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A Model for Wind Farm Management with Option Interactions

Alain Bensoussan

UT Dallas, 800 West Campbell Road, Richardson, TX 75080-3021;
City University of Hong Kong, Tat Chee Avenue, Kowloon, Hong Kong SAR;
alain.bensoussan@utdallas.edu;

Benoît Chevalier-Roignant

emlyon business school, 23 Avenue Guy de Collongue, 69130 Ecully, France; corresponding author;
benoit.chevalier-roignant@outlook.com;

Alejandro Rivera

UT Dallas, 800 West Campbell Road, Richardson, TX 75080-3021;
alejorivera1@gmail.com.

A renewable energy site can expand its power generation capacity by an endogenous amount, but may also want to shut down to save on fixed operating costs and interest payments if the market prospects deteriorate. We model such circumstances and derive managerial implications that help us explain real-world conundrums, illustrating the intricate interactions between the operational decision to build up capacity and the financial decision to exit an industry. Shutting down may be delayed in the hope of expanding capacity upon recovery; expansion may also be delayed in the presence of a valuable exit option. Numerical extensions provide further managerial insights. In particular, the presence of fixed or proportional financing costs may lead the firm to delay its expansion decision, but the scale of investment will only be affected by proportional costs. If herding behavior causes equipment prices to increase (resp., decrease) when electricity prices are high (resp., low), managers should invest earlier (resp., later) and more (resp., less) while equipment prices are low (resp., high). Furthermore, although volume swings (due to capacity decommissionings and expansions) are marked in a homogeneous industry (when the default and expansion thresholds are reached), heterogeneity in the population of wind farms smooths out such effects.

Key words: Wind farm management, real options, capacity expansion, operations-finance interface.

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1 Introduction

Because renewable (wind and solar) energy generation emits virtually no greenhouse gases, states often encourage private investment in these technologies. The free access to these energy sources, coupled with state-led stimulus programs (e.g., in the form of a tax credit in the US or feed-in tariffs in Germany), have made the economics of renewable energy generation an attractive proposition. Renewable energy capacity has been on the rise for several decades and is expected to increase further. In the 2010s, nearly \$2.7trn was invested in renewable energy globally (including \$1.4trn in solar and

\$1.1trn in wind)—principally by China, the US, Japan, Germany, and the UK—accounting for 13.4% of global power generation as of 2019 (UNEP 2020). To reach the targets set by 87 governments around the world, an additional 721 gigawatts (GW) of renewable energy capacity is needed by 2030, including 460GW in solar photovoltaic energy (“Solar PV”) and 223GW in wind (UNEP 2020). Moreover, to appeal to environmentally friendly customers, investors, and potential recruits, an increasing number of corporations (including Apple, Facebook, and Microsoft) have pledged to source 100% of their power from renewables by 2030 (the “RE100 initiative”)—leading to an additional construction need of 105GW. Taken together, these commitments by governments and corporations (826GW in total) will *not* suffice to curb the rise in global temperatures to “well below 2°C,” as targeted by nearly 200 states under the Paris agreement (UNEP 2020, p.12). Further investments are required. The Glasgow Financial Alliance for Net Zero, representing \$130bn in assets under management, has recently pledged to contribute to a shift from fossil fuels to clean energy sources (see Economist 2021).

The overarching research question in this paper is how project finance (which permits a default if the economics of a site deteriorate) plays a role in encouraging or discouraging renewables investment, a key question for the literature at the interface of finance, operations, and risk management (“iFORM”). In 2019, although 65% of renewables investment was financed “on balance sheet” by utilities, energy companies, and developers, 35% was funded via “*project finance*” (see UNEP 2020, p. 35), a share that is likely to increase. Project finance helps fund the development and operations of a set of assets on a non-recourse basis via Special Purpose Vehicles (SPVs). The SPV owns all the affiliated rights, assets, and liabilities, while most of the financing is provided for by bank loans, with debt-to-capital ratios that can exceed 85% (Raikar and Adamson 2020, p. 23). In the renewables market, the largest project finance deal to date is the \$4.3bn “Al Maktoum IV complex” in Dubai. The deal, which closed in March 2019 (for 700MW in solar thermal power and 250MW in solar PV), is 60% financed via debt (see UNEP 2020, p. 46). When a bank funds a project under a project finance arrangement, it cannot make claims against the “sponsor” (or the SPV’s shareholders); its recourse is limited to the SPV’s assets. In other words, project finance effectively allows the sponsor to renege on the SPV’s obligations if the economics of a particular site deteriorate. Because of long and costly development processes (including site identification, land acquisition, permit application, project design, connection to the power grid), which can exceed 10 years (Raikar and Adamson 2020, p. 59), and the “not-in-my-backyard” reluctance of neighboring communities, it is often easier for a sponsor to retrofit or redeploy an existing site than start a project from scratch. The fixed costs of financing and operating a wind farm contribute to the credit risk borne by the SPV and the overall attractiveness of the redeployment project. We contribute to the extant literature by treating credit

risk endogenously: we consider the sponsor's propensity to renege on the SPV's obligations and study how this (endogenous) credit risk affects the sponsor's operational decision to expand capacity.

This paper models the situation faced by the sponsor of a renewable energy project (e.g., Equinor, Iberdrola, Ørsted, or Enel Green Power). The SPV incurs various fixed running costs (e.g., debt servicing, maintenance, property tax). The sponsor can decide to expand power generation capacity at a time and by a size of its choice (if the SPV's revenues attain a sufficiently large level) or let the SPV die to save on fixed running costs (should the revenues fall significantly). The bankruptcy of Suntech Power, a large solar panel manufacturer, is a reminder that the renewable energy sector is not safe from adverse developments. Using the terminology of real options (see Dixit and Pindyck 1994, Trigeorgis 1996, for an overview), the decision to expand capacity (resp., let the SPV die) is akin to exercising an expansion (resp., exit) option. The options here are nonstandard because (a) the sequence of decisions is not set *ex ante*, but is at the discretion of the decision maker (see, also, Kwon 2010) and also because (b) the sponsor decides on the investment time and the capacity investment amount (see Dangl 1999, Bensoussan and Chevalier-Roignant 2018, Bensoussan, Chevalier-Roignant, and Rivera 2021). This leads to interesting interactions between the operational and financial decisions because the SPV's initial, endogenous credit risk (which is linked to the sponsor's decision to stop covering the SPV's losses) depends on the potential future benefits for the sponsor from expanding capacity, which itself depends on the degree of credit risk borne by the SPV following the investment.

The stylized model developed in this paper is designed to address the following research questions (RQs). A) What is the SPV's *credit risk* arising from its sponsor's financial decision to stop covering the SPV's losses? B) How does such endogenous credit risk affect the sponsor's *operational decision* to expand capacity? C) How do the future likely benefits from the capacity expansion affect the sponsor's *initial credit risk*? D) How does the possibility of financing the capacity expansion by *raising new funds* affect the sponsor's expansion decision and its initial credit risk? E) In an *industry context*, does cost heterogeneity in a population of wind farms help reduce the observed volume swings and how do herding behaviors (which affect equipment prices) influence a wind farm's operational and financial decisions?

While addressing these research questions, we obtain novel managerial insights that explain real-world conundrums. To address research question A, we model the sponsor's decision to stop covering the SPV's losses as an optimal stopping problem and determine a cut-off electricity price below which the sponsor will declare bankruptcy. Default is less likely if the SPV is less exposed to merchant risk and if its current generation capacity is larger. We study research question B and establish that, if the

credit risk is limited, e.g., because the SPV secures large revenues from a power purchase agreement (PPA), the sponsor will invest if the electricity price exceeds another cut-off level that is increasing in the current capacity. Interestingly, if the SPV faces significant merchant price risk and is hence less exposed to credit risk, the cut-off level turns out to be *nonmonotonic*: for low initial capacity, the sponsor may agree to finance a capacity expansion to circumvent the SPV's default, while for large initial capacity, credit risk has a less significant effect on the sponsor's operational decision to expand. If the financial prospects remain gloomy despite the possibility of circumventing bankruptcy via a capacity expansion, the sponsor will stop covering the SPV's losses. Furthermore, a greater initial capacity leads to a delay in capacity expansion, but has a lesser impact on the SPV's credit risk. A delay *and* an increase in the scale of redeployment may be caused by higher fixed costs, a stronger price buildup, or a more volatile environment. As part of our investigation of RQ C, we show that the option to expand capacity is valuable, so the sponsor will be more patient before liquidating the SPV if its revenues are less secured.

We build on the base model to address our other research questions and derive new insights numerically. In particular, we explore the effect of financial constraints on the sponsor's decisions (RQ D) and find that additional financing costs tend to delay the expansion decision, but that they only affect the investment scale if the costs are proportional. While we began by considering a wind farm in isolation, we expand our analysis by considering it as part of an industry (RQ E). We capture herding behavior, when greater equipment demand leads to inflated prices, by embedding a stochastic dependency between the equipment cost and the electricity price and observe that, to circumvent herding and preempt inflated equipment prices, the sponsor may expand earlier and is less likely to default since downturns offer investment opportunities. Finally, we study how cost heterogeneity in a population of wind farms reduces the volume swings observed when farms are decommissioned or expanded.

2 Literature review

Our project contributes to the rapidly developing field of research at the interface of finance, operations, and risk management ("iFORM") in the operations management literature, which was initially developed to challenge the assumed classical separation between operational and financial decisions in the finance literature (see, e.g., Seshadri and Subrahmanyam 2005, Birge, Kouvelis, and Seppi 2007, Birge 2015, Babich and Kouvelis 2018, Wang, Zhao, and Huchzermeier 2021). Among numerous topics, the iFORM literature explores how the trading of various financial securities by firms (including the issuance of corporate debt and the associated exposure to financial distress) affects their decisions to build up inventory (e.g., Gaur and Seshadri 2005, Li, Shubik, and Sobel 2013,

Iancu, Trichakis, and Tsoukalas 2017, Alan and Gaur 2018) or to invest in production capacities (e.g., Gaur, Seshadri, and Subrahmanyam 2011, Boyabath and Toktay 2011, Chod and Zhou 2014, de Véricourt and Gromb 2018). We explore such interactions between the financial decision to let the SPV default and the operational decision to expand capacity in the energy context, drawing on notions from real options analysis (ROA) (see, e.g., Dixit and Pindyck 1994, Trigeorgis 1996).

There is a long tradition in the operations management literature of studying a firm’s operational decisions about its capacity, which leverages—often explicitly— notions from ROA. Such capacity-choice situations have been cast as multistage stochastic programs (e.g., Birge 2000, Van Mieghem 2003, Miller and Park 2005, Sting and Huchzermeier 2014, Kouvelis and Tian 2014, Chevalier-Roignant, Flath, and Trigeorgis 2019, Trivella, Nadarajah, Fleten, Mazieres, and Pisinger 2021), as classical optimal stopping problems involving a fixed capacity increment (e.g., Gollier, Prout, Thais, and Walgenwitz 2005, Wang, Liu, Zhang, and Li 2019), or as problems of continuous (e.g., Davis, Dempster, Sethi, and Vermes 1987) or impulse control (e.g., Bensoussan and Chevalier-Roignant 2018). Some papers have adjusted these techniques to account for the idiosyncrasies of the energy sector (e.g., Brennan and Schwartz 1985, Erlenkotter, Sethi, and Okada 1989, Smith and McCardle 1999, Gollier et al. 2005, Secomandi 2010a,b, Lai, Margot, and Secomandi 2010, Secomandi and Wang 2012, Nadarajah, Margot, and Secomandi 2015, Trivella et al. 2021). Within this literature, our model is closest to the optimal stopping problems formulated by Dangl (1999), Bensoussan and Chevalier-Roignant (2013), and Chronopoulos, Hagspiel, and Fleten (2017), which allow for a decision about capacity size at the time of investment. Conversely, our overarching research question is about the interactions between the firm’s operational decision to expand capacity and its sensitivity to financial distress induced by the level of fixed costs (including debt servicing), the latter aspect being omitted by design in the aforementioned papers.

In contrast to the extant iFORM literature (see Wang et al. 2021, Section 6.6), we consider the output price to be a primary source of uncertainty and endogenize the credit risk by considering, in a multiperiod model, the shareholders’ incentive to stop covering project losses in the spirit of Leland (1994) (RQ A). Our setup thus shares similarities with Kwon’s (2010) model, which considers that the investor may exit the industry before or after an investment. However, we go further in that we determine the circumstances under which, given such ex post financial distress, a firm should expand production capacity and the scale of this investment (RQ B) and assess how the ex ante credit risk is reduced by this upside potential (RQ C). We also consider the possibility of raising new funds to finance the expansion (RQ D) and study the effect of cost heterogeneity and herding behaviors in an industry context (RQ E). (Another difference is that Kwon’s paper considers profit, following an arithmetic Brownian motion, as a driver of uncertainty, while our driver is a commodity price that

follows geometric Brownian motion as per the empirical evidence in the energy sector (see Pindyck 1999, Schwartz and Smith 2000).)

The possibility in the energy sector of suspending production to respond to changes in energy prices or governmental policy and the incentives of producers to turn to renewable energy are known to be under-researched topics in the iFORM literature (see Wang et al. 2021, Section 4.4.2). Our model helps us derive novel insights in line with the stylized features of this sector.

3 Renewables farm economics

As the equityholder of an SPV owning a renewable energy site (e.g., the Al Maktoum IV complex), the sponsor (e.g., a renewable energy developer, private equity firm, utility) is the residual claimant of the SPV's profit and faces the highest risk exposure. We review below the main components that affect the sponsor's profit (and ultimately its operational and financing decisions). Our goal is to design a stylized model that can be studied analytically and to provide numerical extensions to address our research questions.

Output. Consider a wind farm project with the specifications of Lazard (2020, p. 14). Given 175MW-capacity turbines, the yearly capacity is $x_0 = 175\text{MW} \times 8,760\text{h/year} = 1,533\text{GWh}$. This capacity does not necessarily convert into power generation because the output depends on weather patterns (e.g., wind speeds, cloud cover) and may be reduced owing to equipment-specific inefficiencies. As we later consider a capacity optimization problem (with a linear investment cost for the capacity addition), we want to ensure that the firm's profit is concave in the capacity x (to ensure a finite capacity). Following the structural calibration of Li, Whited, and Wu (2016) for the energy sector, we set an exponent $\epsilon = 0.57 \in (0, 1)$ in the production function $x \mapsto x^\epsilon$ to capture decreasing returns to scale.

Offtake strategies & power prices. Changes to power prices may be a source of risk. The exposure to price uncertainty is determined by the offtake strategy chosen by the SPV:

Power Purchase Agreements (PPAs). The SPV can sign a long-term contract with an "offtaker" who will be required to purchase the output at a set price. Because the SPV is then exposed to counterparty risk, the offtaker is generally an investment-grade company (Raikar and Adamson 2020, p. 56). Offtakers were traditionally utilities, but are increasingly corporate buyers. For instance, in 2019, Google, Facebook, and Amazon signed PPAs covering 2.7, 1.1, and 0.9GW, respectively (UNEP 2020, p.37). The power price in these contracts can be fixed or indexed, e.g., to inflation.

Spot market & merchant risk. If the SPV has not covered all of its output in a PPA, it will sell power in the open spot market. Merchant deals accounted for 1.3GW of solar power in 2019 and are

expected to become more common (see UNEP 2020, p. 38). In such cases, the clearing price is typically set at the highest bid to generate the power that is necessary to match supply and demand (“merchant price”). As supply and demand change frequently, the merchant price is volatile, even within a single day. In the short run, power prices exhibit mean reversion (Smith and McCardle 1999, Schwartz and Smith 2000), but in the longer run, they vary with other factors, such as natural gas prices in markets relying on combined cycle gas turbine (CCGT) capacity. Geometric Brownian Motion (GBM) is an appropriate model for equilibrium power prices in the longer term (Pindyck 1999, Schwartz and Smith 2000):

$$\begin{cases} Y_0 = y, \\ dY_t = \mu Y_t dt + \sigma Y_t dW_t, \forall t > 0, \end{cases}$$

where $W : \Omega \times \mathbb{R}_+ \rightarrow \mathbb{R}$ is a Brownian motion that generates a filtration $(\mathcal{F}_t; t \geq 0)$ and $\sigma > 0$. We use risk-neutral valuation (Smith 2005, Zhou, Scheller-Wolf, Secomandi, and Smith 2019) and choose a risk-adjusted drift $\mu = 1.15\%$ and a volatility $\sigma = 14.5\%$ (Schwartz and Smith 2000). For capital budgeting purposes, the longer-term price dynamics are the most important factor. A small farm does not wield sufficient market power to influence the equilibrium price and is thus considered a price taker (see, e.g., Zhou et al. 2019). Accounting for strategic interactions would require adjustments (see Chevalier-Roignant, Flath, Huchzermeier, and Trigeorgis 2011, for an overview), which we consider nonessential here. The SPV can hedge power prices through derivative contracts with a financial institution, for a term of 5-10 years (Raikar and Adamson 2020, p. 52).

Residual merchant risk. A farm may have a PPA, yet face significant residual merchant risk if the offtaker buys solely over specific periods, e.g., during the day (leaving some output, e.g., during the night, exposed to merchant risk), if the term of the (e.g., corporate) PPA contract is shorter than the farm’s useful life, or if the farm cannot find a financial institution willing to agree to over-the-counter derivatives contracts with long-term maturities (over 10 years). Consider first an offtake strategy whereby 50% of the power generation is sold via a PPA. By definition, a farm will break even at the Levelized Cost of Electricity (LCOE). We therefore set the fixed PPA price at the LCOE for a wind farm, namely \$54/MWh (Lazard 2020, p. 14). If 50% of the power generated is sold via a PPA agreement, the PPA revenues will be $.5 \times .38 \times 1,533,000 \times 54 = \41.4m . The remaining 50% of the capacity (i.e., 766.5GWh) will be sold at the stochastic merchant price.

Project margin. Project revenues are assessed by multiplying net generation from the project (at a given p -value) by the power price (e.g., from the PPA). If variable inputs are optimized in the short term, we can include the result of this optimization via an exponent $\gamma > 0$ in $y \mapsto y^\gamma$ for the power price y (McDonald and Siegel 1985, p. 335). This is not necessary here because the main production input (wind or sunlight) is free (hence, we set $\gamma = 1$ in eq. (1)).

Fixed expenses (excluding debt servicing). The expenses necessary to keep the project operational are generally fixed: turbine operations & maintenance (O&M), balance-of-plant O&M, land leases, property tax, O&M for communication and transmission equipment, regulatory and professional fees, etc. (Raikar and Adamson 2020, p. 35). We assume fixed O&M costs of \$39.5/kW, which, given a capacity of 175MW, amounts to \$6.9m per year (Lazard 2020, p. 14). Such fixed costs drive the degree of exposure to financial distress (as the sponsor may exploit the SPV’s limited company legal status to stop covering operating losses if the gross margin is no longer sufficient to offset fixed costs), leading to interesting interactions regarding the SPV’s operational decision to expand capacity and by how much.

Debt servicing. Subtracting all the above expenses from the project revenues yields the “cashflows available for debt service.” At the outset, the SPV raises equity or debt to finance the Engineering, Procurement, and Construction (EPC) costs for the wind farm. When an SPV is financed via bank loans (or via bonds, which is less common), the interest rate is generally floating, e.g., LIBOR + 250 bps (Raikar and Adamson 2020, p. 37). Furthermore, debt securities may differ by stage (construction vs term loans), collateral, or seniority (Raikar and Adamson 2020, p. 38). Because outstanding loan agreements are difficult to re-negotiate, such interest commitments increase the SPV’s financial burden and make it more likely that the sponsor will shut down the SPV. Merchant deals are more exposed to risk (and hence less attractive to risk-averse banks). Consequently, merchant deals are often financed “on balance sheet,” as is still commonplace in the EU (Raikar and Adamson 2020, p. 155). We consider two representative cases for the industry:

1. 50% of the power generation is sold via a PPA and 60% of the project is financed via a (single, consol-type) debt instrument, at a yearly (continuously compounded) interest rate of 8.0% (Lazard 2020, p.14). Given 175MW turbine capacity and EPC costs of \$1,450/kW (Lazard 2020, p.14), this leads to yearly interest payments of \$12.2m.
2. If, instead, the farm sells 100% of its power generation on the spot market (as is increasingly common), debt financing is assumed to be precluded, and the firm makes no interest payments.

Sponsor’s profit. The sponsor is entitled to the residual profit once all expenses, including debt servicing, have been paid for. For simplicity, we ignore adjustments to working capital or a buildup in the debt service reserve account. The sponsor thus earns a profit of:

$$\pi(y, x) := a + by^\gamma x^\epsilon. \quad (1)$$

The term a in (1) captures the sum of fixed/PPA revenues minus the fixed (running) costs, net of the 40% corporate tax rate (Lazard 2020, p. 14). In the base case 1 (resp., 2) with 50% (resp., 0%) of the power generation sold via a PPA, the value for the parameter a is $.6 \times (41.4 - 6.9 - 12.2) \approx \13.4m

(resp., $.6 \times -6.9 \approx -\4.2m). The factor $b = 0.6$ allows us to adjust for tax at 40%. The function $x \mapsto \pi(y, x)$ is monotone increasing and concave in the farm’s generation capacity x . Whenever $a < 0$, the SPV is exposed to financial distress because the sponsor’s profit will become negative under specific circumstances.

Stimulus programs. Historically, renewables investments have been driven to some extent by government incentives (e.g., the German “Erneuerbare-Energien-Gesetz”), which have come in different flavors: feed-in tariffs, green certificates, tax incentives, etc. The impact of stimulus programs on renewables investments has been studied by Boomsma, Meade, and Fleten (2012) among others. As the total cost (or levelized cost of energy) of wind and solar power declines and renewables projects gradually become viable on their own, governments are phasing out their incentive programs. For instance, in 2018, the Chinese central government announced that it would soon end subsidies to solar PV. It made a similar announcement for wind in 2019. Because these incentive programs are country-specific and are becoming less critical to ensure the viability of renewables projects, we set aside such programs in our baseline cases, to focus on the interface between financial and operational decisions. However, we account for them via comparative statics of the parameters a and b in eq. (1). Naturally, such programs contribute to reducing the SPV’s exposure to financial distress.

EPC costs. Sponsors of renewable projects typically hire engineers to design a farm meeting a set of contractual (e.g., PPA) and legal requirements, source the main equipment (e.g., wind turbines, solar panels, inverters) from the manufacturers, and purchase all other supporting components in a separate agreement called a “balance-of-plants” contract (Raikar and Adamson 2020, p. 57). This approach helps to mitigate cost overruns. The EPC costs amount to $k = \$1,450/\text{kW}$ for the average farm (Lazard 2020, p.14), but may differ significantly between wind farms owing to sponsors’ varying abilities to negotiate better terms with equipment and services providers.

The project’s useful life. Without retrofitting, the useful life of a wind farm is 20 years on average (Lazard 2020, p. 14). Instead of considering a fixed lifetime (which is unreasonable and less tractable), we model equipment decay at an exponential rate of $1/20$ years = 5% p.a. (see Dixit and Pindyck 1994, pp. 199-207). We assume that the retrofitted equipment has the same decay rate as the older technology. Adopting risk-neutral valuation (Smith 2005, Zhou et al. 2019), we discount at the risk-free rate, which is currently (as of February 2021) 1.7% for a 20-year US Treasury bond. The effective discount rate we use is their sum, namely $r = 5\% + 1.7\% = 6.7\%$ p.a. (see Dixit and Pindyck 1994, pp. 200-204).

Information & agency considerations. Information frictions are less critical in such a setting because extensive data is available on the energy sector (see, e.g., Lazard 2020), and in real time. In addition, under project finance, the sponsor and the lender exchange in order to settle on reasonable assumptions for their financial models, in some cases hiring independent advisors (Raikar and

Adamson 2020, p. 31). Agency problems may, however, arise as the sponsor effectively manages the asset, with the lender having a more limited say in key decisions, except indirectly via covenants. An example of such a problem is the sponsor's unilateral decision to let the SPV go bankrupt if it is in its interest to do so, leading to endogenous default risk.

Table 1 below summarizes the main parameters and symbols used in this manuscript and specifies restrictions on the state and control variables. Table 2 presents the values used for the parameters in our illustrations.

Table 1 Nomenclature		
	SYMBOL	DESCRIPTION
States	y	State variable capturing the commodity price
	Y	\mathbb{F} -adapted, commodity price process
	x	State variable capturing the firm's production capacity
	$X^{x,\nu}$	\mathbb{F} -adapted, capacity stock process
Primitives	$\gamma \in (0, \beta_2)$	Exponent to y in the profit function π
	$\epsilon \in (0, 1)$	Exponent to x in the profit function π
	$a, A \leq 0$	Fixed cost parameters
	$b, B > 0$	Variable cost parameters
	$k > 0$	Cost per unit of extra capacity
	$\mu, \sigma > 0$	Drift and volatility parameters
Decision variables	θ	Investment moment, a \mathbb{F} -stopping time
	ξ	Investment lump, a \mathcal{F}_θ -measurable control variable
	Θ	Exit moment, a \mathbb{F} -stopping time
	ν	Firm strategy $\nu = \{\theta, \xi, \Theta\}$
Values & payoffs	F	Firm value incl. all real options in (14) [decision on ν]
	φ	Firm value incl. exit option in (4) [decision on Θ]
	Φ	Obstacle in (9) [decision on ξ]
	ψ	Perpetuity value of profits in (3)
(Free) boundaries	$y_0(\cdot)$	Exit's free boundary for the problem F
	$y_1(\cdot) \geq y_0(\cdot)$	Exit's free boundary for the problem φ in (5b)
	$x_1(\cdot)$	Inverse of $y_1(\cdot)$
	$x_2(y) \geq x_1(\cdot)$	Maximum of the function $\frac{\partial \varphi}{\partial x}(y, \cdot)$ in (10)
	$x_3(y) \geq x_2(\cdot)$	Local maximum of $x \mapsto \varphi(y, x) - kx$ for $y \geq y^*$
	$y_3(x) \geq y_2(\cdot)$	Inverse of $x_3(\cdot)$
	$\bar{x}_3(\cdot)$	Indifference point defined in (12)
	$\bar{y}_3(\cdot)$	Inverse of $\bar{x}_3(\cdot)$
	$y_4(\cdot) \geq y_3(\cdot)$	Lowest value such that $g(y, x) < 0$ for all $y > y_4(x)$
$y_5(\cdot) \geq y_4(\cdot)$	Expansion's free boundary for the problem F	
Secondary parameters	π	Firm's flow profit in (1)
	$\mathcal{Q}(\cdot)$	"Fundamental quadratic" in (2)
	β_1, β_2	Negative and positive roots of $\mathcal{Q}(\cdot)$ in (2)
	λ	Constant defined in (5b)
	ρ	Constant defined in (10)
	y^*	Constant defined in (11)

		BASE CASE 1	BASE CASE 2
Business model	Offtake strategy	50% PPA	0% PPA
	Debt financing share	60%	0%
	Fixed component a	\$13.4m	-\$4.2m
	Variable factor b		0.6
	Price exponent γ		1
Merchant price	Growth rate μ		0.015
	Volatility σ		0.145
Equipment	Total generation capacity		1,533GWh
	Share to merchant market	50%	100%
	Generation capacity x	766.5GWh	1,533GWh
	“Capacity factor” ϵ		0.93
	EPC cost k		\$1,450/kW

Table 2 Model calibration. Sources, which include Raikar and Adamson (2020), Schwartz and Smith (2000), and Lazard (2020), are detailed in Section 3.

4 Stylized models with single options

We first develop simple models to address our research questions and gain some insights into the interactions between the operational decision to expand and the financial decision to shut down operations.

4.1 Net present value over the equipment’s useful lifetime

Assume that the SPV operates the farm, making no adjustments until the equipment decays. At the end of the project’s operating life, the SPV may be required to dismantle the equipment, but such costs are relatively small for solar and wind projects (Raikar and Adamson 2020, p. 161) and are hence omitted for simplicity. The sponsor is thus entitled to the net present value (NPV), given by

$$\psi(y, x) := \mathbb{E} \left[\int_0^\infty e^{-rt} \pi(Y_t, x) dt \mid Y_0 = y \right],$$

with the sponsor’s profit π given in eq. (1). If $r > \mu$, let $\beta_1 < 0$ and $\beta_2 > 1$ denote the roots of

$$\gamma \mapsto \mathcal{Q}(\gamma) := r - \gamma\mu - \frac{1}{2}\gamma(\gamma - 1)\sigma^2. \quad (2)$$

As we assume that γ belongs to $(0, \beta_2)$ throughout, the sponsor’s NPV becomes

$$\psi(y, x) = A + By^\gamma x^\epsilon, \quad \text{with } A := a/r \text{ and } B := b/\mathcal{Q}(\gamma) > 0. \quad (3)$$

Because the merchant price follows a GBM, the sponsor’s NPV is always positive when $a \geq 0$ (in the base case 1), but will be negative when $a < 0$ (in the base case 2) if the merchant price y falls below the breakeven point $(-Ax^{-\epsilon}/B)^{1/\gamma}$.

4.2 Standalone financial decision to default

We first consider research question A (page 3) in isolation. Assume that the sponsor can decide not to back the SPV if the latter incurs losses. If the SPV defaults, the lenders and the (e.g., O&M) contractors will foreclose on the collateral (to recover the outstanding liabilities, interest, penalties, etc.) and the sponsor will receive nothing. The sponsor's payoff thus differs from the NPV ψ in eq. (3) and is instead given by

$$\varphi(y, x) := \sup_{\Theta} \mathbb{E} \left[\int_0^{\Theta} e^{-rt} \pi(Y_t, x) dt \middle| Y_0 = y \right] \quad (4)$$

We distinguish the following two cases:

If $a \geq 0$ (as in base case 1 with large PPA revenues), the integrand in (4) is always positive, so the sponsor will never default. The sponsor's payoff in (4) is then identical to the NPV in (3).

If $a < 0$ (as in the base case 2 characterized by a larger merchant risk exposure), the sponsor's profit may be negative, but its payoff φ is always positive (as can be seen from arbitrarily setting $\Theta = 0$ in eq. (4)). Theorem 1 provides an explicit expression for the sponsor's payoff. It is optimal for the sponsor to default if the merchant price y falls below a level, namely $y_1(x)$, strictly lower than the breakeven point $(-A^{-\epsilon}/B)^{-1/\gamma}$, indicating that the sponsor waits for the option to be "deep in the money." Furthermore, an SPV exploiting higher generation capacity x or endowed with better turbines, i.e., a greater capacity factor ϵ , is less likely to be shut down (as $\partial y_1 / \partial x < 0$ and $\partial y_1 / \partial \epsilon = -\ln(x)y_1(x)/\gamma < 0$ for $x > 1$).

Theorem 1 (Value with embedded shutdown option for $a < 0$) *Assume that $\mu > \sigma^2/2$. The value function in (4) can be expressed explicitly as*

$$\varphi(y, x) = \begin{cases} 0, & y < y_1(x), \\ A + By^\gamma x^\epsilon + \frac{A\gamma}{\beta_1 - \gamma} \left(\frac{y}{y_1(x)} \right)^{\beta_1}, & y \geq y_1(x), \end{cases} \quad (5a)$$

where the cut-off power price $y_1(x)$ is given by

$$y_1(x) := (\lambda x^{-\epsilon})^{\frac{1}{\gamma}}, \quad \text{with} \quad \lambda := -\frac{A}{B} \frac{\beta_1}{\beta_1 - \gamma} \geq 0. \quad (5b)$$

Furthermore, the sponsor is better off with a walk-away option (i.e., $\varphi \geq \psi$ for ψ given in (3)).

Importantly, when some risks are removed, the sponsor is more likely to default (i.e., the cut-off value $y_1(x)$ increases) as the default option becomes less valuable.

4.3 Standalone operational decision to expand capacity

To develop a benchmark to study RQ B, consider the firm's operational decision to expand capacity independently of the financial risk. If repairs or capital expenditures for wind farms are not conducted promptly, the farm may not generate as much power as initially planned (and may ultimately have trouble covering fixed costs). Banks typically want the SPV to have sufficient reserves on its balance sheet to meet critical expenditures (e.g., operating & maintenance reserves, capital expenditure reserves, and debt service reserves) in case of foreseen or unforeseen circumstances (Raikar and Adamson 2020, p. 48). Assume now that the SPV can replace or add wind turbines (with the same productivity parameter $\epsilon = 0.57$) to the farm (acquired at the unit cost $k = \$1.450/\text{kW}$). This expansion is financed thanks to the sponsor's commitment, e.g., via the SPV's cash reserves. We extend this analysis to consider cases in which the expansion is funded by newly raised equity (RQ D) and in which the EPC price k is affected by herding behaviors. We distinguish the following cases:

$a \geq 0$: When PPA revenues are high (so that $a > 0$), the flexibility to shut down operations is of no value, as per the discussion above. In this case, the sponsor's capacity-choice problem,

$$\Psi(y, x) := \sup \{ \psi(y, x + \xi) - k\xi; \xi \geq 0 \}, \quad (6)$$

with ψ being the NPV given in eq. (3), admits an explicit solution given by

$$\Psi(y, x) = \begin{cases} A + By^\gamma x^\epsilon, & y \leq \hat{x}^{-1}(x), \\ A + By^\gamma \hat{x}(y)^\epsilon - k[\hat{x}(y) - x], & y > \hat{x}^{-1}(x), \end{cases} \quad \text{where } \hat{x}(y) := \left(\frac{\epsilon B}{k} y^\gamma \right)^{\frac{1}{1-\epsilon}}. \quad (7)$$

The sponsor's timing problem is of the form:

$$\sup_{\tau} \mathbb{E} \left[\int_0^{\tau} e^{-rt} \pi(Y_t, x) dt + e^{-r\tau} \Psi(Y_{\tau}, x) \middle| Y_0 = y \right]. \quad (8)$$

Related problems have been studied, e.g., by Dangl (1999) and Bensoussan and Chevalier-Roignant (2013). We do not construct the solution of problem (8) directly, but instead obtain it later by limit considerations.

$a < 0$: When the farm is exposed to large merchant risk, the SPV is prone to financial distress. The operational decision to expand will thus interact with the financial decision to default, leading to a more intricate problem that we deal with in the next section.

5 Interactions between financial and operational decisions

The interactions between the financial decision to let the SPV go under and the operational decision to expand production capacity when $a < 0$ due to a larger merchant risk exposure are central to our study.

5.1 Operational decision to expand capacity under (endogenous) credit risk

We reconsider RQ B but, unlike the discussion in Section 4.3, we now explicitly consider the effect of (endogenous) credit risk on the operational decision to expand capacity. Instead of the capacity-choice problem in eq. (6), the sponsor now solves the problem

$$\Phi(y, x) := \sup_{\xi \geq 0} \left\{ \varphi(y, x + \xi) - k\xi \right\}, \quad (9)$$

for the function φ given in (4) or (5a), which embeds the later financial decision to stop backing the SPV if the commodity price were to fall significantly. The problem in eq. (9) is more involved than the one in eq. (6) because (a) Ψ in (7) can be negative, while Φ cannot and (b) although $\psi(y, \cdot)$ in (3) is concave, the function $\varphi(y, \cdot)$ is *not*, as per Lemma 1 below. For this purpose, we define

$$x_1(y) := (\lambda y^{-\gamma})^{\frac{1}{\epsilon}}, \quad \rho := \left[\frac{\gamma - \beta_1 \epsilon}{(1 - \epsilon)\gamma} \right]^{\frac{\gamma}{\gamma - \beta_1}} \quad \text{and} \quad x_2(y) := (\rho \lambda y^{-\gamma})^{\frac{1}{\epsilon}}. \quad (10)$$

Lemma 1 *The function $x \mapsto \frac{\partial \varphi}{\partial x}(y, x)$ vanishes on $(0, x_1(y))$, increases on $[x_1(y), x_2(y)]$, and decreases on $(x_2(y), \infty)$, attaining a maximum at $x_2(y)$.*

Because the sponsor incurs a negative fixed flow a , it faces two opposing forces. First, increasing power generation reduces the operating leverage, leads to larger profits, and effectively helps reduce the SPV's financial distress. This effect prevails when the generation capacity is limited, i.e., $x_1(y) < x < x_2(y)$, because the investment can be used as an effective means of avoiding financial distress. Second, for sufficiently large generation $x > x_2(y)$, the payoff $\varphi(y, \cdot)$ inherits the concavity of the profit $\pi(y, \cdot)$ in (1), with the decision to expand being mostly driven by operational considerations rather than financial ones. Because $x_2(y) \rightarrow 0$ as $a \rightarrow 0$, the pattern established in Lemma 1 and the convex-concave shape for $\varphi(y, \cdot)$ on $(0, \infty)$ relate to a negative fixed flow a and the related financial distress if the sponsor can decide to stop covering the SPV's losses. Such considerations are misplaced in the case of large PPA revenues because the sponsor has no financial incentive to let the SPV go bankrupt. This explains the fact that, in reality, banks are more willing to lend large amounts of money to SPVs that contracted PPAs covering a large share of their energy production with investment-grade offtakers. They do so because such loan agreements are less likely to lead to a conflict of interest with the SPV's sponsor.

We want to determine the conditions under which expanding power generation creates NPV for the sponsor. Theorem 2 studies the function $x \mapsto \varphi(y, x) - kx$ on $[0, \infty)$, which is meaningful because we can write (9) as $\Phi(y, x) = kx + \sup \{ \varphi(y, \xi) - k\xi; \xi \geq x \}$.

Theorem 2 (Global maximum at $x \mapsto \varphi(y, x) - kx$ for a given $y > 0$.) We distinguish the following cases:

A) If the merchant price is low, namely

$$\underline{y < y^*} := \left[\frac{k}{B} \frac{\gamma - \epsilon \beta_1}{\gamma - \beta_1} \frac{(\lambda \rho)^{\frac{1-\epsilon}{\epsilon}}}{\epsilon^2} \right]^{\frac{\epsilon}{\gamma}}, \quad (11)$$

then 0 is the unique local and global maximum of $x \mapsto \varphi(y, x) - kx$ in $[0, \infty)$.

If the merchant price is higher, i.e., $y \geq y^*$, $x \mapsto \varphi(y, x) - kx$ has two local maxima on $[0, \infty)$, 0 and the unique root $x_3(y)$ of $x \mapsto \frac{\partial \varphi}{\partial x}(y, x) - k$ in the interval $(x_2(y), \infty)$. The function $y \mapsto \varphi(y, x_3(y)) - kx_3(y)$ has a unique root—denoted as y^{**} —on its domain (y^*, ∞) . Then,

B) 0 is the global maximum of $x \mapsto \varphi(y, x) - kx$ on $[0, \infty)$ if the merchant price is in an intermediate range $\underline{y^* \leq y < y^{**}}$. In this case, there is a unique solution $\bar{x}_3(y)$ to the equation

$$\varphi(y, \bar{x}_3(y)) - k\bar{x}_3(y) = \varphi(y, x_3(y)) - kx_3(y). \quad (12)$$

C) $x_3(y)$ is the global maximum of $x \mapsto \varphi(y, x) - kx$ on $[0, \infty)$ if the merchant price is even higher, i.e., $\underline{y \geq y^{**}}$.

Figure 1 depicts the three cases identified in Theorem 2 under high merchant price exposure (i.e., $a < 0$) and significant (endogenous) credit risk. The marginal value of capacity is negative for a low merchant price $y < y^*$ in Panel A: adding turbines destroys value. If the merchant price is in the intermediate range (y^*, y^{**}) , adding capacity may create value at the margin, but not sufficiently so for the sponsor to break even. Finally, for a sufficiently high merchant price $y \geq y^{**}$ in Panel C, the added value exceeds the opportunity cost: the sponsor should raise power generation, from 0 to $x_3(y)$. We study the functions $x_3(\cdot)$ and $\bar{x}_3(\cdot)$ in Corollary 1.

Corollary 1 The function $x_3(\cdot)$ increases on $[y^*, \infty)$ from $x^* := (\lambda \rho y^{*- \gamma})^{\frac{1}{\epsilon}} > 0$ to ∞ and is asymptotically equivalent to $\hat{x}(\cdot)$ in (7). The function $\bar{x}_3(\cdot)$ decreases on (y^*, y^{**}) from x^* to 0.

It is not surprising that a higher merchant price $y \geq y^*$ leads to a larger expansion, i.e., $x'_3(\cdot) > 0$. Furthermore, because the fixed flow a becomes relatively negligible as the merchant price increases, the sponsor's financial incentive to shut down the SPV gradually vanishes, with the SPV expanding as if the sponsor's shut-down option is worthless, reaching the capacity level $\hat{x}(y)$ of eq. (7). Consequently, banks may be open to the possibility of financing an SPV with merchant risk exposure if the merchant price is significantly high. Let $y_3(\cdot)$ (resp., $\bar{y}_3(\cdot)$) denote the inverse of $x_3(\cdot)$ (resp., $\bar{x}_3(\cdot)$) whose domain is (x^*, ∞) (resp., $(0, x^*)$). Corollary 2 specifies the optimal (static) expansion policy:

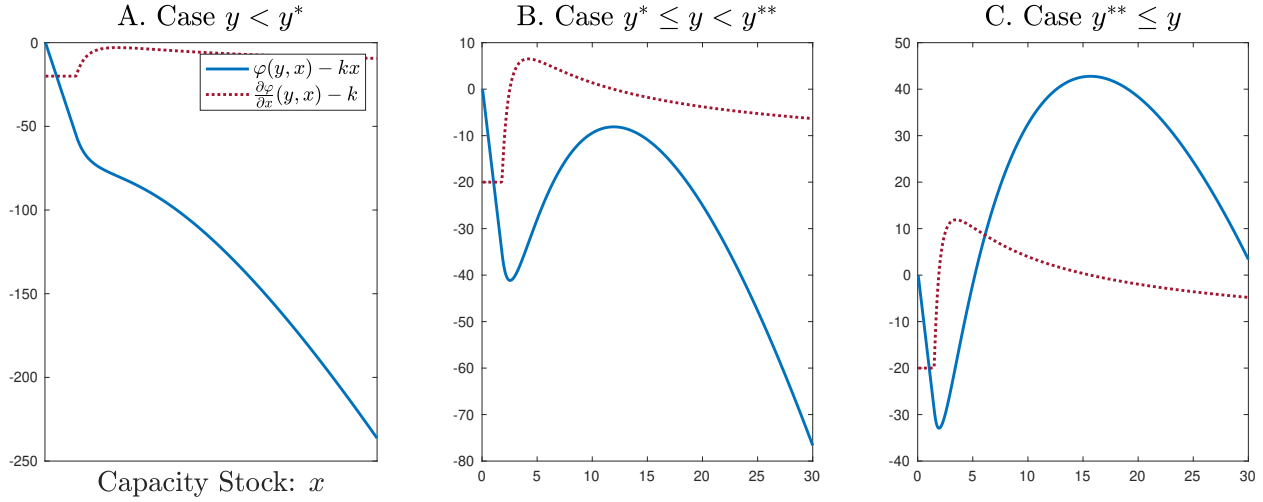


Figure 1 Study of the function $x \mapsto \varphi(y, x) - kx$ for a given $y > 0$. We assume that $r = 0.067$, $\sigma = 0.145$, $\mu = 0.015$, $k = 1450$, $a = -13.4$, $b = 0.6$, $\gamma = 1$, and $\epsilon = 0.57$.

Corollary 2 (Static optimization problem for $a < 0$) *The expansion NPV Φ in (9) simplifies to*

$$\Phi(y, x) = \begin{cases} \varphi(y, x), & \text{Limited existing capacity: } x < x^* \\ \varphi(y, x_3(y)) - k[x_3(y) - x], & y \leq \bar{y}_3(x), \\ \varphi(y, x_3(y)) - k[x_3(y) - x], & y > \bar{y}_3(x) \end{cases} \quad \begin{cases} \text{Larger existing capacity: } x \geq x^* \\ \varphi(y, x), & y \leq y_3(x), \\ \varphi(y, x_3(y)) - k[x_3(y) - x], & y > y_3(x). \end{cases}$$

The gain function $\Phi(\cdot, x)$ is differentiable if $x \geq x^$, but not if $x < x^*$.*

Following Corollary 2, if the existing capacity x is below x^* (resp., above x^*), there is a cut-off merchant price $\bar{y}_3(x)$ (resp., $y_3(x)$) above which expanding capacity from x to $x_3(y)$ creates value, i.e., $\Phi(y, x) > \varphi(y, x)$. For $a < 0$, the set $\mathcal{R}_a := \{(y, x) \in \mathbb{R}_+^2 \mid \Phi(y, x) > \varphi(y, x)\}$ specifies the conditions under which expanding generation capacity creates NPV. Corollary 3 establishes that a larger share of PPA revenues (i.e., an increase in $a \in (-\infty, 0)$) will make it more attractive for the sponsor to expand capacity. In other words, merchant price risk exposure makes the SPV less likely to invest.

Corollary 3 (Comparative statics with respect to the fixed cost) *The inequality $a < \alpha < 0$ implies the set inclusion $\mathcal{R}_a \subset \mathcal{R}_\alpha \subset \mathcal{R}_0 := \{(y, x) \in \mathbb{R}_+^2 \mid x \geq \hat{x}(y)\}$ for $\hat{x}(\cdot)$ in (7).*

Figure 2 illustrates Corollaries 2 and 3. Panel A plots the cut-off merchant prices $\bar{y}_3(\cdot)$ and $y_3(\cdot)$ above which expanding generation capacity creates NPV for the sponsor; these correspond to the boundary $\partial\mathcal{R}_a$ of the set \mathcal{R}_a . For merchant prices y below y^* , expanding capacity will destroy value. If the merchant price is higher, the question of whether the sponsor can create NPV by adding capacity will depend on the initial capacity x . Below the minimum of the black and red curves, walking away would mean that the sponsor could avoid covering the SPV's losses. Below the black curve and above the red curve, expansion will not create NPV as $\Phi(y, x) = \varphi(y, x)$. A tricky situation may arise

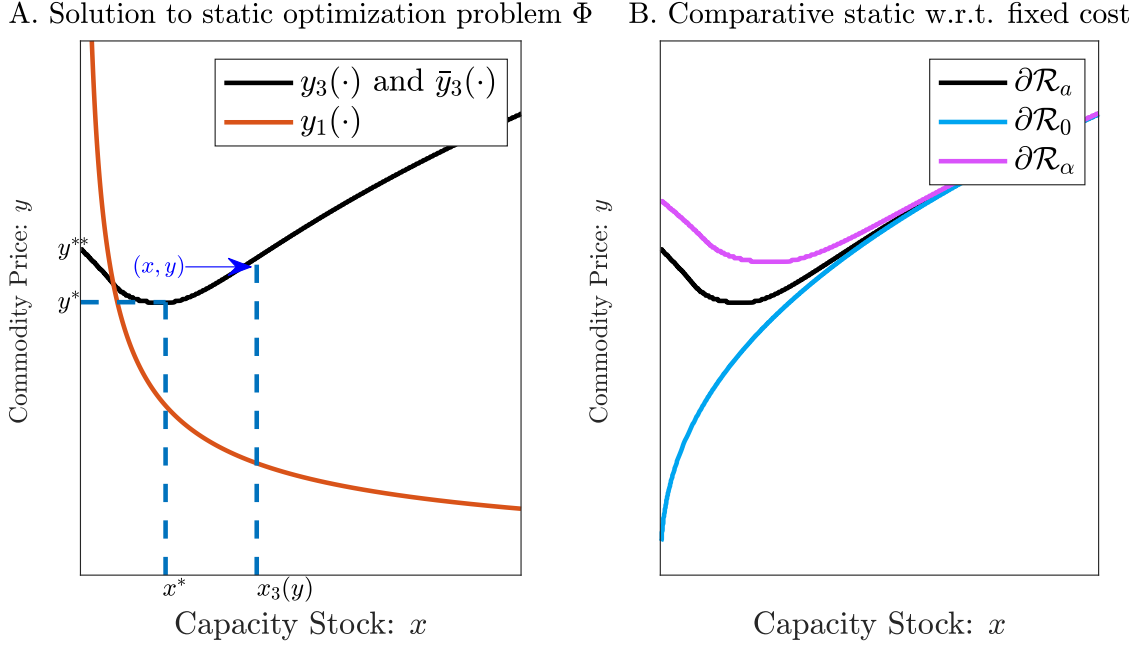


Figure 2 Regions (y, x) in which $\Phi(y, x) = \varphi(y, x)$ or $\Phi(y, x) > \varphi(y, x)$. We assume that $r = 0.067$, $\sigma = 0.145$, $\mu = 0.015$, $k = 1450$, $a = -13.4$, $\alpha = -18.4$, $b = 0.6$, $\gamma = 1$, and $\epsilon = 0.57$.

when the initial generation capacity is limited (i.e., $x < x^*$) because the sponsor may want to walk away even after helping to finance an expansion. Panel B shows that the conditions under which expanding capacity creates NPV become less restrictive as PPA revenues increase (i.e., a increases and approaches 0 from the left). (If the PPA revenues are even larger, i.e., $a \geq 0$, problem (9) is reduced to the simpler problem (6), solved in (7).)

5.2 Expanding capacity vs shutting down

We now address research question C, considering that the (possible) financial decision to shut down the SPV depends ex ante on the future, likely benefits the sponsor could gain from its operational decision to expand the wind farm's capacity in the future. This research question is addressed by considering the timing of the two options under high merchant price risk exposure (i.e., $a < 0$). The sponsor can raise the farm's generation capacity at a time θ by an extra ξ units and/or shut down the SPV at a time Θ . The power generation capacity $X^{x,\nu}$ depends on the initial capacity x and the sponsor's strategy $\nu := \{\theta, \xi, \Theta\}$:

$$X_t^{x,\nu}(\omega) = \begin{cases} x + \xi(\omega), & \theta(\omega) < t \leq \Theta(\omega), \\ x, & \text{otherwise.} \end{cases} \quad (13)$$

The sponsor chooses a strategy $\hat{\nu}$ to maximize the residual NPV:

$$F(y, x) := \sup_{\nu} \mathbb{E} \left[\int_0^{\Theta} e^{-rt} \pi(Y_t, X_t^{x,\nu}) dt - e^{-r\theta} k \xi \mathbf{1}_{[0,\Theta)}(\theta) \middle| Y_0 = y \right]. \quad (14)$$

The first right-hand side (RHS) term in (14) corresponds to the present value of the sponsor's profits until it walks away at time Θ . The second RHS term is the present value of the capacity expansion cost, which is incurred only if an expansion occurs before the sponsor walks away.

Because the control variable ξ is \mathcal{F}_θ -measurable (see Table 1), it follows from the strong Markov property, $\Phi \geq 0$ for Φ in (9) (obtained by setting $\xi = 0$ in (9) and then $\Theta = 0$ in (4)), and finally a change of function $\tau = \theta \wedge \Theta$ that

$$F(y, x) = \sup_{\tau} \mathbb{E} \left[\int_0^{\tau} e^{-rt} \pi(Y_t, x) dt + e^{-r\tau} \Phi(Y_{\tau}, x) \middle| Y_0 = y \right]. \quad (14')$$

Following the expression in eq. (14'), the sponsor receives a profit flow $\pi(Y_t, x)$ up until time τ , at which point it either expands capacity or defaults and then receives the amount $\Phi(Y_{\tau}, x)$. Interestingly, the problems of (a) expanding capacity and receiving Φ or (b) exiting and receiving 0 have the same mathematical formulation (14'). To provoke a situation where the sponsor reneges on the SPV's obligations, it can install a trivial amount (0 extra capacity) and then walk away. (The situation would be different if expansion entailed a one-time fixed cost.)

We study problem (14') using dynamic programming. First, the sponsor can exercise the options immediately, so $F \geq \Phi$. Second, if the sponsor defers any decision for a period of length $h > 0$, it follows from Bellman's (1957) "principle of optimality" that

$$F(y, x) \geq \mathbb{E} \left[\int_0^h e^{-rt} \pi(Y_t, x) dt + e^{-rh} F(Y_h, x) \middle| Y_0 = y \right].$$

Under sufficient smoothness (not yet proven), this inequality becomes $\mathcal{L}F \geq \pi$ as $h \downarrow 0$, with \mathcal{L} being the second-order differential operator $\mathcal{L}f(y) := rf(y) - \mu y f'(y) - \frac{1}{2} \sigma^2 y^2 f''(y)$. Third, the two decisions above are mutually exclusive. These three conditions can be expressed in the form of a variational inequality (VI), namely

$$0 = \min\{F - \Phi; \mathcal{L}F - \pi\} \quad \text{for almost every } y > 0, \quad (15)$$

with boundary conditions at 0 and ∞ obtained after further study of $\Phi(\cdot, x)$. It can be proven that if Φ is differentiable, then the solution of the VI (15) coincides with the value function (14) (see, e.g., Bensoussan and Lions 1982, Theorem 3.1, p. 305). The problem is more delicate if Φ is nondifferentiable, a situation we face for low initial capacity $x < x^*$ according to Corollary 2.

Case with large initial generation capacity $x \geq x^$.* If a sponsor delays its decision, it achieves an *economic profit* or *loss* $g := \pi - \mathcal{L}\Phi$. The function $\chi := F - \Phi$ captures the "option value" in the sense that it is the difference between the value of the sponsor's claim including the real options in (14)

and the NPV from immediately exercising the options in (9). (We drop the use of x in the notation when no confusion arises.) Using g and χ , we can re-write the VI (15) as

$$0 = \min\{\chi; \mathcal{L}\chi - g\}, \quad \text{for almost every } y > 0, \quad (16)$$

a formulation that is more tractable. Economic arguments suggest that the option value vanishes when the merchant price reaches zero, i.e., $\chi(0) = 0$. Lemma 2 summarizes key results about g :

Lemma 2 (Behavior of g .) *For a given $x \geq x^*$, the function g , capturing the instantaneous economic profit from delaying, is continuous, except at the cut-off merchant prices y_1 and y_3 . It is negative on $(0, y_1)$ and vanishes on (y_1, y_3) . It is positive to the right of y_3 and goes to $-\infty$ if $\frac{\gamma}{1-\epsilon} < \beta_2$ or to $+\infty$ if $\frac{\gamma}{1-\epsilon} > \beta_2$ as $y \rightarrow \infty$.*

The analytic study of g over (y_3, ∞) is delicate. The behavior of g at ∞ established in Lemma 2 allows us to rule out $\gamma/[1-\epsilon] > \beta_2$ as a case of interest (see Corollary 4 in the Appendix). The condition $\gamma/[1-\epsilon] < \beta_2$ —which is satisfied in the base cases—holds when the discount rate r is large and the growth rate μ is small and/or when the power parameters γ and ϵ are low: a decrease in μ , γ , or ϵ leads to a greater economic profit from delaying, whereas a larger discount rate r leads to a lower present value and a greater incentive to exercise the American option sooner. This condition is aligned with key results in financial economics that an American option will not be exercised before its maturity unless the underlying asset’s growth rate is restricted relative to the discount rate (Samuelson 1965, Merton 1973).

To solve the VI (16), the online appendix (in Section EC.8) determines mild conditions on g to ensure that the sponsor will delay its decision whenever the merchant price is in the interval (y_0, y_5) . This section first establishes (sufficient and necessary) conditions on g to solve an ordinary differential equation (ODE) with free boundaries y_0 and y_5 and states another condition to ensure that the ODE’s solution solves (16). Let y_4 denote the smallest merchant price above which g is negative. Given g ’s behavior, it seems natural that the sponsor accumulates losses when it does not walk away for low merchant prices $y < y_1$ or when it does not expand generation capacity for high prices $y > y_4$. The following question arises: How large a loss is the sponsor willing to endure? If the merchant price is in the interval (y_0, y_1) (resp., (y_4, y_5)), the sponsor entertains temporary losses in the hope (resp., fear) of a merchant price recovery (resp., cutback). The sponsor will waver until the losses are sufficiently material to justify making an irreversible decision, i.e., below y_0 and above y_5 respectively. As $y_0 < y_1$ (resp., $y_4 < y_5$), the option needs to be “deep in the money” (not just “in the money”) to justify its exercise. The prices y_0 and y_5 specified in the appendix are such that the present value

of the economic profits exactly offsets the present value of the economic losses. In some scenarios ω , the merchant price $t \mapsto Y_t^y(\omega)$ will fall below the cut-off price y_0 , with the sponsor walking away. In other scenarios ω , the price will rise above y_5 , so the SPV will replace or install new turbines until it reaches a capacity of $x_3(\max\{y; y_5\}) > x$ GWh. The sponsor may then walk away if the SPV cannot sell the residual output in the spot market above a price $y_1(x_3(\max\{y; y_5\}))$.

Case with low initial generation capacity $x < x^$.* In this case, we know from Corollary 2 that the NPV Φ is nondifferentiable at \bar{y}_3 . Because of this, the mathematical procedure for determining the sponsor's optimal policy differs, defining the economic profit or loss $g := \pi - \mathcal{L}\Phi$ only on the interval (\bar{y}_3, ∞) . The farm faces losses from deferring an expansion $g(y) < 0$ if the merchant price exceeds the level $y_4 > \bar{y}_3$. The online appendix (in Section EC.9) specifies conditions for this case under which the continuation region is of the form (y_0, y_5) . We later consider the sensitivity of the thresholds y_0 and y_5 on model primitives such as the merchant price's growth rate μ and volatility σ .

We now come back to the simpler problem in eq. (8). The following proposition establishes that the free boundary for problem (8) is obtained by limit considerations of the problem with $a < 0$.

Proposition 1 (Optimal stopping problem (8) for $a \geq 0$) *The solution $G(y, x)$ of the optimal stopping problem (8) can be written as $G(y, x) = a/r + \lim_{a \rightarrow 0} F(y, x)$, with F defined in (14'). If $a \geq 0$, a sponsor facing problem (8) should never default and should install generation capacity if the merchant price exceeds the level $\lim_{a \rightarrow 0} y_5(x)$.*

If the SPV has large PPA revenues (i.e., $a \geq 0$), timing the option exercise is simpler because there is no interaction between the expansion and exit options (as the exit option is worthless). Our model illustrates the subtle effect of the subsequent financial decision to shut down operations (when doing so is meaningful) on the sponsor's capacity investment policy: the optimal policy involves two thresholds (instead of only one), one below which the sponsor defaults and another one above which it expands generation capacity.

5.3 Comparative statics

This section investigates (numerically) the sensitivity of the SPV's credit risk (RQs A and C), and of the sponsor's decision to expand (RQ B), to changes in the economic environment.

Change in initial generation capacity. Figure 3A plots the NPV at exercise Φ and the value function F in eq. (14') for the base case 2 with $x_0 = 1,533$ GWh initial capacity ($x_0 > x^*$). The sponsor should shut down the SPV (resp., replace and install new turbines) if the merchant price falls below $y_0(x)$ (resp., rises above $y_5(x)$). In Panel B, we observe that a sponsor is less likely to default if the farm has more generation capacity x and will instead replace and install turbines at the site.

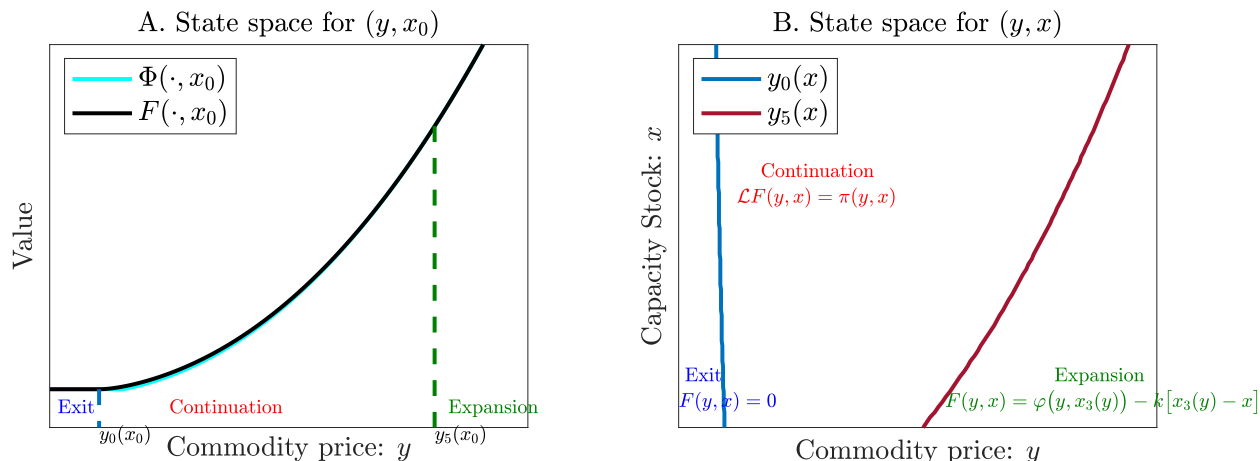


Figure 3 Exit, hysteresis, and expansion regions. We assume that $r = 0.067$, $\sigma = 0.145$, $\mu = 0.015$, $a = -13.4$, $b = 0.6$, $k = 1450$, $\epsilon = 0.57$, $x_0 = 1533$, and $\gamma = 1$.

Change in the farm’s profitability parameters a and b , e.g., owing to stimulus programs. According to Figure 4A, lower secured PPA revenues (a lower $a < 0$) lead to delayed expansion and put higher pressure on the sponsor to default. These two results generalize insights established in Corollary 3 to a setting with a timing decision. Panel C shows the impact of the fixed flow a on the number of turbines refurbished/added $x_3(y) - x$. Lower PPA revenues make an expansion less attractive and also reduce the investment size. In Panel B, a larger variable contribution b provides an incentive for the manager to replace or install generation capacity earlier. Panel D depicts comparative statics for $x_3(y) - x$ with respect to b . Higher b makes investment more attractive, thereby leading to a larger investment lump. Consequently, to favor renewable energies, a government should reduce corporate taxes or offer tax rebates on such SPVs, thereby contributing to an acceleration in capacity installments and increasing the overall renewable generation capacity.

Change in merchant price dynamics. Figure 5 illustrates how changes to the drift μ and volatility σ of the merchant price affect the sponsor’s timing decisions (in Panel A) and the expansion lumps (in Panel B). When faced with higher merchant price *volatility* σ , a sponsor should delay both the default and expansion decisions, out of caution. The effect of merchant risk on the sponsor’s value $F(y, x)$ and the size of the expansion $x_3(y) - x$ is ambiguous: higher volatility σ increases the option value, but it also increases the probability that the sponsor will default. If the merchant price tends to appreciate more, the sponsor should delay its decisions to default (because it is hoping for a prompter recovery) and to replace and install turbines (as the opportunity cost of “killing” the expansion option becomes larger). As the price grows stronger, the SPV should add more turbines (with $x_3(y) - x$ being larger).

Change in equipment productivity and EPC costs. Using similar illustrations, it can be shown that a lower concavity ϵ of the production function makes the investment more attractive, so that the

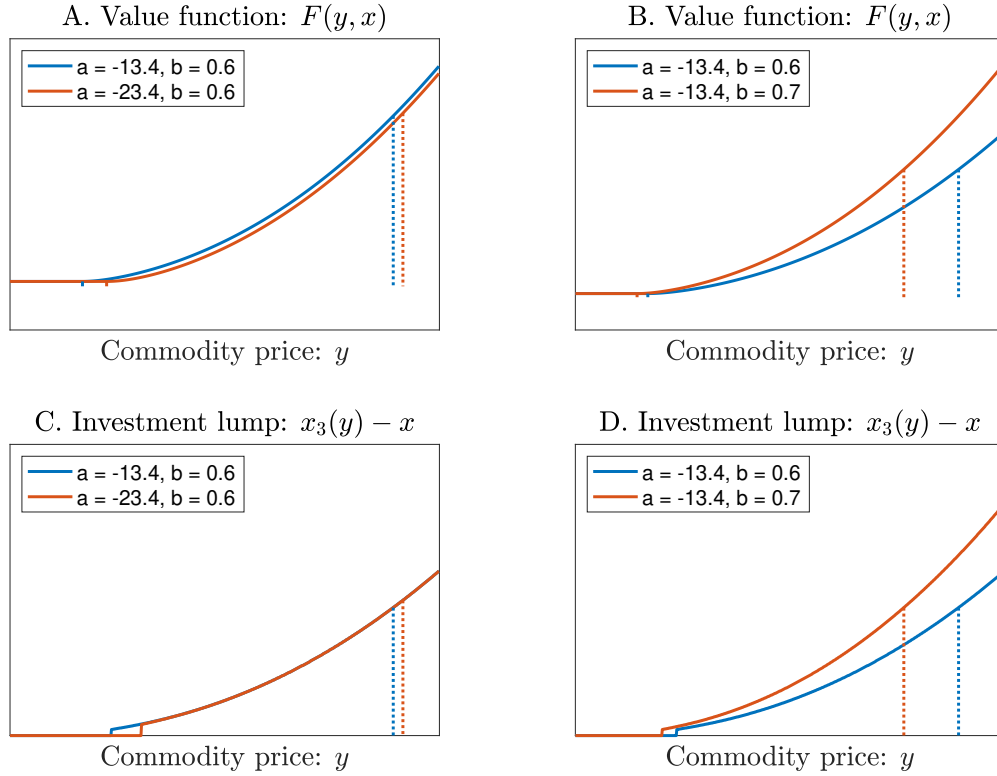


Figure 4 Comparative statics for $F(\cdot, x)$ and $y \mapsto x_3(y) - x$ with respect to a and b . We assume that $r = 0.067$, $\sigma = 0.145$, $\mu = 0.015$, $k = 1450$, $\epsilon = 0.57$, $x = 1533$, $\gamma = 1$.

sponsor will hasten capacity expansion and invest a larger amount. In contrast, a larger EPC cost k (e.g., due to the increased price of particular components as a result of industry concentration) renders the expansion less attractive, so the sponsor delays further capacity expansion and invests less. Sponsors may have privileged access to EPC, giving them an edge vs other wind farms.

6 Refined models

This section presents refined models that capture further realistic features and help us address research questions D and E (page 3).

6.1 Capital budgetary restrictions

As implied in research question D on page 3, there may be insufficient cash reserves to finance an expansion, meaning that an equity injection is required. Following Bolton, Chen, and Wang (2011), we assume that the total cost of raising the investment amount $k\xi$ is $\rho_0 + (1 + \rho_1)k\xi$, with $\rho_0 \geq 0$ capturing a fixed financing fee and $\rho_1 \geq 0$ a proportional fee. Combining a fixed and a proportional issuance fee allows the model to capture the stylized fact that direct issuance costs decrease proportionally

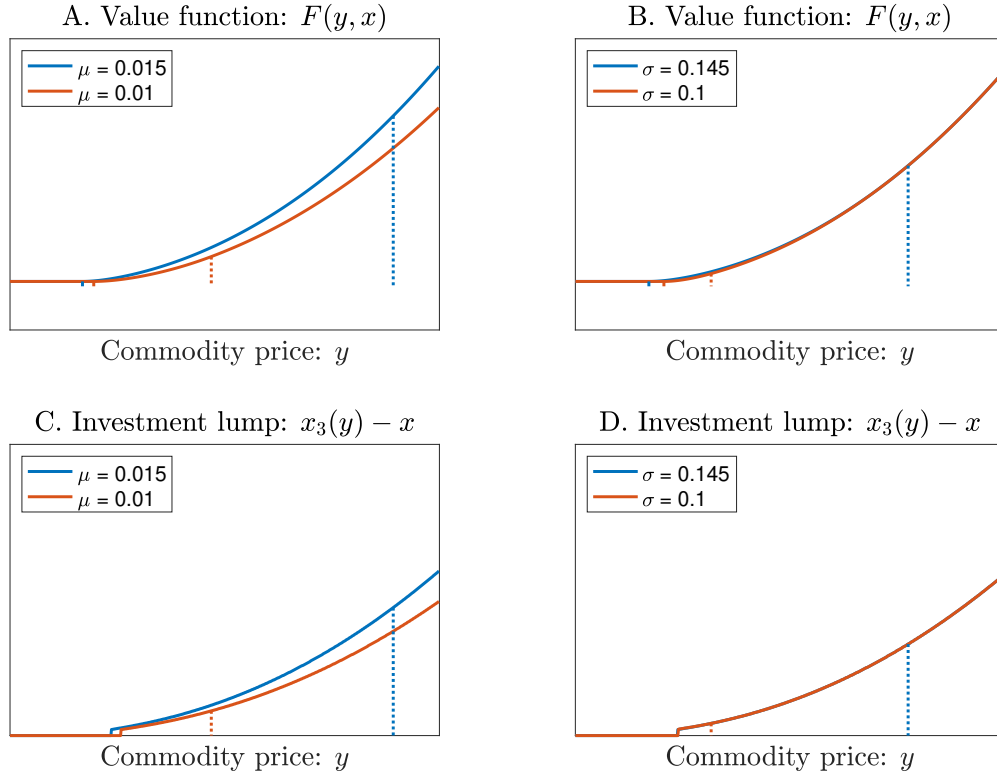


Figure 5 Comparative statics for the value function $F(y, x)$ and investment lump $x_3(y) - x$ with respect to σ and μ . The solid (resp. dotted) lines correspond to the exit thresholds $y_0(x)$ (resp. investment thresholds $y_5(x)$). We assume that $r = 0.067$, $\sigma = 0.145$, $\mu = 0.015$, $a = -13.4$, $b = 0.6$, $k = 1450$, $x = 1533$, $\epsilon = 0.57$, $\gamma = 1$.

with respect to the gross proceeds from the equity issuance (see Brealey, Myers, Allen, and Mohanty 2012, Figure 15.5). The problem when PPA revenues are low (i.e., $a < 0$) in eq. (9) now reads

$$\Phi(y, x) := \sup_{\xi \geq 0} \left\{ \varphi(y, x + \xi) - (\rho_0 + (1 + \rho_1)k\xi) \right\}. \quad (17)$$

Figure 6 illustrates the corresponding value function (obtained by replacing Φ in eq. (9) by Φ in eq. (17) in the eq. (15)) and highlights the point above (resp., below) which the sponsor will expand capacity (resp., default). First, if the sponsor faces a larger fixed financing fee ρ_0 , it should delay the expansion further due to hysteresis (while the investment size for a given electricity price remains unaffected) and should default earlier (see Panels A and C). Second, under a larger proportional financing fee ρ_1 , the sponsor should delay the expansion, reduce the investment amount, and exit earlier (see Panels B and D).

6.2 Default and expansion decisions in an industry context

We now develop two model extensions to address research question E.

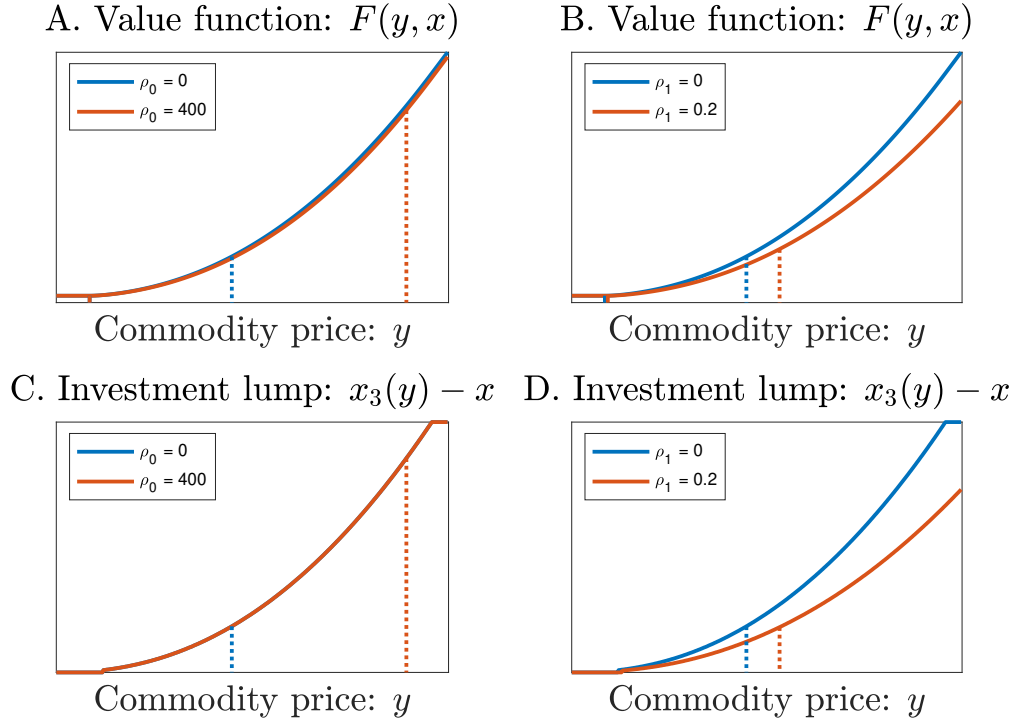


Figure 6 Comparative statics for the value function $F(y, x)$ and investment lump $x_3(y) - x$ with respect to ρ_0 and ρ_1 . The solid (resp. dotted) lines correspond to the exit thresholds y_0 (resp. investment threshold y_5). We assume that $r = 0.067$, $\sigma = 0.145$, $\mu = 0.015$, $a = -13.4$, $b = 0.6$, $k = 1450$, $\epsilon = 0.57$, $\gamma = 1$.

Changing equipment prices/EPC costs. Equipment prices are subject to two main counteracting forces. First, there are continuous efforts to manufacture bigger, more powerful, and more effective equipment at lower unit cost (due to economies of scale). For instance, floating offshore wind turbines (used, e.g., at the Hywind Scotland wind farm) are less customized than traditional turbines, require less steel, and are easier to install. Although sponsors are slow to embrace new technologies (because lenders are averse to unproven technologies), such developments at the industry level are still likely to lead to reduced equipment prices. Second, the industry demand for wind turbines tends to be positively correlated with power prices. A reason for this trend is that an increase in carbon prices (which eventually pushes power prices up) leads to a relative improvement in the LCOEs of renewable technologies, enticing the energy sector to invest more in this renewable technology.

We model the above forces using a two-state Markov chain for the EPC cost ($K_t; t \geq 0$): this cost can transition from \underline{k} to $\bar{k} > \underline{k}$ with probability λ_{LH} and from \bar{k} to \underline{k} with probability λ_{HL} . Another way to model EPC costs is via a Poisson process (see, e.g., Murto 2007), but this approach does not readily accommodate a correlation with the electricity price. Assume that the transition probabilities are affine in the electricity price y , given by

$$\lambda_{LH}(y) = \frac{2\lambda_0 - \lambda_1 y^*}{2} + \lambda_1 y \geq 0 \quad \text{and} \quad \lambda_{HL}(y) = \max \left\{ \frac{2\lambda_0 + \lambda_1 y^*}{2} - \lambda_1 y; 0 \right\} \geq 0, \quad (18)$$

where y^* corresponds to an upper bound on the price of electricity (e.g., the historic maximum price). A higher intercept $\lambda_0 \geq 0$ implies more regular changes in the EPC costs (ceteris paribus). By contrast, λ_1 captures herding: when electricity prices are high (resp., low), the EPC cost is more likely to increase or remain high (resp., low). Because the EPC cost is a stochastic process, the problem becomes two-dimensional. We identify the sponsor’s optimal decisions by solving the corresponding dynamic programming equation numerically.

Figure 7 illustrates the effects of increasing transition probabilities (via λ_0) on the sponsor’s decisions. It follows from Panel A that, fearing increasing (resp., hoping for declining) EPC costs, the sponsor has an incentive to expand capacity sooner (resp., later) if the EPC cost is currently low (resp., high). Indeed, uncertainty with respect to the equipment cost can either increase the value of delaying investment (if the EPC cost is currently high) or decrease it (if the cost is currently low). Interestingly, following Panel B, the sponsor should default earlier (resp., later) when the EPC cost is low (resp., high) because a recovery episode with affordable expansion is less (resp., more) likely.

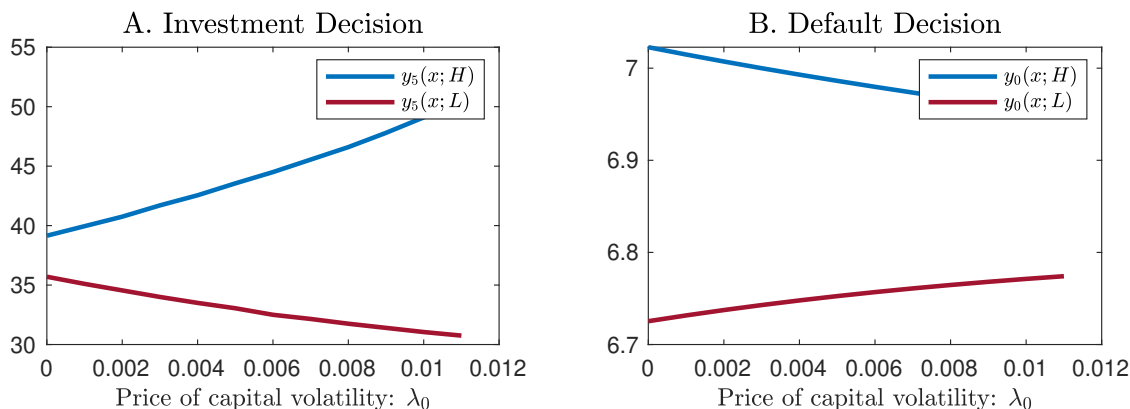


Figure 7 Comparative statics for default and investment decisions with respect to λ_0 . The blue dots (red circles) correspond to the high (low) equipment costs \bar{k} (\underline{k}) case. We assume that $r = 0.067$, $\sigma = 0.145$, $\mu = 0.015$, $a = -13.4$, $b = 0.6$, $\underline{k} = 1450$, $\bar{k} = 1740$, $\epsilon = 0.57$, $\gamma = 1$.

Figure 8 illustrates the effect of herding (via λ_1) on the sponsor’s decisions. In line with Panel B, as herding becomes more intense, the sponsor should expand capacity sooner to preempt a rise in EPC costs. A depressed electricity market leads to a more affordable expansion as λ_1 increases. The sponsor is thus less likely to close up shop, as confirmed by Panel A.

Effect of cost heterogeneity on industry capacity. We now study the effect of wind farm heterogeneity on industry capacity. Consider an industry—characterized by a unit measure of firms—that features heterogeneity (only) with respect to the EPC cost parameter k . Such heterogeneity may stem from sponsors’ differing access to key equipment and services providers. This is likely to be the case as large utilities may negotiate better terms with these providers than a financial investor

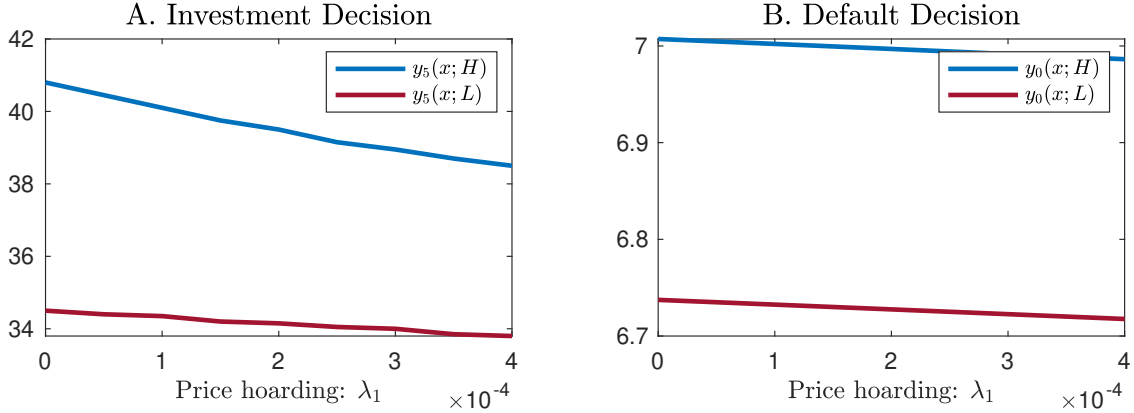


Figure 8 Comparative statics for default and investment decisions with respect to λ_1 . The blue dots (red circles) correspond to the high (low) equipment costs \bar{k} (\underline{k}) case. We assume that $r = 0.067$, $\sigma = 0.145$, $\mu = 0.015$, $a = -13.4$, $b = 0.6$, $\underline{k} = 1450$, $\bar{k} = 1740$, $\epsilon = 0.57$, $\gamma = 1$.

focused on a one-time deal. Specifically, we assume that the EPC cost k is normally distributed with mean $\bar{k} = 1450$ and variance σ_k^2 . If the sponsor characterized by a parameter k follows the optimal policy $\hat{\nu}$, it will hold a capacity $X_t^{x, \hat{\nu}}(k)$ according to eq. (13). If $y \notin (y_0, y_5)$, the sponsor will stay put at the outset, but will adjust its capacity (at time 0+) otherwise. We denote the capacity of firm k after this possible adjustment as $X_{0+}^{x, \hat{\nu}}(k)$.

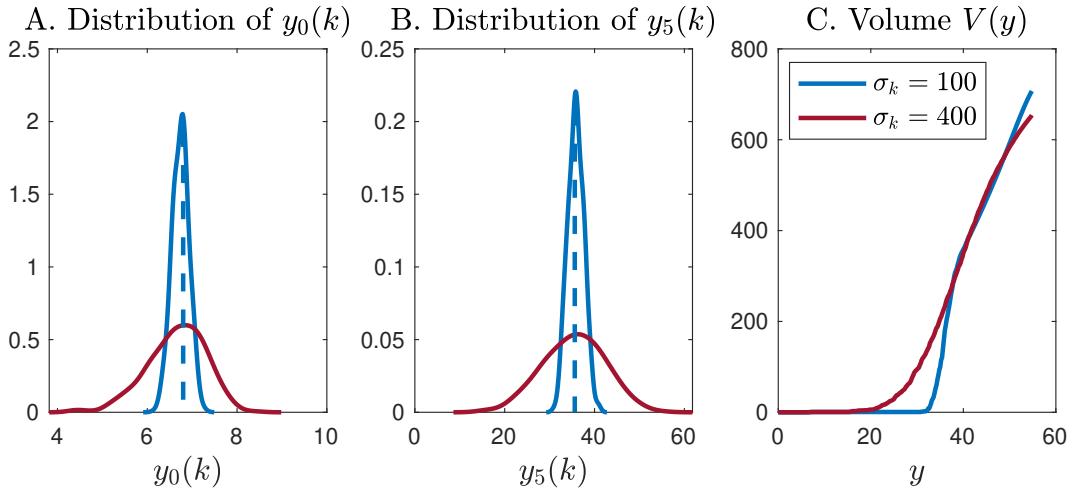


Figure 9 Firm heterogeneity in the investment cost parameter k . We assume that $r = 0.067$, $\sigma = 0.145$, $\mu = 0.015$, $a = -13.4$, $b = 0.6$, $\bar{k} = 1450$, $x = 1533$, $\epsilon = 0.57$, $\gamma = 1$.

Figure 9 illustrates the situations in which the heterogeneity in k is large (red line) versus small (blue line). In particular, it depicts the distribution of the default threshold $y_0(k)$ (resp., investment

threshold $y_5(k)$ in Panel A (resp., Panel B) and the industry output as a function of the commodity price y , i.e.,

$$V(y) = \int_{-\infty}^{\infty} X_{0+}^{x,\nu}(k) \frac{1}{\sigma_k \sqrt{2\pi}} e^{-\frac{1}{2} \frac{(k-\bar{k})^2}{\sigma_k^2}} dk, \quad (19)$$

in panel C. In line with intuition, as some sponsors have more privileged access to equipment and services providers (i.e., the distribution of k becomes more disperse), the disadvantaged (resp., advantaged) sponsors become more (resp., less) likely to default and will want the electricity price to rise (resp., fall) to justify a capacity expansion. This leads to a larger dispersion of the default and investment thresholds around their means (illustrated with dashed vertical lines), as observed in Panels A and B. The sponsors paying above-average (resp., below-average) EPC costs b are at the right (resp., left) of the mean $y_0(\bar{k})$ in Panel A (resp., $y_5(\bar{k})$ in Panel B). Panel C of Figure 9 highlights large volume swings at the industry level, with abrupt capacity decommissioning (resp., buildup) close to the threshold $y_0(\bar{k})$ (resp., $y_5(\bar{k})$). Interestingly, greater heterogeneity ($\sigma_k = 400$ vs 100) partially mitigates such volume swings because capacity decommissionings (resp., buildups) arise over a larger range of electricity prices. Figure 10 focuses the analysis on the effect of heterogeneity with respect to the fixed profit component a in eq. (1), assuming that it follows a normal distribution again. Here, the smoothing benefits of cost heterogeneity are less pronounced.

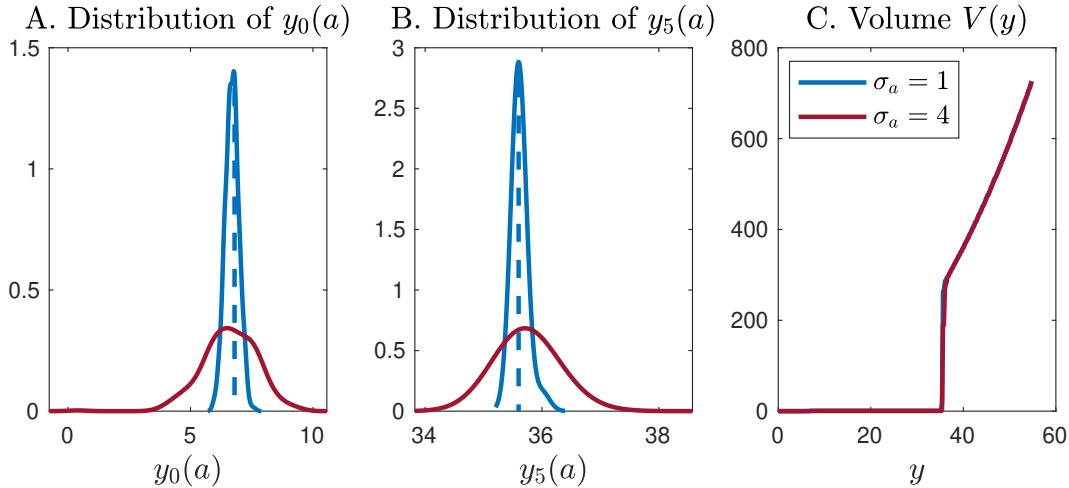


Figure 10 Firm heterogeneity in a . We assume that $r = 0.067$, $\sigma = 0.145$, $\mu = 0.015$, $a = -13.4$, $\bar{b} = 0.6$, $k = 1450$, $x = 1533$, $\epsilon = 0.57$, $\gamma = 1$.

7 Conclusion

This paper studies, for a problem related to the sponsor of a wind farm, the intricate interactions between the operational decision to expand generation capacity (and by how much) and the financial decision to default. To explore this essential question from the perspective of the iFORM literature, we developed a nonstandard real options model in which the sequence of decisions is not set ex ante

and in which the sponsor decides on capacity installment and investment time. We characterize the sponsor's decisions in terms of an interval outside of which the sponsor intervenes: it will default for low merchant prices and renew/install turbines for high prices.

The managerial insights from this model are numerous. For instance, when faced with significant merchant risk, the sponsor may install more capacity to avoid shutting down the SPV when the merchant price is low, but will virtually disregard the exit option for higher merchant prices. Furthermore, because the sponsor can either shut down operations or renew/install turbines, it will not default as soon as the merchant price falls below the cut-off level of the (stand-alone) exit option: it will exert more caution before killing both real options. Our numerical extensions provide further insights. Financing costs lead to delayed expansion, but the scale is only affected by proportional costs. Herding leads to an equipment price increase (resp., decrease) when the merchant price is high (resp., low), so the sponsor may hasten or delay investment to benefit from better procurement terms. Cost heterogeneity helps reduce observed volume swings as sponsors default or expand capacities over a larger range of electricity prices.

Like any model, our model has certain limitations. Some of them can be addressed within our general model framework; others are left for future research. For example, the efficiency of the power generation technology is assumed to be constant over time, thus ruling out the possibility that the sponsor may upgrade the technology when renewing/installing turbines. Furthermore, letting an SPV go bankrupt may lead to reputational damage for the sponsor, e.g., an incumbent utility, which is not captured here. In addition, the flow of energy from wind and sunlight is not constant; modeling their random availability (via a Weibull distribution) would add another layer of complexity, which is also omitted in our model. Most importantly, we considered one segment of the energy sector in isolation, disregarding the fact that wind farms are only one class of productive assets among a pool comprising base (e.g., nuclear) and peak-load assets (e.g., CCGT). The effect of wind farm decommissionings and capacity expansions on overall electricity prices is difficult to assess as these other generation technologies can offset such effects. If the economics of wind farms were to deteriorate (resp., improve), investments may be channeled into (resp., diverted away from) other flexible technologies (e.g., CCGT), which would mitigate the effect on the electricity price in the long term. However, unless significant storage capacities are installed, the increased investments in renewables may increase short-term supply fluctuations, which may exacerbate the volatility of merchant prices, at least in the short term.

We believe, however, that our model has helped us address key research questions of relevance for the energy sector. The insights we revealed may be carried over to other industries where project

finance plays a key role, e.g., for funding infrastructure such as bridges, tunnels, and toll roads or for financing long-term healthcare facilities such as hospitals.

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Online appendices

EC.1 Proof of Theorem 1

VI. Since Bensoussan and Lions (1982), it is known that the dynamic programming equation of optimal stopping is a VI. For the optimal stopping problem (4), the VI is

$$\min \left\{ \varphi(y, x); \mathcal{L}\varphi(y, x) - \pi(y, x) \right\} = 0, \quad \text{a.e. } y > 0. \quad (\text{EC.1a})$$

It seems natural to introduce the boundary condition

$$\liminf_{x \uparrow \infty} \left\{ \varphi(y, x) - \psi(y, x) \right\} = 0, \quad (\text{EC.1b})$$

with $\psi(\cdot, x)$ given in (3). To apply the operator \mathcal{L} , we conjecture that φ is regular in the sense that

$$\varphi(\cdot, x) \text{ is continuously differentiable with second-order derivatives almost everywhere.} \quad (\text{EC.1c})$$

We now turn to proving that the solution to a FBP solves the VI (EC.1a)–(EC.1c).

FBP. We conjecture a continuation set of the form $\{(y, x) \in \mathbb{R}_+^2 \mid y > y_1(x)\}$. Given this conjecture, we consider the FBP

$$\varphi(y, x) = 0, \quad \forall y < y_1(x), \quad (\text{EC.2a})$$

$$\mathcal{L}\varphi(y, x) = \pi(y, x), \quad \forall y \geq y_1(x) \quad (\text{EC.2b})$$

$$\liminf_{x \uparrow \infty} \left\{ \varphi(y, x) - \psi(y, x) \right\} = 0, \quad (\text{EC.2c})$$

$$\varphi(y_1(x), x) = 0, \quad (\text{value matching}) \quad (\text{EC.2d})$$

$$\frac{\partial \varphi}{\partial y}(y_1(x), x) = 0. \quad (\text{smooth pasting}) \quad (\text{EC.2e})$$

We want to solve the FBP (EC.2a)–(EC.2e) explicitly. We know that ψ in (3) is a particular solution to (EC.2b). It is immediate that $y \mapsto y^\gamma$ solves the *homogenous ODE* $\mathcal{L}y^\gamma = 0$ if γ is a root of the function $\mathcal{Q}(\cdot)$ in (2).

For completeness, we study the variation of $\mathcal{Q}(\cdot)$. By differentiation, $\mathcal{Q}'(\gamma) \leq 0$ iff $\gamma \geq \gamma^* := -[\mu - \sigma^2/2]/\sigma^2$. Besides, the quadratic function $\mathcal{Q}(\cdot)$ is asymptotically equivalent to $\gamma \mapsto -\frac{1}{2}\gamma^2\sigma^2$ at ∞ and $-\infty$; hence, $\mathcal{Q}(\infty) = \mathcal{Q}(-\infty) = -\infty$. It follows that $\mathcal{Q}(\cdot)$ is strictly increasing on $(-\infty, \gamma^*)$ from $-\infty$ to $\mathcal{Q}(\gamma^*)$ and strictly decreasing on (γ^*, ∞) from $\mathcal{Q}(\gamma^*)$ to $-\infty$. The maximum, $\mathcal{Q}(\gamma^*) = r + \frac{1}{2}[\mu - \sigma^2/2]^2/\sigma^2$, is strictly positive because we assume $r > 0$. It follows that the function $\mathcal{Q}(\cdot)$ has two distinct roots, β_1 and $\beta_2 > \beta_1$, given by

$$\beta_1, \beta_2 := -\frac{\mu - \sigma^2/2}{\sigma^2} \pm \sqrt{\left(\frac{\mu - \sigma^2/2}{\sigma^2}\right)^2 + \frac{2r}{\sigma^2}}. \quad (\text{EC.3})$$

We have $0 < \mathcal{Q}(0) = r \leq \mathcal{Q}(\gamma^*)$. Because $\mathcal{Q}(\cdot)$ is monotone increasing on $(-\infty, \gamma^*)$, we have $\beta_1 < 0$. Because $\mathcal{Q}(\cdot)$ is monotone decreasing on $(\gamma^*, \infty) \subset (0, \infty)$, $\beta_2 > 0$. Therefore, $\beta_1 < 0 < \beta_2$. The assumption $\gamma \in (0, \beta_2)$ implies that $\mathcal{Q}(\gamma) > 0$. Finally we are interested in *comparative statics* of $\beta_{1/2}$ with respect to the parameters $p \in \{r, \mu, \sigma\}$. We note that \mathcal{Q} is a function of both p and γ and that $\beta_{1/2}$ is a root of \mathcal{Q} which also depends on p . We now write

$$\mathcal{Q}\left(p, \beta_{1/2}(p)\right) = 0.$$

By total differentiation and the chain rule,

$$\frac{d\beta_{1/2}}{dp}(p) = -\frac{\frac{\partial \mathcal{Q}}{\partial p}\left(p, \beta_{1/2}(p)\right)}{\frac{\partial \mathcal{Q}}{\partial \gamma}\left(p, \beta_{1/2}(p)\right)}, \quad p \in \{r, \mu, \sigma\},$$

We already know that $\gamma \mapsto \frac{\partial \mathcal{Q}}{\partial \gamma}(p, \gamma)$ is positive at $\beta_1 < 0$ and negative at $\beta_2 > \gamma_*$. Besides, from (2),

$$\frac{\partial \mathcal{Q}}{\partial r}(r, \gamma) > 0, \quad \frac{\partial \mathcal{Q}}{\partial \mu}(\mu, \gamma) < 0 \quad \text{and} \quad \frac{\partial \mathcal{Q}}{\partial \sigma}(\sigma, \gamma) < 0.$$

It follows that

$$\frac{d\beta_2}{dr}(r) > 0, \quad \frac{d\beta_2}{d\mu}(\mu) < 0, \quad \frac{d\beta_2}{d\sigma}(\sigma) < 0 \quad \frac{d\beta_1}{dr}(r) < 0, \quad \frac{d\beta_1}{d\mu}(\mu) > 0, \quad \text{and} \quad \frac{d\beta_1}{d\sigma}(\sigma) > 0. \quad (\text{EC.4})$$

The ODE (EC.2b) admits a solution of the form

$$\varphi(y, x) = A + By^\gamma x^\epsilon + C_1(x)y^{\beta_1} + C_2(x)y^{\beta_2}, \quad y \geq y_1(x). \quad (\text{EC.5})$$

The unknowns $C_1(\cdot)$, $C_2(\cdot)$ and $y_1(\cdot)$ are obtained from (EC.2c)–(EC.2e). To ensure that (EC.2c) is satisfied, we set $C_2(\cdot) \equiv 0$. For a given $x > 0$, it obtains from (EC.2d)–(EC.2e) that $y_1(x)$ is a root of $x \mapsto \frac{\beta_1 - \gamma}{\beta_1} By^\gamma x^\epsilon + A$ in \mathbb{R}_+ . Because $\gamma \in (0, \beta_2)$, $\frac{\beta_1 - \gamma}{\beta_1} B\gamma y^{\gamma-1} x^\epsilon > 0$ and thus $y \mapsto \frac{\beta_1 - \gamma}{\beta_1} By^\gamma x^\epsilon + A$ increases on \mathbb{R}_+ from $A < 0$ to ∞ , proving the uniqueness of a root in (5b). By differentiation, $y'_1(x) < 0$, whence $y_1(\cdot)$ is monotone decreasing on \mathbb{R}_+ from ∞ to 0 and invertible, with its monotone increasing inverse function $x_1(\cdot) := y_1^{-1}(\cdot)$. It now obtains by differentiation that

$$\frac{\partial \varphi}{\partial y}(y, x) = \gamma By^{\gamma-1} x^\epsilon + \beta_1 C_1(x) y^{\beta_1-1}, \quad y \geq y_1(x), \quad (\text{EC.6})$$

so that $C_1(x)$ obtains from (EC.2e) as being equal to

$$C_1(x) = \frac{\gamma A}{\beta_1 - \gamma} \left(\frac{x^\epsilon}{\lambda}\right)^{\frac{\beta_1}{\gamma}} \geq 0. \quad (\text{EC.7})$$

In summary, we just established that the function φ in (5a) solves the FBP (EC.2a)–(EC.2e).

Verification of the VI. For the solution φ to (EC.2a)–(EC.2e) to solve (EC.1a)–(EC.1c), it remains to verify the inequalities

$$\varphi(y, x) \geq 0, \quad \forall y \geq y_1(x), \quad (\text{EC.8a})$$

$$\mathcal{L}\varphi(y, x) \geq \pi(y, x), \quad \forall y < y_1(x). \quad (\text{EC.8b})$$

We start with (EC.8a). We obtain for φ in (5a)–(5b) that

$$\frac{\partial \varphi}{\partial x}(y, x) = \epsilon B x^{\epsilon-1} y^\gamma \left[1 - \lambda \frac{\gamma - \beta_1}{\gamma} x^{\epsilon \frac{\beta_1 - \gamma}{\gamma}} y^{\beta_1 - \gamma} \right] \mathbf{1}_{[x_1(y), \infty)}(x). \quad (\text{EC.9})$$

Hence, $\frac{\partial \varphi}{\partial x}(y, x) \geq 0$ for all $x \geq x_1(y)$. It follows that $\varphi(y, \cdot)$ is monotone increasing on $(x_1(y), \infty)$ and bounded below by $\varphi(y, x_1(y)) = 0$, because of (EC.2d). The inequality (EC.8a) is thus satisfied.

We want to prove (EC.8b). We already know that, by definition of $y_1(x)$, $rA + r \frac{\beta_1 - \gamma}{\beta_1} B y^\gamma x^\epsilon \leq 0$ for all $y \leq y_1(x)$. It thus suffices to show that

$$\zeta(\gamma) := r(\beta_1 - \gamma) - \beta_1 \mathcal{Q}(\gamma) = \beta_1 \left[\left(\mu - \frac{1}{2} \sigma^2 \right) \gamma + \frac{1}{2} \sigma^2 \gamma^2 \right] - r\gamma$$

is negative for all $\gamma \in (0, \beta_2)$. The function $\zeta(\cdot)$ is $C^2(\mathbb{R}_+^2)$. We have $\zeta(0) = 0$. Besides, $\zeta'(\gamma) = \beta_1 \left[\mu + \frac{2\gamma-1}{2} \sigma^2 \right] - r$ and $\zeta''(\gamma) = \beta_1 \sigma^2 < 0$. It follows that $\zeta'(\cdot)$ is monotone decreasing on $[0, \infty)$ from $\zeta'(0) = \frac{\beta_1}{r} \left[\mu - \frac{1}{2} \sigma^2 \right] - 1$ to $-\infty$. If $\mu > \frac{1}{2} \sigma^2$, then $\zeta'(0) < 0$ and $\zeta'(\gamma) < 0$ on $[0, \infty)$. So $\zeta(\cdot)$ is monotone decreasing on \mathbb{R}_+ from $\zeta(0) = 0$ to $-\infty$, taking negative values. We thus proved that $\varphi(\cdot, x)$ in (5a) is a solution to the VI (EC.1a)–(EC.1c).

Verification theorem. To prove that the solution of VI (EC.1a)–(EC.1c) coincides with the value function (4) requires a proof known as a *verification theorem*. We omit this proof relying instead on general verification theorems (see, e.g., Bensoussan and Lions 1982).

EC.2 Proof of Lemma 1

If we define $x_2(y)$ as in (10), then we can express the derivative of $x \mapsto \frac{\partial \varphi}{\partial x}(y, x)$ in (EC.9) as

$$\frac{\partial^2 \varphi}{\partial x^2}(y, x) = \epsilon(\epsilon - 1) B x^{\epsilon-2} y^\gamma \left[1 - \left(\frac{x_2(y)}{x} \right)^{\epsilon \frac{\gamma - \beta_1}{\gamma}} \right] \mathbf{1}_{[x_1(y), \infty)}(x). \quad (\text{EC.10})$$

The pattern for the function $x \mapsto \frac{\partial \varphi}{\partial x}(y, x)$ described in Lemma 1 is now immediate.

EC.3 Proof of Theorem 2

From (EC.9) and Lemma 1, the global maximum of $x \mapsto \frac{\partial \varphi}{\partial x}(y, x)$ is

$$\frac{\partial \varphi}{\partial x}(y, x_2(y)) = \epsilon^2 B(\lambda \rho)^{\frac{\epsilon-1}{\epsilon}} y^{\frac{\gamma}{\epsilon}} \frac{\gamma - \beta_1}{\gamma - \epsilon \beta_1}.$$

We define the points y^* in (11) and $x^* = x_2(y^*)$. We distinguish several cases:

- If $y \leq y^*$, then $\frac{\partial \varphi}{\partial x}(y, x_2(y)) \leq k$. Consequently, $x \mapsto \varphi(y, x) - kx$ is decreasing on $[0, \infty)$.
- If $y > y^*$, $\frac{\partial \varphi}{\partial x}(y, x) = k$ has two solutions in $[x_1(y), \infty)$, which we note $x_3(y)$ and $\tilde{x}_3(y)$ with

$$x_3(y) > x_2(y) > \tilde{x}_3(y), \quad y > y^*. \quad (\text{EC.11})$$

The function $x \mapsto \frac{\partial \varphi}{\partial x}(y, x)$ increases at $\tilde{x}_3(y)$ and decreases at $x_3(y)$ and so the function $x \mapsto \varphi(y, x) - kx$ attains a minimum at $\tilde{x}_3(y)$ and a local maximum at $x_3(y)$. We introduce

$$\phi(y) := \varphi(y, x_3(y)) - kx_3(y), \quad (\text{EC.12})$$

Because $x \mapsto \varphi(y, x) - kx$ vanishes at $x = 0$, it must be that $\phi(y) > 0$ for the local maximum $x_3(\cdot)$ to be a global maximum.

We now study the sign of $\phi(\cdot)$ on (y^*, ∞) . It follows from total differentiation and definition of $x_3(y)$ that $\phi'(y) = \frac{\partial \varphi}{\partial y}(y, x_3(y))$. It is immediate from (5a) that $\phi'(y) \geq 0$ and, hence, that $\phi(\cdot)$ is monotone increasing. We note from (EC.9) and the definition of $x_3(y)$ that

$$x_3(y)^{\epsilon-1} y^\gamma - \lambda^{\frac{\gamma-\beta_1}{\gamma}} x_3(y)^{-1+\epsilon \frac{\beta_1}{\gamma}} y^{\beta_1} = \frac{k}{\epsilon B}. \quad (\text{EC.13})$$

It follows from (EC.13) and (5a) that

$$\phi(y) = A + B(1 - \epsilon)x_3(y)^\epsilon y^\gamma + \frac{A(\gamma - \epsilon\beta_1)}{\beta_1 - \gamma} \left(\frac{x_3(y)^\epsilon y^\gamma}{\lambda} \right)^{\frac{\beta_1}{\gamma}}, \quad y > y^*. \quad (\text{EC.14})$$

We are interested in the behavior of $x_3(\cdot)$. By implicit differentiation,

$$x_3'(y) = - \frac{\frac{\partial^2 \varphi}{\partial x \partial y}(y, x_3(y))}{\frac{\partial^2 \varphi}{\partial x^2}(y, x_3(y))}. \quad (\text{EC.15})$$

Because $x_3(y)$ is a maximum, $\frac{\partial^2 \varphi}{\partial x^2}(y, x_3(y)) < 0$. Furthermore, by differentiating (EC.9),

$$\frac{\partial^2 \varphi}{\partial x \partial y}(y, x) = \epsilon \gamma B y^{\gamma-1} x^{\epsilon-1} \left[1 + (\gamma - 1 - \beta_1) \left(\frac{\lambda}{x^\epsilon} \right)^{\frac{\gamma-\beta_1}{\gamma}} x^{-\gamma+\beta_1} \right], \quad (\text{EC.16})$$

which is positive. So, $x'_3(y)$ in (EC.15) is strictly positive and $x_3(\cdot)$ is monotone increasing on (y^*, ∞) . We note that $\tilde{x}_3(y^*) = x_2(y^*) = x_3(y^*)$ and denote this value x^* . We recall (EC.13) and note for $y > y^*$ the relationship

$$x_3(y) > x^* \implies 0 < \lambda^{\frac{\gamma-\beta_1}{\gamma}} x_3(y)^{-1+\epsilon\frac{\beta_1}{\gamma}} y^{\beta_1} < \lambda^{\frac{\gamma-\beta_1}{\gamma}} (x^*)^{-1+\epsilon\frac{\beta_1}{\gamma}} y^{\beta_1}. \quad (\text{EC.17})$$

It follows from (EC.17) that

$$\lambda^{\frac{\gamma-\beta_1}{\gamma}} x_3(y)^{-1+\epsilon\frac{\beta_1}{\gamma}} y^{\beta_1} \rightarrow 0 \quad \text{as } y \rightarrow \infty.$$

and from (EC.13) that

$$\frac{x_3(y)^{1-\epsilon}}{y^\gamma} \rightarrow \frac{\epsilon B}{k} \quad \text{as } y \rightarrow \infty. \quad (\text{EC.18})$$

Because $x_3(y) \rightarrow \infty$ as $y \rightarrow \infty$, it follows that the function $\phi(\cdot)$ in (EC.14) diverges to ∞ as y goes to ∞ .

We recall that $x_3(y^*) = x^* := (\lambda \rho x^{*\gamma})^{\frac{1}{\epsilon}}$. It follows that

$$\phi(y^*) = A + B(1-\epsilon)\lambda\rho + \frac{A(\gamma - \epsilon\beta_1)}{\beta_1 - \gamma} \rho^{\frac{\beta_1}{\gamma}}.$$

Hence, from (5b),

$$\phi(y^*) = A \left[1 - \frac{(1-\epsilon)\rho}{\beta_1 - \gamma} \left(\beta_1 - \frac{\gamma - \epsilon\beta_1}{1-\epsilon} \rho^{\frac{\beta_1 - \gamma}{\gamma}} \right) \right],$$

which from (11) simplifies to $\phi(y^*) = A[1 - (1-\epsilon)\rho]$. We obtain from the definition of ρ in (11) the equivalence

$$(1-\epsilon)\rho < 1 \iff 1 - \epsilon \frac{\beta_1}{\gamma} < (1-\epsilon) \frac{\beta_1}{\gamma}. \quad (\text{EC.19})$$

The statement on the RHS of (EC.19) holds true because $\epsilon \mapsto (1-\epsilon) \frac{\beta_1}{\gamma}$ is convex. Because $A < 0$, it follows that $\phi(y^*) < 0$. Consequently $\phi(\cdot)$ has a unique root y^{**} on its domain (y^*, ∞) . This completes the proof.

EC.4 Proof of Corollary 1

The properties for the function $x_3(\cdot)$ are immediate for the proof of Theorem 2; in particular, the asymptotic equivalence between $x_3(\cdot)$ and $\hat{x}(\cdot)$ is stated in (EC.18). As $y \mapsto \varphi(y, x_3(y)) - kx_3(y)$ increases on (y^*, ∞) , it is intuitive that the point $\bar{x}_3(\cdot)$ defined on (y^*, y^{**}) by (12) decreases on its domain of definition.

EC.5 Proof of Corollary 2

Expression for Φ . We now want to determine Φ in (9), distinguishing several cases:

1) $y \leq y^*$. It follows Theorem 2 that

$$\Phi(y, x) = \varphi(y, x), \quad \text{if } y \leq y^*. \quad (\text{EC.20})$$

2a) $y^* < y < y^{**}$. From Corollary 1, equation (12) admits a unique $\bar{x}_3(y)$ such that

$$\begin{cases} \varphi(y, x) - kx > \phi(y), & \text{if } 0 < x < \bar{x}_3(y) \\ \varphi(y, x) - kx \leq \phi(y), & \text{if } \bar{x}_3(y) \leq x \leq x_3(y). \end{cases} \quad (\text{EC.21})$$

We have the ordering

$$\max\{x_1(y), \bar{x}_3(y)\} < \tilde{x}_3(y) < x_2(y) < x_3(y). \quad (\text{EC.22})$$

We recall the expression for Φ in (9) and can assert because of (EC.21) that

$$\Phi(y, x) = \begin{cases} \varphi(y, x), & 0 < x \leq \bar{x}_3(y), \\ \phi(y) + kx, & \bar{x}_3(y) < x < x_3(y). \end{cases} \quad (\text{EC.23a})$$

If $x \geq x_3(y)$, we use the fact that $x \mapsto \varphi(y, x) - kx$ decreases on $(x_3(y), \infty)$ to assert that $\Phi(y, x) = \varphi(y, x)$.

2b) $y \geq y^{**}$. Here, $\phi(y) \geq 0$. Therefore,

$$\Phi(y, x) = \begin{cases} \phi(y) + kx, & x < x_3(y), \\ \varphi(y, x), & x \geq x_3(y). \end{cases} \quad (\text{EC.23b})$$

We note that $\bar{y}_3(\cdot)$, which is defined on $[0, x^*]$, decreases from y^{**} to y^* on its domain. We have $\Phi(y, x) = \varphi(y, x)$ for $y \leq y^*$. For $y > y^*$, there are two cases depending on whether x is lower or larger than x^* :

2a') $x < x^*$. Here, $\bar{y}_3(x)$ is defined and larger than y^* . Then, from (EC.23a),

$$\Phi(y, x) = \begin{cases} \varphi(y, x), & y^* < y \leq \bar{y}_3(x), \\ \phi(y) + kx, & y > \bar{y}_3(x). \end{cases} \quad (\text{EC.24a})$$

2b') $x \geq x^*$. Here, $y_3(x)$ is defined and larger than y^* . We have from (EC.23b)

$$\Phi(y, x) = \begin{cases} \varphi(y, x), & y^* < y \leq y_3(x), \\ \phi(y) + kx, & y > y_3(x). \end{cases} \quad (\text{EC.24b})$$

For the case where $x > x^*$, we prefer the formulation (EC.24a)-(EC.24b) to the formulation (EC.23a)-(EC.23b). We extend the definition of the function $y \mapsto \Phi(y, x)$ in (EC.24a) and (EC.24b) to the interval $[0, y^*]$ by using the relationship (EC.20).

Regularity of Φ . We now study the regularity of $\Phi(\cdot, x)$.

- For the case 2a') with $x < x^*$, it obtains from (EC.24a) that $\Phi(\bar{y}_3(x), x) = \varphi(\bar{y}_3(x), x)$, while the right limit is

$$\Phi(\bar{y}_3(x)+, x) = \phi(\bar{y}_3(x)) + kx.$$

Using the definitions of $\bar{x}_3(\cdot)$ in (12) and of $\bar{y}_3(\cdot) \in (y^*, y^{**})$ as the inverse of $\bar{x}_3(\cdot)$, we then have

$$\begin{aligned} \Phi(\bar{y}_3(x)+, x) &= \varphi\left(\bar{y}_3(x), \bar{x}_3(\bar{y}_3(x))\right) - k\bar{x}_3(\bar{y}_3(x)) + kx \\ &= \varphi(\bar{y}_3(x), x). \end{aligned}$$

It is now immediate that, for $x < x^*$, $\Phi(\cdot, x)$ is continuous at $\bar{y}_3(x)$ and actually on $[y^*, \infty)$. We consider the continuity of the first-order derivative at $\bar{y}_3(x)$. On the one hand, $\frac{\partial \Phi}{\partial y}(\bar{y}_3(x)-, x) = \frac{\partial \varphi}{\partial y}(\bar{y}_3(x), x)$. On the other hand, by definitions of $\phi(\cdot)$ in (EC.12) and of $x_3(\cdot)$ in Theorem 2, we have

$$\frac{\partial \Phi}{\partial y}(\bar{y}_3(x)+, x) = \frac{\partial \varphi}{\partial y}\left(\bar{y}_3(x), x_3(\bar{y}_3(x))\right).$$

However, $x < x^* < x_3(\bar{y}_3(x))$. Because $y \mapsto \frac{\partial \varphi}{\partial y}(y, x)$ increases, we obtain $\frac{\partial \Phi}{\partial y}(\bar{y}_3(x)-, x) < \frac{\partial \Phi}{\partial y}(\bar{y}_3(x)+, x)$, which proves that $\Phi(\cdot, x)$ is not continuously differentiable.

- In case 2b') with $x \geq x^*$, we see from (EC.24b) that $\Phi(\cdot, x)$ is continuous; it is also continuously differentiable with second-order derivatives almost everywhere by the definitions of $\phi(\cdot)$ in (EC.12) and of $x_3(\cdot)$ in Theorem 2.

This completes the proof of Corollary 2.

EC.6 Proof of Corollary 3

Let $a < \alpha < 0$. We now highlight the dependence on the parameter a . It is immediate from (5b) and (11) that

$$x^*(a) = \left[k \frac{\gamma - \epsilon \beta_1}{\gamma - \beta_1} \frac{\left(-\frac{A\rho\beta_1}{\beta_1 - \gamma}\right)^{\frac{1-\epsilon}{\epsilon}}}{\epsilon^2} \right]^{\frac{\epsilon}{\gamma}} B^{-\frac{1}{\gamma}} \geq 0. \quad (\text{EC.25})$$

It follows from (EC.25) that $a \mapsto x^*(a)$ is continuous and decreases on $(-\infty, 0]$ from $+\infty$ to 0. To prove the inclusion, it remains to prove that

- (i) $0 < y_3(x; \alpha) < y_3(x, a)$ for $x \geq x^*(a)$. We rewrite (EC.14) in the form

$$x^{\epsilon-1} y_3(x, a)^\gamma - \lambda(a) \frac{\gamma - \beta_1}{\gamma} x^{-1 + \epsilon \frac{\beta_1}{\gamma}} y_3(x, a)^{\beta_1} = \frac{k}{\epsilon B}.$$

By total differentiation in a ,

$$\frac{\partial y_3}{\partial a}(x, a) \times \underbrace{\left[x^{\epsilon-1} \gamma y_3(x, a)^{\gamma-1} - \beta_1 \lambda(a) \frac{\gamma - \beta_1}{\gamma} x^{-1 + \epsilon \frac{\beta_1}{\gamma}} y_3(x, a)^{\beta_1-1} \right]}_{>0} = \underbrace{\frac{\gamma - \beta_1}{\gamma} \lambda'(a) x^{-\frac{\beta_1}{\gamma}} x^{-1 + \epsilon \frac{\beta_1}{\gamma}} y_3(x, a)^{\beta_1}}_{<0}.$$

because we know from (5b) that $\lambda'(a) < 0$. Hence, $a \mapsto y_3(x, a)$ necessarily decreases on $(-\infty, 0]$, so

$$a < \alpha < 0 \implies y_3(x, a) > y_3(x, \alpha) \text{ for } x \geq x^*(a).$$

(ii) $0 \leq \bar{y}_3(x, \alpha) \leq \bar{y}_3(x, a)$ for $x \leq x^*(\alpha)$. We rewrite (12) as

$$\varphi(\bar{y}_3(x, a), x, a) - kx = \phi(\bar{y}_3(x, a), a).$$

By total differentiation in a ,

$$\frac{\partial \varphi}{\partial y}(\bar{y}_3(x, a), x, a) \times \frac{\partial \bar{y}_3}{\partial a}(x, a) + \frac{\partial \varphi}{\partial a}(\bar{y}_3(x, a), x, a) = \frac{\partial \phi}{\partial y}(\bar{y}_3(x, a), a) \times \frac{\partial \bar{y}_3}{\partial a}(x, a) + \frac{\partial \phi}{\partial a}(\bar{y}_3(x, a)).$$

By the definitions of $\phi(\cdot)$ in (EC.12) and of $x_3(y)$ in Theorem 2, we have $\frac{\partial \phi}{\partial a}(y, a) = \frac{\partial \varphi}{\partial a}(y, x_3(y), a)$ and $\frac{\partial \phi}{\partial y}(y, a) = \frac{\partial \varphi}{\partial y}(y, x_3(y))$. It follows

$$\frac{\partial \bar{y}_3}{\partial a}(x, a) = - \frac{\int_x^{(x_3 \circ \bar{y}_3)(x)} \frac{\partial^2 \varphi}{\partial y \partial a}(\bar{y}_3(x, a), \xi, a) d\xi}{\int_x^{(x_3 \circ \bar{y}_3)(x)} \frac{\partial^2 \varphi}{\partial y \partial x}(\bar{y}_3(x, a), \xi, a) d\xi}. \quad (\text{EC.26})$$

Besides, $x < x^*(a)$ implies $(x_3 \circ \bar{y}_3)(x, a) > x$. Because we know from (EC.16) that $\frac{\partial^2 \varphi}{\partial x \partial y}(y, x) > 0$, it remains to establish the sign of $\frac{\partial^2 \varphi}{\partial y \partial a}(y, x, a)$. It obtains from (EC.6) that

$$\frac{\partial^2 \varphi}{\partial y \partial a}(y, x, a) = \beta_1 \frac{\partial C_1}{\partial a}(x, a) y^{\beta_1 - 1}, \quad y \geq y_1(x), \quad (\text{EC.27})$$

From (EC.7),

$$\frac{\partial C_1}{\partial a}(x, a) = \frac{\gamma}{\beta_1 - \gamma} x^{\epsilon \frac{\beta_1}{\gamma}} \left[\frac{\lambda(a)^{-\frac{\beta_1}{\gamma}}}{r} - \frac{\beta_1}{\gamma} A \lambda'(a) \lambda(a)^{-\frac{\beta_1}{\gamma} - 1} \right] < 0$$

and so $\frac{\partial^2 \varphi}{\partial y \partial a}(y, x, a) \geq 0$. It now follows that $\frac{\partial \bar{y}_3}{\partial a}(x, a) \leq 0$ and $\bar{y}_3(x, a) \geq \bar{y}_3(x, \alpha)$ for $x \leq x^*(a)$.

(iii) $\bar{y}_3(x, a) \geq y_3(x, \alpha)$ for $x^*(\alpha) < x < x^*(a)$. We know that the function $x \mapsto \bar{y}_3(x, a)$ [resp., $x \mapsto y_3(x, \alpha)$] is continuous and monotone decreasing [resp., increasing] on the interval $(x^*(\alpha), x^*(a))$. Further, we know that $\bar{y}_3(x^*(\cdot), \cdot) = y_3(x^*(\cdot), \cdot) = y^*(\cdot)$. We established in (ii) that

$$\bar{y}_3(x^*(\alpha), a) \geq \bar{y}_3(x^*(\alpha), \alpha) = y_3(x^*(\alpha), \alpha)$$

and in (i) that

$$\bar{y}_3(x^*(a), a) = y_3(x^*(a), a) > y_3(x^*(a), \alpha).$$

The result now readily obtains by continuity and monotonicity.

The above proves that $a < \alpha < 0 \implies \mathcal{R}_a \subset \mathcal{R}_\alpha$.

Because $\lambda \rightarrow 0$ as $a \rightarrow 0$, it follows from (EC.17) that

$$\lambda^{\frac{\gamma-\beta_1}{\gamma}} x_3(y)^{-1+\epsilon\frac{\beta_1}{\gamma}} y^{\beta_1} \rightarrow 0 \quad \text{as } a \rightarrow 0,$$

and from (EC.13) that

$$\frac{x_3(y)^{1-\epsilon}}{y^\gamma} \rightarrow \frac{\epsilon B}{k} \quad \text{as } a \rightarrow 0.$$

Recalling the definitions of \mathcal{R}_0 and $\hat{x}(\cdot)$ in (7) to establish that $x_3(\cdot)$ converges to $\hat{x}(\cdot)$ as $a \uparrow 0$.

EC.7 Proof of Lemma 2

We recall the definition of \mathcal{L} . We introduce the functions

$$\chi(y, x) := F(y, x) - \Phi(y, x) \quad \text{and} \quad g(y, x) := \pi(y, x) - \mathcal{L}\Phi(y, x).$$

The function g is defined because $\Phi(\cdot, x)$ is continuously differentiable with second-order derivatives almost everywhere. For $y < y_1(x)$, it is immediate from Theorems 1 and 2 that the function g reads

$$g(y, x) = rA + B\mathcal{Q}(\gamma)y^\gamma x^\epsilon, \quad y < y_1(x).$$

For $y_1(x) < y < y_3(x)$, we further need to note that $\mathcal{Q}(\beta_1) = 0$ by definition of β_1 to obtain

$$g(y, x) = 0, \quad y_1(x) < y < y_3(x).$$

The case $y > y_3(x)$ requires tedious calculations. By total differentiation of $\phi(\cdot)$ in (EC.12) and definition of $x_3(y)$, we obtain

$$\phi'(y) = \frac{\partial \varphi}{\partial y}(y, x_3(y)) \quad \text{and} \quad \phi''(y) = \frac{\partial^2 \varphi}{\partial y^2}(y, x_3(y)) + \frac{\partial^2 \varphi}{\partial x \partial y}(y, x_3(y)) \times x'_3(y).$$

We recall that $[y_3(y), \infty)$ is a subset of $[y_1(y), \infty)$, the continuation region of the optimal stopping problem (4) given in Theorem 1. It follows that

$$\mathcal{L}\phi(y) = rA + \mathcal{Q}(\gamma)y^\gamma x_3(y)^\epsilon - rkx_3(y) - \frac{1}{2}\sigma^2 y^2 \frac{\partial^2 \varphi}{\partial x \partial y}(y, x_3(y)) \times x'_3(y).$$

Because $\Phi(y, x) = \phi(y) + kx$ in this region, it now obtains from (EC.15) that

$$g(y, x) = -\mathcal{Q}(\gamma)By^\gamma [x_3(y)^\epsilon - x^\epsilon] + rk[x_3(y) - x] - \frac{1}{2}\sigma^2 y^2 \frac{\frac{\partial^2 \varphi}{\partial x \partial y}(y, x_3(y))^2}{\frac{\partial^2 \varphi}{\partial x^2}(y, x_3(y))}, \quad y > y_3(x). \quad (\text{EC.28})$$

To summarize, the function g can be written as

$$g(y, x) = \begin{cases} a + B\mathcal{Q}(\gamma)y^\gamma x^\epsilon, & y < y_1(x), \\ 0, & y_1(x) < y < y_3(y), \\ -\mathcal{Q}(\gamma)By^\gamma [x_3(y)^\epsilon - x^\epsilon] + rk[x_3(y) - x] - \frac{1}{2}\sigma^2 y^2 \frac{\frac{\partial^2 \varphi}{\partial x \partial y}(y, x_3(y))^2}{\frac{\partial^2 \varphi}{\partial x^2}(y, x_3(y))}, & y > y_3(x). \end{cases}$$

Discontinuities arise because $x \mapsto \Phi(y, x)$ is not C^2 at $y_1(x)$ and $y_3(x)$.

We now want to study the function $g(\cdot, x)$ further. First, $g(\cdot, x)$ is monotone increasing on $(0, y_1(x))$. It now obtains from (5b) and (5b) that for $x < y_1(x)$

$$\begin{aligned} g(y, x) &< rA + B\mathcal{Q}(\gamma)y_1(x)^\gamma x^\epsilon, \\ &< A \left[r - \frac{\beta_1}{\beta_1 - \gamma} \mathcal{Q}(\gamma) \right] \end{aligned}$$

It follows now from (2) that

$$g(y, x) < \frac{A\gamma}{\beta_1 - \gamma} \left[-r + \beta_1 \left(\mu + \frac{\gamma - 1}{2} \sigma^2 \right) \right] < 0, \quad y < y_1(x).$$

Further, it is immediate from (EC.28) that

$$g(y_3(x)+, x) = -\frac{1}{2} \sigma^2 y^2 \frac{\frac{\partial^2 \varphi}{\partial x \partial y}(y, x_3(y))^2}{\frac{\partial^2 \varphi}{\partial x^2}(y, x_3(y))} > 0.$$

We want to study the behavior of $g(\cdot, x)$ at ∞ . We have from (EC.10) and (EC.16) that

$$\begin{aligned} \frac{\frac{\partial^2 \varphi}{\partial x \partial y}(y, x_3(y))^2}{\frac{\partial^2 \varphi}{\partial x^2}(y, x_3(y))} &= y^{\gamma-2} \frac{\epsilon \gamma^2 B}{\epsilon - 1} x_3(y)^\epsilon \frac{\left(1 + (\gamma - 1 - \beta_1) [\lambda x^{-\gamma} x_3(y)^{-\epsilon}]^{\frac{\gamma - \beta_1}{\gamma}} \right)^2}{1 - [\rho \lambda x^{-\gamma} x_3(y)^{-\epsilon}]^{\frac{\gamma - \beta_1}{\gamma}}} \\ &= y^{\gamma-2} \frac{\epsilon \gamma^2 B}{\epsilon - 1} x_3(y)^\epsilon \frac{\left(1 + (\gamma - 1 - \beta_1) \left(\frac{x_1(y)}{x_3(y)} \right)^{\frac{\gamma - \beta_1}{\gamma}} \right)^2}{1 - \rho^{\frac{\gamma - \beta_1}{\gamma}} \left(\frac{x_1(y)}{x_3(y)} \right)^{\frac{\gamma - \beta_1}{\gamma}}} \end{aligned} \quad (\text{EC.29})$$

We recall that $x_1(\cdot)$ vanishes as $y \rightarrow \infty$ while $x_3(\cdot)$ goes to ∞ . From (EC.13) and (EC.29) we conclude

$$\frac{g(y, x)}{Bx_3^\epsilon(y)y^\gamma} \rightarrow -c_\infty, \quad \text{as } y \rightarrow \infty, \quad (\text{EC.30})$$

where

$$c_\infty := \mathcal{Q}(\gamma) - r\epsilon - \frac{1}{2} \sigma^2 \frac{\epsilon}{1 - \epsilon} \gamma^2. \quad (\text{EC.31})$$

We combine the limits in (EC.30) and (EC.18) to conclude that

$$\frac{g(y, x)}{y^{\frac{\gamma}{1-\epsilon}}} \rightarrow -\left(\frac{\epsilon}{k}\right)^{\frac{\epsilon}{1-\epsilon}} B^{\frac{1}{1-\epsilon}} c_\infty \quad \text{as } x \rightarrow \infty. \quad (\text{EC.32})$$

It follows from (EC.31) and (2),

$$\begin{aligned} c_\infty &= r(1 - \epsilon) - \mu\gamma - \frac{1}{2} \sigma^2 \gamma \left[\gamma - 1 + \frac{\epsilon}{1 - \epsilon} \gamma \right] \\ &= (1 - \epsilon) \left[r - \mu \frac{\gamma}{1 - \epsilon} - \frac{1}{2} \sigma^2 \frac{\gamma}{1 - \epsilon} \left(\frac{\gamma}{1 - \epsilon} - 1 \right) \right] \\ &= (1 - \epsilon) \mathcal{Q} \left(\frac{\gamma}{1 - \epsilon} \right). \end{aligned}$$

We have $\frac{\gamma}{1-\epsilon} > \gamma$. From the study of $\mathcal{Q}(\cdot)$, we know that

$$\mathcal{Q}\left(\frac{\gamma}{1-\epsilon}\right) \leq 0 \iff 0 < \frac{\gamma}{1-\epsilon} \leq \beta_2.$$

From (EC.32) we can now assert that $\frac{g(y,x)}{y^{\frac{\gamma}{1-\epsilon}}}$ goes to a negative constant if $\frac{\gamma}{1-\epsilon} < \beta_2$ and to a positive constant if $\frac{\gamma}{1-\epsilon} > \beta_2$. To simplify the discussion, we exclude the case $\frac{\gamma}{1-\epsilon} = \beta_2$, which would require an analysis of the sign of $g(\cdot)$ as x goes to ∞ . This concludes the proof of Lemma 2.

We recall the probabilistic interpretation (EC.36) of the solution χ to the VI (16). Similarly, the probabilistic interpretation of the solution $\underline{\chi}$ to the second-order ODE

$$\mathcal{L}\underline{\chi}(y, x) = g(y, x), \quad x \geq x^* \tag{EC.33}$$

is

$$\underline{\chi}(y, x) = \mathbb{E}\left[\int_0^\infty e^{-rt} g(Y_t, x) dt \mid Y_0 = y\right]. \tag{EC.34}$$

if one exists. While the firm must keep options alive in (EC.34), the firm can exercise these options in (EC.36). Given optionality, $\chi \geq \underline{\chi}$. From the behavior of $g(\cdot, x)$ described in (the proof of) Lemma 2, the function $\underline{\chi}$ in (EC.34) is finite iff

$$\mathbb{E}\left[\int_0^\infty e^{-rt} (Y_t)^{\frac{\gamma}{1-\epsilon}} dt \mid Y_0 = y\right] < \infty. \tag{EC.35}$$

Given standard statistics on the geometric Brownian motion

$$\mathbb{E}\left[(Y_t^y)^{\frac{\gamma}{1-\epsilon}}\right] = y^{\frac{\gamma}{1-\epsilon}} \exp\left(\mu\left(\frac{\gamma}{1-\epsilon}\right) + \frac{1}{2}\sigma^2 \frac{\gamma}{1-\epsilon}\left(\frac{\gamma}{1-\epsilon} - 1\right)\right)t.$$

It follows that

$$\mathbb{E}\left[\int_0^\infty e^{-rt} (Y_t^y)^{\frac{\gamma}{1-\epsilon}} dt \mid Y_0 = y\right] = y^{\frac{\gamma}{1-\epsilon}} \int_0^\infty e^{-\mathcal{Q}(\frac{\gamma}{1-\epsilon})t} dt$$

and that the condition (EC.35) is satisfied iff $\mathcal{Q}(\frac{\gamma}{1-\epsilon}) > 0$, i.e., iff $\frac{\gamma}{1-\epsilon} < \beta_2$. This proves the

Corollary 4 *If $\beta_2 < \frac{\gamma}{1-\epsilon}$, the solution χ to the VI (16) has no probabilistic interpretation*

$$\chi(y) = \sup_\tau \mathbb{E}\left[\int_0^\tau e^{-rt} g(Y_t) dt \mid Y_0 = y\right]. \tag{EC.36}$$

EC.8 Solution of the VI (16) for $x > x^*$

Lemma 2 specifies properties of the function g . We now want to express *mild* necessary and sufficient conditions on g (so indirectly on the model primitives) to ensure that the Ansatz holds:

Ansatz 1 The set $\{y > 0 | \chi(y) > 0\}$ is of the form (y_0, y_5) where $y_0 < y_5$ solve the FBP:

$$\mathcal{L}\chi(y) = g(y), \quad y_0 < y < y_5, \quad (\text{EC.37a})$$

$$\chi(y_0) = \chi(y_5) = 0, \quad [\text{value matching } (\times 2)] \quad (\text{EC.37b})$$

$$\chi'(y_0) = \chi'(y_5) = 0. \quad [\text{smooth pasting } (\times 2)]. \quad (\text{EC.37c})$$

As usual, we guess for the ODE (EC.37a) a solution of the form

$$\chi(y) = C_1(y)y^{\beta_1} + C_2(y)y^{\beta_2}, \quad \text{with} \quad C_1'(y)y^{\beta_1} + C_2'(y)y^{\beta_2} \equiv 0 \quad (\text{EC.38})$$

in the interval (y_0, y_5) . It follows that

$$\chi'(y) = \beta_1 C_1(y)y^{\beta_1-1} + \beta_2 C_2(y)y^{\beta_2-1}, \quad (\text{EC.39})$$

$$\chi''(y) = \beta_1(\beta_1 - 1)C_1(y)y^{\beta_1-2} + \beta_2(\beta_2 - 1)C_2(y)y^{\beta_2-2} + \beta_1 C_2'(y)y^{\beta_1-1} + \beta_2 C_1'(y)y^{\beta_2-1}.$$

Because β_2 and β_1 are the roots of $\mathcal{Q}(\cdot)$ in (2), the function $\chi(\cdot)$ in (EC.38) will indeed solve (EC.37a) if

$$\begin{pmatrix} y^{\beta_2} & y^{\beta_1} \\ \beta_2 y^{\beta_2+1} & \beta_1 y^{\beta_1+1} \end{pmatrix} \begin{pmatrix} C_2'(y) \\ C_1'(y) \end{pmatrix} = \begin{pmatrix} 0 \\ -\frac{2g(y,x)}{\sigma^2} \end{pmatrix} \quad (\text{EC.40})$$

for any $y \in (y_0, y_5)$ or from Cramer's rule if

$$\begin{pmatrix} C_2'(y) \\ C_1'(y) \end{pmatrix} = \begin{pmatrix} -\frac{2}{(\beta_2 - \beta_1)\sigma^2} g(y)y^{-\beta_2+1} \\ \frac{2}{(\beta_2 - \beta_1)\sigma^2} g(y)y^{-\beta_1-1} \end{pmatrix}. \quad (\text{EC.41})$$

Furthermore, it follows from (EC.37b) and (EC.38) that $C_1(y_0)y_0^{\beta_1} + C_2(y_0)y_0^{\beta_2} = 0$ and $C_1(y_5)y_5^{\beta_1} + C_2(y_5)y_5^{\beta_2} = 0$, and from (EC.37c) and (EC.39) that

$$\beta_1 C_1(y_0)y_0^{\beta_1} + \beta_2 C_2(y_0)y_0^{\beta_2} = 0 \quad \text{and} \quad \beta_1 C_1(y_5)y_5^{\beta_1} + \beta_2 C_2(y_5)y_5^{\beta_2} = 0.$$

For these four conditions to be satisfied, it must be that $C_1(y_0) = C_2(y_0) = C_1(y_5) = C_2(y_5) = 0$. By integration of (EC.41), it follows that y_0 and y_5 solve

$$\int_{y_0}^{y_5} g(\xi)\xi^{-\beta_1-1} d\xi = 0 \quad \text{and} \quad \int_{y_0}^{y_5} g(\xi)\xi^{-\beta_2-1} d\xi = 0. \quad (\text{EC.42})$$

Further, recalling (EC.36), for $(0, y_0) \cup (y_5, \infty)$ to be the stopping region, it must be that

$$\begin{cases} g(y) < 0, & 0 < y < y_0, \\ g(y) < 0, & y > y_5. \end{cases} \quad (\text{EC.43})$$

For (EC.43) to obtain, it is necessary that

$$\text{there exist } y_1 > 0 \text{ and } y_4 > y_1 \text{ such that } \begin{cases} g(y) < 0, & \forall y \in (0, y_1), \\ g(y) < 0, & \forall y \in (y_4, \infty). \end{cases} \quad (\text{EC.44})$$

and such that $0 < y_0 < y_1 < y_4 < y_5$. We can re-write (EC.42) as

$$\begin{cases} \int_{y_0}^{y_1} g(\xi)\xi^{-\beta_1-1}d\xi + \int_{y_1}^{y_4} g(\xi)\xi^{-\beta_1-1}d\xi + \int_{y_4}^{y_5} g(\xi)\xi^{-\beta_1-1}d\xi = 0, \\ \int_{y_0}^{y_1} g(\xi)\xi^{-\beta_2-1}d\xi + \int_{y_1}^{y_4} g(\xi)\xi^{-\beta_2-1}d\xi + \int_{y_4}^{y_5} g(\xi)\xi^{-\beta_2-1}d\xi = 0, \end{cases}$$

Because of (EC.44), for the above system to have a solution, it is necessary that

$$\int_{y_1}^{y_4} g(\xi)\xi^{-\beta_1-1}d\xi > 0, \quad \int_{y_1}^{y_4} g(\xi)\xi^{-\beta_2-1}d\xi > 0 \quad \int_0^\infty g(\xi)\xi^{-\beta_1-1}d\xi < 0 \quad \text{and} \quad \int_0^\infty g(\xi)\xi^{-\beta_1-1}d\xi < 0. \quad (\text{EC.45})$$

If these conditions hold, we can then define

$$y_\dagger := \inf \left\{ z \mid 0 \leq z < y_1 \text{ and } \int_z^{y_4} \frac{g(\xi)}{\xi^{\beta_1+1}} d\xi > 0 \right\} \text{ and } y^\dagger := \sup \left\{ z \mid 0 < z \leq y_1 \text{ and } \int_z^\infty \frac{g(\xi)}{\xi^{\beta_1+1}} d\xi < 0 \right\}. \quad (\text{EC.46})$$

The terms y_\dagger (resp., y^\dagger) can take the values 0 (resp., y_1) if $\int_0^{y_4} g(\xi)\xi^{-\beta_1-1}d\xi > 0$ (resp., $\int_{y_1}^\infty g(\xi)\xi^{-\beta_1-1}d\xi < 0$). Because $y_\dagger < y_1$ and $y^\dagger > 0$ by definition, it is clear that $y_\dagger < y^\dagger$ if $y_\dagger = 0$ and $y^\dagger = y_1$. If $y_\dagger > 0$ and $y^\dagger < y_1$, then by definition

$$\int_{y_\dagger}^{y_4} g(\xi)\xi^{-\beta_1-1}d\xi = 0 \quad \text{and} \quad \int_{y^\dagger}^\infty g(\xi)\xi^{-\beta_1-1}d\xi = 0;$$

this implies that $0 \leq y_\dagger < y^\dagger \leq y_1$. Consequently, if $u \in (y_\dagger, y^\dagger)$, then

$$\int_u^{y_4} g(\xi)\xi^{-\beta_1-1}d\xi \geq 0 \geq \int_u^\infty g(\xi)\xi^{-\beta_1-1}d\xi.$$

For a given $u \in (y_\dagger, y^\dagger)$, the function $v \mapsto \int_u^v g(\xi, x)\xi^{-\beta_1-1}d\xi$ is monotone decreasing on $[y_4(x), \infty)$.

From the previous inequality, there is a unique $v(u) \in [y_4(x), \infty)$ such that

$$\int_u^{v(u)} g(\xi)\xi^{-\beta_1-1}d\xi = 0. \quad (\text{EC.47})$$

The function $v(\cdot)$ is monotone increasing.

We define

$$G(u) := \int_u^{v(u)} g(\xi)\xi^{-\beta_2-1}d\xi. \quad (\text{EC.48})$$

We can check that, for $y_\dagger < u_1 < u_2 < y^\dagger$ that

$$G(u_2) - G(u_1) = \int_{u_1}^{u_2} g(\xi)\xi^{-\beta_2-1} \left\{ \left(\frac{v(\xi)}{\xi} \right)^{\beta_1-\beta_2} - 1 \right\} d\xi.$$

If $g(\cdot)$ is continuous on (y_\dagger, y^\dagger) , this is easily checked by differentiation of $G(\cdot)$ and $v(\cdot)$. If $g(\cdot)$ is not continuous on this interval, then one can use a more technical proof involving an approximation of derivatives. Because $g(\xi) < 0$ on $(0, y_1)$, $v(\xi) > \xi$ and $\beta_2 > 0 > \beta_1$, we have $G(u_2) > G(u_1)$, so $G(\cdot)$ is monotone increasing on (y_\dagger, y^\dagger) .

From (EC.42), there must exist a u , necessarily unique in (y_\dagger, y^\dagger) , such that $G(u) = 0$. Because $G(\cdot)$ is monotone increasing on (y_\dagger, y^\dagger) , it is necessary and sufficient that $G(y_\dagger) \leq 0 \leq G(y^\dagger)$. We compute $G(y_\dagger)$ and $G(y^\dagger)$ respectively:

- If $y_{\dagger} > 0$, then, by definition of y_{\dagger} and $v(\cdot)$, $v(y_{\dagger}) = y_{\dagger}$ and $G(y_{\dagger}) = \int_{y_{\dagger}}^{y_{\dagger}} g(\xi)\xi^{-\beta_2-1}d\xi$. If $y_{\dagger} = 0$, then $v(0) = \inf \{z > y_4 \mid \int_0^z g(\xi)\xi^{-\beta_1-1}d\xi \leq 0\}$ and $G(0) = \int_0^{v(0)} g(\xi)\xi^{-\beta_2-1}d\xi$.
- If $y_{\dagger} < y_1$, then by definition of y_{\dagger} and $v(\cdot)$ $v(y_{\dagger}) = \infty$ and $G(y_{\dagger}) = \int_{y_{\dagger}}^{\infty} g(\xi)\xi^{-\beta_2-1}d\xi$. If $y_{\dagger} = y_1$, then $v(y_1) = \sup \{x > y_4 \mid \int_{y_1}^x g(\xi)\xi^{-\beta_1-1}d\xi \geq 0\}$ and $G(y_1) = \int_{y_1}^{v(y_1)} g(\xi)\xi^{-\beta_2-1}d\xi$.

We then conclude:

Lemma 3 *There exists a pair of points $\{y_0, y_5\}$ satisfying eqs. (EC.42) and (EC.43) if and only if:*

- *the function $g(\cdot)$ satisfies eqs. (EC.44) and (EC.45);*
- *the inequalities $G(y_{\dagger}) \leq 0 \leq G(y^{\dagger})$ hold, whereby y_{\dagger} and y^{\dagger} are defined in eq. (EC.46) and $G(y_{\dagger})$ and $G(y^{\dagger})$ are defined by*

$$G(y_{\dagger}) = \begin{cases} \int_0^{v(0)} g(\xi)\xi^{-\beta_2-1}d\xi, & y_{\dagger} = 0, \\ \int_{y_{\dagger}}^{y_{\dagger}} g(\xi)\xi^{-\beta_2-1}d\xi, & y_{\dagger} > 0, \end{cases} \quad \text{and} \quad G(y^{\dagger}) = \begin{cases} \int_{y^{\dagger}}^{\infty} g(\xi)\xi^{-\beta_2-1}d\xi, & y^{\dagger} < y_1, \\ \int_{y_1}^{v(y_1)} g(\xi)\xi^{-\beta_2-1}d\xi, & y^{\dagger} = y_1. \end{cases}$$

with the function $v(\cdot)$ defined in (EC.47).

The pair $\{y_0, y_5\}$, when it exists, is unique.

The conditions in the Lemma ensures that the function

$$y \mapsto \chi(y) = \begin{cases} 0, & 0 < y \leq y_0, \\ \frac{2}{(\beta_2 - \beta_1)\sigma^2} \left[\int_{y_0}^y \frac{g(\xi)}{\xi^{\beta_1+1}} d\xi + \int_y^{y_5} \frac{g(\xi)}{\xi^{\beta_2+1}} d\xi \right], & y_0 \leq y \leq y_5, \\ 0, & y \geq y_5. \end{cases} \quad (\text{EC.49})$$

is continuously differentiable with bounded second-order derivatives that have left and right limits and solves the FBP (EC.37a)–(EC.37c). We now want to prove that this function coincides with the solution to the VI (16) and that the Ansatz 1 holds. It remains to prove that

$$\chi(y) \geq 0 \text{ for } y \in (y_0, y_5) \quad \text{and} \quad \mathcal{L}\chi(y) \geq g(y) \text{ for } y \in (0, y_0) \cup (y_5, \infty).$$

The second inequality is immediate because $\chi(y) = 0$ and $g(y) < 0$ in the stopping regions. It remains to prove $\chi(y) \geq 0$.

- We apply the first theorem of calculus to $t \mapsto \chi(y_0 + \varepsilon)$ and $u \mapsto \chi'(y_0 + t\varepsilon)$ on $[0, 1]$ to obtain

$$\chi(y_0 + \varepsilon) = \varepsilon^2 \int_0^1 \int_0^1 t\chi''(y_0 + t\varepsilon) dt du, \quad \varepsilon > 0.$$

It follows from (EC.37b)–(EC.37c) that

$$\mathcal{L}\chi(y_0+) = -\frac{1}{2}\sigma^2 y_0^2 \chi''(y_0+).$$

Besides, $\mathcal{L}\chi(y_0+) = g(y_0) < 0$. Hence, $\chi''(y_0+) > 0$. Therefore, for ε sufficiently small, $\chi''(y_0 + tu\varepsilon) > 0$ with $t, u \in (0, 1)$. Hence $\chi(y) > 0$ in the right vicinity of $y_0(x)$. Necessarily, $\chi(\cdot)$ has positive local maxima, possibly a maximum. A local maximum, say z , cannot be in (y_0, y_1) because $g(y) < 0$ for $y \in (y_0, y_1)$ and necessarily $\mathcal{L}\chi(z) > 0$. Thus, $\chi(\cdot)$ increases and is positive on (y_0, y_1) ; in particular, $\chi(y_1) > 0$.

- We follow a similar reasoning. We have

$$\chi(y_5 - \varepsilon) = \varepsilon^2 \int_0^1 \int_0^1 t\chi''(y_5 - tu\varepsilon) dt du, \quad \varepsilon > 0.$$

We proceed similarly obtaining that $\chi(y) > 0$ in the left vicinity of y_5 . Because $\mathcal{L}\chi(y) = g(y) < 0$ on (y_4, y_5) , it follows from the maximum principle that there is no local maximum in (y_4, y_5) .

Consequently, $\chi(\cdot)$ decreases and is positive on (y_4, y_5) ; in particular $\chi(y_4) > 0$.

- It follows from (EC.37a) and the inequality $y_0 < y_1 < y_4 < y_5$ that

$$\mathcal{L}\chi(y) = g(y) \text{ for all } y \in (y_1, y_4) \quad \text{with } \chi(y_1) > 0 \text{ and } \chi(y_4) > 0.$$

Given the definition of $\Gamma(\cdot)$ in Theorem 3, it is immediate that $\chi(\cdot) > \Gamma(\cdot)$ on (y_1, y_4) . Assume that the solution Γ to the Dirichlet problem

$$\begin{cases} \Gamma(y_1) = 0, \\ \mathcal{L}\Gamma(y) = g(y), \quad y_1 < y < y_4 \\ \Gamma(y_4) = 0, \end{cases} \quad (\text{EC.50})$$

is strictly positive on (y_1, y_4) . It immediately follows that χ is positive on (y_1, y_4) .

We conclude that the Ansatz 1 holds and that the FBP's solution coincides with the VI's.

Theorem 3 (Solution of the VI in (15) for $x \geq x^*$ and $\frac{\gamma}{1-\epsilon} < \beta_2$) *Assume that the conditions in Lemma 3 are satisfied. Further, assume that the solution Γ of the Dirichlet problem in eq. (EC.50) is strictly positive. Then, the function χ in eq. (EC.49) is differentiable and solves the VI (16).*

We omit the verification theorem, leveraging on the known connection between a VI's solution and the value function of optimal stopping.

EC.9 Solution of the VI (15) for $x < x^*$

Here, we introduce $g(y) := \pi(y, x) - \mathcal{L}\Phi(y)$, $y > \bar{y}_3$ and define $y_4 := \inf\{y > \bar{y}_3 \mid g(y) < 0 \text{ for } y > y_4\}$. If we assume that

$$y_4 > \bar{y}_3 > y_1, \quad (\text{EC.51})$$

then $g(y_4) = 0$.

We postulate the existence and uniqueness of a solution F to the FBP

$$F(y) = 0, \quad y < y_0, \quad (\text{EC.52a})$$

$$\mathcal{L}F(y) = \pi(y, x), \quad y_0 < y < y_5, \quad (\text{EC.52b})$$

$$F(y) = \Phi(y), \quad y > y_5, \quad (\text{EC.52c})$$

$$F(y_0) = 0 \text{ and } F'(y_0) = 0, \quad [\text{smooth fit at } y_0] \quad (\text{EC.52d})$$

$$F(y_5) = \Phi(y_5) \text{ and } F'(y_5) = \Phi'(y_5). \quad [\text{smooth fit at } y_5] \quad (\text{EC.52e})$$

Further, we assume that the following inequalities are satisfied:

$$y_0 < y_1 < \bar{y}_3 < y_4 < y_5. \quad (\text{EC.53})$$

To prove that the solution F solves the VI (15), it remains to prove that

$$\mathcal{L}F(y) \geq rA + \mathcal{Q}(\gamma)By^\gamma x^\epsilon, \quad y < y_0(x), \quad (\text{EC.54a})$$

$$\mathcal{L}F(y) \geq rA + \mathcal{Q}(\gamma)By^\gamma x^\epsilon, \quad y > y_5 \quad (\text{EC.54b})$$

$$F(y) \geq 0, \quad y_0 < y < y_1, \quad (\text{EC.54c})$$

$$F(y) \geq \Phi(y) \quad y_1 < y < y_4, \quad (\text{EC.54d})$$

$$F(y) \geq \Phi(y), \quad y_4 < y < y_5. \quad (\text{EC.54e})$$

It follows from the ordering $y_0 < y_1$ and from the definition of y_1 in (5b) that $rA + \mathcal{Q}(\gamma)By^\gamma x^\epsilon \leq 0$ for $y < y_0$. Because $\mathcal{L}F(y) = 0$ for $y < y_0$, it is immediate that the inequality (EC.54a) is satisfied. It follows that for $y > y_5 > \bar{y}_3$, $\mathcal{L}F(y) = \mathcal{L}\Phi(y) = rA + \mathcal{Q}(\gamma)By^\gamma x^\epsilon - g(y)$. Yet, $g(y) < 0$ for $y > y_5 > y_4$, so (EC.54b) is satisfied. The function $F(\cdot)$ is positive in the right vicinity of y_0 . Because $\mathcal{L}F(y) \leq 0$ on (y_0, y_1) , it follows from the maximum principle that $F(\cdot)$ has no local maximum in (y_0, y_1) : it is thus increasing, with (EC.54c) necessarily satisfied. Similarly, the function $(F - \Phi)(\cdot)$ is positive in the left vicinity of y_5 . Because $\mathcal{L}(F - \Phi)(y) = g(y) < 0$ on (y_4, y_5) , it has no local maximum in (y_4, y_5) and is decreasing on (y_4, y_5) taking positive values. It immediately follows that (EC.54e) is satisfied. Finally, assume that $\mathcal{L}\Gamma(y) = \pi(y, x)$ on (y_1, y_4) with boundary conditions $\Gamma(y_1) = 0$ and $\Gamma(y_4) = \Phi(y_4)$ has a unique solution, noted Γ and that $\Gamma(y) \geq \varphi(y)$ for $y \in (y_1, \bar{y}_3)$ and $\Gamma(y) \geq \Phi(y)$ for $y \in (\bar{y}_3, y_4)$. Because $F(\cdot)$ satisfies (EC.52b) on (y_1, y_4) with $F(y_1) > \Gamma(y_1)$ and $F(y_4) > \Gamma(y_4)$, we have $F(y) \geq \Gamma(y)$. As Γ is above the obstacle Φ , the function F is also above the obstacle, which proves (EC.54d) and more generally establishes that F solves the VI (15). Consequently, the interval (y_0, y_5) coincides with the continuation set $\{y > 0 | F(y) > \Phi(y)\}$.

EC.10 Proof of Proposition 1

Let π^a , φ^a , Φ^a and F^a denote the terms introduced in eqs. (1), (5a), (9) and (14'), respectively. It follows from (5b) that $y_1 \rightarrow 0$ for any x as $a \rightarrow 0$, so $\varphi^0(y, x) = By^\gamma x^\epsilon$ for any (x, y) . We can thus re-write ψ in eq. (3) as $\psi(y, x) = a/r + \varphi^0(y, x)$ and Ψ in (6) as

$$\Psi(y, x) = \frac{a}{r} + \sup_{\xi \geq 0} \{ \varphi^0(y, x + \xi) - k\xi \} = \frac{a}{r} + \Phi^0(y, x),$$

from eq. (9). Further, $\pi^a(y, x) = a + \pi^0(y, x)$. We can thus write G in eq. (8) as

$$G(y, x) = \sup_{\tau} \mathbb{E} \left[\int_0^{\tau} e^{-rt} a dt + \int_0^{\tau} e^{-rt} \pi^0(Y_t, x) dt + e^{-r\tau} \frac{a}{r} + e^{-r\tau} \Psi^0(Y_{\tau}, x) \middle| Y_0 = y \right].$$

It holds pathwise that $e^{-r\tau} = 1 - \int_0^{\tau} r e^{-rt} dt$. It thus follows from eq. (14') that

$$G(y, x) = \frac{a}{r} + \sup_{\tau} \mathbb{E} \left[\int_0^{\tau} e^{-rt} \pi^0(Y_t, x) dt + e^{-r\tau} \Psi^0(Y_{\tau}, x) \middle| Y_0 = y \right] = \frac{a}{r} + F^0(y, x).$$

Further, we note that $0 < y_0 < y_1$ while $y_1 \rightarrow 0$ as $a \rightarrow 0$. It follows that the lower free boundary y_0 vanishes. This completes the proof.