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# SYSTOLICALLY EXTREMAL NONPOSITIVELY CURVED SURFACES ARE FLAT WITH FINITELY MANY SINGULARITIES

MIKHAIL G. KATZ AND STÉPHANE SABOURAU

**ABSTRACT.** The regularity of systolically extremal surfaces is a notoriously difficult problem already discussed by M. Gromov in 1983, who proposed an argument toward the existence of  $L^2$ -extremizers exploiting the theory of  $r$ -regularity developed by P. A. White and others by the 1950s. We propose to study the problem of systolically extremal metrics in the context of generalized metrics of nonpositive curvature. A natural approach would be to work in the class of Alexandrov surfaces of finite total curvature, where one can exploit the tools of the completion provided in the context of Radon measures as studied by Reshetnyak and others. However the generalized metrics in this sense still don't have enough regularity. Instead, we develop a more hands-on approach and show that, for each genus, every systolically extremal nonpositively curved surface is piecewise flat with finitely many conical singularities. This result exploits a decomposition of the surface into flat systolic bands and nonsystolic polygonal regions, as well as the combinatorial/topological estimates of Malestein–Rivin–Theran, Przytycki, Aougab–Biringer–Gaster and Greene on the number of curves meeting at most once, combined with a kite excision move. The move merges pairs of conical singularities on a surface of genus  $g$  and leads to an asymptotic upper bound  $g^{4+\epsilon}$  on the number of singularities.

## 1. INTRODUCTION

The *systole* of a Riemannian manifold  $M$ , denoted  $\text{sys}(M)$ , is the least length of a noncontractible loop in  $M$ . A seminal text in this area is Gromov's paper *Filling Riemannian manifolds* [14]. It deals in particular with the problem of the existence of systolically extremal surfaces, *i.e.*, surfaces with maximal systole for a fixed area, or equivalently minimal area for a fixed systole. There is a discussion of systolically extremal surfaces without curvature assumptions in [14, pp. 64–65]. The proposed existence of the surfaces in question is only in a weak sense as it relies on the theory of  $r$ -regular convergence of P. A. White and others, introduced in the thirties; see [41]. More precisely, systolically extremal surfaces are endowed with a length metric structure along with a (possibly vanishing)  $L^2$ -limit of the conformal

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factors of some approximating Riemannian metrics. Despite this preliminary result, the existence of more regular systolically extremal surfaces without curvature assumptions remains an open problem, except for the torus [21], the projective plane [31] and the Klein bottle [5], where systolically extremal metrics have been determined (for other optimal Loewner-type inequalities see [4], [19], [22]). No conjecture is available for other surfaces, except in genus 3 where Calabi constructed nonpositively curved piecewise flat metrics with systolically extremal-like properties; see [12], [34] (and [35] for related systolic-like properties in genus 2). Partial results concerning systolically optimal metrics were obtained by Bryant [8] using PDE techniques, assuming regularity.

**1.1. Statement of the problem.** We will study the extremality problem in the context of surfaces endowed with a Riemannian metric of nonpositive curvature. The *systolic area*  $\sigma$  of a surface  $M$  with a fixed metric is defined as

$$\sigma(M) = \frac{\text{area}(M)}{\text{sys}(M)^2}.$$

The optimal systolic area in genus  $g$  for nonpositive curvature is defined as

$$\sigma_{\mathcal{H}}(g) = \inf_M \sigma(M) \tag{1.1}$$

where the infimum is taken over all nonpositively curved genus  $g$  surfaces  $M$ . Here, the subscript  $\mathcal{H}$  alludes to Hadamard as the surfaces considered are locally CAT(0). For a recent study of Hadamard spaces see Bačák [3].

For surfaces of nonpositive curvature of genus  $g = 2$ , we showed in [23] that the metric realizing the infimum  $\sigma_{\mathcal{H}}(2)$  of the systolic area is flat with finitely many conical singularities, in the conformal class of the smooth completion of the affine complex algebraic curve  $w^2 = z^5 - z$ , and one has  $\sigma_{\mathcal{H}}(2) = 3 \tan(\frac{\pi}{8})$ .<sup>1</sup> A similar result holds for the metric realizing the infimum of the systolic area among all nonpositively curved metrics on the surface homeomorphic to the connected sum of three projective planes, also known as Dyck's surface; see [24].

The purpose of the present text is to extend this result to surfaces of arbitrary genus. We will need a few more definitions to cover the case of local infima, and not just global infima.

**Definition 1.1.** A closed surface  $M$  with a Riemannian metric with conical singularities is locally isometric to the complex plane endowed with the metric

$$ds^2 = e^{2u(z)} |z|^{2\beta} |dz|^2$$

where  $\beta > -1$  and  $u: \mathbb{C} \rightarrow \mathbb{R}$  is a continuous function, smooth everywhere except possibly at the origin.

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<sup>1</sup>The same conformal class contains an optimal metric for a related first eigenvalue problem; see [28]. This optimal metric similarly has finitely many conical singularities.

See Troyanov [38] for a detailed description. Here, the point of  $M$  corresponding to the origin in  $\mathbb{C}$  is a conical singularity of total angle  $\theta = 2\pi(\beta + 1)$ .

**Example 1.2.** Gluing together  $n$  Euclidean angular sectors of angle  $\theta_i, i = 1, \dots, n$  side by side in circular order gives rise to a conical singularity of total angle  $\theta_1 + \dots + \theta_n$ .

Such a surface  $M$  is nonpositively curved (in Alexandrov's sense) if and only if the Gaussian curvature of  $M$  is nonpositive away from the conical singularities and the total angle at each conical singularity is greater than  $2\pi$ .

**Definition 1.3.** The space  $\mathcal{H}_g$  consists of nonpositively curved Riemannian metrics (possibly with conical singularities) on a genus  $g$  surface. This space will be endowed with either of the following nonequivalent distances, namely, the Gromov–Hausdorff distance or the Lipschitz distance; see [16]. It fibers over the conformal moduli space  $\mathcal{M}_g$  (see [38]):

$$\begin{array}{c} \mathcal{H}_g \\ \downarrow \\ \mathcal{M}_g \end{array}$$

where the base is  $(6g - 6)$ -dimensional and the fiber infinite-dimensional.

**Definition 1.4.** A *local infimum* of the systolic area on  $\mathcal{H}_g$  is a real number  $\mu > 0$  such that there exists an open set  $\mathcal{U} \subseteq \mathcal{H}_g$  satisfying a strict inequality

$$\mu = \inf_{M \in \mathcal{U}} \sigma(M) < \inf_{M \in \partial \mathcal{U}} \sigma(M). \quad (1.2)$$

Note that, though we use the term *local*, this definition is not entirely local as the strict inequality (1.2) may hold for some open set  $\mathcal{U}$ , but fail for arbitrarily small ones.

**Definition 1.5.** A nonpositively curved surface  $M \in \mathcal{H}_g$  (possibly with conical singularities) is *locally extremal* for the systolic area if there exists an open set  $\mathcal{U} \subseteq \mathcal{H}_g$  containing  $M$  such that

$$\sigma(M) = \inf_{M \in \mathcal{U}} \sigma(M) < \inf_{M \in \partial \mathcal{U}} \sigma(M).$$

In such case we say that the local infimum  $\mu = \inf_{M \in \mathcal{U}} \sigma(M)$  is *attained* by  $M$ .

**1.2. Main results.** We can now state our main result concerning the existence of systolically extremal metrics.

**Theorem 1.6.** *Every local infimum of the systolic area on the space  $\mathcal{H}_g$  of nonpositively curved genus  $g$  surfaces (possibly with conical singularities) is attained by a nonpositively curved piecewise flat metric with at most  $\mathcal{N}_0 \leq 2^{25} g^4 \log^2(g)$  conical singularities.*

**Remark 1.7.** The upper bound  $\mathcal{N}_0$  can be expressed in terms of the maximal number  $\bar{Q}(g)$  of systolic homotopy classes on a closed nonpositively curved surface of genus  $g$  with finitely many conical singularities; see Theorem 8.1. In turn, the quantity  $\bar{Q}(g)$  can be bounded in terms of the number  $Q(g)$  of pairwise nonhomotopic simple closed curves on a genus  $g$  surface; see Theorem 4.5 and Proposition 4.7.

Thus the number of conical singularities is uniformly bounded for all locally extremal metrics of nonpositive curvature on a surface of fixed genus. By Theorem 1.6 and the conformal representation of piecewise flat surfaces, see [37, §5], every locally extremal metric has a well-defined conformal class and a well-defined continuous conformal factor (with finitely many zeros). Moreover, the space of locally extremal nonpositively curved metrics is finite-dimensional of dimension at most  $2^{25} (6g - 6) g^4 \log^2(g)$ .

**Corollary 1.8.** *For every genus  $g$ , the global infimum  $\sigma_{\mathcal{H}}(g)$  is attained by a nonpositively curved piecewise flat metric on a genus  $g$  surface with at most  $\mathcal{N}_0 \leq 2^{25} g^4 \log^2(g)$  conical singularities.*

This type of result is apparently of interest in closed string theory; see the work by Zwiebach and coauthors [42], [43], [17], [18], [27].

Systolically extremal surfaces without curvature assumptions, if they are sufficiently regular (for instance, if they have bounded integral curvature in Alexandrov's sense; see [1], [33], [40]), are covered by their systolic loops. This may no longer be the case for locally extremal nonpositively curved surfaces. Still, by analyzing the geometry and shape of nonsystolic domains of locally extremal nonpositively curved surfaces in Section 9, we obtain the following.

**Corollary 1.9.** *The union of the systolic loops of a locally extremal nonpositively curved surface is path-connected.*

**Remark 1.10.** A important tool in our study is the kite excision trick which merges pairs of conical singularities, while keeping the systole fixed and strictly decreasing the area. More precisely, this trick consists in excising a flat kite from a surface and identifying pairs of adjacent sides of the excised surface; see Section 6 for details.

**1.3. Some open problems.** We conclude this introduction with a few open problems of varying levels of difficulty.

- (1) By Corollary 1.9, the systolic region of a locally extremal nonpositively curved surface is connected. Does this result still hold for the interior of the systolic region?
- (2) As we show in this article, the set of locally extremal nonpositively curved metrics on a given genus  $g$  surface lies in a finite-dimensional space. A natural question to ask is whether the set of locally extremal surfaces is finite, as it is the case for hyperbolic surfaces, see [6] for a general setting.

- (3) Continuing with the previous item: do isosystolic deformations (i.e., deformations preserving the systolic area) of locally extremal non-positively curved surfaces exist?
- (4) To what extent can one relax the nonpositive curvature condition? For instance, what can be said about extremal metrics of curvature at most  $\varepsilon$  with unit systole for small  $\varepsilon > 0$ ?

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## 2. STRATEGY

Let us comment on the strategy of the proof of Theorem 1.6. We first discuss the general idea on a sufficiently smooth extremal surface pointing out the main difficulties. After presenting a natural attempt to overcome these difficulties in the context of Alexandrov surfaces, we finally develop the strategy of the proof, sketching the argument.

**2.1. Local perturbation of extremal surfaces.** Suppose first that an extremal metric with nonpositive curvature exists on a given surface and that this metric is sufficiently smooth. By the flat strip theorem (see [7, §II.2.13]), two homotopic systolic loops on a nonpositively curved surface bound a flat annulus foliated by systolic loops. Away from these flat systolic bands, in regions where no systolic loop passes, the extremal surface must be flat, for otherwise its curvature would be negative and its area could be decreased by a local perturbation of the conformal factor, affecting neither the systole, nor the sign of curvature, and contradicting the extremality of the surface. Of course, this argument only holds for regions where the metric is smooth enough. In particular, it does not shed any light on the nature of the singularities of the extremal metric, which necessarily exist in genus at least two, otherwise the extremal surface would be flat. Thus, though appealing, this argument does not prove anything if we cannot establish the existence of a smooth enough extremal metric *a priori*, which amounts to a classical issue in the calculus of variations.

The existence of extremal metrics in a given conformal class can be derived from compactness results on the conformal factor using its log-subharmonicity in nonpositive curvature. However, the regularity of the metrics thus obtained is too weak for our purposes. Moreover, it is unknown whether the systolically optimal metric in a given conformal class has finitely many singularities or not. The advantage of our technique based on the kite excision move (see Section 6) is that it has the mobility of moving about freely in the moduli space of conformal classes and is not constrained to a single class.

**2.2. Alexandrov surfaces.** From a different (more geometric) point of view, the theory of Alexandrov surfaces with bounded integral curvature provides the desired features regarding curvature measure, compactness results and conformal representation; see [1], [33], [40]. Loosely speaking, every Alexandrov surface with bounded integral curvature can be described by its conformal structure, represented by a (hyperbolic) Riemannian metric  $h$  of curvature  $K_h$ , and a curvature measure

$$d\omega = K_h dA_h + d\mu$$

where  $\mu$  is a Radon measure of total mass zero. Here, the function  $u$  in the conformal factor  $e^{2u}$  of the surface satisfies  $\Delta_h u = \mu$  in the distribution sense. Therefore, it is determined by the inverse of the Laplacian on  $(M, h)$  given by the Green function  $G$ , namely

$$u(x) = \int_M G(x, y) d\mu. \quad (2.1)$$

Arguing as before, we seek to show that the curvature measure vanishes in a neighborhood of a point where no systolic loop passes, by a perturbation of the conformal factor. For this purpose, we consider variations of the Radon measure  $\mu$  in a neighborhood of this point, leaving us with the following problem: even though the support of the measure variation is localized in this neighborhood, we have no control on the support of the variation of the conformal factor given by (2.1). This could affect the value of the systole and, therefore, the validity of the argument. If the conformal factor is modified in a given region the curvature measure is affected only in this region, but it is not clear how to read off this property from the curvature measure variation.

**2.3. A priori bounds.** We will follow a different strategy enabling us to establish *a priori* upper bounds on the number of conical singularities. The argument proceeds as follows. In Section 3, we first recall that every nonpositively curved surface can be approximated by a nonpositively curved piecewise flat surface with conical singularities. We also show that the systolic area defines a proper functional when restricted to the moduli space of nonpositively curved piecewise flat metrics whose number of conical singularity is uniformly bounded. This compactness result will allow us to derive the existence of locally extremal surfaces from an *a priori* upper bound on the number of conical singularities of an almost locally extremal piecewise flat surface. Such an upper bound follows from a polynomial bound on the number of systolic loops up to homotopy; see Section 4.

More specifically, flat systolic bands and isolated systolic loops decompose any nonpositively curved piecewise flat surface into nonsystolic polygonal regions whose number of edges is related to the number of systolic homotopy classes, and is therefore uniformly bounded; see Section 5. We introduce the kite excision trick in Section 6. We exploit the trick to deduce that each nonsystolic polygonal region of an almost locally extremal piecewise flat

surface has at most one conical singularity; see Sections 8 and 9. It follows that the number of conical singularities of this surface is uniformly bounded as desired.

The kite excision trick has the effect of moving a pair of singularities lying in the same nonsystolic region closer and closer until they merge into a single nonpositively curved conical singularity, while keeping the systole fixed and strictly decreasing the area. More precisely, it consists in excising a flat kite from a surface and identifying pairs of adjacent sides of the excised surface; see Sections 6 and 7. Moreover, this construction gives a way of reaching a locally extremal surface.

We will assume throughout that all surfaces are of genus at least 2 to avoid the torus case where the extremal systolic problem was completely solved by Loewner; see [21, Theorem 5.4.1].

### 3. METRIC APPROXIMATION AND COMPACTNESS

We present a few classical results which will be used in the proof of the existence of locally extremal nonpositively curved piecewise flat metrics for each genus  $g$ ; see Theorem 8.1. We start with the following metric approximation result.

**Proposition 3.1.** *For every genus  $g \geq 2$ , the infimum of the systolic area over the following three spaces yields the same value  $\sigma_{\mathcal{H}}(g)$ :*

- (1) *over all nonpositively curved Riemannian metrics;*
- (2) *over all nonpositively curved Riemannian metrics with conical singularities in the sense of Definition 1.1;*
- (3) *over all nonpositively curved piecewise flat metrics with conical singularities.*

*Proof.* A metric  $M \in \mathcal{H}_g$  (e.g., a nonpositively curved metric with conical singularities) can be approximated by a smooth one of nonpositive curvature by smoothing out each of the conical singularities, without significantly affecting the area and the systole.

Next, rescale the smooth surface  $M$  (without changing the systolic area) so that its Gaussian curvature  $K$  satisfies  $-1 \leq K \leq 0$ . For every  $\varepsilon > 0$ , we can partition  $M$  into sufficiently small right-angled geodesic triangles  $\Delta \subseteq M$  so that  $\text{area}(\Delta) \geq (1 - \varepsilon) \text{area}(\Delta_0)$ , where  $\Delta_0$  is the corresponding flat triangle with the same sidelengths. Indeed, by the Alexandrov–Toponogov comparison theorem, comparing  $M$  with the spaceform of *smaller* constant curvature (namely,  $-1$ ), the area of  $\Delta$  is at least the area of the comparison right-angled hyperbolic triangle. The area of the hyperbolic triangle is  $2 \arctan(\tanh \frac{a}{2} \tanh \frac{b}{2})$  where  $a, b$  are the two sides. Thus the lower bound on  $\text{area}(\Delta)$  can be made as close to  $\frac{1}{2}ab$  as we wish for  $a, b$  small enough, exploiting the developments of  $\arctan$  and  $\tanh$ .

By the Alexandrov–Toponogov comparison theorem, comparing with the spaceform of *greater* constant curvature (namely,  $0$ ), the angles of  $\Delta_0$  are



no smaller than the corresponding angles of  $\Delta$ . Hence the total angles of the conical singularities of the piecewise flat surface  $M_0$  obtained from  $M$  by replacing each  $\Delta$  by  $\Delta_0$  are at least  $2\pi$ , so that  $M_0$  has nonpositive curvature.

Since  $\text{area}(\Delta_0) \leq \frac{1}{2}ab$ , replacing  $\Delta$  by  $\Delta_0$  increases the area by a factor at most  $1+\varepsilon$ , so that we have tight control on the area of  $M_0$ . Meanwhile, each loop in  $M_0$  decomposes into paths where each path is contained in a suitable triangle  $\Delta_0 \subseteq M_0$  with endpoints on the boundary of  $\Delta_0$ . The corresponding path in  $\Delta \subseteq M$  is necessarily shorter by the Alexandrov–Toponogov comparison of  $M$  with the spaceform of *greater* constant curvature (namely, 0) and therefore  $\text{sys}(M_0) \geq \text{sys}(M)$  and thus  $\sigma(M) \geq (1-\varepsilon)\sigma(M_0)$ .  $\square$

More generally, one has the following result on metric approximation, announced by Reshetnyak [32] and proved by Yu. Burago [9, Lemma 6], in the more general setting of Alexandrov surfaces.

**Proposition 3.2.** *Let  $M$  be a surface of genus  $g$ . Every Riemannian metric with conical singularities on  $M$  is bilipschitz close to a piecewise flat metric with conical singularities. In particular, the systole and the area of the two metrics are close.*

*Proof.* The argument of [9, Lemma 6] proceeds as follows. Construct a suitable partition  $\mathcal{T}$  of  $M$  into small geodesic triangles, where the conical singularities of  $M$  are located at the vertices and where each triangle of  $\mathcal{T}$  is bilipschitz close to its comparison flat triangle with the same side lengths. Replacing each triangle of  $\mathcal{T}$  with its comparison flat triangle gives rise to a piecewise flat metric with conical singularities on  $M$ . Putting together the bilipschitz maps between triangles yields a bilipschitz map between the two metrics on  $M$ , with bilipschitz constant close to 1.  $\square$

**Corollary 3.3.** *In Proposition 3.2, if the Riemannian metric with conical singularities on  $M$  is nonpositively curved then the piecewise flat metric can be assumed to be nonpositively curved, as well.*

*Proof.* Examining the construction in the proof of Proposition 3.2, we note that if the initial Riemannian metric with conical singularities on  $M$  is nonpositively curved, then an Alexandrov–Toponogov comparison of triangle angles shows that the associated piecewise flat metric has nonpositive curvature as at the end of the proof of Proposition 3.1.  $\square$

**Remark 3.4.** As mentioned earlier, the metric constructions in the proofs of Proposition 3.2 and Corollary 3.3 go through in the class of Alexandrov surfaces. Thus, these two results still hold for Alexandrov surfaces. In particular, the optimal systolic area  $\sigma_{\mathcal{H}}(g)$  for nonpositive curvature agrees with the infimum of the systolic area over all genus  $g$  surfaces with nonpositive curvature in the sense of Alexandrov, or equivalently CAT(0) surfaces.

The next result on conformal compactness (Proposition 3.6) is due to Gromov [14, §5]. We present the proof since the original arguments are somewhat scattered. We first state a result used in the proof.

**Theorem 3.5** (The collar theorem). *On a closed hyperbolic surface  $M$ , a simple closed geodesic  $\gamma$  of hyperbolic length  $\ell$  admits a tubular neighborhood*

$$\mathcal{C} = \{x \in M \mid d_{\text{hyp}}(x, \gamma) < w\}$$

*of width*

$$w = \operatorname{arcsinh} \left( \frac{1}{\sinh(\frac{\ell}{2})} \right) \quad (3.1)$$

*diffomorphic to an annulus.*

A proof can be found in [10, Theorem 4.1.1].

**Proposition 3.6.** *Let  $M$  be a closed surface of genus  $g \geq 2$ , and let  $K > 0$ . The space of conformal classes of Riemannian metrics (possibly with conical singularities) on  $M$  with systolic area at most  $K$  is a compact set in the conformal moduli space  $\mathcal{M}_g$ .*

*Proof.* The capacity of an annulus  $\mathcal{C}$  endowed with a Riemannian metric possibly with conical singularities is a conformal invariant defined as

$$\operatorname{Cap}(\mathcal{C}) = \inf_F \int_{\mathcal{C}} |dF|^2 dA$$

where  $F$  is a smooth function on  $\mathcal{C}$  equal to 0 on one boundary component of  $\mathcal{C}$  and to 1 on the other. The capacity of an annulus  $\mathcal{C} \subseteq M$  around a noncontractible simple closed curve of  $M$  is related to the systolic area of  $M$  by the inequality

$$\operatorname{Cap}(\mathcal{C}) \geq \frac{1}{\sigma(M)}; \quad (3.2)$$

see [14, §5] and [15, §2.D.6] for further detail.

Let  $\gamma$  be a systolic loop of length  $\ell$  for the hyperbolic metric conformally equivalent to the metric  $M$ . By Buser–Sarnak [11, (3.4)], the capacity of the collar provided by Theorem 3.5 is given by

$$\operatorname{Cap}(\mathcal{C}) = \frac{\ell}{\pi - 2\theta_0}$$

where

$$\theta_0 = \arcsin \left( \frac{1}{\cosh(w)} \right)$$

with  $w$  as in (3.1). Therefore, the capacity of the annulus  $\mathcal{C}$  tends to zero with the (hyperbolic) length of  $\gamma$ .

Since the capacity is a conformal invariant, we deduce from (3.2) and the assumption on the systolic area of  $M$ , that the systole  $\ell$  of the hyperbolic metric conformally equivalent to  $M$  is bounded away from zero. By Mumford [26], this implies that the conformal hyperbolic metric, and so the

conformal class of  $M$ , varies through a compact set in the conformal moduli space  $\mathcal{M}_g$ .  $\square$

As a consequence of the previous result, we obtain that the systolic area function is proper on a suitable moduli space, in the following sense.

**Proposition 3.7.** *Let  $N$  be an arbitrary natural number. The space of piecewise flat nonpositively curved metrics with at most  $N$  conical singularities on a closed surface  $M$  of genus  $g \geq 2$  of systole normalized to 1 and area bounded above is compact.*

*Proof.* By Proposition 3.6, the conformal classes of the metrics considered in Proposition 3.7 lie in a compact set  $\mathcal{K} \subseteq \mathcal{M}_g$ .

By Troyanov [37, §5], each conformal class of  $M$  carries a piecewise flat conformal metric with at most  $N$  prescribed conical singularities  $p_i$  of given total angles  $\theta_i$ , provided that the Gauss–Bonnet relation

$$\sum_{i=1}^N (\theta_i - 2\pi) = 4\pi(g - 1). \quad (3.3)$$

(see [37, §3]) is satisfied. This metric is unique upon normalization to unit systole. Furthermore, the dependence on parameters is continuous (see also [39]).

As the metric is nonpositively curved, the angles  $\theta_i$  are at least  $2\pi$  and so lie in the interval  $[2\pi, (4g - 2)\pi]$ . Since the Gauss–Bonnet relation (3.3) is closed, the  $N$ -tuple  $(\theta_1, \dots, \theta_N)$  ranges through a compact set  $L \subseteq \mathbb{R}^N$ . Thus, the space of piecewise flat nonpositively curved metrics with at most  $N$  conical singularities on a closed genus  $g$  surface of systole normalized to 1 and area bounded above is homeomorphic to a compact subset of  $\mathcal{K} \times M^N \times L$ .  $\square$

#### 4. SYSTOLIC BANDS

The goal of this section is to present some geometric properties related to the notion of systolic bands, based of the flat strip theorem and Greene’s results [13].

The following result is immediate from the flat strip theorem; see Bridson–Haefliger [7, §II.2.13].

**Lemma 4.1.** *Let  $M$  be a closed nonpositively curved surface with finitely many conical singularities. Then every pair of homotopic simple closed geodesics bounds a flat annulus in  $M$ .*

**Definition 4.2.** A curve passing through a singular  $p \in M$  splits the total angle  $\theta_p > 2\pi$  into two *rotation angles*  $R_p$  and  $L_p$  with  $R_p + L_p = \theta_p$ .

The local condition defining a geodesic encountering a singular point requires that the rotation angles satisfy  $R_p \geq \pi$  and  $L_p \geq \pi$ .

**Definition 4.3.** Let  $M$  be a closed nonpositively curved surface with finitely many conical singularities. A *systolic homotopy class* of  $M$  is a free homotopy class of loops containing a systolic loop. Every systolic homotopy class  $\mathcal{C}$  of  $M$  gives rise to a closed *systolic band* in  $M$  defined as the union of the systolic loops in  $\mathcal{C}$ .

By Lemma 4.1, there are two possibilities for a systolic band.

**Definition 4.4.** A systolic band is formed of

- (1) either an *isolated systolic loop* when there is only one systolic loop in the corresponding systolic homotopy class; or
- (2) a flat open annulus bounded by two *limit systolic loops* (which are not necessarily disjoint) and foliated by systolic loops.<sup>2</sup>

In case (2), we refer to the systolic band as a *fat systolic band*.

Observe that a closed fat systolic band is not always homeomorphic to a closed annulus as its two limit systolic loops are not necessarily disjoint.

We will need the following result of Greene [13].

**Theorem 4.5** (Greene). *Let  $M$  be a closed surface of genus  $g \geq 2$ . Then the number of pairwise nonhomotopic simple closed curves on  $M$  meeting each other at most once is bounded by*

$$Q(g) \leq 2^9 g^2 \log g. \quad (4.1)$$

The multiplicative constant  $2^9$  does not appear in [13]. However, going through the argument of [13], we obtain the bound  $8x \log_2 x$  for  $Q(g)$ , where  $x = 8(g-1)(2g-1)$ , which leads to the multiplicative constant  $2^9$  in (4.1).

Now, we are ready to prove the following proposition.

**Proposition 4.6.** *Let  $M$  be a closed nonpositively curved surface of genus  $g \geq 2$  with finitely many conical singularities. Then*

- (1) *Each pair of intersecting systolic loops of  $M$  meet exactly at one or two points, or along an arc;*
- (2) *When two systolic loops meet at two points exactly, this pair of points decomposes each of the two systolic loops into geodesic arcs of the same length.*

*Proof.* To show (1) and (2), let  $\alpha$  and  $\beta$  be two systolic loops that meet in a single connected component, then they meet either in one point or along an arc.

Suppose now that their intersection  $\alpha \cap \beta$  has at least two connected components. Then there exist two subarcs  $\beta_1 \subseteq \beta$  and  $\beta_2 \subseteq \beta$  (in the complement of  $\alpha \cap \beta$ ) with disjoint interior meeting  $\alpha$  only at their endpoints.

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<sup>2</sup>The case of the torus is exceptional and was already excluded from the outset.

The endpoints of the arc  $\beta_i$  decompose  $\alpha$  into two arcs denoted  $\alpha'_i$  and  $\alpha''_i$ . Observe that none of the four loops  $\alpha'_1 \cup \beta_1$ ,  $\alpha''_1 \cup \beta_1$ ,  $\alpha'_2 \cup \beta_2$  and  $\alpha''_2 \cup \beta_2$  is contractible, otherwise two distinct geodesic arcs with the same endpoints would be homotopic, which is impossible on a nonpositively curved surface. Thus, each of these four loops is of length at least  $\text{sys}(M)$ . The sum of their lengths is at least  $4\text{sys}(M)$  and at most twice the total length of  $\alpha$  and  $\beta$ :

$$4\text{sys}(M) \leq |\alpha'_1| + |\alpha''_1| + |\alpha'_2| + |\alpha''_2| + 2|\beta_1| + 2|\beta_2| \leq 2|\alpha| + 2|\beta|. \quad (4.2)$$

Hence both inequalities in (4.2) are equalities and the same holds for the four inequalities involved in the sum. It follows that each of the arcs  $\alpha'_i$ ,  $\alpha''_i$  and  $\beta_i$  is of length  $\frac{1}{2}\text{sys}(M)$ . Therefore, the only intersection points between  $\alpha$  and  $\beta$  are the two endpoints  $p$  and  $q$  which are necessarily antipodal points of both of these loops.  $\square$

**Proposition 4.7.** *Let  $M$  be a closed nonpositively curved surface of genus  $g \geq 2$  with finitely many conical singularities. Then the number  $\bar{Q}(g)$  of systolic homotopy classes is at most*

$$\bar{Q}(g) \leq 32(g-1)^2 + Q(g) \leq 2^{10} g^2 \log g.$$

*Proof.* Let us estimate first the number of homotopy classes of systolic loops meeting at exactly two points. Let  $\alpha, \beta$  be two such loops. The case of equality in inequality (4.2) implies that the systolic loops  $\alpha$  and  $\beta$  meeting at points  $p$  and  $q$  decompose into four distinct length-minimizing arcs of length  $\frac{1}{2}\text{sys}(M)$  joining  $p$  and  $q$ .

Let  $a$  and  $b$  be two length-minimizing arcs of length  $\frac{1}{2}\text{sys}(M)$  joining  $p$  and  $q$ . As  $M$  is nonpositively curved, the two geodesic arcs  $a$  and  $b$  are nonhomotopic and form a systolic loop. By first variation, the angle at  $p$  between the arcs  $a$  and  $b$  is at least  $\pi$ . It follows that  $p$  is a conical singularity of total angle  $\theta_p \geq 4\pi$  and similarly for  $q$ .

The lower bound on the angles between the length-minimizing arcs joining  $p$  to  $q$  imply that there exist at most  $\lfloor \theta_p/\pi \rfloor$  such length-minimizing arcs. This shows that there are at most

$$\sum_{\theta_p \geq 4\pi} \binom{\lfloor \theta_p/\pi \rfloor}{2} \leq \frac{1}{2\pi^2} \sum_{\theta \geq 4\pi} \theta_p^2 \leq \frac{1}{2\pi^2} \left( \sum_{\theta \geq 4\pi} \theta_p \right)^2 \quad (4.3)$$

systolic loops  $\alpha \subseteq M$  meeting another systolic loop  $\beta$  at exactly two points. the Gauss–Bonnet formula (3.3) implies that whenever  $\theta_p \geq 4\pi$ , we have  $\theta_p \leq 2(\theta_p - 2\pi)$ . Since  $\theta_i > 2\pi$ , it follows that

$$\sum_{\theta_p \geq 4\pi} \theta_p \leq 2 \sum_{\theta_p \geq 4\pi} (\theta_p - 2\pi) \leq 2 \sum_{i=1}^k (\theta_i - 2\pi) = 8\pi(g-1).$$

We derive from (4.3) that the number of systolic loops intersecting another systolic loop at exactly two points is bounded by

$$\frac{1}{2\pi^2} [8\pi(g-1)]^2 = 32(g-1)^2$$

and so is at most quadratic.

It remains to estimate the number of homotopy classes of systolic loops that do not meet any other loop in more than one point. We choose representative loops from these remaining classes, and deform them so that each pair of loops meet at most at a single point. By Theorem 4.5, the number of pairwise nonhomotopic loops intersecting each other at most once is bounded by  $Q(g) \leq 2^9 g^2 \log g$ .  $\square$

**Remark 4.8.** Theorem 4 in [13] also provides an upper bound on the number of pairwise nonhomotopic loops intersecting at most twice. Directly applying this result would yield an  $O(g^5 \log g)$  upper bound on the number of systolic homotopy classes in  $M$ . We obtained a better almost quadratic bound by analyzing the special structure of systolic loops intersecting twice, combined with the almost quadratic bound of Theorem 4.5 on the number of pairwise nonhomotopic loops meeting at most once. See Przytycki [30] and Aougab-Biringer-Gaster [2] for earlier polynomial bounds, and Juvan-Malnič-Mohar [20] or Malestein-Rivin-Theran [25] for even earlier exponential ones. Greene's almost quadratic upper bound  $O(g^2 \log g)$  for pairwise nonhomotopic loops intersecting at most once can be improved in the hyperbolic case to a subquadratic one  $O\left(\frac{g^2}{\log g}\right)$  due to Parlier [29]. A *lower* bound of type  $O\left(g^{\frac{4}{3}-\epsilon}\right)$  is due to Schmutz-Schaller [36].

**Example 4.9.** On smooth surfaces curves can be shortened by smoothing them out by first variation, implying that systolic loops meet each other at most once. However, on a singular surface they may intersect twice, even in nonpositive curvature. For example, consider the standard sphere along with four meridians joining the two poles. Replace each of the four lune-shaped spherical regions bounded by the meridians by a flat cylinder of circumference  $\pi$  and altitude at least  $\frac{\pi}{4}$ , where the bottom of each cylinder is glued in isometrically along the boundary of each of the lunes. The resulting surface  $X$  is a flat four-holed sphere with two conical singularities (at former poles) of total angle  $4\pi$ , which can be turned into a nonpositively curved piecewise flat genus 3 surface  $M = X \cup_{\partial X} X$  by gluing another copy to it. The surface  $M$  obtained in this way has pairs of systolic loops meeting transversely twice.

## 5. SYSTOLIC DECOMPOSITION

In this section, we describe the systolic decomposition of a closed non-positively curved piecewise flat surface  $M$  of genus  $g$ . By the curvature condition, each total angle  $\theta_i$  at a conical singularity is greater than  $2\pi$ .

**Definition 5.1.** A conical singularity  $p \in M$  is said to be *large* if the angle at  $p$  is at least  $3\pi$ , and *small* otherwise.

**Lemma 5.2.** *Let  $M$  be a closed nonpositively curved piecewise flat surface. There are at most  $4(g-1)$  large conical singularities on  $M$ , each of total angle at most  $2\pi(2g-1)$ .*

*Proof.* This is immediate from the Gauss–Bonnet formula (3.3) since large conical singularities satisfy  $\theta_i - 2\pi \geq \pi$  and small conical singularities satisfy  $\theta_i - 2\pi > 0$  as  $M$  is nonpositively curved.  $\square$

We will need a few more definitions.

**Definition 5.3.** A conical singularity  $p \in M$  is *special* if every point in a neighborhood of  $p$  lies in a (fat) systolic band. In more detail, a special singularity lies on the boundary of several closed fat systolic bands in such a way that the union of these bands contains an open neighborhood of the singularity.

**Definition 5.4.** The *systolic decomposition* of  $M$  is a partition

$$M = (\sqcup_i S_i) \sqcup (\sqcup_j D_j)$$

of  $M$  into systolic domains  $S_i$  and nonsystolic domains  $D_j$  where

- (1) each *systolic domain*  $S_i$  is a connected component of the union of the systolic bands of  $M$  (see Definition 4.3);
- (2) each *nonsystolic domain*  $D_j$  is a connected component of the complementary set in  $M$  of the systolic bands of  $M$ .

The intersection pattern of the systolic bands of  $M$  described in Proposition 4.6.(1) shows that every systolic and nonsystolic domain has a finite geodesic polygonal structure described as follows.

**Definition 5.5.** The *vertices* of the polygonal structure are of two types:

- (1) the intersection points between pairs of either isolated or limit systolic loops when they meet at one or two points (see Definition 4.4);
- (2) if systolic loops meet along a segment  $I \subseteq M$  (see Proposition 4.6.(1)) then the endpoints of  $I$  (which are also conical singularities) are also taken to be vertices.

The *edges* of a systolic or nonsystolic domain  $D$  are the connected components of  $\partial D$  minus the vertices of  $\partial D$ .

**Remark 5.6.** The vertices of a systolic or nonsystolic domain  $D$  are not necessarily located at conical singularities and conical singularities may lie in the interior of the edges of  $D$ .

We now describe the structure of the systolic decomposition of  $M$ . Recall that  $\bar{Q}$  is the maximal number of systolic homotopy classes; see Proposition 4.7.

**Proposition 5.7.** *Let  $M$  be a piecewise flat nonpositively curved surface of genus  $g \geq 2$ . Let  $\mathcal{N} = 4\bar{Q}(g)^2 \leq 2^{22}g^4 \log^2(g)$ . Then*

- (1) *the corners of every nonsystolic domain at nonsingular points are convex, i.e., their angles are at most  $\pi$ .*

- (2) *the surface has at most  $\mathcal{N}$  special conical singularities;*
- (3) *the surface decomposes into at most  $\mathcal{N}$  nonsystolic domains;*
- (4) *the surface has a total of at most  $\mathcal{N}$  edges.*

*Proof.* Let  $D$  be a nonsystolic domain of  $M$ . By definition (see Definition 5.5), the corners of  $D$  and the special conical singularities<sup>3</sup> of  $M$  correspond to intersections either

- (1) between two systolic bands, giving rise to at most eight corners; or
- (2) between two *homotopic* limit systolic loops meeting at one or two points or along an arc, giving rise to at most four corners.

Since there are at most  $\bar{Q}(g) \leq 2^{10} g^2 \log g$  systolic bands by Proposition 4.7, this yields at most

$$\mathcal{N} = 8 \binom{\bar{Q}(g)}{2} + 4 \bar{Q}(g) = 4 \bar{Q}(g)^2 \leq 2^{22} g^4 \log^2(g)$$

corners and special conical singularities. In particular, the surface  $M$  decomposes into at most  $\mathcal{N}$  nonsystolic domains with a total number of at most  $\mathcal{N}$  edges.  $\square$

## 6. THE KITE EXCISION TRICK

In this section, we describe the kite excision trick, a key tool in the proof of Theorem 1.6. Consider a nonpositively curved piecewise flat surface  $M$ . Let  $p, q \in M$  be two conical singularities connected by a geodesic arc  $[p, q]$  with no conical singularity lying in the interior  $(p, q)$ . Denote by  $\theta_p$  and  $\theta_q$  the total angles at  $p$  and  $q$ .

**Definition 6.1** (Kite). Let  $r \in M$  (not on  $[p, q]$ ) be a point such that the triangle  $pqr$  is flat. Consider the reflection  $pqr'$  of triangle  $pqr$  with respect to  $[p, q]$ . Define the *kite*  $K = prqr'$  as the union of the two symmetric triangles; see Figure 1. The two opposite vertices  $p$  and  $q$  of  $K$  are referred to as the *main vertices* of the kite. The *width*  $w$  of  $K$  is the length of the diagonal  $[r, r'] \subseteq K$ .

**Definition 6.2** (Admissible kite). The kite  $K$  is *admissible* if all its angles are less than  $\pi$  and its angles at the main vertices  $p$  and  $q$  are related to the angle excesses of the conical singularities  $p$  and  $q$  as follows:

$$\begin{aligned} \angle rpr' &\leq \min\{\theta_p - 2\pi, \pi\} \\ \angle rqr' &\leq \min\{\theta_q - 2\pi, \pi\}. \end{aligned}$$

**Definition 6.3** (Exact and diamond kites). When  $p$  is a small conical singularity, an admissible kite  $K$  is *exact at  $p$*  if the following equality is satisfied:

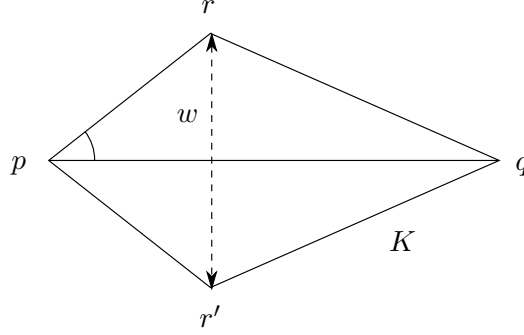
$$\angle rpr' = \theta_p - 2\pi < \pi.$$

$K$  is called a *diamond kite* if  $|pr| = |qr|$ .

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<sup>3</sup>Though not required for our argument, note that the special conical singularities of  $M$  can be thought of as degenerate nonsystolic domains.



FIGURE 1. The flat kite  $K$  of width  $w$ 

**Remark 6.4.** By construction, every admissible kite  $K$  is convex.

**Definition 6.5** (Excised surface  $M_w$ ). Let  $K_w \subseteq M$  be an admissible kite of width  $w = |rr'|$ . We perform a cut-and-paste procedure on  $M$  as follows. We excise the kite  $K_w \subseteq M$  and introduce identifications on the boundary of  $M \setminus K$  by setting  $[p, r] \sim [p, r']$  and  $[q, r] \sim [q, r']$ . The result of the surgery is a piecewise flat surface

$$M_w = (M \setminus K_w) / \sim \quad (6.1)$$

of genus  $g$  with conical singularities.

Note that  $\text{area}(M_w) < \text{area}(M)$ .

**Definition 6.6.** The quotient map

$$\pi_w : M \rightarrow M_w \quad (6.2)$$

is obtained by collapsing each segment of  $K_w$  parallel to the diagonal  $[r, r']$  to a point.

The map  $\pi_w$  is a homotopy equivalence.

**Proposition 6.7.** *If  $K_w$  is an admissible kite then the excised surface  $M_w$  is nonpositively curved with at most one more conical singularity than  $M$ . Furthermore, if  $K_w$  is exact at one of its main vertices, then the surface  $M_w$  has at most as many conical singularities as  $M$ .*

*Proof.* The first statement follows by analyzing the total angles of the points corresponding to the vertices of  $K_w$  and showing that they are at least  $2\pi$ . More precisely, the total angles at the points  $p$  and  $q$  in the excised surface  $M_w$  are  $\theta_p - \angle rpr'$  and  $\theta_q - \angle rqr'$ , both of which are at least  $2\pi$  since the kite is admissible. Similarly, the total angle at the point  $r = r' \in M_w$  is  $2\pi + \angle rpr' + \angle rqr'$ .

For the second statement, even if the point  $r$  (identified with  $r'$ ) creates a new conical singularity, the point  $p$  of total angle  $2\pi$  is no longer a singularity in the new surface  $M_w$ .  $\square$

**Proposition 6.8.** *Let  $p, q \in M$ . Consider an admissible kite  $K_w$  with main diagonal  $[p, q]$  which is either a diamond or an exact kite at  $p$ . Then the excised surface  $M_w$  converges to  $M$ , both for the Gromov–Hausdorff distance and the Lipschitz distance, as the width  $w$  of  $K_w$  tends to zero.*

*Proof.* Fix an admissible diamond  $K_D = pr_0qr'_0$  with main diagonal  $[p, q]$ . Consider a smaller admissible diamond  $K_w = prqr'$  of width  $w$ , and build the excised surface  $M_w = (M \setminus K_w)/\sim$  as in (6.1). Let  $s$  be the midpoint of  $[p, q]$ , so that  $r \in (r_0, s)$ . The diamond is the union of two triangles,  $pr_0r$  and  $qr_0r$ . We will need a map  $\phi_w$  defined as follows.

**Definition 6.9.** Consider the linear map  $\phi_w: pr_0r \rightarrow pr_0s$  (respectively,  $\phi_w: qr_0r \rightarrow qr_0s$ ) fixing  $[p, r_0]$  and mapping the triangle  $pr_0r$  (respectively,  $qr_0r$ ) to the right-angle triangle  $pr_0s$  (respectively,  $qr_0s$ ). We extend the linear map to a continuous map

$$\phi_w: M_w \rightarrow M \quad (6.3)$$

by the identity map on the complement in  $M_w$  of  $K_D \setminus K_w$ .

The map  $\phi_w$  is clearly  $(1+\epsilon)$ -bilipschitz (i.e., the bilipschitz constant tends to 1 as  $w$  tends to zero). It follows that the quadrilateral  $pr_0qr$  is  $(1+\epsilon)$ -bilipschitz with the triangle  $pr_0q$ . By symmetry, the same holds with the quadrilateral  $pr'_0qr'$  and the triangle  $pr'_0q$ . Thus the map  $\phi_w$  is  $(1+\epsilon)$ -bilipschitz. The surfaces are therefore also Gromov–Hausdorff close.

Now consider the case of a kite  $K_E = pr_0qr'_0$  exact at  $p$ . Consider a point  $p_*$  close to  $p$  such that  $p_*$  is on a geodesic extension  $p_*q$  of  $[p, q]$  so that the rotation angle of  $p_*q$  at  $p$  is equal to  $\frac{\theta_p}{2} \geq \pi$  on either side of the segment  $p_*q$ . Since the kite  $K_E$  is exact at  $p$ , we have  $p \in [p_*, r_0]$ . Consider the segment  $[p_*, r_0]$  containing  $p$ . Fix a circular arc  $\widehat{p_*r_0} \subseteq M \setminus K_E$  bounding a flat region  $\mathcal{R}$  together with the segment  $[p_*, r_0]$  containing  $p$ . Take a smaller kite  $K_w = prqr' \subseteq K_E$  of width  $w$  and exact at  $p$ , where  $r \in (p, r_0)$ . Let  $p_r \in [p_*, p]$  with  $|pp_r| = |pr|$ . Note that the rotation angle of  $\mathcal{R}$  at  $p$  is precisely  $\pi$  by the exactness hypothesis. The rotation angle is also  $\pi$  at  $r$  and  $p_r$  by construction. There exists a  $(1+\epsilon)$ -bilipschitz homeomorphism

$$h_r: \mathcal{R} \rightarrow \mathcal{R} \quad (6.4)$$

which fixes the circular arc  $\widehat{p_*r_0}$  pointwise and linearly maps  $[r_0, r]$ ,  $[r, p]$ , and  $[p, p_*]$  to  $[r_0, p]$ ,  $[p, p_r]$ , and  $[p_r, p_*]$ , respectively, where  $\epsilon$  tends to 0 as  $r$  approaches  $p$ .

**Definition 6.10.** We combine the map  $h_r$  of (6.4) with the  $(1+\epsilon)$ -bilipschitz linear map from  $rr_0q$  to  $pr_0q$  fixing  $[r_0, q]$ , and perform a symmetric construction on the other half of the kite, to produce a map

$$\phi_w: M_w \rightarrow M \quad \text{where} \quad M_w = (M \setminus K_E)/\sim. \quad (6.5)$$

The resulting map  $\phi_w: M_w \rightarrow M$  is  $(1+\epsilon)$ -bilipschitz, as in the diamond case.  $\square$

## 7. SYSTOLE COMPARISON

Let  $M$  be a nonpositively curved piecewise flat surface of genus  $g$ . Consider a nonsystolic domain  $D \subseteq M$ . Let  $p$  and  $q$  be conical singularities in the closure  $\overline{D}$  of  $D$ , joined by a geodesic arc  $[p, q] \subseteq \overline{D}$ . Note that the arc may start and end at the same point  $p = q$  in the cases  $(D_1)$  and  $(D_1'')$  below. We can assume that no conical singularity lies in the interior  $(p, q)$ , by picking a different pair of conical singularities along the arc, if necessary. Recall that the set  $D \subseteq M$  is open.

We will now choose an admissible kite  $K_w \subseteq M$  of width  $w$  constructed by symmetry with respect to  $[p, q]$  (see Definition 6.1) in one of the following ways; see Figures 2 through 5.

- $(D_1)$  if  $[p, q] \subseteq D$ , take a diamond  $K_w$  of sufficiently small width so that it lies in  $D$ ;
- $(D_1')$  if  $[p, q] \subseteq D$  where  $q \in \partial D$  and the angle of  $D$  at the point  $q$  is greater than  $\pi$ , take a diamond  $K_w$  of sufficiently small width so that  $K_w \setminus \{q\}$  lies in  $D$ ;
- $(D_1'')$  if  $(p, q) \subseteq D$  with  $p, q \in \partial D$  and the angles of  $D$  at  $p$  and  $q$  are greater than  $\pi$ , take a diamond  $K_w$  of sufficiently small width so that  $K_w \setminus \{p, q\}$  lies in  $D$ ;
- $(E_1)$  if  $[p, q] \subseteq D$  and  $p$  is a small conical singularity, take  $K_w$  exact at  $p$  of sufficiently small width so that it lies in  $D$ ;
- $(E_1')$  if  $[p, q] \subseteq D$  with  $p$  a small conical singularity and  $q \in \partial D$ , and the angle of  $D$  at  $q$  is greater than  $\pi$ , take  $K_w$  exact at  $p$  of sufficiently small width so that  $K_w \setminus \{q\}$  lies in  $D$ ;
- $(E_2)$  if  $[p, q]$  is contained in the interior of an edge of  $\partial D$  and  $p$  is a small conical singularity, take  $K_w$  exact at  $p$  of sufficiently small width so that the part of every systolic loop passing through  $K_w$  is parallel to  $[p, q]$ .<sup>4</sup>

Consider the surface  $M_w = (M \setminus K_w)/\sim$  where the kite  $K_w$  is admissible and satisfies one of the previous hypotheses. Only the properties for kites satisfying  $(E_1)$ ,  $(E_1')$ , and  $(E_2)$  are required for the proof of our main result, Theorem 8.1. In the proofs of Proposition 7.1 and Proposition 7.3, the width  $w$  of  $K_w$  will need to be chosen even smaller to satisfy further restrictions.

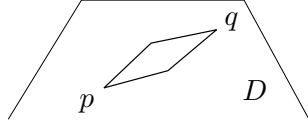
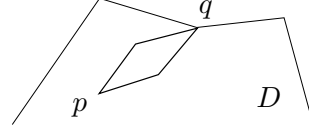
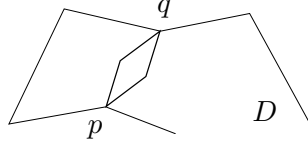
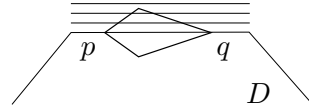
Since the quotient map  $\pi_w: M \rightarrow M_w$  of (6.2) is a nonexpanding homotopy equivalence, we have  $\text{sys}(M_w) \leq \text{sys}(M)$ . In the following, we will show that the reverse inequality holds in the cases  $(D_1)$ ,  $(E_1)$ ,  $(D_1')$ ,  $(E_1')$ ,  $(D_1'')$  and  $(E_2)$  as well.

7.1. Analysis of cases  $(D_1)$  and  $(E_1)$ .

**Proposition 7.1.** *Consider an admissible kite  $K_w \subseteq D$  satisfying  $(D_1)$  or  $(E_1)$ , so that  $K_w$  is either an admissible diamond or an exact kite at  $p$ .*

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<sup>4</sup>Recall that, by definition of a nonsystolic domain  $D$ , no systolic loop meets the interior of an edge of  $\partial D$  unless it contains this edge, which ensures the existence of such kites.

FIGURE 2.  $(D_1), (E_1)$ FIGURE 3.  $(D'_1), (E'_1)$ FIGURE 4.  $(D''_1)$ FIGURE 5.  $(E_2)$ 

If the width  $w$  of  $K_w$  is sufficiently small, then

$$\text{sys}(M_w) = \text{sys}(M).$$

*Proof.* We first consider a diamond kite  $K_D$  lying in a nonsystolic domain  $D \subseteq M$  as in the proposition. Consider the function  $x \mapsto \text{sys}(M, x)$  on  $M$ , where  $\text{sys}(M, x)$  represents the least length of a shortest noncontractible loop based at the point  $x \in M$ . Since  $K_D$  is a compact subset of a nonsystolic domain, there exists an  $\varepsilon > 0$  such that

$$\forall x \in K_D, \text{sys}(M, x) > \text{sys}(M) + \varepsilon. \quad (7.1)$$

Consider a subkite  $K_w \subseteq K_D$ . We construct the map  $\phi_w: M_w \rightarrow M$  using the pair  $K_w \subseteq K_D$  as in Definition 6.9. As  $w$  tends to 0, the bilipschitz constant of  $\phi_w$  tends to 1. Since the deformation  $M_w$  of  $M = M_0$  is continuous with respect to the bilipschitz distance, inequality (7.1) implies that for  $w_0$  sufficiently small, each noncontractible loop  $\gamma \subseteq M_{w_0}$  based at a point of  $\phi_{w_0}^{-1}(K_D)$  satisfies

$$|\gamma| \geq \text{sys}(M). \quad (7.2)$$

Meanwhile if a loop  $\gamma \subseteq M_{w_0}$  is disjoint from  $\phi_{w_0}^{-1}(K_D)$  then

$$|\gamma| = |\phi_w^{-1}(\gamma)| \geq \text{sys}(M) \quad (7.3)$$

since  $\phi_{w_0}$  is an isometry outside of  $K_D$ . The bounds (7.2) and (7.3) prove the proposition in the case of a diamond kite.

For an exact kite  $K_E$ , we follow a similar procedure with  $\phi_w$  of Definition 6.10.  $\square$

## 7.2. Analysis of cases $(D'_1)$ , $(D''_1)$ , and $(E'_1)$ .

**Proposition 7.2.** *Consider an admissible kite  $K_w$  with main diagonal  $[p, q]$  with  $K_w \setminus \{p, q\} \subseteq D$  in one of the following three cases:*

- $(D'_1)$   $K_w$  is an admissible diamond with  $p \in D$  such that the internal angle of  $D$  at  $q \in \partial D$  is greater than  $\pi$ ;

- ( $D_1''$ )  $K_w$  is an admissible diamond,  $p, q \in \partial D$ , and the internal angles of  $D$  at  $p, q \in \partial D$  are greater than  $\pi$ ;  
 ( $E_1'$ )  $K_w$  is an exact kite at  $p \in D$  such that the internal angle of  $D$  at  $q \in \partial D$  is greater than  $\pi$ .

If the width  $w$  of  $K_w$  is sufficiently small, then

$$\text{sys}(M_w) = \text{sys}(M).$$

*Proof.* We will focus on the case ( $E_1'$ ) required for the proof of our main theorem. The proof in the other cases is similar.

The set  $\mathcal{C}$  of conjugacy classes in  $\pi_1(M)$  of systolic loops of  $M = M_0$  is finite. Denote by  $\lambda(M_w, \mathcal{C})$  the least length of a loop of  $M_w$  from a nontrivial conjugacy class *not* in  $\mathcal{C}$ . Clearly,  $\lambda(M, \mathcal{C}) > \text{sys}(M)$ . By continuity, we still have  $\lambda(M_w, \mathcal{C}) > \text{sys}(M) \geq \text{sys}(M_w)$  for sufficiently small  $w$ . Therefore, a systolic loop  $\gamma_w \subseteq M_w$  necessarily represents a class  $C \in \mathcal{C}$ .

Consider a systolic loop  $\gamma \subseteq M$  representing the class  $C$ . Recall that  $\gamma$  does not meet  $D$  and observe that  $|\pi_w(\gamma)| = |\gamma|$  where  $\pi_w$  is the homotopy equivalence (6.2).

Suppose  $\gamma$  does not pass through the singularity  $q \in \partial D$ . Then the loop  $\gamma$  is disjoint from  $K_w$ . Then the projection  $\pi_w(\gamma)$  remains a closed geodesic in  $M_w$ , and  $|\pi_w(\gamma)| = |\gamma|$ . Thus the systolic loop  $\gamma_w$  and the loop  $\pi_w(\gamma)$  are freely homotopic closed geodesics in  $M_w$ . By the flat strip theorem, we obtain  $|\gamma_w| = |\gamma|$  and therefore  $\text{sys}(M_w) = \text{sys}(M)$ .

Now assume  $q \in \gamma$ . Since the angle of  $D$  at  $q$  is greater than  $\pi$ , the rotation angle  $L_q$  (see Definition 4.2) of  $\gamma \subseteq M$  at  $q$  from the side of  $D$  is also greater than  $\pi$ . Note that the angle  $\angle rqr'$  at  $q$  of the excised exact kite  $K_w$  tends to zero. Choosing

$$\angle rqr' < L_q - \pi,$$

we ensure that the rotation angle at  $q$  is still greater than  $\pi$  for the projected loop  $\pi_w(\gamma) \subseteq M_w$ . By the local characterisation of geodesics, the projected loop  $\pi_w(\gamma) \subseteq M_w$  is still a closed geodesic. Since  $|\pi_w(\gamma)| = |\gamma|$ , the loops  $\gamma_w$  and  $\pi_w(\gamma)$  are freely homotopic closed geodesics in  $M_w$ . By the flat strip theorem, we conclude that  $|\gamma_w| = |\gamma|$  and hence  $\text{sys}(M_w) = \text{sys}(M)$ , as required.  $\square$

### 7.3. Analysis of case ( $E_2$ ).

**Proposition 7.3.** *Consider an admissible kite  $K_w$  with main diagonal  $[p, q]$  satisfying ( $E_2$ ). Namely,  $[p, q]$  is contained in the interior of an edge of  $\partial D$  and  $K_w$  is an exact kite at  $p$  so that the part of every systolic loop passing through  $K_w$  is parallel to  $[p, q]$ . If the width  $w$  of  $K_w$  is sufficiently small, then*

$$\text{sys}(M_w) = \text{sys}(M).$$

*Proof.* As in the proof of Proposition 7.2, a systolic loop  $\gamma_w \subseteq M_w$  necessarily represents a class  $C$  in the set  $\mathcal{C}$  of conjugacy classes of systolic loops

of  $M$ . Consider the segment  $I_w \subseteq M_w$  defined by

$$I_w = \pi_w(K_w) = \pi_w([p, q]),$$

where  $\pi_w: M \rightarrow M_w$  is the projection (6.2). Let  $c_w = \gamma_w \setminus I_w$  in  $M_w$ . Consider a systolic loop  $\gamma \subseteq M$  representing the class  $C$ .

If  $\gamma$  is disjoint from the segment  $[p, q]$  (and hence from  $K_w$  if  $w$  is sufficiently small) then its projection  $\pi_w(\gamma)$  remains a closed geodesic in  $M_w$  of the same length as  $\gamma$ . By the flat strip theorem, we conclude that  $|\gamma_w| = |\gamma|$  and  $\text{sys}(M_w) = \text{sys}(M)$ .

Thus, we can assume that the class  $C$  contains a single (isolated) systolic loop  $\gamma \subseteq M$ , which meets the segment  $[p, q]$  and therefore must contain  $[p, q]$  by condition  $(E_2)$ . Now Proposition 7.3 will result from the following lemma.

**Lemma 7.4.** *Let  $\gamma_w \subseteq M_w$  be a systolic loop. Let  $I_w = \pi_w(K_w) \subseteq M_w$  be the segment given by the image of the kite  $K_w$ . Let  $c_w = \gamma_w \setminus I_w$ . Then  $c_w \subseteq M_w$  is a connected open segment.*

*Proof.* Recall that the map  $\phi_w: M_w \rightarrow M$  of (6.5) is  $(1 + \epsilon)$ -bilipschitz. Since  $\text{sys}(M_w) \leq \text{sys}(M)$ , the loop  $\phi_w(\gamma_w) \subseteq M$  homotopic to  $\gamma$  is of length at most  $(1 + \epsilon)|\gamma|$ . Since the systolic loop  $\gamma \subseteq M$  is isolated, the loop  $\phi_w(\gamma_w)$  necessarily converges to  $\gamma$ .

Since  $\gamma$  is a simple loop containing the main diagonal of  $K_w$ , the part of  $\gamma$  lying outside  $K_w \subseteq M$  necessarily consists of a single arc for  $w$  small enough. It follows that there is a single subarc of  $\phi_w(\gamma_w)$  lying outside some small open neighborhood  $U$  of  $K_w$  such that  $K_w$  is a deformation retract of  $U$ . The image of this subarc by  $\phi_w^{-1}$  lies in an open subarc  $\alpha_w$  of  $\gamma_w$  lying in  $M_w \setminus I_w$  with endpoints in  $I_w$ . Since  $M$  is nonpositively curved and  $I_w$  is convex, all the other geodesic subarcs of  $\gamma_w$  with endpoints in  $I_w$ , which lie in a small neighborhood of  $I_w$ , in fact lie in  $I_w$ . Thus,  $\alpha_w$  is the only subarc of  $\gamma_w$  lying outside  $I_w$ , that is,  $c_w = \alpha_w$ .  $\square$

We continue with the proof of Proposition 7.3. Let  $\sigma_w \subseteq M$  be the closure of  $\pi_w^{-1}(c_w)$  in  $M$ .

Suppose one of the endpoints of  $\sigma_w$  is one of the main vertices of the kite, say  $p$ . Let  $y$  be the other endpoint. The segment  $[p, y] \subseteq K_w$  projects to the path of  $I_w \subseteq M_w$  connecting  $\pi_w(p)$  and  $\pi_w(y)$ . Then the loop  $\bar{\gamma}_w = \sigma_w \cup [p, y] \subseteq M$  in the homotopy class  $C$  satisfies  $|\bar{\gamma}_w| \leq |\gamma_w|$ . Thus,  $\text{sys}(M) \leq \text{sys}(M_w)$ , providing the required bound. Therefore, we can assume that the endpoints of  $\sigma_w$  are disjoint from  $\{p, q\} \subseteq M$ .

Suppose one of the endpoints of the path  $\sigma_w$  in  $\partial K_w \subseteq M$  is a point other than  $r$  and  $r'$ . In such case, the minimizing loop  $\gamma_w \subseteq M_w$  meets the interval  $I_w$  transversely at a regular (i.e., non-singular) point of  $M_w$ . It follows that the endpoints of  $\sigma_w$  project to the same point on the closed geodesic  $\gamma \subseteq M$ . Hence the nearest-point projection of  $\sigma_w$  to  $\gamma$  closes up. By the assumption of nonpositive curvature, the projection map is distance-decreasing. Therefore  $|\gamma_w| \geq |\sigma_w| \geq |\gamma| = \text{sys}(M)$  in this case, as well.

Thus we can assume that the endpoints of  $\sigma_w$  are the points  $r, r' \in M$ . In this case also the nearest-point projection of  $\sigma_w$  to the loop  $\gamma \subseteq M$  closes up. Hence  $|\gamma_w| \geq |\sigma_w| \geq |\gamma| = \text{sys}(M)$ , proving the proposition.  $\square$

## 8. EXPLOITING THE KITE EXCISION TRICK

We proceed to the proof of the existence of nonpositively curved piecewise flat locally extremal metrics on every genus  $g$  surface.

Recall that a local infimum of the systolic area on the space  $\mathcal{H}_g$  of non-positively curved Riemannian metrics (possibly with conical singularities) on a genus  $g$  surface is a real number  $\mu > 0$  such that there exists an open set  $\mathcal{U} \subseteq \mathcal{H}_g$  satisfying a strict inequality

$$\mu = \inf_{M \in \mathcal{U}} \sigma(M) < \inf_{M \in \partial \mathcal{U}} \sigma(M), \quad (8.1)$$

as in Definition 1.4. Recall that  $\bar{Q}(g)$  is the maximal number of systolic homotopy classes; see Proposition 4.7.

**Theorem 8.1.** *Let  $\mathcal{U}$  be an open set in the space  $\mathcal{H}_g$  of nonpositively curved genus  $g$  surfaces (possibly with conical singularities) defining a local infimum. Then there exists a nonpositively curved piecewise flat metric  $\mathcal{G}_0$  in  $\mathcal{U}$  with at most*

$$\mathcal{N}_0 = 20 \bar{Q}(g)^2 \leq 2^{25} g^4 \log^2(g)$$

*conical singularities whose systolic area is the infimum of the systolic area of any nonpositively curved Riemannian metric  $\mathcal{G} \in \mathcal{U}$  with conical singularities, i.e.,  $\sigma(\mathcal{G}_0) \leq \sigma(\mathcal{G})$ .*

*Proof.* Let  $\mathcal{G} \in \mathcal{U}$  be a nonpositively curved Riemannian metric with conical singularities such that

$$\sigma(\mathcal{G}) < \inf_{\partial \mathcal{U}} \sigma. \quad (8.2)$$

By metric approximation (see Proposition 3.2) we can assume that  $\mathcal{G}$  is a nonpositively curved piecewise flat metric with conical singularities. Denote by  $N$  the number of conical singularities of  $\mathcal{G}$ . By compactness (see Proposition 3.7) and the strict inequality (8.2), there exists a metric  $\mathcal{G}_1$  in  $\mathcal{U}$  with minimal systolic area among all nonpositively curved piecewise flat metrics in  $\mathcal{U}$  with at most  $N$  conical singularities. By Proposition 5.7 and Lemma 5.2, the metric  $\mathcal{G}_1$  (as any nonpositively curved piecewise flat metric on  $M$ ) has at most

$$\mathcal{N} = 4 \bar{Q}(g)^2 \leq 2^{22} g^4 \log^2(g) \quad (8.3)$$

special conical singularities and large conical singularities.

It remains to find a similar upper bound on the number of small non-special conical singularities for  $\mathcal{G}_1$  by relying on its local extremality among all nonpositively curved piecewise flat surfaces of  $\mathcal{U}$  with at most  $N$  singularities. From now on, the surface  $M$  will be endowed with the metric  $\mathcal{G}_1$ . Recall that the (small) nonspecial conical singularities lie in (the closure of) the nonsystolic domains of  $M$ .

The next pair of lemmas exploiting the kite excision trick provide such an upper bound.

**Lemma 8.2.** *Let  $M$  be a local extremum of the systolic area relative to an open set  $\mathcal{U} \subseteq \mathcal{H}_g$  among all nonpositively curved piecewise flat genus  $g$  surfaces in  $\mathcal{U}$  with at most  $N$  conical singularities as in (8.3). Then every nonsystolic domain  $D \subseteq M$  contains at most one small conical singularity.*

*Proof.* We argue by contradiction. Suppose  $p$  and  $q'$  are two conical singularities in  $D$  with  $p$  small. Let  $[p, q']$  be a length-minimizing arc in the closure of  $D$  joining the two points. We consider the following two cases.

- (1) If  $[p, q']$  lies in the open domain  $D$ , we denote by  $q$  the first conical singularity along  $(p, q']$  from  $p$ .
- (2) Otherwise, the arc  $[p, q']$  meets  $\partial D$ , and the first point of intersection of  $[p, q']$  with  $\partial D$  from  $p$  is a point, denoted  $q$ , at which  $D$  is strictly concave.

In the second case, the angle of  $D$  at  $q$  is greater than  $\pi$ , which shows that  $q$  is a conical singularity.

In either case, we apply the kite excision trick to  $[p, q]$  with an exact kite  $K_w$  at  $p$ , see Definition 6.1, of width  $w$  small enough to satisfy  $(E_1)$  in the first case (when  $q$  lies in  $D$ ) and  $(E'_1)$  in the second case (when  $q$  lies in  $\partial D$ ); see Section 7. We also choose  $w$  small enough to ensure that the resulting piecewise flat surface  $M_w$  lies in  $\mathcal{U}$ ; see Proposition 6.8. By Proposition 6.7, the surface  $M_w$  is nonpositively curved and has no more conical singularities than  $M$ . By Proposition 7.1, the systole of  $M_w$  is equal to the systole of  $M$ . As the area of  $M_w$  is less than the area of  $M$ , this contradicts the local extremality of  $M$  among all nonpositively curved piecewise flat genus  $g$  surfaces in  $\mathcal{U}$  with at most  $N$  conical singularities.  $\square$

**Lemma 8.3.** *Let  $M$  be a local extremum of the systolic area relative to an open set  $\mathcal{U} \subseteq \mathcal{H}_g$  among all nonpositively curved piecewise flat genus  $g$  surfaces in  $\mathcal{U}$  with at most  $N$  conical singularities as in (8.3). Then the interior of every edge  $\mathcal{E}$  of a nonsystolic domain  $D$  of  $M$  contains at most one small conical singularity.*

*Proof.* We argue by contradiction. Let  $p$  be a small conical singularity in the interior of the edge  $\mathcal{E}$ . Let  $q$  be a conical singularity in the interior of  $\mathcal{E}$  adjacent to  $p$ . Note that the conical singularity  $q$  may be large. Apply the kite excision trick to  $[p, q]$  with an exact kite  $K_w$  at  $p$  (see Definition 6.1) of width  $w$  small enough to satisfy  $(E_2)$  (see Section 7) and to ensure that the resulting piecewise flat surface  $M_w$  lies in  $\mathcal{U}$ ; see Proposition 6.8. We obtain a contradiction by arguing as in the proof of Lemma 8.2 by applying Proposition 7.3.  $\square$

**Remark 8.4.** Technically speaking, we show stronger results in the proofs of the two previous lemmas. Namely, if a small conical singularity lies in a nonsystolic domain of  $M$  then this domain contains no other conical singularity (small or large). Similarly, if a small conical singularity lies in the



interior of an edge of  $M$  then the interior of this edge contains no other conical singularity (small or large).

We conclude the proof of Theorem 8.1 as follows. Proposition 5.7 provides an upper bound on the total number of nonsystolic domains and edges (and so vertices). Combined with Lemma 8.2 and Lemma 8.3, this shows that the surface  $M$  has at most  $3\mathcal{N}$  small nonspecial conical singularities. Along with our previous estimates on the number of special and large conical singularities, this shows that the metric  $\mathcal{G}_1$  has at most

$$\mathcal{N}_0 = 5\mathcal{N} = 20\bar{Q}(g)^2 \leq 2^{25}g^4 \log^2(g)$$

conical singularities. In particular, the open set  $\mathcal{U}$  contains nonpositively curved piecewise flat metrics on  $M$  with at most  $\mathcal{N}_0$  conical singularities.

By compactness (see Proposition 3.7), and since the systolic area of  $\mathcal{G}_1$  is less than  $\inf_{\partial\mathcal{U}} \sigma$ , there exists a metric  $\mathcal{G}_0$  in  $\mathcal{U}$  with minimal systolic area among all nonpositively curved piecewise flat metrics in  $\mathcal{U}$  with at most  $\mathcal{N}_0$  conical singularities. By definition, the metric  $\mathcal{G}_0$  does not depend on  $\mathcal{G}$  (nor on  $\mathcal{G}_1$ ) and satisfies

$$\sigma(\mathcal{G}_0) \leq \sigma(\mathcal{G}_1) \leq \sigma(\mathcal{G})$$

for every nonpositively curved Riemannian metric  $\mathcal{G}$  with conical singularities in  $\mathcal{U}$ .  $\square$

We immediately deduce the existence of locally extremal nonpositively curved piecewise flat metrics on every genus  $g$  surface.

**Corollary 8.5.** *Every local infimum of the systolic area on the space  $\mathcal{H}_g$  of nonpositively curved genus  $g$  surfaces (possibly with conical singularities) is attained by a nonpositively curved piecewise flat metric.*

## 9. SHAPE AND SINGULARITIES OF NONSYSTOLIC DOMAINS

In this section, we provide a more precise description of nonsystolic domains of a locally extremal nonpositively curved surface  $M$  of genus  $g$ , whose existence was established in Corollary 8.5. We also show that the systolic part of  $M$  is connected.

**Lemma 9.1.** *Every nonsystolic domain  $D$  of a locally extremal surface  $M$  contains at most one conical singularity.*

*Proof.* We argue as in the proof of Lemma 8.2, relying now on the local extremality of  $M$ . Assume by contradiction that there are two conical singularities  $p$  and  $q'$  in  $D$ . If the length-minimizing arc  $[p, q']$  joining  $p$  to  $q'$  in the closure of  $D$  lies in  $D$ , we denote by  $q$  the first conical singularity along  $(p, q']$  from  $p$ . Otherwise, the arc  $[p, q']$  meets  $\partial D$  and its first point of intersection (from  $p$ ) is a conical singularity, denoted  $q$ .

In either case, take an admissible diamond  $K_w$  with diagonal  $[p, q]$  of width  $w$  small enough to satisfy  $(D_1)$  in the former case and  $(D'_1)$  in the latter case; see Section 7. Apply the kite excision trick to  $K_w$ . The resulting

piecewise flat surface  $M_w$  may have more conical singularities than  $M$ , but is still nonpositively curved; see Proposition 6.7. Taking the width of  $K_w$  small enough as in Proposition 6.8, we can further ensure that the surface  $M_w$  lies in the open set  $\mathcal{U}$  of  $\mathcal{H}_g$  involved in the definition of a locally extremal nonpositively curved metric on  $M$ ; see Definition 1.4. By Proposition 7.1, the systole of  $M_w$  is greater or equal to the systole of  $M$ . As the area of  $M_w$  is less than the area of  $M$ , this contradicts the local extremality of  $M$  among all nonpositively curved piecewise flat genus  $g$  surfaces in  $\mathcal{U}$ , establishing the lemma.  $\square$

**Lemma 9.2.** *A nonsystolic non-simply-connected domain  $D$  of a locally extremal surface  $M$  is necessarily convex.*

*Proof.* Assume that  $D$  is nonconvex. Then there is a conical singularity  $x \in \partial D$  where the angle of  $D$  is greater than  $\pi$ . Consider a length-minimizing noncontractible loop  $\gamma$  based at  $x$  in the closure of  $D$ , which contains an arc  $[p, q]$  with  $p, q \in \partial D$ , whose interior  $(p, q)$  lies in  $D$ . The angles of  $D$  at the points  $p$  and  $q$  are greater than  $\pi$ , which implies that these two points are conical singularities. Note that the points  $p$  and  $q$  may agree. We apply the kite excision trick to an admissible diamond with diagonal  $[p, q]$  of width small enough to satisfy  $(D'_1)$ , or  $(D''_1)$  if  $p = q$ , and derive a contradiction as in the proof of Lemma 9.1. This shows that such a domain  $D$  must be convex.  $\square$

**Proposition 9.3.** *Every nonsystolic domain  $D$  of a locally extremal surface  $M$  is homeomorphic to a disk.*

*Proof.* Arguing by contradiction, we suppose that  $D \subseteq M$  is nonsimply connected. By Lemma 9.2,  $D$  must be convex. Assume that  $D$  contains a conical singularity  $p$ . By Lemma 9.1, this is the only conical singularity in  $D$ . Since  $D$  is convex (and nonsimply connected), there is a length-minimizing noncontractible loop  $\gamma$  based at  $p$  lying in  $D$ . We apply the kite excision trick to an admissible diamond with diagonal the geodesic arc  $\gamma$ , starting and ending at  $p$ , of width small enough to satisfy  $(D_1)$ , and derive a contradiction as in the proof of Lemma 9.1. This shows that  $D$  has no conical singularity.

By the Gauss–Bonnet formula for surfaces with boundary, the Euler characteristic of the (orientable) flat surface  $D \subseteq M$  with convex boundary is nonnegative. This implies that  $D$  is a flat cylinder (by assumption, it is not a disk). In this case, it also follows from the Gauss–Bonnet formula that the cylinder  $D = S^1 \times I$  has geodesic boundary components. Since the cylinder is nonsystolic, its boundary loops are systolic geodesics, and any closed geodesic not parallel to the boundary must have length strictly greater than  $\text{sys}(M)$ . Therefore we can slightly shrink the height  $I$  of the cylinder without affecting the systole of  $M$ , and respecting the condition of nonpositive curvature on  $M$ . This contradicts the local extremality of the surface. Hence, the domain  $D$  is simply connected and so is a disk.  $\square$

As a consequence of Proposition 9.3, the systolic part of  $M$ , defined as the union of its systolic loops, is obtained by removing finitely many open disks from the surface. In particular, we obtain the following corollary.

**Corollary 9.4.** *The systolic part of a locally extremal nonpositively curved surface is path-connected.*

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