}^2$). More precisely, the area of g grows roughly as D . Note that $2R \leq \text{diam}(g) \leq 2D + \frac{L}{2}$. This implies that the remainder term λ is asymptotically linear in (1.3).

Let $\lambda(t)$ be the infimum of the difference $\text{area} - \frac{2}{\pi} \text{sys}^2$ among nonnegatively curved metric with $\frac{R-r}{\text{sys}} \geq t$. From the previous discussion, we can assume that the diameter of g is bounded when considering $\lambda(t)$. Since the space $\mathfrak{M}(n, D, v)$ of n -dimensional compact Alexandrov spaces of diameter at most D and of volume at least $v > 0$ is compact for the Gromov-Hausdorff topology (see Theorem 10.7.2 and Corollary 10.10.11 of [3]), it follows from Perelman's Stability Theorem (see [3, Theorem 10.10.5]) that this infimum is attained by an extremal metric with nonnegative curvature in Alexandrov's sense. Such a (singular) metric can be written as $g = e^u g_0$, where g_0 is the canonical metric on $\mathbb{R}\mathbb{P}^2$ and u is the difference of two subharmonic functions;

see Section 2. Suppose that $\lambda(t) = 0$. By the characterization of the equality case in Pu's inequality (see Corollary 2.2), the extremal metric g is internally isometric to a metric of constant Gaussian curvature. By Pogorelov's rigidity theorem, such a convex surface is congruent to the standard round sphere, so that $R - r = 0$. Therefore, $\lambda(t) > 0$ for $t > 0$. \square

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