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Spectral characteristics of Harmonizable VARMA processes

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Abstract

Harmonizable processes form a wide class of nonstationary processes, which admit a convenient Fourier analysis and have spectral distributions characterized by correlated components. They are proved to be useful in many fields of application, for example, recently they were successfully applied in the analysis of replicated ElectroEncephaloGram signals for studying the brain connectivity. In this paper, we introduce a parametric form for these harmonizable processes, namely Harmonizable Vector AutoRegressive and Moving Average models (HVARMA). In the same spirit as of standard VARMA models, they are derived as a unique solution of a difference equation based on a properly defined concept of harmonizable noise. We exhibit their spectral characteristics and provide a way to generate realizations from a given HVARMA model.

keywords: Harmonizable processes, Loève spectrum, time series, VARMA models.

1 Introduction

Stationary processes are commonly used because a wide set of tools is available for their time and frequency domain analysis. However, there are many problems for which stationarity is an unacceptable limitation, and for which it is of interest to maintain a convenient and easy-to-interpret spectral analysis (e.g., econometrics, telecommunication, vibroacoustics, mechanics). Sometimes nonstationarity can be modeled by assuming that stationarity holds locally. This can be done, for example, using piecewise stationary or locally stationary processes. ARMA models with time-varying coefficients (Dahlhaus, 2012) provide a very powerful and convenient parametric class flexible enough to describe many real-life signals. Nevertheless, those models do not allow for capturing many complex dependencies that characterize some signals such as dependence in the frequency domain.

Harmonizable processes are a natural extension of stationary processes with a spectral (frequency) distribution, whose components are correlated. Their covariance function is bivariate and admits a two-dimensional Fourier decomposition (Loève spectrum). Hence, the study of this class of time series is possible through Fourier analysis methods.

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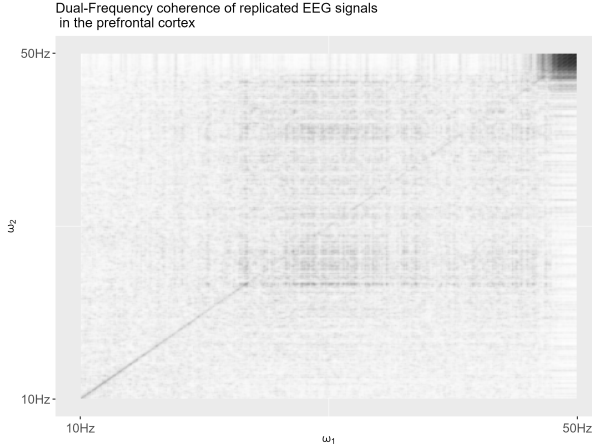


Figure 1: Nonparametric estimate of the dual-frequency coherence computed on replicated EEG data. The darker the color, the higher the coherence value.

Harmonizable processes have been studied in the literature over the past 70 years. We refer the reader in particular to the works of Loève (1948-1963) (see Cramér, 1961; Loève, 1978; Rozanov, 1959). Numerous applications of the concept of a harmonizable process have been developed in signal theory, in communications, in mechanics, etc. (see, e.g., Gardner et al., 2006; Napolitano, 2019; Serpedin et al., 2005; Setoodeh & Haykin, 2017). That is to say, in fields where filtering problems are common, and frequency domain methods are highly regarded by practitioners for describing or inferring the properties of noisy signals.

Recent applications of harmonizable time series have opened new research directions for EEG data (Aston et al., 2023; Gorrostieta et al., 2019). In these papers, the authors introduced nonparametric methods for estimating the Loève spectrum, along with some generalizations, including the spatially time-localized Loève spectrum (see Aston et al., 2023). In addition, they estimated dual-frequency coherence, a measure that captures the linear relationship between two frequencies and a pair of time series. This quantity, derived from Loève spectrum, provides insightful analysis of replicated EEG time series (Ombao & Pinto, 2022). Figure 1 shows this spectral dependence measure computed on EEG signals recorded from electrodes monitoring the prefrontal cortex activity (for details on the data, see Morán & Soriano, 2018)). We can observe many non-zero off-diagonal values that may be the key to understanding the complex dependence structure of these signals. The high-frequency content in the Beta (13-30Hz) and Gamma (above 32Hz) oscillation ranges appear to be potentially correlated with each other. This clearly suggests that models allowing for frequency domain dependencies are required to capture the complexity of these signals.

It should be noted, that the calculation of the Loève spectrum requires replicated observations. Moreover, there is currently no way to generate harmonizable time series with known spectral characteristics, which is essential to assess the numerical performance of new methods based on harmonizable processes. Developing parametric models and related statistical inference methods for harmonizable processes could address both challenges. Using parametric models can significantly reduce the number of replicated time series required for real data analysis compared to nonparametric methods. Furthermore, it could facilitate data generation whenever the spectral characteristics of these models are available. Such advancements would significantly expand the use of methods based on harmonizable processes. However, despite several publications on linear harmonizable time series (Mehlman, 1991, 1992), no parametric models have been

developed for these processes. This gap deprives researchers and practitioners of a robust set of tools for estimation, forecasting, data generation, and more.

Taking the above into account, this paper makes four key contributions (denoted later in this section as Contributions 1-4). First, we address a gap in the literature by introducing the first parametric VARMA-type model for harmonizable processes, which we refer to as HVARMA. Second, we exhibit their spectral characteristics. Third, we tackle the challenge of generating non-trivial realizations of harmonizable time series with a specified Loève spectrum. Finally, we present preliminary results of how to perform parameter estimation of a model with harmonizable noise.

Contributions 1 and 2 are addressed in the first four sections of this paper, where we lay the mathematical foundation for defining the HVARMA models (see Definition 5.1. Our approach draws inspiration from the work of Brockwell and Davis (1991). The HVARMA model is obtained as a solution of a difference equation based on appropriately defined harmonizable noise, which serves as innovations (see Section (6.5)). This harmonizable noise is the fundamental building block of our linear model. We assume that it is uncorrelated, but its variance is not constant over time, and the vector components of a multivariate harmonizable noise at any fixed time moment can be correlated. The formal definition of harmonizable noise, provided in Theorem (3.1), differs from that used in Mehlmán (1991, 1992) (see Definition 6.1 in Mehlmán (1991) and Definition 2.15 in Mehlmán (1992)).

Dealing with such a definition requires distinguishing the subtle differences between weakly and strongly harmonizable processes (see Proposition 2.1 and related remark). To ensure that we obtain an interpretable spectrum for our HVARMA model, we pay great attention to the spectral characterization of harmonizable noise in Section 3, and its filtered version (moving average) in Section 4, resulting in the multivariate linear harmonizable model. Furthermore, we establish necessary and sufficient conditions for strong harmonizability and the existence of a stochastic spectral density in Proposition 3.1 and Proposition 3.2.

Finally, in Theorem 5.1 we state the spectral characterization of HVARMA models. It gives the analytical form of the Loève spectral density matrix of the HVARMA model. Interestingly, it combines the spectral characteristics of the harmonizable noise and the spectral characteristics of the filter that are used to obtain the $\text{HVARMA}(p,q)$ model.

Section 6 refers to Contribution 3, in which we provide a methodology for generating realizations of harmonizable time series based on known spectral characteristics. In this section, we highlight the technical subtleties that must be addressed to generate harmonizable noise on a finite group. In particular, we discuss how to get harmonizable noise using a certain $\text{VARMA}(p',q')$ process. We will then use it to construct a $\text{HVARMA}(p,q)$ process. This gives us a subclass of HVARMA models, which we denote as $\text{HVARMA}(p,q) - (p',q')$. In Section 6.5, we describe in detail how to generate realizations from $\text{HVARMA}(p,q) - (p',q')$ and compare them with standard VARMA models.

Moreover, Figure 2 presents a realization of an usual $\text{VARMA}(3,2)$ process (top left panel), which was used to obtain three $\text{HVARMA}(3,2) - (p',q')$ series with different incremental processes (see the other three panels). As can easily be seen, HVARMA realizations based on the same VARMA can vary significantly from each other (as well as from the input VARMA process) and consequently can be used to model very different phenomena. They also exhibit complex nonstationary behaviors.

Contribution 4 is addressed in Section 5.2, where we discuss how to estimate the parameters of the HVARMA model on a simple example of an univariate HVARMA (1,0) model. Here we do not assume any model for the harmonizable noise except its heteroskedasticity. We do not consider the estimation problem of the variance of the harmonizable noise (see White, 1958). This is the subject of another

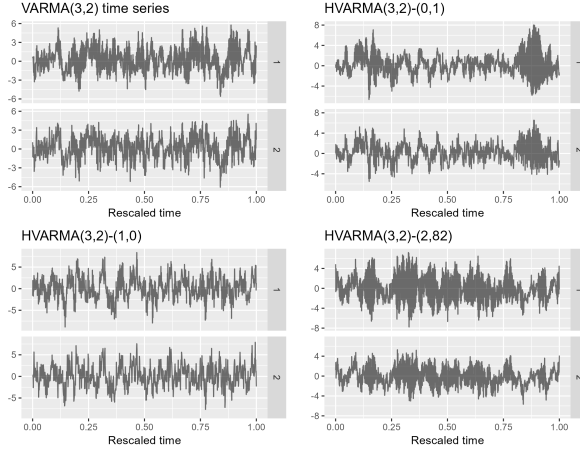


Figure 2: Example realizations of an usual $VARMA(3, 2)$ of dimension 2 and its modified versions $HVARMA(3, 2) - (p', q')$ with different incremental processes.

project.

In summary, this paper is organized as follows. In Section 2 a brief description of the spectral representation of multivariate harmonizable time series is presented. In Section 3, the notion of multivariate harmonizable noise (or innovation) is introduced and its spectrum characterized. This is the basic building block of linear models. Additionally, necessary and sufficient conditions for strong harmonizability and for the existence of a stochastic spectral density are given. Section 4 is devoted to defining multivariate linear harmonizable time series as the moving average of harmonizable noise. Section 5 deals with multivariate harmonizable autoregressive moving average (VARMA) models. Section 5.2 introduces the idea of estimating the parameters of an univariate HVARMA model. Section 6 details a proper methodology for generating realizations of harmonizable time series from known spectral characteristics. Finally, a short discussion is provided in Section 7. All proofs are deferred to Appendix.

2 Background

In this section, we introduce the basic notations and recall some definitions and properties of harmonizable processes. For more details, we refer the reader to Loève (1978), Mehlmán (1991), Rao (1982), and Rozanov (1959).

2.1 Notation

Let d be a fixed non-zero integer. From now on, \mathbb{C}^d denotes the set of d -dimensional complex column vectors and $\mathbb{T} \stackrel{\text{def}}{=} \mathbb{R}/2\pi$ represented by $(-\pi, \pi]$ is the topological dual group of \mathbb{Z} . In this paper, we assume that the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ is sufficiently large so that there exists a sequence of uncorrelated random d -dimensional vectors with a variance matrix equal to the identity matrix of size $d \times d$ denoted as $\mathbb{I}_{d \times d}$, and which are not correlated with the d -dimensional time series $(\mathbf{Z}_n)_{n \in \mathbb{Z}}$ studied in the following. Otherwise, one can always replace any probability space $(\Omega, \mathcal{F}, \mathbb{P})$ by an enlarged probability space with this property. For the construction of such an enlargement, we refer the reader to Loève (1978).

Hereafter, we consider only complex-valued random variables defined on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$, that have zero mean and are mean square integrable. For simplicity, in what follows, when we write "time

series” we mean second order time series with zero mean, i.e., a sequence of square integrable random vectors with zero mean (or variables) on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$. We denote by $\mathcal{H} = L^2_{\mathbb{C}}(\mathbb{P})$ the complex Hilbert space of square integrable complex-valued random variables with scalar product $\langle x|y \rangle_{\mathcal{H}} = \mathbb{E}(x\bar{y})$. The inflated space $\mathcal{H}^d \stackrel{\text{def}}{=} \oplus_1^d \mathcal{H} = L^2_{\mathbb{C}^d}(\mathbb{P})$ is a Hilbert space with the scalar product $\langle \mathbf{x}|\mathbf{y} \rangle_{\mathcal{H}^d} \stackrel{\text{def}}{=} \mathbb{E}(\mathbf{x}^\top \bar{\mathbf{y}})$. If \mathbf{x} and \mathbf{y} are two zero mean random vectors in $L^2_{\mathbb{C}^d}(\mathbb{P})$, then the covariance between \mathbf{x} and \mathbf{y} is the $d \times d$ -matrix $\text{Cov}(\mathbf{x}, \mathbf{y}) = \mathbb{E}(\mathbf{x}\mathbf{y}^*) = (\text{cov}(x_i, y_j))_{1 \leq i, j \leq d}$, where $\mathbf{x} = (x_1, \dots, x_d)^\top$ and $\mathbf{y} = (y_1, \dots, y_d)^\top \in \mathcal{H}^d$.

Here and in the following, the superscript \top denotes the transpose operator, and the superscript $'^*$ is the adjoint operator, i.e., the conjugate transpose operator.

Let $\mathbf{X} = (\mathbf{X}_n)_{n \in \mathbb{Z}}$ be a d -variate time series and let $\mathcal{H}(\mathbf{X}_n)$ be the Hilbert subspace of \mathcal{H} generated by $X_{n,1}, \dots, X_{n,d}$, i.e., $\mathcal{H}(\mathbf{X}_n) \stackrel{\text{def}}{=} \{\mathbf{a}^\top \mathbf{X}_n : \mathbf{a} \in \mathbb{C}^d\}$, where $\mathbf{X}_n = (X_{n,1}, \dots, X_{n,d})^\top$. Notice that $\dim \mathcal{H}(\mathbf{X}_n) \leq d$. Moreover, let $\mathcal{H}^d(\mathbf{X}_n)$ be the Hilbert subspace of \mathcal{H}^d generated by \mathbf{X}_n , i.e., $\mathcal{H}^d(\mathbf{X}_n) \stackrel{\text{def}}{=} \{\mathbf{A}\mathbf{X}_n : \mathbf{A} \in \mathbb{C}^{d \times d}\}$, $\mathcal{H}^d(\mathbf{X}_n) = \mathcal{H}(\mathbf{X}_n) \oplus \dots \oplus \mathcal{H}(\mathbf{X}_n)$ and $\dim \mathcal{H}^d(\mathbf{X}_n) = d \times \dim \mathcal{H}(\mathbf{X}_n) \leq d^2$.

In the following, we also consider $\mathcal{H}(\mathbf{X})$ a Hilbert subspace of \mathcal{H} generated by the Hilbert subspaces $\mathcal{H}(\mathbf{X}_n)$, $n \in \mathbb{Z}$, and $\mathcal{H}^d(\mathbf{X})$ the Hilbert subspace of \mathcal{H}^d generated by the subspaces $\mathcal{H}^d(\mathbf{X}_n)$, $n \in \mathbb{Z}$. If we replace \mathbf{X} by a d -variate white noise $\boldsymbol{\varepsilon} = (\boldsymbol{\varepsilon}_n)_{n \in \mathbb{Z}}$, then the Hilbert subspaces $\mathcal{H}(\boldsymbol{\varepsilon}_n)$, $n \in \mathbb{Z}$, are orthogonal with the same dimension equal to the rank of the covariance matrix $\boldsymbol{\Sigma}_{\boldsymbol{\varepsilon}}^2 = \text{Var}(\boldsymbol{\varepsilon}_n)$. If, in addition, the d -variate white noise is pure (see Definition 3.1), then the family $\{\boldsymbol{\varepsilon}_{n,1}, \dots, \boldsymbol{\varepsilon}_{n,d}\}$ is the orthonormal basis of $\mathcal{H}(\boldsymbol{\varepsilon}_n)$. Thus, $\dim \mathcal{H}(\boldsymbol{\varepsilon}_n) = d$ and $\dim \mathcal{H}^d(\boldsymbol{\varepsilon}_n) = d^2$.

2.2 Multivariate harmonizable time series

Definition 2.1 A d -variate time series $\mathbf{X} = (\mathbf{X}_n)_{n \in \mathbb{Z}}$ is called harmonizable when there exists a (σ -additive) vector measure $\boldsymbol{\mu}_{\mathbf{X}} : \mathcal{B}(\mathbb{T}) \rightarrow \mathcal{H}^d$, such that

$$\mathbf{X}_n = \int_{\mathbb{T}} e^{in\lambda} \boldsymbol{\mu}_{\mathbf{X}}(d\lambda).$$

The measure $\boldsymbol{\mu}_{\mathbf{X}}$ is unique, it is called stochastic spectral measure of the time series \mathbf{X} . The integral with respect to the vector measure $\boldsymbol{\mu}_{\mathbf{X}}$ is in the sense of Dunford and Schwartz (see chapter IV, section 10, Dunford & Schwartz, 1957).

In the following, for a harmonizable time series \mathbf{X} , we will consider the Hilbert subspace $\mathcal{H}(\boldsymbol{\mu}_{\mathbf{X}})$ of \mathcal{H} generated by $\mu_{\mathbf{X},j}(A)$, $j = 1, \dots, d$, $A \in \mathcal{B}(\mathbb{T})$. Then $\mathcal{H}(\boldsymbol{\mu}_{\mathbf{X}}) = \mathcal{H}(\mathbf{X})$ and $\mathcal{H}^d(\boldsymbol{\mu}_{\mathbf{X}}) = \mathcal{H}^d(\mathbf{X})$ (Rao, 1982).

A harmonizable time series can also be characterized by a dependence structure in the frequency domain. More specifically, we have the following result.

Proposition 2.1 (Rao, 1982; Rozanov, 1959) A d -variate time series \mathbf{X} is harmonizable if and only if there exists a positive semi-definite bimeasure $\mathbf{M}_{\mathbf{X}} : \mathcal{B}(\mathbb{T}) \times \mathcal{B}(\mathbb{T}) \rightarrow \mathbb{C}^{d \times d}$ (the set function $\mathbf{M}_{\mathbf{X}}$ is σ -additive with respect to each of its components) such that,

$$\mathbb{E}(\mathbf{X}_m \mathbf{X}_n^*) = \int \int_{\mathbb{T}^2} e^{i(m\lambda_1 - n\lambda_2)} \mathbf{M}_{\mathbf{X}}(d\lambda_1, d\lambda_2).$$

The bimeasure $\mathbf{M}_{\mathbf{X}}$ is unique, and it is called the spectral bimeasure of the time series \mathbf{X} . The above integral with respect to the bimeasure $\mathbf{M}_{\mathbf{X}}$ is a Morse-Transue integral (Rao, 1982; Rozanov, 1959). Furthermore,

$$\mathbf{M}_{\mathbf{X}}(A, B) = \text{Cov}(\boldsymbol{\mu}_{\mathbf{X}}(A), \boldsymbol{\mu}_{\mathbf{X}}(B)) = \mathbb{E}(\boldsymbol{\mu}_{\mathbf{X}}(A) \boldsymbol{\mu}_{\mathbf{X}}^*(B))$$

for any A and $B \in \mathcal{B}(\mathbb{T})$.

Remark 2.1 • The bimeasure $\mathbf{M}_{\mathbf{X}}$ is not necessarily extendible as a $\mathbb{C}^{d \times d}$ -valued measure on \mathbb{T}^2 .

When the bimeasure $\mathbf{M}_{\mathbf{X}}$ is extendible as a $\mathbb{C}^{d \times d}$ -valued measure on \mathbb{T}^2 , the time series is called strongly harmonizable (Loève harmonizable), otherwise it is called weakly harmonizable.

- Stationary (wide-sense or second order) time series are strongly harmonizable. A harmonizable time series is stationary if and only if its stochastic spectral measure is orthogonally scattered, i.e., its spectral bimeasure is concentrated on the main diagonal of \mathbb{T}^2 : $\mathbf{M}_{\mathbf{X}}(A, B) = \mathbb{E}(\boldsymbol{\mu}_{\mathbf{X}}(A)\boldsymbol{\mu}_{\mathbf{X}}^*(B)) = \mathbf{0}_{d \times d}$, when $A \cap B = \emptyset$, $A, B \in \mathcal{B}(\mathbb{T})$. Then the spectral measure $\mathbf{m}_{\mathbf{X}}$ on \mathbb{T} of the stationary time series \mathbf{X} satisfies $\mathbf{m}_{\mathbf{X}}(A) = \mathbf{M}_{\mathbf{X}}(A, \mathbb{T}) = \mathbf{M}_{\mathbf{X}}^*(\mathbb{T}, A)$ for any $A \in \mathcal{B}(\mathbb{T})$.

3 Multivariate harmonizable noise or innovation

For the sake of clarity, we first recall the definition of a white noise $\boldsymbol{\varepsilon} \stackrel{\text{def}}{=} (\boldsymbol{\varepsilon}_n)_{n \in \mathbb{Z}}$. Then, we state a necessary and sufficient condition for a time series $\mathbf{Z} \stackrel{\text{def}}{=} (\mathbf{Z}_n)_{n \in \mathbb{Z}}$ to be a harmonizable noise, and we characterize its stochastic spectral measure. Finally, we establish necessary and sufficient conditions for \mathbf{Z} to be strongly harmonizable, and to admit a stochastic spectral density.

Definition 3.1 (White noise) *A d -variate time series $\boldsymbol{\varepsilon} \stackrel{\text{def}}{=} (\boldsymbol{\varepsilon}_n)_{n \in \mathbb{Z}}$ is a white noise when it has zero mean and $\text{Cov}(\boldsymbol{\varepsilon}_m, \boldsymbol{\varepsilon}_n) = \mathbb{E}(\boldsymbol{\varepsilon}_m \boldsymbol{\varepsilon}_n^*) = \boldsymbol{\Sigma}_{\boldsymbol{\varepsilon}}^2 \mathbf{1}_{\{m=n\}}$ for some hermitian positive semi-definite $d \times d$ -matrix $\boldsymbol{\Sigma}_{\boldsymbol{\varepsilon}}^2$. A d -variate white noise is a stationary time series. In the following, a d -variate white noise is said to be a d -variate pure white noise when $\boldsymbol{\Sigma}_{\boldsymbol{\varepsilon}}^2 = \mathbb{I}_{d \times d}$ (identity $d \times d$ -matrix).*

We consider the following condition for a d -variate time series $\mathbf{Z} \stackrel{\text{def}}{=} (\mathbf{Z}_n)_{n \in \mathbb{Z}}$, whose covariance matrix is time-varying.

- (Ha)** $\mathbb{E}(\mathbf{Z}_n) = \mathbf{0}_d$ and $\text{Cov}(\mathbf{Z}_m, \mathbf{Z}_n) = \mathbb{E}(\mathbf{Z}_m \mathbf{Z}_n^*) = \boldsymbol{\Sigma}_{\mathbf{Z}_n}^2 \mathbf{1}_{\{m=n\}}$,
where the components of the covariance matrices $\boldsymbol{\Sigma}_{\mathbf{Z}_n}^2$, $n \in \mathbb{Z}$, are uniformly bounded, that is,
 $\sup_{n,j} \mathbb{E}(|Z_{n,j}|^2) < \infty$.

Below we establish a fundamental building block for our results, stating a necessary and sufficient condition for the time series to satisfy **(Ha)**.

Lemma 3.1 (Fundamental Lemma) *A d -variate time series $\mathbf{Z} \stackrel{\text{def}}{=} (\mathbf{Z}_n)_{n \in \mathbb{Z}}$ fulfills condition **(Ha)** if and only if there exists a d -variate white noise $\boldsymbol{\varepsilon} \stackrel{\text{def}}{=} (\boldsymbol{\varepsilon}_n)_{n \in \mathbb{Z}}$ and a family of $d \times d$ -matrices $\mathbf{S}_n = (s_{n,j,k})_{j,k=1,\dots,d}$ such that $\sup_{n,j,k} |s_{n,j,k}| < \infty$ and $\mathbf{Z}_n = \mathbf{S}_n \boldsymbol{\varepsilon}_n$, $n \in \mathbb{Z}$.*

Note that (\mathbf{Z}_n) is not necessarily white noise, since the variance matrix of \mathbf{Z}_n may depend on n .

Remark 3.1 (i) In Lemma 3.1 the white noise $(\boldsymbol{\varepsilon}_n)_{n \in \mathbb{Z}}$ is not unique and it can be selected as a d -variate pure white noise.

- (ii) $\text{Var}(\mathbf{Z}_n) = \mathbf{S}_n \boldsymbol{\Sigma}_{\boldsymbol{\varepsilon}}^2 \mathbf{S}_n^*$. In the case of a pure white noise, we have $\text{Var}(\mathbf{Z}_n) = \mathbf{S}_n \mathbf{S}_n^*$ and $\mathbb{E}(\mathbf{Z}_n^\top \overline{\mathbf{Z}_n}) = \text{tr}(\mathbf{S}_n \mathbf{S}_n^*)$.

Below we state the harmonizability of the time series $(\mathbf{Z}_n)_{n \in \mathbb{Z}}$ satisfying the condition **(Ha)**. The proof of the following theorem relies on the construction of a bounded linear operator $\Xi : \mathcal{H}^d(\boldsymbol{\varepsilon}) \rightarrow \mathcal{H}^d(\mathbf{Z})$ such that $\mathbf{Z}_n = \Xi \boldsymbol{\varepsilon}_n$, where the time series $(\boldsymbol{\varepsilon}_n)_{n \in \mathbb{Z}}$ is stationary. For more details, we refer the reader to Section 8, proof of Theorem 3.1.

Theorem 3.1 (Harmonizable noise or innovation) *A d -variate time series $(\mathbf{Z}_n)_{n \in \mathbb{Z}}$ fulfilling condition (Ha) is harmonizable. Its stochastic spectral measure $\boldsymbol{\mu}_{\mathbf{Z}}$ defined on $\mathcal{B}(\mathbb{T})$ satisfies*

$$\boldsymbol{\mu}_{\mathbf{Z}}(A) = \frac{1}{2\pi} \sum_{n \in \mathbb{Z}} \left(\int_A e^{-in\lambda} d\lambda \right) \mathbf{Z}_n, \quad (1)$$

where the convergence is in $\mathcal{H}^d(\mathbf{Z})$. Conversely, any d -variate harmonizable time series $(\mathbf{Z}_n)_{n \in \mathbb{Z}}$ whose stochastic spectral measure satisfies relation (1), satisfies the condition (Ha). In what follows, such a time series will be referred to as d -variate harmonizable noise.

Remark 3.2 (i) Note that the spectral bimeasure $\mathbf{M}_{\mathbf{Z}}$ of $(\mathbf{Z}_n)_{n \in \mathbb{Z}}$ satisfies

$$\mathbf{M}_{\mathbf{Z}}(A, B) = \frac{1}{4\pi^2} \sum_{n \in \mathbb{Z}} \left(\int_A \int_B e^{-in(\lambda_1 - \lambda_2)} d\lambda_1 d\lambda_2 \right) \boldsymbol{\Sigma}_{\mathbf{Z}_n}^2,$$

where the convergence of the series is in \mathbb{R} .

(ii) If $\boldsymbol{\varepsilon} \stackrel{\text{def}}{=} (\boldsymbol{\varepsilon}_n)_{n \in \mathbb{Z}}$ is a d -variate white noise, then it is harmonizable noise and it is stationary. The stochastic spectral measure $\boldsymbol{\mu}_{\boldsymbol{\varepsilon}}$ is orthogonally scattered in the sense that $\mathbb{E}[\boldsymbol{\mu}_{\boldsymbol{\varepsilon}}(A)\boldsymbol{\mu}_{\boldsymbol{\varepsilon}}^*(B)] = \mathbf{0}_{d \times d}$ when $A \cap B = \emptyset$. It admits a spectral measure $\mathbf{m}_{\boldsymbol{\varepsilon}}$ defined on $\mathcal{B}(\mathbb{T})$ which satisfies

$$\mathbf{m}_{\boldsymbol{\varepsilon}}(A \cap B) = \mathbf{M}_{\boldsymbol{\varepsilon}}(A, B) = \mathbb{E}[\boldsymbol{\mu}_{\boldsymbol{\varepsilon}}(A)\boldsymbol{\mu}_{\boldsymbol{\varepsilon}}^*(B)], \quad A, B \in \mathcal{B}(\mathbb{T}).$$

(iii) When $\boldsymbol{\varepsilon} \stackrel{\text{def}}{=} (\boldsymbol{\varepsilon}_n)_{n \in \mathbb{Z}}$ is a d -variate pure white noise, its spectral measure $\mathbf{m}_{\boldsymbol{\varepsilon}}$ has a density function with respect to Lebesgue measure on \mathbb{T} , which coincides with the $\mathbb{C}^{d \times d}$ -valued constant function $\frac{1}{2\pi} \mathbb{I}_{d \times d}$. Every function $f \in L^2_{\mathbb{C}}(\mathbb{T})$ is $\boldsymbol{\mu}_{\boldsymbol{\varepsilon}}$ -integrable and

$$\left\| \int_{\mathbb{T}} f(\lambda) \boldsymbol{\mu}_{\boldsymbol{\varepsilon}}(d\lambda) \right\|_{\mathcal{H}^d}^2 = \text{tr} \left(\mathbb{E} \left[\left(\int_{\mathbb{T}} f(\lambda) \boldsymbol{\mu}_{\boldsymbol{\varepsilon}}(d\lambda) \right) \left(\int_{\mathbb{T}} f(\lambda) \boldsymbol{\mu}_{\boldsymbol{\varepsilon}}(d\lambda) \right)^* \right] \right) = \frac{d}{2\pi} \int_{\mathbb{T}} |f(\lambda)|^2 d\lambda.$$

(iv) Consider a d -variate harmonizable noise \mathbf{Z} and a d -variate pure white noise $\boldsymbol{\varepsilon}$ associated to \mathbf{Z} in Lemma 3.1. Applying the previous remark (iii) and Dunford and Schwartz (Theorem 3.2.19 1957), every function $f \in L^2(\mathbb{T})$ is integrable with respect to $\boldsymbol{\mu}_{\mathbf{Z}} = \Xi \boldsymbol{\mu}_{\boldsymbol{\varepsilon}}$, where Ξ is the bounded linear operator defined in the proof of Theorem 3.1. Furthermore,

$$\left\| \int_{\mathbb{T}} f(\lambda) \boldsymbol{\mu}_{\mathbf{Z}}(d\lambda) \right\|_{\mathcal{H}^d}^2 \leq \sup_{n,j,k} |s_{n,j,k}|^2 \left\| \int_{\mathbb{T}} f(\lambda) \boldsymbol{\mu}_{\boldsymbol{\varepsilon}}(d\lambda) \right\|_{\mathcal{H}^d}^2 = \frac{\zeta^2 d}{2\pi} \int_{\mathbb{T}} |f(\lambda)|^2 d\lambda. \quad (2)$$

where $\zeta^2 = \sup_{n,j,k} |s_{n,j,k}|^2$. Consequently, the stochastic spectral measure $\boldsymbol{\mu}_{\mathbf{Z}}$ is absolutely continuous with respect to Lebesgue measure on \mathbb{T} . Nevertheless, this does not imply that there exists a function $\boldsymbol{\Delta}_{\mathbf{Z}} : \mathbb{T} \rightarrow \mathcal{H}^d$ that is integrable in some sense (Pettis or Bochner, Dunford and Schwartz (see 1957)) and such that

$$\boldsymbol{\mu}_{\mathbf{Z}}(A) = \int_A \boldsymbol{\Delta}_{\mathbf{Z}}(\lambda) d\lambda.$$

Neither that the spectral bimeasure $\mathbf{M}_{\mathbf{Z}}$ is extendible to a $\mathbb{C}^{d \times d}$ -valued measure on \mathbb{T}^2 (see Proposition 3.1).

Before presenting other theoretical results, we give some elementary examples of harmonizable noises.

Example 3.1 Let $\boldsymbol{\varepsilon} \stackrel{\text{def}}{=} (\boldsymbol{\varepsilon}_n)_{n \in \mathbb{Z}}$ be a d -variate white noise. Consider the following time series:

- (i) $\mathbf{Z}_n^{(1)} \stackrel{\text{def}}{=} \boldsymbol{\varepsilon}_n$ for any $n \in \mathbb{Z}$.
- (ii) $\mathbf{Z}_n^{(2)} \stackrel{\text{def}}{=} \boldsymbol{\varepsilon}_n$ for $n_1 \leq n \leq n_2$, and $\mathbf{Z}_n^{(2)} \stackrel{\text{def}}{=} 0$ otherwise, $n_1 < n_2$ being fixed in \mathbb{Z} (Rao, 1982).
- (iii) $\mathbf{Z}_n^{(3)} \stackrel{\text{def}}{=} \boldsymbol{\varepsilon}_n$ for $n \geq 0$, and $\mathbf{Z}_n^{(3)} \stackrel{\text{def}}{=} 0$ otherwise (Rao, 1982).
- (iv) Let $d = 2$ and $\mathbf{S}_n^{(4)} \stackrel{\text{def}}{=} \begin{pmatrix} 1 & 0 \\ 0 & 1 + (-1)^n \end{pmatrix}$, $\mathbf{S}_n^{(5)} \stackrel{\text{def}}{=} \begin{pmatrix} 1 & 0 \\ 0 & 1 \wedge |n|^{-1} \end{pmatrix}$. Set $\mathbf{Z}_n^{(i)} \stackrel{\text{def}}{=} \mathbf{S}_n^{(i)} \boldsymbol{\varepsilon}_n$, for $i = 4, 5$.
- (v) Let $\mathbf{X} \stackrel{\text{def}}{=} (\mathbf{X}_n)_{n \in \mathbb{Z}}$ be a $L_{\mathbb{C}^d}^2(\mathbb{P})$ -bounded time series. Consider the innovation series $\mathbf{Z}^{(6)} = (\mathbf{Z}_n^{(6)})_{n \in \mathbb{Z}}$ defined by $\mathbf{Z}_{n,j}^{(6)} \stackrel{\text{def}}{=} X_{n,j} - P_{n-1} X_{n,j}$, where $P_{n-1} X_{n,j}$ is the one-step prediction of $X_{n,j}$, i.e., the orthogonal projection of $X_{n,j}$ onto the sub-Hilbert space $\mathcal{H}_{n-1}(\mathbf{X})$ of $\mathcal{H}(\mathbf{X}) \subset L_{\mathbb{C}}^2(\mathbb{P})$ generated by $\{X_{p,k} : p \leq n-1, k = 1, \dots, d\}$.

Then the series $\mathbf{Z}^{(i)} = (\mathbf{Z}_n^{(i)})_{n \in \mathbb{Z}}$, $i = 1, \dots, 6$ are d -variate harmonizable noises.

Next, we state a condition for the d -variate time series $(\mathbf{Z}_n)_{n \in \mathbb{Z}}$ to be strongly harmonizable.

Proposition 3.1 (Strong harmonizability) *A d -variate harmonizable noise $(\mathbf{Z}_n)_{n \in \mathbb{Z}}$ is strongly harmonizable if and only if*

$$\boldsymbol{\Sigma}_{\mathbf{Z}_n}^2 = \int_{\mathbb{T}} e^{in\lambda} \mathbf{n}_{\mathbf{Z}}(d\lambda), \quad (3)$$

where $\mathbf{n}_{\mathbf{Z}}$ is some $\mathbb{C}^{d \times d}$ -valued measure on \mathbb{T} . In this case, the measure $\mathbf{n}_{\mathbf{Z}}$ is unique, for every $A \in \mathcal{B}(\mathbb{T})$

$$\mathbf{n}_{\mathbf{Z}}(A) = \iint_{\mathbb{T}^2} \mathbb{I}_A(\lambda_1 + \lambda_2) \mathbf{M}_{\mathbf{Z}}(d\lambda_1, d\lambda_2), \quad (4)$$

and for every $E \in \mathcal{B}(\mathbb{T}^2)$ we have

$$\mathbf{M}_{\mathbf{Z}}(E) = \frac{1}{2\pi} \iint_{\mathbb{T}^2} \mathbb{I}_E(\lambda - \alpha, \alpha) \mathbf{n}_{\mathbf{Z}}(d\lambda) d\alpha. \quad (5)$$

\mathbb{I}_E is the indicator function of the set E , that is, $\mathbb{I}_E(\lambda) = 1$ if $\lambda \in E$ and $\mathbb{I}_E(\lambda) = 0$ otherwise.

Remark 3.3 (i) The measure $\mathbf{n}_{\mathbf{Z}}$ in Proposition 3.1 satisfies

$$\mathbf{n}_{\mathbf{Z}}^*(A) = \mathbf{n}_{\mathbf{Z}}(-A) = \mathbf{n}_{\mathbf{Z}}(A), \quad A \in \mathcal{B}(\mathbb{T}).$$

Indeed

$$\begin{aligned} \mathbf{M}_{\mathbf{Z}}^*(d\lambda_1, d\lambda_2) &= \mathbf{M}_{\mathbf{Z}}(d\lambda_2, d\lambda_1) \implies \\ \mathbf{n}_{\mathbf{Z}}^*(A) &= \iint_{\mathbb{T}^2} \mathbb{I}_A(\lambda_1 + \lambda_2) \mathbf{M}_{\mathbf{Z}}^*(d\lambda_1, d\lambda_2) = \iint_{\mathbb{T}^2} \mathbb{I}_A(\lambda_1 + \lambda_2) \mathbf{M}_{\mathbf{Z}}(d\lambda_2, d\lambda_1) = \mathbf{n}_{\mathbf{Z}}(A) \end{aligned}$$

and

$$\begin{aligned} 0 \leq \mathbf{v} \boldsymbol{\Sigma}_{\mathbf{Z}_n}^2 \mathbf{v}^* &= (\mathbf{v} \boldsymbol{\Sigma}_{\mathbf{Z}_n}^2 \mathbf{v}^*)^* \implies \\ \int_{\mathbb{T}} e^{in\lambda} (\mathbf{v} \mathbf{n}_{\mathbf{Z}}(d\lambda) \mathbf{v}^*) &= \int_{\mathbb{T}} e^{-in\lambda} (\mathbf{v} \mathbf{n}_{\mathbf{Z}}(d\lambda) \mathbf{v}^*)^* = \int_{\mathbb{T}} e^{-in\lambda} (\mathbf{v} \mathbf{n}_{\mathbf{Z}}^*(d\lambda) \mathbf{v}^*) = \int_{\mathbb{T}} e^{in\lambda} (\mathbf{v} \mathbf{n}_{\mathbf{Z}}^*(-d\lambda) \mathbf{v}^*), \quad n \in \mathbb{Z}. \end{aligned}$$

- (ii) A white noise $(\boldsymbol{\varepsilon}_n)_{n \in \mathbb{Z}}$ is strongly harmonizable, and the measure $\mathbf{n}_{\boldsymbol{\varepsilon}}$ coincides with $\boldsymbol{\Sigma}_{\boldsymbol{\varepsilon}}^2 \delta_0$, where δ_0 is the Dirac measure at 0.

(iii) Following the notations of Examples 3.1, the d -variate time series $\mathbf{Z}^{(3)}$ is weakly harmonizable and not strongly harmonizable (Rao (1982, Sec.2 p.301) and Helson and Lowdenslager (1958, p.183)), while $\mathbf{Z}^{(1)}$ is a white noise. Applying Proposition 3.1, we establish that the d -variate time series $\mathbf{Z}^{(2)}$, $\mathbf{Z}^{(4)}$ and $\mathbf{Z}^{(5)}$ are strongly harmonizable noises.

– For $\mathbf{Z}^{(2)}$, let $f_{\mathbf{Z}^{(2)}}(\lambda) \stackrel{\text{def}}{=} \sum_{k=n_1}^{n_2} e^{i\lambda k}$, then $\text{Var}(\mathbf{Z}_n^{(2)}) = \frac{1}{2\pi} \int_{\mathbb{T}} e^{-i\lambda n} f_{\mathbf{Z}^{(2)}}(\lambda) d\lambda$ and hence $\mathbf{n}_{\mathbf{Z}^{(2)}}(d\lambda) = \frac{1}{2\pi} f_{\mathbf{Z}^{(2)}}(\lambda) d\lambda$.

– For $\mathbf{Z}^{(4)}$, notice that

$$(1 + (-1)^n)^2 = (1 + \cos(\pi n))^2 = 2 + 2\cos(\pi n) = 2 + e^{i\pi n} + e^{-i\pi n}.$$

Then we obtain that

$$\mathbf{n}_{\mathbf{Z}^{(4)}} = \Sigma_{\varepsilon}^2 \begin{pmatrix} \delta_0 \\ 2\delta_0 + \delta_{\pi} + \delta_{-\pi} \end{pmatrix}.$$

– For $\mathbf{Z}^{(5)}$ we have

$$(1 \wedge |n|^{-1})^2 = 1 \wedge |n|^{-2}$$

Consider the function $f_{\mathbf{Z}^{(5)}}(\lambda) \stackrel{\text{def}}{=} 1 + 2 \sum_{n=1}^{\infty} \frac{1}{n^2} \cos(\lambda n)$ and the real valued measure defined on \mathbb{T} by $\mathbf{n}_{\mathbf{Z}^{(5)}}(A) \stackrel{\text{def}}{=} \frac{1}{2\pi} \int_A f_{\mathbf{Z}^{(5)}}(\lambda) d\lambda$. Then we obtain that

$$\mathbf{n}_{\mathbf{Z}^{(5)}} = \Sigma_{\varepsilon}^2 \begin{pmatrix} \delta_0 \\ \mathbf{n}_{\mathbf{Z}^{(5)}} \end{pmatrix}.$$

(iv) For any integrable function $\mathbf{G} : \mathbb{T} \rightarrow \mathbb{C}^{d \times d}$, $\mathbf{G} \in L^1_{\mathbb{C}^{d \times d}}(\mathbb{T})$, the ('convolution') function

$$\mathbf{D}(\lambda) \stackrel{\text{def}}{=} \int_{\mathbb{T}} \mathbf{G}(\alpha) \mathbf{G}^*(\alpha + \lambda) d\alpha.$$

The $d \times d$ -matrix $\int_{\mathbb{T}} e^{in\lambda} \mathbf{D}(\lambda) d\lambda$ is positive semi-definite for any n . It is therefore the covariance matrix of a zero mean d -variate random vector \mathbf{Z}_n : $\Sigma_{\mathbf{Z}_n}^2 = \int_{\mathbb{T}} e^{in\lambda} \mathbf{D}(\lambda) d\lambda$. Moreover, we can enlarge the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ in such a way that we can select the zero mean d -valued random vectors \mathbf{Z}_n , $n \in \mathbb{Z}$, so that they are uncorrelated: $\text{Cov}(\mathbf{Z}_m, \mathbf{Z}_n) = \Sigma_n^2 \mathbb{1}_{\{m=n\}}$. In this case, the d -variate time series $\mathbf{Z} = (\mathbf{Z}_n)_{n \in \mathbb{Z}}$ is a strongly harmonizable noise, so that $\mathbf{n}_{\mathbf{Z}}(d\lambda) = \mathbf{D}(\lambda) d\lambda$.

In the following proposition based on a general result of Szökefalvi-Nagy and Foias (1970, pp.183–184) (see also Niemi, 1978), we obtain a characterization for the stochastic measure of a d -variate harmonizable noise admitting a stochastic Radon-Nikodym density function with respect to the Lebesgue measure, i.e., for which there exists a strongly measurable Lebesgue-integrable function $\Delta_{\mathbf{Z}} : \mathbb{T} \rightarrow \mathcal{H}^d(\mathbf{Z})$ such that

$$\mu_{\mathbf{Z}}(A) = \int_A \Delta_{\mathbf{Z}}(\lambda) d\lambda.$$

Proposition 3.2 (Stochastic spectral density) *The stochastic spectral measure $\mu_{\mathbf{Z}}$ of a d -variate harmonizable noise \mathbf{Z} admits a density function with respect to the Lebesgue measure on \mathbb{T} , if and only if the total energy of the d -variate time series \mathbf{Z} is finite:*

$$\sum_n \text{tr} \Sigma_{\mathbf{Z}_n}^2 = \sum_n \text{tr} \text{Var}(\mathbf{Z}_n) = \sum_n \mathbb{E}(\mathbf{Z}_n^{\top} \mathbf{Z}_n) = \sum_n \sum_j \sigma_{Z_{n,j}}^2 < \infty. \quad (6)$$

In this case, the stochastic spectral density function $\Delta_{\mathbf{Z}}$ on \mathbb{T} satisfies

$$\Delta_{\mathbf{Z}}(\lambda) = \frac{1}{2\pi} \sum_{n \in \mathbb{Z}} e^{-in\lambda} \mathbf{Z}_n, \quad (7)$$

where the convergence of the series (7) is in \mathcal{H}^d .

Moreover, the d -variate harmonizable noise \mathbf{Z} is strongly harmonizable. The spectral bimeasure $\mathbf{M}_{\mathbf{Z}}$ is extendible to a $\mathbb{C}^{d \times d}$ -valued measure on \mathbb{T}^2 with density function

$$\mathbf{F}_{\mathbf{Z}}(\lambda_1, \lambda_2) = \mathbb{E}(\Delta_{\mathbf{Z}}(\lambda_1) \Delta_{\mathbf{Z}}^*(\lambda_2)) = \frac{1}{4\pi^2} \sum_{n \in \mathbb{Z}} e^{-in(\lambda_1 - \lambda_2)} \Sigma_{\mathbf{Z}_n}^2.$$

Thus, $\mathbf{n}_{\mathbf{Z}}(d\lambda) = \mathbf{D}_{\mathbf{Z}}(\lambda) d\lambda$, where

$$\mathbf{D}_{\mathbf{Z}}(\lambda) = \frac{1}{4\pi^2} \sum_{n \in \mathbb{Z}} e^{-in\lambda} \Sigma_{\mathbf{Z}_n}^2. \quad (8)$$

Remark 3.4 • Under the condition 6, the stochastic spectral density function $\Delta_{\mathbf{Z}}$ is stationary with respect to the frequencies: $\mathbf{F}_{\mathbf{Z}}(\lambda_1, \lambda_2) = \mathbb{E}(\Delta_{\mathbf{Z}}(\lambda_1) \Delta_{\mathbf{Z}}^*(\lambda_2)) = \mathbf{D}_{\mathbf{Z}}(\lambda_1 - \lambda_2)$.

- A (non-zero) white noise $\boldsymbol{\varepsilon} = (\boldsymbol{\varepsilon}_n)_{n \in \mathbb{Z}}$ has no stochastic spectral density with respect to the Lebesgue measure on \mathbb{T} . Indeed, the condition (6) is not satisfied for $(\boldsymbol{\varepsilon}_n)$, because $\Sigma_{\boldsymbol{\varepsilon}_n}^2$ is constant and different from the zero $d \times d$ -matrix $\mathbf{0}_{d \times d}$. Recall, however, that the non-zero white noise $\boldsymbol{\varepsilon}$ has a spectral density with respect to the Lebesgue measure on \mathbb{T} which is constant and equal to $\frac{1}{2\pi} \mathbb{I}_{d \times d}$.

4 Multivariate harmonizable linear model

In this section, we show that multivariate time series defined as moving average of a harmonizable noise is harmonizable. This is a key result that allows us to introduce harmonizable ARMA time series in the next section.

Let $\mathbf{Z} = (\mathbf{Z}_n)_{n \in \mathbb{Z}}$ be a d -variate harmonizable noise and $\mathbf{A} \stackrel{\text{def}}{=} (\mathbf{A}_n)_{n \in \mathbb{Z}}$ be a family of complex $d \times d$ -matrices such that $\sum_n \text{tr}(\mathbf{A}_n \mathbf{A}_n^*) < \infty$. Then the d -variate time series $\mathbf{X} = (\mathbf{X}_n)_{n \in \mathbb{Z}}$ is well-defined by

$$\mathbf{X}_n = \sum_{m \in \mathbb{Z}} \mathbf{A}_{n-m} \mathbf{Z}_m = \sum_{m \in \mathbb{Z}} \mathbf{A}_m \mathbf{Z}_{n-m}, \quad (9)$$

where the convergence is in $\mathcal{H}(\mathbf{Z}) \subset L_{\mathbb{C}^d}^2(\mathbb{P})$ (see Theorem 4.1 below, for stationary case see Brockwell and Davis (1991)). Notice that the convergence in the sum (9) is almost sure when $\sum_n |\mathbf{A}_n| < \infty$. The d -variate time series \mathbf{X} is the transformation of the d -variate harmonizable noise \mathbf{Z} by a time invariant linear filter and

$$\text{Cov}(\mathbf{X}_n, \mathbf{X}_m) = \sum_{j \in \mathbb{Z}} \mathbf{A}_{n-j} \Sigma_{\mathbf{Z}_j}^2 \mathbf{A}_{m-j}^*.$$

Thanks to Plancherel theorem (Rudin, 1967, see Theorem 1.6.1) the Fourier transform $\widehat{\mathbf{A}} : \mathbb{T} \rightarrow \mathbb{C}^{d \times d}$ of the sequence $\mathbf{A} = (\mathbf{A}_n)_{n \in \mathbb{Z}}$ is well-defined, and is square integrable on \mathbb{T} , i.e., the $\mathbb{C}^{d \times d}$ -valued function $\lambda \mapsto \widehat{\mathbf{A}}(\lambda) \widehat{\mathbf{A}}^*(\lambda)$ is integrable with respect to the Lebesgue measure on \mathbb{T} . Furthermore, since the topological group \mathbb{T} is compact, the Fourier transform $\widehat{\mathbf{A}}$ is integrable and

$$\mathbf{A}_n = \frac{1}{2\pi} \int_{\mathbb{T}} \widehat{\mathbf{A}}(\lambda) e^{-in\lambda} d\lambda.$$

When the sequence \mathbf{A} is absolutely summable ($\sum_n |\mathbf{A}_n| < \infty$), the Fourier transform $\widehat{\mathbf{A}}$ is equal to

$$\widehat{\mathbf{A}}(\lambda) = \sum_{n \in \mathbb{Z}} \mathbf{A}_n e^{in\lambda}.$$

In the following theorem, we state that the time series $(\mathbf{X}_n)_{n \in \mathbb{Z}}$ given by (9) is harmonizable.

Theorem 4.1 *Let $\mathbf{Z} \stackrel{\text{def}}{=} (\mathbf{Z}_n)_{n \in \mathbb{Z}}$ be a d -variate harmonizable noise, and let $\mathbf{A} \stackrel{\text{def}}{=} (\mathbf{A}_n)_{n \in \mathbb{Z}}$ be a family of complex $d \times d$ -matrices such that $\sum_n \text{tr}(\mathbf{A}_n \mathbf{A}_n^*) < \infty$. Then the d -variate time series $\mathbf{X} \stackrel{\text{def}}{=} (\mathbf{X}_n)_{n \in \mathbb{Z}}$ defined by the moving average series (9) is well-defined in $\mathcal{H}^d(\mathbf{Z})$ and it is harmonizable. Moreover, the Fourier transform $\widehat{\mathbf{A}}$ of \mathbf{A} is $\mu_{\mathbf{Z}}$ -integrable and the stochastic spectral measure $\mu_{\mathbf{X}}$ of \mathbf{X} satisfies*

$$\mu_{\mathbf{X}}(A) = \frac{1}{2\pi} \sum_{n \in \mathbb{Z}} \left(\int_A e^{-in\lambda} \widehat{\mathbf{A}}(-\lambda) d\lambda \right) \mathbf{Z}_n = \int_A \widehat{\mathbf{A}}(-\lambda) \mu_{\mathbf{Z}}(d\lambda), \quad (10)$$

where $A \in \mathcal{B}(\mathbb{T})$. Therefore, the spectral bimeasure of \mathbf{X} can be expressed by

$$\begin{aligned} \mathbf{M}_{\mathbf{X}}(A, B) &= \frac{1}{4\pi^2} \sum_{n \in \mathbb{Z}} \left(\int_A e^{-in\lambda_1} \widehat{\mathbf{A}}(-\lambda_1) d\lambda_1 \right) \Sigma_{\mathbf{Z}_n}^2 \left(\int_B e^{in\lambda_2} \widehat{\mathbf{A}}^*(-\lambda_2) d\lambda_2 \right) \\ &= \int_A \int_B \widehat{\mathbf{A}}(-\lambda_1) \mathbf{M}_{\mathbf{Z}}(d\lambda_1, d\lambda_2) \widehat{\mathbf{A}}^*(-\lambda_2), \end{aligned} \quad (11)$$

where $A, B \in \mathcal{B}(\mathbb{T})$. The last integral is in the sense of an integral with respect to a bimeasure (see (Rao, 1982), see also (Rozanov, 1959)).

Remark 4.1 (i) If $(\mathbf{Z}_n)_{n \in \mathbb{Z}}$ is strongly harmonizable, then $(\mathbf{X}_n)_{n \in \mathbb{Z}}$ is strongly harmonizable. The spectral bimeasure $\mathbf{M}_{\mathbf{X}}$ is extendible to a $\mathbb{C}^{d \times d}$ -valued measure on \mathbb{T}^2 and the integral in equality (11) is the Lebesgue integral with respect to the $\mathbb{C}^{d \times d}$ -valued measure $\mathbf{M}_{\mathbf{Z}}$.

Indeed, since $\sum_n \text{tr}(\mathbf{A}_n \mathbf{A}_n^*) < \infty$, Proposition 3.1 implies that

$$\text{Cov}(\mathbf{X}_n, \mathbf{X}_m) = \sum_{j \in \mathbb{Z}} \int_{\mathbb{T}} e^{ij\lambda} (\mathbf{A}_{n-j} \mathbf{n}_{\mathbf{Z}}(d\lambda) \mathbf{A}_{m-j}^*)$$

Applying the Parseval's equality to each of the components of $\text{Cov}(\mathbf{X}_n, \mathbf{X}_m)$ we obtain that

$$\text{Cov}(\mathbf{X}_n, \mathbf{X}_m) = \frac{1}{2\pi} \int_{\mathbb{T}} \left(\int_{\mathbb{T}} e^{i(n(\alpha+\lambda)-m\lambda)} \widehat{\mathbf{A}}(\alpha-\lambda) \mathbf{n}_{\mathbf{Z}}(d\alpha) \widehat{\mathbf{A}}^*(-\lambda) \right) d\lambda,$$

and the spectral bimeasure $\mathbf{M}_{\mathbf{X}}$ is extendible to a vector valued measure on \mathbb{T}^2 , and

$$\mathbf{M}_{\mathbf{X}}(A \times B) = \frac{1}{2\pi} \int_B \left(\int_A \widehat{\mathbf{A}}(\lambda_1 - \lambda_2) \mathbf{n}_{\mathbf{Z}}(d\lambda_1 - \lambda_2) \widehat{\mathbf{A}}^*(-\lambda_2) \right) d\lambda_2.$$

(ii) If $(\mathbf{Z}_n)_{n \in \mathbb{Z}}$ has a stochastic spectral density $\Delta_{\mathbf{Z}}$ (Proposition 3.2), then $(\mathbf{X}_n)_{n \in \mathbb{Z}}$ has a stochastic spectral density which coincides with $\Delta_{\mathbf{X}}(\lambda) = \widehat{\mathbf{A}}(-\lambda) \Delta_{\mathbf{Z}}(\lambda)$:

$$\mu_{\mathbf{X}}(A) = \int_A \widehat{\mathbf{A}}(-\lambda) \Delta_{\mathbf{Z}}(\lambda) d\lambda.$$

Thus, the spectral bimeasure $\mathbf{M}_{\mathbf{X}}$ is extendible to a measure on \mathbb{T}^2 with a density function

$$\mathbf{F}_{\mathbf{X}}(\lambda_1, \lambda_2) = \widehat{\mathbf{A}}(-\lambda_1) \mathbf{D}_{\mathbf{Z}}(\lambda_1 - \lambda_2) \widehat{\mathbf{A}}^*(-\lambda_2)$$

and

$$\begin{aligned} \text{Cov}(\mathbf{X}_n, \mathbf{X}_m) &= \int_{\mathbb{T}} \left(\sum_{n \in \mathbb{Z}} e^{ij\lambda} \mathbf{A}_{n-j} \mathbf{D}_{\mathbf{Z}}(\lambda) \mathbf{A}_{m-j}^* \right) d\lambda \\ &= \int_{\mathbb{T}^2} e^{-i(n\lambda_1 - m\lambda_2)} \widehat{\mathbf{A}}(-\lambda_1) \mathbf{D}_{\mathbf{Z}}(\lambda_1 - \lambda_2) \widehat{\mathbf{A}}^*(-\lambda_2) d\lambda_1 d\lambda_2. \end{aligned}$$

(iii) If $(\mathbf{Z}_n)_{n \in \mathbb{Z}}$ has a discrete stochastic spectral measure, then $(\mathbf{X}_n)_{n \in \mathbb{Z}}$ has a discrete stochastic spectral measure: $\boldsymbol{\mu}_{\mathbf{X}}(\lambda) = \widehat{\mathbf{A}}(-\lambda)\boldsymbol{\mu}_{\mathbf{Z}}(\lambda)$ and $\mathbf{M}_{\mathbf{X}}(\lambda_1, \lambda_2) = \widehat{\mathbf{A}}(-\lambda_1)\mathbf{M}_{\mathbf{Z}}(\lambda_1, \lambda_2)\widehat{\mathbf{A}}^*(-\lambda_2)$.

5 Harmonizable VARMA model

Our definition of harmonizable noise allows the construction of HVARMA models following the ideas of Brockwell and Davis (1991, sect. 3.1), resulting in Definition 5.1. Theorem 5.1 below describes the Loève spectral matrices of HVARMA(p, q) processes. This is the starting point for many possible studies on their estimation, prediction, etc., as well as a key results that will allow us to propose a method for generating realizations of the HVARMA time series later in the paper.

Definition 5.1 (HVARMA(p, q) processes) *The process $(\mathbf{X}_n)_{n \in \mathbb{Z}}$ is said to be a HVARMA(p, q) process if for every $n \in \mathbb{Z}$,*

$$\mathbf{X}_n + \phi_1 \mathbf{X}_{n-1} + \cdots + \phi_p \mathbf{X}_{n-p} = \mathbf{Z}_n + \theta_1 \mathbf{Z}_{n-1} + \cdots + \theta_q \mathbf{Z}_{n-q}, \quad (12)$$

where $(\mathbf{Z}_n)_{n \in \mathbb{Z}}$ is a d -variate harmonizable noise as in Lemma 3.1, and $\phi_1, \dots, \phi_p, \theta_1, \dots, \theta_q \in \mathbb{C}^{d \times d}$ are the AR and MA coefficient matrices, respectively. The model equation (12) can be written in the compact form:

$$\Phi(B)\mathbf{X}_n = \Theta(B)\mathbf{Z}_n,$$

where B is the backward operator, $\Phi(z) = \mathbb{I}_{d \times d} + \phi_1 z + \cdots + \phi_p z^p$ and $\Theta(z) = \mathbb{I}_{d \times d} + \theta_1 z + \cdots + \theta_q z^q$ are the polynomial matrices associated to the difference equation (12).

Theorem 5.1 (Loève spectral matrices of HVARMA(p, q) processes) *Let $(\mathbf{X}_n)_{n \in \mathbb{Z}}$ be a HVARMA(p, q) process (see Definition 5.1). Assume that the polynomial functions $\det \Phi(\cdot)$ and $\det \Theta(\cdot)$ have no common zeroes, and that the harmonizable noise $(\mathbf{Z}_n)_{n \in \mathbb{Z}}$ has a stochastic spectral density $\Delta_{\mathbf{Z}}$ (defined in Proposition 3.2). Then $(\mathbf{X}_n)_{n \in \mathbb{Z}}$ has a stochastic spectral density $\Delta_{\mathbf{X}}(\lambda) = \Phi(e^{-i\lambda})^{-1} \Theta(e^{-i\lambda}) \Delta_{\mathbf{Z}}(\lambda)$, and the Loève spectral density matrix is given by*

$$m(\lambda_1, \lambda_2) = \frac{1}{4\pi^2} \Phi(e^{-i\lambda_1})^{-1} \Theta(e^{-i\lambda_1}) \mathbf{D}_{\mathbf{Z}}(\lambda_1 - \lambda_2) \Theta^*(e^{-i\lambda_2}) \Phi^*(e^{-i\lambda_2})^{-1}, \quad (\lambda_1, \lambda_2) \in \mathbb{T}^2.$$

5.1 Spectrum

Consider a d -variate white noise $\boldsymbol{\varepsilon}$ associated to the d -variate noise \mathbf{Z} as in Lemma 3.1: $\mathbf{Z}_n = \mathbf{S}_n \boldsymbol{\varepsilon}_n = \Xi(\boldsymbol{\varepsilon}_n)$, with $\sup_{n,j,k} |s_{n,j,k}| < \infty$. Let \mathbf{Y} be a solution of the difference equation

$$\mathbf{Y}_n + \phi_1 \mathbf{Y}_{n-1} + \cdots + \phi_p \mathbf{Y}_{n-p} = \boldsymbol{\varepsilon}_n + \theta_1 \boldsymbol{\varepsilon}_{n-1} + \cdots + \theta_q \boldsymbol{\varepsilon}_{n-q}. \quad (13)$$

The operator Ξ , introduced in the proof of Theorem 3.1, which is a bounded operator on $\mathcal{H}^d(\boldsymbol{\varepsilon})$, can be applied to relation (13).

Thus,

$$\Xi(\mathbf{Y}_n) + \phi_1 \Xi(\mathbf{Y}_{n-1}) + \cdots + \phi_p \Xi(\mathbf{Y}_{n-p}) = \mathbf{Z}_n + \theta_1 \mathbf{Z}_{n-1} + \cdots + \theta_q \mathbf{Z}_{n-q}.$$

Consequently, the d -variate time series $(\mathbf{X}_n)_{n \in \mathbb{Z}} = (\Xi(\mathbf{Y}_n))_{n \in \mathbb{Z}}$ is well-defined, and is the solution of the VARMA equation:

$$\mathbf{X}_n + \phi_1 \mathbf{X}_{n-1} + \cdots + \phi_p \mathbf{X}_{n-p} = \mathbf{Z}_n + \theta_1 \mathbf{Z}_{n-1} + \cdots + \theta_q \mathbf{Z}_{n-q}, \quad (14)$$

where $\mathbf{Z} = (\mathbf{Z}_n)_{n \in \mathbb{Z}}$ is a d -variate harmonizable noise and $\phi_1, \dots, \phi_p, \theta_1, \dots, \theta_q \in \mathbb{C}^{d \times d}$. We define the polynomial matrices $\Phi(z) = \mathbb{I}_{d \times d} + \phi_1 z + \dots + \phi_p z^p$ and $\Theta(z) = \mathbb{I}_{d \times d} + \theta_1 z + \dots + \theta_q z^q$ associated to the difference equation (14).

From now on, we assume that $\det \Phi(e^{i\lambda}) \neq 0$ for any λ . Then the difference equation (14) admits a unique d -variate harmonizable solution.

$$\mathbf{X}_n = \int_{\mathbb{T}} e^{in\lambda} \Phi(e^{-i\lambda})^{-1} \Theta(e^{-i\lambda}) \boldsymbol{\mu}_{\mathbf{Z}}(d\lambda) = \sum_{j \in \mathbb{Z}} \mathbf{A}_j \mathbf{Z}_{n-j}, \quad (15)$$

where the matrix \mathbf{A}_j is the coefficient of z^j in the Laurent expansion

$$\Phi(z)^{-1} \Theta(z) = \sum_{j \in \mathbb{Z}} \mathbf{A}_j z^j.$$

Since $\det \Phi(e^{i\lambda}) \neq 0$ for any λ , the Laurent series is absolutely convergent in some ring (annulus) $r^{-1} < z < r$ for some $r > 1$, and thus $\sum_{n \in \mathbb{Z}} |\mathbf{A}_n| < \infty$ (see, for e.g., Mel'nyk, 2023) and $\hat{\mathbf{A}}(\lambda) = \Phi(e^{-i\lambda})^{-1} \Theta(e^{-i\lambda})$. The relation (10) implies that the stochastic spectral measure $\boldsymbol{\mu}_{\mathbf{X}}$ verifies

$$\boldsymbol{\mu}_{\mathbf{X}}(A) = \int_A \Phi(e^{-i\lambda})^{-1} \Theta(e^{-i\lambda}) \boldsymbol{\mu}_{\mathbf{Z}}(d\lambda) = \frac{1}{2\pi} \sum_{n \in \mathbb{Z}} \left(\int_A e^{-in\lambda} \Phi(e^{-i\lambda})^{-1} \Theta(e^{-i\lambda}) d\lambda \right) \mathbf{Z}_n.$$

Furthermore,

$$\begin{aligned} \text{Cov}(\mathbf{X}_n, \mathbf{X}_m) &= \sum_{j \in \mathbb{Z}} \mathbf{A}_{n-j} \boldsymbol{\Sigma}_{\mathbf{Z}_j}^2 \mathbf{A}_{m-j}^* \\ &= \iint_{\mathbb{T}^2} e^{i(n\lambda_1 - m\lambda_2)} (\Phi(e^{-i\lambda_1})^{-1} \Theta(e^{-i\lambda_1})) \mathbf{M}_{\mathbf{Z}}(d\lambda_1, d\lambda_2) (\Theta^*(e^{-i\lambda_2}) \Phi^*(e^{-i\lambda_2})^{-1}) \end{aligned}$$

and the spectral bimeasure can be expressed as

$$\begin{aligned} \mathbf{M}_{\mathbf{X}}(A, B) &= \iint_{A \times B} (\Phi(e^{-i\lambda_1})^{-1} \Theta(e^{-i\lambda_1})) \mathbf{M}_{\mathbf{Z}}(d\lambda_1, d\lambda_2) (\Theta^*(e^{-i\lambda_2}) \Phi^*(e^{-i\lambda_2})^{-1}) \\ &= \frac{1}{4\pi^2} \sum_{n \in \mathbb{Z}} \left(\int_A e^{-in\lambda} \Phi(e^{-i\lambda})^{-1} \Theta(e^{-i\lambda}) d\lambda \right) \boldsymbol{\Sigma}_{\mathbf{Z}_n}^2 \left(\int_B e^{in\lambda} \Theta^*(e^{-i\lambda}) \Phi^*(e^{-i\lambda})^{-1} d\lambda \right). \end{aligned} \quad (16)$$

In the remark below, we give some important properties of HVARMA models.

Remark 5.1 (i) If $(\mathbf{Z}_n)_{n \in \mathbb{Z}}$ is a strongly harmonizable noise, then $(\mathbf{X}_n)_{n \in \mathbb{Z}}$ is strongly harmonizable.

(ii) If the harmonizable noise $(\mathbf{Z}_n)_{n \in \mathbb{Z}}$ has a stochastic spectral density $\boldsymbol{\Delta}_{\mathbf{Z}}$, then $(\mathbf{X}_n)_{n \in \mathbb{Z}}$ has a stochastic spectral density $\boldsymbol{\Delta}_{\mathbf{X}}(\lambda) = \Phi(e^{-i\lambda})^{-1} \Theta(e^{-i\lambda}) \boldsymbol{\Delta}_{\mathbf{Z}}(\lambda)$, and

$$\begin{aligned} \text{Cov}(\mathbf{X}_n, \mathbf{X}_m) &= \frac{1}{4\pi^2} \iint_{\mathbb{T}^2} e^{i(n\lambda_1 - m\lambda_2)} \Phi(e^{-i\lambda_1})^{-1} \Theta(e^{-i\lambda_1}) \boldsymbol{D}_{\mathbf{Z}}(\lambda_1 - \lambda_2) \Theta^*(e^{-i\lambda_2}) \Phi^*(e^{-i\lambda_2})^{-1} d\lambda_1 d\lambda_2. \end{aligned}$$

(iii) Causality (Brockwell & Davis, 1991, Theorem 3.1.1) :

$$\det \Phi(z) \neq 0 \quad \text{for } |z| \leq 1 \quad \implies \quad \mathbf{X}_n = \sum_{j=-\infty}^n \mathbf{A}_{n-j} \mathbf{Z}_j = \sum_{j=0}^{\infty} \mathbf{A}_j \mathbf{Z}_{n-j}.$$

(iv) Invertibility (Brockwell & Davis, 1991, Theorem 3.1.2) :

$$\det \Theta(z) \neq 0 \quad \text{for } |z| \leq 1 \quad \implies \quad \mathbf{Z}_n = \sum_{j=-\infty}^n \boldsymbol{\Pi}_{n-j} \mathbf{X}_j = \sum_{j=0}^{\infty} \boldsymbol{\Pi}_j \mathbf{X}_{n-j}.$$

Here $\Theta(z)^{-1} \Phi(z) = \sum_{j \in \mathbb{Z}} \boldsymbol{\Pi}_j z^j$.

(v) In (iii) and (iv), we have equivalence whenever the polynomial functions $\det \Phi(\cdot)$ and $\det \Theta(\cdot)$ have no common zeroes.

Finally, we can define the dual-frequency coherence of a HVARMA(p,q) process.

Definition 5.2 (Dual-frequency coherence) *Let $(X_n)_{n \in \mathbb{Z}}$ be a d -variate HVARMA(p,q) process (see Definition 5.1) satisfying conditions of Theorem 5.1. Denote its Loève spectral density matrix as $m_X(\lambda_1, \lambda_2)$, $(\lambda_1, \lambda_2) \in \mathbb{T}^2$. Then its dual-frequency coherence matrix is defined as:*

$$\rho_X(\lambda_1, \lambda_2) = \left(\frac{|m_{i,j}(\lambda_1, \lambda_2)|^2}{m_{i,i}(\lambda_1, \lambda_1)m_{j,j}(\lambda_2, \lambda_2)} \right)_{i,j=1,\dots,d}.$$

5.2 Estimation - Preliminary results

In this section, we consider the HVARMA(1,0) model. Using this simple model we show how the model parameter can be estimated. The main results are given in Propositions 5.1 and 5.2. For the self-reliant lecture of the paper, we give sketch of the proofs in Appendix.

Let

$$X_n - \phi X_{n-1} = Z_n.$$

The time series $\{X_n\}$ is a nonhomogeneous Markov chain. From now on, we assume that the random variables X_0 and Z_n , $n \geq 1$, are square-integrable, $E(Z_n) = E(X_0) = 0$, and

$$Z_n \perp\!\!\!\perp Z_k, \quad Z_n \perp\!\!\!\perp X_0 \quad \text{for } n \neq k,$$

where $\perp\!\!\!\perp$ denotes the independence of the random variables.

The time series $\{S_n \stackrel{\text{def}}{=} \phi^{-n} X_n - X_0\}$ is a martingale with respect to the filtration $\{\mathcal{F}_n\}$ generated by $\{Z_n\}$ and X_0 . Denote $\sigma_n^2 \stackrel{\text{def}}{=} \text{var}(Z_n) = E(Z_n^2)$. Then

$$\text{var}(X_n) = E(X_n^2) = \phi^{2n} \sum_{k=1}^n \phi^{-2k} \sigma_k^2 + \phi^{2n} E(X_0^2).$$

Notice that when $|\phi| > 1$ and $\sup_n \sigma_n^2 < \infty$, the time series $\{S_n\}$ converges in q.m. and a.e. to some random variable S_∞ as $n \rightarrow \infty$ and $E(S_\infty) \stackrel{\text{def}}{=} 0$ and $E(S_\infty^2) = \sum_n \phi^{-2n} \sigma_n^2 < \infty$. Thus,

$$\lim_{n \rightarrow \infty} \phi^{-n} X_n = S_\infty + X_0 \quad \text{in q.m. and a.e.}$$

Now we define the least squares estimator of $\phi \in \mathbb{R}$ as follows

$$\hat{\phi}_n \stackrel{\text{def}}{=} \arg \min_{\phi} \sum_{k=1}^n (X_k - \phi X_{k-1})^2.$$

Before studying its consistency, we introduce some additional notation. We define

$$\mu_n \stackrel{\text{def}}{=} \frac{1}{n} \sum_{k=1}^n \sigma_k^2 \quad \text{and} \quad \eta_{k,n} \stackrel{\text{def}}{=} \frac{1}{n} \sum_{j=1}^{n-k} \sigma_j^2 \sigma_{j+k}^2.$$

Notice that $\mu_n \leq \sup_k \sigma_k^2$ and $\eta_{k,n} \leq \sup_j \sigma_j^4$. Below, applying the strong law of large numbers for martingales (see Theorem 2.18 in Hall & Heyde, 1980), we state the consistency of the least square estimator $\hat{\phi}_n$.

Proposition 5.1 (Consistency) *Let $Z_n \perp\!\!\!\perp \mathcal{F}_{n-1}$ for $n \geq 1$, $\sup_n \sigma_n^2 < \infty$, $\sup_n \mathbb{E}(Z_n^4) < \infty$ and $\mathbb{E}(X_0^4) < \infty$. In addition assume that (i) when $|\phi| \leq 1$, $\liminf_n \mu_n > 0$, (ii) when $|\phi| > 1$, $\mathbb{P}(S_\infty + X_0 = 0) = 0$. Then*

$$\lim_{n \rightarrow \infty} \widehat{\phi}_n = \phi \text{ a.e.}$$

Next, the central limit theorem for square-integrable martingales (see Theorem 2.8.41 in Dacunha-Castelle & Duflo, 1993) allows establishing the asymptotic normality of $\widehat{\theta}_n$ when $|\phi| < 1$ (see Corollary 3.1 in Hall & Heyde, 1980).

Proposition 5.2 ($|\phi| < 1$ - Asymptotic normality) *Let $|\phi| < 1$, $Z_n \perp\!\!\!\perp \mathcal{F}_{n-1}$ for $n \geq 1$, $\sup_n \sigma_n^2 < \infty$, $\sup_n \mathbb{E}(Z_n^4) < \infty$, $\mathbb{E}(X_0^4) < \infty$. In addition, assume that the limits $\mu \stackrel{\text{def}}{=} \lim_n \mu_n$ and $\eta_k \stackrel{\text{def}}{=} \lim_n \eta_{k,n}$ for any $k \geq 1$, exist and are positive. Then $\sqrt{n}(\widehat{\phi}_n - \phi)$ converges in distribution to the normal law $\mathcal{N}(0, \Gamma_\phi)$, where $\Gamma_\phi = (1 - \phi^2)^2 \mu^{-2} \sum_k \phi^{2(k-1)} \eta_k$.*

6 Generation of harmonizable time series with a given Loève spectrum

In this section, we give tools to generate finite time sample realizations of a HVARMA process with the known Loève spectrum, while ensuring they provide a correct approximation of the associated time series defined on \mathbb{Z} .

We first show, in Section 6.1, how to construct a periodic harmonizable time series on \mathbb{Z} from its discrete Loève spectrum. However, this method does not allow constructing harmonizable noise, which is the basic building block of our HVARMA model. In Section 6.2 we describe how to overcome this difficulty by defining harmonizable noise and time series directly on the finite group \mathbb{Z}_N . Next, results of Section 6.3 ensure that the restriction of harmonizable noise and time series on \mathbb{Z} to \mathbb{Z}_N provides a proper approximation. Consequently, we consider in Section 6.4 the extension by zeros of harmonizable noise and time series defined on the finite group \mathbb{Z}_N to (the full space) \mathbb{Z} . We prove that in general this does not produce a harmonizable time series, but a harmonizable noise on \mathbb{Z} , and a linear harmonizable time series. On that basis, we can construct a HVARMA time series on \mathbb{Z} as described in Section 6.5. Finally, we provide some numerical examples of realizations from HVARMA process in Section 6.6.

Hereafter, N is a fixed integer, $N \geq 1$. We denote as $\mathbb{Z}_N \stackrel{\text{def}}{=} \mathbb{Z}/(2N+1)$ the finite group of integer numbers modulo $2N+1$ that we represent as $\{-N, \dots, N\}$. Its dual group \mathbb{T}_N can be represented by $\{\omega_j : j \in \mathbb{Z}_N\}$, where $\omega_j \stackrel{\text{def}}{=} \frac{2\pi j}{2N+1}$.

6.1 Construction of a periodic harmonizable time series

We consider a d -variate time series $(\mathbf{X}_n)_{n \in \mathbb{Z}}$ defined as

$$\mathbf{X}_n \stackrel{\text{def}}{=} \sum_{j=-N}^N e^{in\omega_j} \boldsymbol{\mu}_j,$$

where the random vectors (called increments) $\boldsymbol{\mu}_j$ are extracted from a d -variate zero mean stationary time series $(\boldsymbol{\mu}_n)_{n \in \mathbb{Z}}$ with covariance function

$$\mathbf{K}_\boldsymbol{\mu}(r) \stackrel{\text{def}}{=} \text{Cov}(\boldsymbol{\mu}_{j+r}, \boldsymbol{\mu}_j).$$

$(\mathbf{X}_n)_{n \in \mathbb{Z}}$ is periodic with period $2N + 1$, zero mean, strongly harmonizable with a stochastic spectral measure $\boldsymbol{\mu}_{\mathbf{X}}$, which has a discrete support at frequencies ω_j , $j = -N, \dots, N$, $\boldsymbol{\mu}_{\mathbf{X}}(\{\omega_j\}) = \boldsymbol{\mu}_j$. Furthermore, its spectral bimeasure $\mathbf{M}_{\mathbf{X}}$ is a discrete measure on \mathbb{T}^2

$$\mathbf{M}_{\mathbf{X}}(\{\omega_{j_1}, \omega_{j_2}\}) = \text{Cov}(\boldsymbol{\mu}_{\mathbf{X}}(\omega_{j_1}), \boldsymbol{\mu}_{\mathbf{X}}(\omega_{j_2})) = \text{Cov}(\boldsymbol{\mu}_{j_1}, \boldsymbol{\mu}_{j_2}) = \mathbf{K}_{\boldsymbol{\mu}}(j_1 - j_2).$$

Additionally,

$$\text{Cov}(\mathbf{X}_{n+k}, \mathbf{X}_n) = \sum_{j_1=-N}^N \sum_{r=j_1-N}^{j_1+N} e^{ik\omega_{j_1}} e^{in\omega_r} \mathbf{K}_{\boldsymbol{\mu}}(r).$$

Assume that $\mathbf{K}_{\boldsymbol{\mu}}$ is periodic with period $2N + 1$.

Then we have

$$\text{Cov}(\mathbf{X}_{n+k}, \mathbf{X}_n) = \sum_{j_1=-N}^N \sum_{r=-N}^N e^{ik\omega_{j_1}} e^{in\omega_r} \mathbf{K}_{\boldsymbol{\mu}}(r).$$

Therefore, $\text{Cov}(\mathbf{X}_{n+k}, \mathbf{X}_n) = \mathbf{0}_{d \times d}$ for $k \not\equiv 0 \pmod{2N + 1}$. While, for $k \equiv 0 \pmod{2N + 1}$, we have

$$\text{Cov}(\mathbf{X}_{n+k}, \mathbf{X}_n) = \text{Var}(\mathbf{X}_n) = (2N + 1) \sum_{r=-N}^N e^{in\omega_r} \mathbf{K}_{\boldsymbol{\mu}}(r).$$

Hence, \mathbf{X} cannot be a non-identically null harmonizable noise. In other words, due to its periodicity, the process \mathbf{X} cannot be a harmonizable noise.

6.2 Time series indexed by a finite group

Similarly to Sections 2.2 and 3, we can consider harmonizable time series and harmonizable noise defined on \mathbb{Z}_N by replacing \mathbb{Z} by \mathbb{Z}_N and \mathbb{T} by \mathbb{T}_N in Definitions 2.1 and 3.1 and condition (Ha). In this case Proposition 2.1, Lemma 3.1 as well as Theorem 3.1 are valid. Additionally, since the group \mathbb{Z}_N and its dual group \mathbb{T}_N are finite, any zero mean second order time series $\boldsymbol{\chi} = (\boldsymbol{\chi}_n)_{n \in \mathbb{Z}_N}$ is strongly harmonizable with a discrete stochastic spectral measure:

$$\boldsymbol{\mu}_{\boldsymbol{\chi}}(\omega_j) = \frac{1}{2N + 1} \sum_{n \in \mathbb{Z}_N} e^{-in\omega_j} \boldsymbol{\chi}_n = \frac{1}{2N + 1} \sum_{n \in \mathbb{Z}_N} e^{-inj\omega_1} \boldsymbol{\chi}_n, \quad (17)$$

and a discrete spectral bimeasure

$$\mathbf{M}_{\boldsymbol{\chi}}(\omega_{j_1}, \omega_{j_2}) = \frac{1}{(2N + 1)^2} \sum_{n_1 \in \mathbb{Z}_N} \sum_{n_2 \in \mathbb{Z}_N} e^{-i(n_1 j_1 - n_2 j_2)\omega_1} \mathbf{E}(\boldsymbol{\chi}_{n_1} \boldsymbol{\chi}_{n_2}^*).$$

Hence,

$$\boldsymbol{\chi}_n = \sum_{j \in \mathbb{Z}_N} e^{inj\omega_1} \boldsymbol{\mu}_{\boldsymbol{\chi}}(\omega_j),$$

and

$$\text{Cov}(\boldsymbol{\chi}_{n_1}, \boldsymbol{\chi}_{n_2}) = \sum_{j_1 \in \mathbb{Z}_N} \sum_{j_2 \in \mathbb{Z}_N} e^{-i(n_1 j_1 - n_2 j_2)\omega_1} \mathbf{M}_{\boldsymbol{\chi}}(\omega_{j_1}, \omega_{j_2}).$$

Notice that any (second order) time series $\boldsymbol{\chi}$ can be extended periodically to a strongly harmonizable time series $\tilde{\boldsymbol{\chi}}$ on \mathbb{Z} with the discrete stochastic spectral measure $\boldsymbol{\mu}_{\tilde{\boldsymbol{\chi}}}(\omega_j) = \boldsymbol{\mu}_{\boldsymbol{\chi}}(\omega_j)$ for $j = -N, \dots, N$, and $\boldsymbol{\mu}_{\tilde{\boldsymbol{\chi}}}(\omega) = \mathbf{0}_d$ otherwise.

6.2.1 Harmonizable noise on \mathbb{Z}_N

Let $\boldsymbol{\mu} = (\boldsymbol{\mu}_j)_{j \in \mathbb{Z}_N}$ be a zero mean stationary d -variate time series on \mathbb{Z}_N with covariance matrix function $\mathbf{K}_\boldsymbol{\mu}(r) \stackrel{\text{def}}{=} \text{Cov}(\boldsymbol{\mu}_{j+r}, \boldsymbol{\mu}_j) = \mathbb{E}(\boldsymbol{\mu}_{j+r} \boldsymbol{\mu}_j^*)$. An example can be found in Section 6.2.3. Then, the time series $\boldsymbol{\zeta} = (\boldsymbol{\zeta}_n)_{n \in \mathbb{Z}_N}$ defined by

$$\boldsymbol{\zeta}_n \stackrel{\text{def}}{=} \sum_{j \in \mathbb{Z}_N} e^{in\omega_j} \boldsymbol{\mu}_j, \quad (18)$$

is the (strongly) harmonizable noise on \mathbb{Z}_N (see Theorem 3.1) with the stochastic spectral measure $\boldsymbol{\mu}_\boldsymbol{\zeta}$ defined on \mathbb{T}_N by $\boldsymbol{\mu}_\boldsymbol{\zeta}(\omega_j) = \boldsymbol{\mu}_j$, and

$$\boldsymbol{\Sigma}_{\boldsymbol{\zeta}_n}^2 \stackrel{\text{def}}{=} \text{Var}(\boldsymbol{\zeta}_n) = \sum_{j \in \mathbb{Z}_N} e^{in\omega_j} \mathbf{n}_\boldsymbol{\zeta}(\omega_j) = (2N+1) \sum_{j \in \mathbb{Z}_N} e^{in\omega_j} \mathbf{K}_\boldsymbol{\mu}(j) = \int_{\mathbb{T}_N} e^{in\omega} \mathbf{n}_\boldsymbol{\zeta}(d\omega),$$

where $\mathbf{n}_\boldsymbol{\zeta}(\omega_j) = (2N+1) \mathbf{K}_\boldsymbol{\mu}(j)$ (see relation (3)). Notice that the time series $\boldsymbol{\zeta}$ is a real-valued vector if and only if $\bar{\boldsymbol{\mu}}_j = \boldsymbol{\mu}_{-j}$, $j \in \mathbb{Z}_N$.

Conversely, if $\boldsymbol{\zeta} = (\boldsymbol{\zeta}_n)_{n \in \mathbb{Z}_N}$ is any harmonizable noise on \mathbb{Z}_N , then the time series $\boldsymbol{\mu} = (\boldsymbol{\mu}_j)_{j \in \mathbb{Z}_N}$ defined by

$$\boldsymbol{\mu}_j \stackrel{\text{def}}{=} \boldsymbol{\mu}_\boldsymbol{\zeta}(\omega_j) = \frac{1}{2N+1} \sum_{n \in \mathbb{Z}_N} e^{-in\omega_j} \boldsymbol{\zeta}_n$$

is (second order) stationary on \mathbb{Z}_N with the covariance function

$$\mathbf{K}_{\boldsymbol{\mu}_\boldsymbol{\zeta}}(j) = \frac{1}{2N+1} \mathbf{n}_\boldsymbol{\zeta}(\omega_j) = \frac{1}{2N+1} \sum_{n \in \mathbb{Z}_N} e^{-in\omega_j} \text{Var}(\boldsymbol{\zeta}_n) = \frac{1}{2N+1} \sum_{n \in \mathbb{Z}_N} e^{-in\omega_j} \boldsymbol{\Sigma}_{\boldsymbol{\zeta}_n}^2. \quad (19)$$

6.2.2 Moving average on \mathbb{Z}_N

If $\boldsymbol{\zeta} = (\boldsymbol{\zeta}_n)_{n \in \mathbb{Z}_N}$ is a harmonizable noise on \mathbb{Z}_N and $(\boldsymbol{\Gamma}_n)_{n \in \mathbb{Z}_N}$ is a family of complex $d \times d$ -matrices, then the time series $\boldsymbol{\chi} = (\boldsymbol{\chi}_n)_{n \in \mathbb{Z}_N}$ defined by

$$\boldsymbol{\chi}_n \stackrel{\text{def}}{=} \sum_{m \in \mathbb{Z}_N} \boldsymbol{\Gamma}_{n-m} \boldsymbol{\zeta}_m \quad (20)$$

is (strongly) harmonizable on \mathbb{Z}_N with stochastic spectral measure and spectral bimeasure

$$\boldsymbol{\mu}_\boldsymbol{\chi}(\omega_j) = \widehat{\boldsymbol{\Gamma}}_{-j} \boldsymbol{\mu}_\boldsymbol{\zeta}(\omega_j) \quad \text{and} \quad \mathbf{M}_\boldsymbol{\chi}(\omega_{j_1}, \omega_{j_2}) = \widehat{\boldsymbol{\Gamma}}_{-j_1} \mathbf{K}_{\boldsymbol{\mu}_\boldsymbol{\zeta}}(j_1 - j_2) \widehat{\boldsymbol{\Gamma}}_{-j_2}^*,$$

where

$$\widehat{\boldsymbol{\Gamma}}_j = \sum_{k \in \mathbb{Z}_N} \boldsymbol{\Gamma}_k e^{ik\omega_j}.$$

Notice that in formula (20), $\boldsymbol{\Gamma}_n$ is periodically extended to \mathbb{Z} by setting $\boldsymbol{\Gamma}_{n+kN} = \boldsymbol{\Gamma}_n$.

6.2.3 Moving average with band-limited spectrum

We consider a white noise $(\boldsymbol{\varepsilon}_n)_{n \in \mathbb{Z}_N}$ on \mathbb{Z}_N and a family $(\boldsymbol{\vartheta}_n)_{n \in \mathbb{Z}_N}$ of complex $d \times d$ -matrices. Then the moving average

$$\boldsymbol{\mu}_j = \sum_{k \in \mathbb{Z}_N} \boldsymbol{\vartheta}_{j-k} \boldsymbol{\varepsilon}_k$$

defines a stationary time series $\boldsymbol{\mu} = (\boldsymbol{\mu}_j)_{j \in \mathbb{Z}_N}$ with covariance function

$$\mathbf{K}_\boldsymbol{\mu}(r) = \mathbb{E}(\boldsymbol{\mu}_{j+r} \boldsymbol{\mu}_j^*) = \sum_{k \in \mathbb{Z}_N} \boldsymbol{\vartheta}_{k+r} \boldsymbol{\Sigma}_\boldsymbol{\varepsilon}^2 \boldsymbol{\vartheta}_k^*.$$

The harmonizable noise ζ defined by relation (18) satisfies the equality $\zeta_n = \hat{\theta}_n \hat{\varepsilon}_n$, where

$$\hat{\theta}_n \stackrel{\text{def}}{=} \sum_{k \in \mathbb{Z}_N} e^{in\omega_k} \theta_k \quad \text{and} \quad \hat{\varepsilon}_n \stackrel{\text{def}}{=} \sum_{k \in \mathbb{Z}_N} e^{in\omega_k} \varepsilon_k.$$

Notice that $\hat{\varepsilon}$ is a white noise with variance $\Sigma_{\hat{\varepsilon}}^2 = (2N+1)\Sigma_{\varepsilon}^2$.

Whenever $\theta_k = \mathbf{0}_d$ for $k = \pm q, \dots, \pm N \bmod (2N+1)$, for some $0 < q \leq N/2$ then $\mathbf{K}_{\mu}(r) = \mathbf{0}_{d \times d}$ for $r = \pm 2q, \dots, \pm N \bmod (2N+1)$ and the harmonizable noise ζ has a band-limited (Loève) spectrum: $\mathbf{n}_{\zeta}(\omega_j) = \mathbf{0}_{d \times d}$ for $j = \pm 2q, \dots, \pm N \bmod (2N+1)$. In addition, the harmonizable moving average time series χ defined by relation (20) has a band-limited (Loève) spectrum: $\mathbf{M}_{\chi}(\omega_{j_1}, \omega_{j_2}) = \mathbf{0}_{d \times d}$ for $j_1 - j_2 = \pm 2q, \dots, \pm N \bmod (2N+1)$.

6.2.4 Autoregressive time series on \mathbb{Z}_N

First, notice that the definition of an autoregressive time series with uncorrelated innovations on \mathbb{Z}_N need a constraining condition to justify its existence. Indeed, consider the autoregressive time series $\mu_n = \Phi \mu_{n-1} + \varepsilon_n$, $n \in \mathbb{Z}_N$, where $\{\varepsilon_n\}_{n \in \mathbb{Z}_N}$ is a d -variate white noise and Φ is a complex $d \times d$ -matrix. To justify the existence of such a time series $\{\mu_n\}_{n \in \mathbb{Z}_N}$, we need to assume the constraint : $\mu_{-N} = \Phi \mu_N + \varepsilon_{-N}$. That is, the $d \times d$ -matrix $\mathbb{I}_{d \times d} - \Phi^{2N+1}$ is invertible and

$$\mu_{-N} = (\mathbb{I}_{d \times d} - \Phi^{2N+1})^{-1} (\varepsilon_{-N} + \Phi^{2N} \varepsilon_{-N+1} + \dots + \Phi^{2N-k} \varepsilon_{-N+k+1} + \dots + \Phi \varepsilon_N).$$

In this case,

$$\mu_j = (\mathbb{I}_{d \times d} - \Phi^{2N+1})^{-1} (\varepsilon_j + \Phi^{2N} \varepsilon_{j+1} + \dots + \Phi^{2N-k} \varepsilon_{j+k+1} + \dots + \Phi \varepsilon_{j+2N})$$

for $j = -N, \dots, N$ and where $\varepsilon_{k+N} = \varepsilon_{k-N-1}$ for $k = 1, \dots, 2N$. Then, we can derive the expression for $K_{\mu}(r)$ in terms of Φ and r . For instance

$$K_{\mu}(0) = (\mathbb{I}_{d \times d} - \Phi^{2N+1})^{-1} (\mathbb{I} + (\Phi \Phi^*)^{2N} + \dots + (\Phi \Phi^*)^{2N-k} + \dots + \Phi \Phi^*) (\mathbb{I}_{d \times d} - \Phi^{*2N+1})^{-1}.$$

6.3 Restriction to \mathbb{Z}_N of a harmonizable process on \mathbb{Z}

In this section, we study the link between the spectrum of the time series under analysis and the spectrum of the observation interpreted as a time series on a finite group.

Let $\mathbf{X} = (\mathbf{X}_n)_{n \in \mathbb{Z}}$ be a harmonizable time series on \mathbb{Z} . Then the time series $\chi = (\chi_n)_{n \in \mathbb{Z}_N}$ defined by $\chi_n = \mathbf{X}_n$ for $n = -N, \dots, N$, is a strongly harmonizable on \mathbb{Z}_N with

$$\mu_{\chi}(\omega_j) = \frac{1}{2N+1} \sum_{n \in \mathbb{Z}_N} e^{-in\omega_j} \chi_n = \frac{1}{2N+1} \sum_{n=-N}^N e^{-in\omega_j} \mathbf{X}_n = \int_{\mathbb{T}} \frac{W_N(\lambda - \omega_j)}{2N+1} \mu_{\mathbf{X}}(d\lambda)$$

and

$$\mathbf{M}_{\chi}(\omega_{j_1}, \omega_{j_2}) = \iint_{\mathbb{T} \times \mathbb{T}} \frac{W_N(\lambda_1 - \omega_{j_1}) W_N(-\lambda_2 + \omega_{j_2})}{(2N+1)^2} \mathbf{M}_{\mathbf{X}}(d\lambda_1, d\lambda_2),$$

where

$$W_N(\lambda) = \sum_{n=-N}^N e^{in\lambda} = 1 + 2 \sum_{n=1}^N \cos(n\lambda).$$

Notice that $W_N(0) = 2N+1$ and for $\lambda \neq 0 \bmod 2\pi$ we have $|W_N(\lambda)|^2 = \frac{\sin^2((2N+1)\lambda/2)}{\sin^2(\lambda/2)}$.

Hence, $\mu_{\chi}(\omega_j) \approx \mu_{\mathbf{X}}(\omega_j)$ (in the sense of the L^2 -norm) and $\mathbf{M}_{\chi}(\omega_{j_1}, \omega_{j_2}) \approx \mathbf{M}_{\mathbf{X}}(\omega_{j_1}, \omega_{j_2})$ for large N (Rozaanov (see Theorem 2.2 1959) and Rao (1982)). Moreover, if $\lambda_1 < \lambda'_1$ and $\mu_{\mathbf{X}}(\{\lambda_1\}) = \mu_{\mathbf{X}}(\{\lambda'_1\}) = 0$, we have

$$\lim_{N \rightarrow \infty} \frac{1}{2\pi} \sum_{k=-N}^N \frac{e^{-i\lambda'_1 k} - e^{-i\lambda_1 k}}{-ik} \mathbf{X}_k = \mu_{\mathbf{X}}([\lambda_1, \lambda'_1]) \quad \text{in } L^2\text{-norm.}$$

If $\lambda_1 < \lambda'_1$, $\lambda_2 < \lambda'_2$ and $\mu_{\mathbf{X}}(\{\lambda_1\}) = \mu_{\mathbf{X}}(\{\lambda'_1\}) = \mu_{\mathbf{X}}(\{\lambda_2\}) = \mu_{\mathbf{X}}(\{\lambda'_2\}) = 0$, then we have

$$\lim_{N_1, N_2 \rightarrow \infty} \frac{1}{4\pi^2} \sum_{k_1=-N_1}^{N_1} \sum_{k_2=-N_2}^{N_2} \frac{e^{-i\lambda'_1 k_1} - e^{-i\lambda_1 k_1}}{-ik_1} \times \frac{e^{i\lambda'_2 k_2} - e^{i\lambda_2 k_2}}{ik_2} \mathbb{E}[\mathbf{X}_{k_1} \mathbf{X}_{k_2}^*] = \mathbf{M}_{\mathbf{X}}([\lambda_1, \lambda'_1] \times [\lambda_2, \lambda'_2]).$$

Remark 6.1 *If \mathbf{Z} is a harmonizable noise on \mathbb{Z} , then the time series ζ defined on \mathbb{Z}_N by $\zeta_n = \mathbf{Z}_n$ for $n = -N, \dots, N$, is a harmonizable noise on \mathbb{Z}_N with*

$$\mu_{\zeta}(\omega_j) = \frac{1}{2N+1} \sum_{n=-N}^N e^{-in\omega_j} \mathbf{Z}_n, \quad \mathbf{M}_{\zeta}(\omega_{j_1}, \omega_{j_2}) = \frac{1}{(2N+1)^2} \sum_{n=-N}^N e^{-in(\omega_{j_1} - \omega_{j_2})} \Sigma_{\mathbf{Z}_n}^2$$

and

$$n_{\zeta}(\omega_j) = \frac{1}{(2N+1)^2} \sum_{n=-N}^N e^{-in\omega_j} \Sigma_{\mathbf{Z}_n}^2.$$

6.4 Extension to \mathbb{Z} of a harmonizable process defined on \mathbb{Z}_N

Since we cannot obtain a non-null harmonizable noise on \mathbb{Z} from periodic extension, we discuss hereafter how to obtain a finite time sample, from the extension by zeros. We show that we can obtain harmonizable noise on \mathbb{Z} and characterize its moving average form.

Let χ be a harmonizable time series on \mathbb{Z}_N , and let $\tilde{\mathbf{X}}$ be the time series on \mathbb{Z} defined by $\tilde{\mathbf{X}}_n = \chi_n$ for $n = -N, \dots, N$, and $\tilde{\mathbf{X}}_n = \mathbf{0}_d$ otherwise. Then, in general, we cannot say whether the time series $\tilde{\mathbf{X}}$ is harmonizable or not. Nevertheless, for a harmonizable noise $\zeta = (\zeta_n)_{n \in \mathbb{Z}_N}$, the time series $\tilde{\mathbf{Z}} = (\tilde{\mathbf{Z}}_n)_{n \in \mathbb{Z}}$ defined by $\tilde{\mathbf{Z}}_n = \zeta_n$ for $n = -N, \dots, N$, and $\tilde{\mathbf{Z}}_n = \mathbf{0}_d$ otherwise, is a harmonizable noise on \mathbb{Z} . Indeed, condition (Ha) is fulfilled by $\tilde{\mathbf{Z}}$.

Furthermore, from Proposition 3.2, the stochastic spectral measure $\mu_{\tilde{\mathbf{Z}}}$ admits a density function with respect to the Lebesgue measure on \mathbb{T} :

$$\Delta_{\tilde{\mathbf{Z}}}(\lambda) = \frac{1}{2\pi} \sum_{n=-N}^N e^{-in\lambda} \zeta_n = \frac{1}{2\pi} \sum_{j \in \mathbb{Z}_N} W_N(\omega_j - \lambda) \mu_{\zeta}(\omega_j),$$

and the measure $n_{\tilde{\mathbf{Z}}}$ admits a density function

$$D_{\tilde{\mathbf{Z}}}(\lambda) = \frac{1}{4\pi^2} \sum_{n=-N}^N e^{-in\lambda} \text{Var}(\zeta_n) = \frac{2N+1}{4\pi^2} \sum_{j \in \mathbb{Z}_N} W_N(\omega_j - \lambda) \mathbf{K} \mu_{\zeta}(j).$$

6.4.1 Linear harmonizable time series on \mathbb{Z} with innovation $\tilde{\mathbf{Z}}$

Let $(\mathbf{A}_n)_{n \in \mathbb{Z}}$ be a family of matrices such that $\sum_n \text{tr}(\mathbf{A}_n \mathbf{A}_n^*) < \infty$. Define the time series \mathbf{X} on \mathbb{Z} by

$$\mathbf{X}_n \stackrel{\text{def}}{=} \sum_{m=-N}^N \mathbf{A}_{n-m} \tilde{\mathbf{Z}}_m.$$

Then the time series \mathbf{X} is strongly harmonizable with the stochastic spectral density

$$\begin{aligned}\Delta_{\mathbf{X}}(\lambda) &= \widehat{\mathbf{A}}(-\lambda)\Delta_{\widehat{\mathbf{Z}}}(\lambda) = \frac{1}{2\pi}\widehat{\mathbf{A}}(-\lambda)\sum_{n=-N}^N e^{-in\lambda}\zeta_n \\ &= \frac{1}{2\pi}\widehat{\mathbf{A}}(-\lambda)\sum_{j\in\mathbb{Z}_N} W_N(\omega_j - \lambda)\boldsymbol{\mu}_{\zeta}(\omega_j).\end{aligned}$$

Its spectral bimeasure admits a density function (see Remark (3.4))

$$\begin{aligned}\mathbf{F}_{\mathbf{X}}(\lambda_1, \lambda_2) &= \widehat{\mathbf{A}}(-\lambda_1)\mathbf{F}_{\widehat{\mathbf{Z}}}(\lambda_1, \lambda_2)\widehat{\mathbf{A}}^*(-\lambda_2) = \widehat{\mathbf{A}}(-\lambda_1)\mathbf{D}_{\widehat{\mathbf{Z}}}(\lambda_1 - \lambda_2)\widehat{\mathbf{A}}^*(-\lambda_2) \\ &= \frac{2N+1}{4\pi^2}\sum_{j\in\mathbb{Z}_N} W_N(\omega_j - \lambda_1 + \lambda_2)\widehat{\mathbf{A}}(-\lambda_1)\mathbf{K}_{\boldsymbol{\mu}_{\zeta}}(j)\widehat{\mathbf{A}}^*(-\lambda_2).\end{aligned}\quad (21)$$

Now consider the "restriction" $\boldsymbol{\chi}$ on \mathbb{Z}_N of the time series \mathbf{X} defined by $\boldsymbol{\chi}_n = \mathbf{X}_n$ for $n = -N, \dots, N$. From Section 6.3, we know that $\boldsymbol{\chi}$ is a harmonizable time series in \mathbb{Z}_N with the stochastic spectral measure

$$\boldsymbol{\mu}_{\boldsymbol{\chi}}(\omega_j) = \int_{\mathbb{T}} \frac{W_N(\lambda - \omega_j)}{2N+1} \boldsymbol{\mu}_{\mathbf{X}}(d\lambda) = \frac{1}{2\pi(2N+1)} \sum_{k\in\mathbb{Z}_N} \left(\int_{\mathbb{T}} W_N(\lambda - \omega_j) W_N(\omega_k - \lambda) \widehat{\mathbf{A}}(-\lambda) d\lambda \right) \boldsymbol{\mu}_{\zeta}(\omega_k)$$

and spectral bimeasure

$$\begin{aligned}\mathbf{M}_{\boldsymbol{\chi}}(\omega_{j_1}, \omega_{j_2}) &= \iint_{\mathbb{T}\times\mathbb{T}} \frac{W_N(\lambda_1 - \omega_{j_1}) W_N(-\lambda_2 + \omega_{j_2})}{(2N+1)^2} \mathbf{M}_{\mathbf{X}}(d\lambda_1, d\lambda_2) \\ &= \iint_{\mathbb{T}\times\mathbb{T}} \frac{W_N(\lambda_1 - \omega_{j_1}) W_N(\lambda_2 - \omega_{j_2})}{(2N+1)^2} \widehat{\mathbf{A}}(-\lambda_1) \mathbf{D}_{\widehat{\mathbf{Z}}}(\lambda_1 - \lambda_2) \widehat{\mathbf{A}}^*(-\lambda_2) d\lambda_1 d\lambda_2 \\ &= \frac{1}{4\pi^2(2N+1)} \sum_{j\in\mathbb{Z}_N} \iint_{\mathbb{T}\times\mathbb{T}} W_N(\lambda_1 - \omega_{j_1}) W_N(\lambda_2 - \omega_{j_2}) W_N(\omega_j - \lambda_1 + \lambda_2) \times \\ &\quad \times \widehat{\mathbf{A}}(-\lambda_1) \mathbf{K}_{\boldsymbol{\mu}_{\zeta}}(j) \widehat{\mathbf{A}}^*(-\lambda_2) d\lambda_1 d\lambda_2.\end{aligned}\quad (22)$$

Recall that $W_N(0) = 2N+1$ and for $\lambda \neq 0 \pmod{2\pi}$ we have $W_N(\lambda)^2 = \frac{\sin^2((2N+1)\lambda/2)}{\sin^2(\lambda/2)}$.

6.4.2 Band-limited harmonizable time series

We consider a white noise $(\varepsilon_n)_{n\in\mathbb{Z}_N}$ on \mathbb{Z}_N and a family $(\boldsymbol{\vartheta}_n)_{n\in\mathbb{Z}_N}$ of complex $d \times d$ -matrices. Then the moving average

$$\boldsymbol{\mu}_j = \sum_{k\in\mathbb{Z}_N} \boldsymbol{\vartheta}_{j-k} \varepsilon_k$$

defines a stationary time series $\boldsymbol{\mu} = (\boldsymbol{\mu}_j)_{j\in\mathbb{Z}_N}$ with covariance function

$$\mathbf{K}_{\boldsymbol{\mu}}(r) = \mathbb{E}(\boldsymbol{\mu}_{j+r} \boldsymbol{\mu}_j^*) = \sum_{k\in\mathbb{Z}_N} \boldsymbol{\vartheta}_{k+r} \boldsymbol{\Sigma}_{\varepsilon}^2 \boldsymbol{\vartheta}_k^*.$$

Whenever $\boldsymbol{\vartheta}_k = \mathbf{0}_d$ for $k = \pm q, \dots, \pm N$ for some $0 < q \leq N/2$, then $\mathbf{K}_{\boldsymbol{\mu}}(r) = \mathbf{0}_{d \times d}$ for $r = \pm 2q, \dots, \pm N$, and the harmonizable noise ζ defined by relation (18) has a band-limited (Loève) spectrum: $\boldsymbol{n}_{\zeta}(\omega_j) = \mathbf{0}_{d \times d}$ for $j = \pm 2q, \dots, \pm N$.

Hence, when $\widehat{\mathbf{A}}$ is *smooth enough*, relations (21) and (22) imply that

$$\mathbf{F}_{\mathbf{X}}(\lambda_1, \lambda_2)/(2N+1)^2 \quad \text{and} \quad \mathbf{M}_{\boldsymbol{\chi}}(\omega_{j_1}, \omega_{j_2})/(2N+1)^2$$

are close to 0 for sufficiently large N , and (λ_1, λ_2) , $(\omega_{j_1}, \omega_{j_2})$ are 'far' from the main diagonal.

6.5 Harmonizable VARMA time series on \mathbb{Z}_N

We are now ready to define HVARMA models. First, let us consider the following difference equation:

$$\mathbf{X}_n + \phi_1 \mathbf{X}_{n-1} + \cdots + \phi_p \mathbf{X}_{n-p} = \tilde{\mathbf{Z}}_n + \theta_1 \tilde{\mathbf{Z}}_{n-1} + \cdots + \theta_q \tilde{\mathbf{Z}}_{n-q}, \quad (23)$$

where $\{\tilde{\mathbf{Z}}_n\}$ is extension by zeros of a harmonizable noise on \mathbb{Z}_N . Then the results of the previous section apply with $\tilde{\mathbf{A}}(\lambda) = \Phi(e^{i\lambda})^{-1} \Theta(e^{i\lambda})$.

Then, the generation of HVARMA time series is as follows:

- Generate a realization from the (incremental) process $\boldsymbol{\mu} = (\boldsymbol{\mu}_j)_{j \in \mathbb{Z}_N}$ according to a zero mean VARMA process of order (p', q') defined on \mathbb{Z}_N , denoted as $\text{VARMA}(p', q')$, with matrix polynomials $\tilde{\Phi}(\cdot)$, $\tilde{\Theta}(\cdot)$. Then, the time series $\boldsymbol{\zeta} = (\boldsymbol{\zeta}_n)_{n \in \mathbb{Z}_N}$ defined by

$$\boldsymbol{\zeta}_n \stackrel{\text{def}}{=} \sum_{j \in \mathbb{Z}_N} e^{in\omega_j} \boldsymbol{\mu}_j, \quad \omega_j = \frac{2\pi j}{2N+1},$$

is a (strongly) harmonizable noise on \mathbb{Z}_N .

- Generate a time series realization on \mathbb{Z}_N as $\mathbf{X}_n = \sum_{j \in \mathbb{Z}_N} e^{i\omega_j n} \hat{\mathbf{A}}(\omega_j) \boldsymbol{\mu}_j$, where $\hat{\mathbf{A}}(\lambda)$ defines a VARMA process of order (p, q) , denoted as $\text{VARMA}(p, q)$.

We denote the resulting process as $H\text{VARMA}(p, q) - (p', q')$, where (p, q) is the order of the VARMA process (see the equation 23) and (p', q') is the order of VARMA incremental process. Below we give the formal definition of the $H\text{VARMA}(p, q) - (p', q')$ process.

Definition 6.1 (HVARMA $(p, q) - (p', q')$ process) *The process $(\mathbf{X}_n)_{n \in \mathbb{Z}}$ is said to be a $H\text{VARMA}(p, q) - (p', q')$ process if for every $n \in \mathbb{Z}$,*

$$\mathbf{X}_n + \phi_1 \mathbf{X}_{n-1} + \cdots + \phi_p \mathbf{X}_{n-p} = \tilde{\mathbf{Z}}_n + \theta_1 \tilde{\mathbf{Z}}_{n-1} + \cdots + \theta_q \tilde{\mathbf{Z}}_{n-q}, \quad (24)$$

where $\{\tilde{\mathbf{Z}}_n\}$ is extension by zeros of a harmonizable noise on \mathbb{Z}_N (see Section 6.4), $\tilde{\mathbf{Z}}_n = \xi_n$ for $n = -1, \dots, N$ and $\tilde{\mathbf{Z}}_n = 0$ otherwise. Moreover, $\xi_n = \sum_{j=-N}^N e^{i\omega_j n} \boldsymbol{\mu}_j$, where $\boldsymbol{\mu} = (\boldsymbol{\mu}_j)_{j \in \mathbb{Z}_N}$ follows a $\text{VARMA}(p', q')$ process on \mathbb{Z}_N . Its spectral bimeasure is given by the equation (22).

Remark 6.2 *Theorem 5.1 allows us to understand how harmonizable noise contributes to the spectral properties of $H\text{VARMA}(p, q)$ processes by inducing stationary dependencies in the frequency domain. To generate time series from a discretized spectral representation according to the steps described in Section 6, we chose to use a $\text{VARMA}(p', q')$ model, resulting in the class of $H\text{VARMA}(p, q) - (p', q')$ (see Definition 6.1).*

6.6 Some examples

In this section, we will present some examples of two-dimensional $\text{VARMA}(p, q) - (p', q')$ time series that we generated. They were constructed by combining two different $\text{VARMA}(p, q)$ ($\text{VARMA}(1, 1)$ and $\text{VARMA}(3, 2)$) processes with three different parametrizations for the incremental process $\text{VARMA}(p', q')$ (described below as variants 1-3). These were

- $\text{VARMA}(1, 1)$

$$\Phi = \begin{pmatrix} 0.9 & 0.4 \\ 0.4 & 0 \end{pmatrix}; \Theta = \begin{pmatrix} 0 & 0.4 \\ 0.4 & 0.8 \end{pmatrix};$$

- *VARMA*(3,2)

$$\Phi_1 = \begin{pmatrix} -0.9 & 0.8 \\ 0.8 & 0 \end{pmatrix}; \Phi_2 = \begin{pmatrix} 0.2 & 0.4 \\ 0.4 & 0.1 \end{pmatrix}; \Phi_3 = \begin{pmatrix} 0.5 & 0.2 \\ 0.2 & 0 \end{pmatrix}; \Theta_1 = \begin{pmatrix} 0.4 & 0 \\ 0 & 0.9 \end{pmatrix}; \Theta_2 = \begin{pmatrix} -0.2 & 0 \\ 0 & 0.4 \end{pmatrix};$$

together with

1. Variant 1

$$\tilde{\Phi} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}; \tilde{\Theta} = \begin{pmatrix} 0.8 & 0 \\ 0 & 0.8 \end{pmatrix};$$

2. Variant 2

$$\tilde{\Phi} = \begin{pmatrix} 0.8 & 0 \\ 0 & 0.8 \end{pmatrix}; \tilde{\Theta} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix};$$

3. Variant 3

$$\tilde{\Phi}_1 = \begin{pmatrix} 0.1 & 0.2 \\ 0.2 & 0.1 \end{pmatrix}; \tilde{\Phi}_2 = \begin{pmatrix} 0.9 & 0 \\ 0 & 0.2 \end{pmatrix}; \tilde{\Theta}_1 = \begin{pmatrix} 0.9 & 0 \\ 0 & 0.2 \end{pmatrix}; \tilde{\Theta}_2 = \begin{pmatrix} 0.5 & 0 \\ 0 & 0.2 \end{pmatrix};$$

$$\tilde{\Theta}_a = \begin{pmatrix} 0.4 & 0 \\ 0 & 0.2 \end{pmatrix}; \tilde{\Theta}_b = \begin{pmatrix} 0.4 & 0.9 \\ 0.9 & 0.2 \end{pmatrix},$$

where $a \in \{10, 11, 16, 80, 81\}$ and $b \in \{14, 15, 82\}$.

Single realizations of each of these processes are shown in Figures 3-10 (see top left panels). In addition, the corresponding Loève (auto-)spectra are given in each figure in the top right and bottom right panels. Finally, the bottom left panel presents the dual-frequency coherence (see Definition 5.2). The values of the spectrum and coherence are displayed on a logarithmic scale, with colors ranging from blue through yellow to red to indicate larger values.

Figures 3 and 7 show results for correlated stationary *VARMA*. The obtained Loève spectra and dual-frequency coherence are supported on the main diagonal. The process in Figures 4, 5 and 8, 9 were obtained with specifications of rather simple incremental processes, introducing some dependencies in the frequencies over a relatively restricted domain (see for example the Loève spectrum of the process *HVARMA*(1,1) – (1,0)). Definitely, more sophisticated dependencies were obtained with the variant 3 (see Figures 6 and 10).

7 Summary

This paper describes a novel parametric model for a wide class of nonstationary time series that present correlation between their frequency components. The proposed Harmonizable *VARMA*(p, q) process is obtained as a solution of a difference equation based on a harmonizable noise. We describe its spectral characteristics and propose a least squares estimator in the *HAR*(1) model. We prove its consistency and asymptotic normality. Next, we describe how to generate realizations of such processes. That requires to specify a stationary dependence structure in the frequencies. It gives rise to a class of processes whose

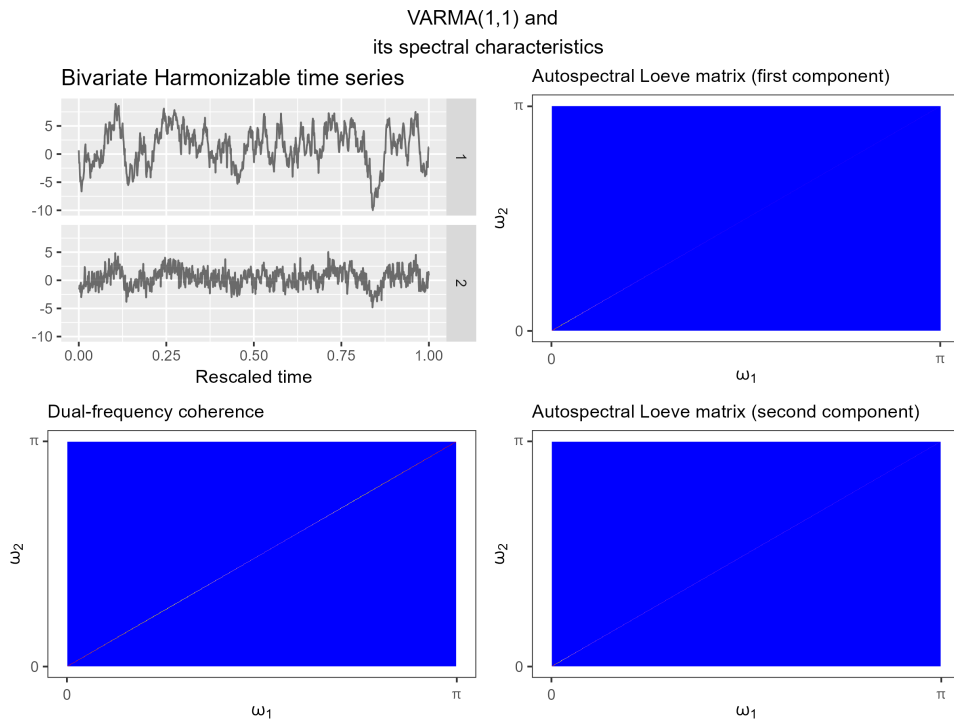


Figure 3: Realization of $VARMA(1,1)$ (top left panel) together with the corresponding Loève (auto-)spectra (top right and bottom right panels) and dual-frequency coherence (bottom left panel). The values of the spectrum and coherence are displayed on a logarithmic scale, with colors ranging from blue through yellow to red to indicate larger values.

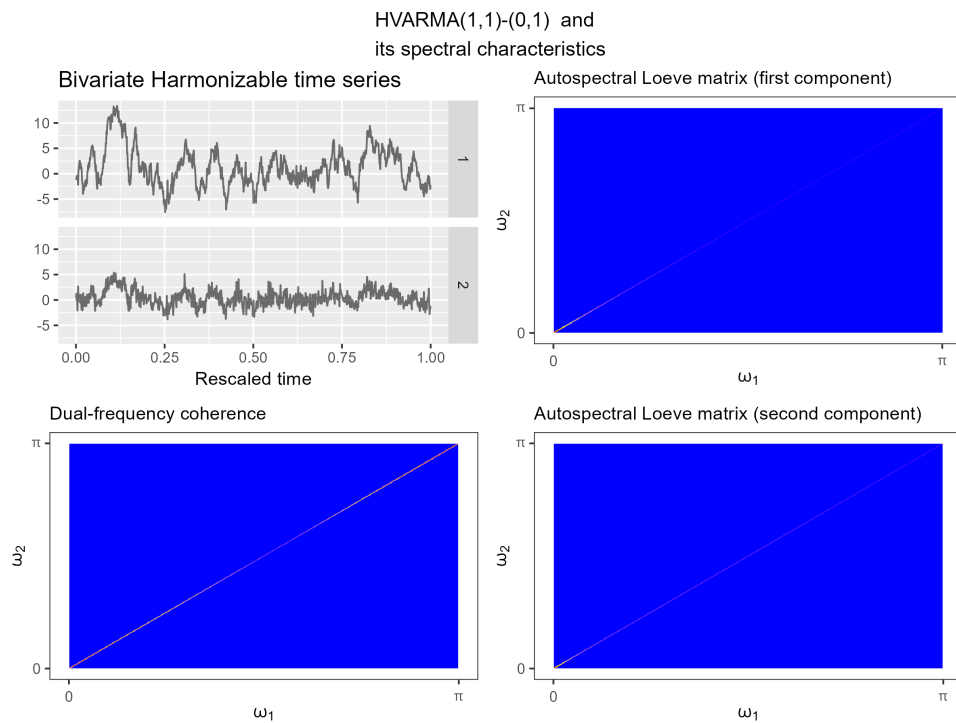


Figure 4: Realization of $HVARMA(1,1) - (0,1)$ (top left panel) together with the corresponding Loève (auto-)spectra (top right and bottom right panels) and dual-frequency coherence (bottom left panel). The values of the spectrum and coherence are displayed on a logarithmic scale, with colors ranging from blue through yellow to red to indicate larger values.

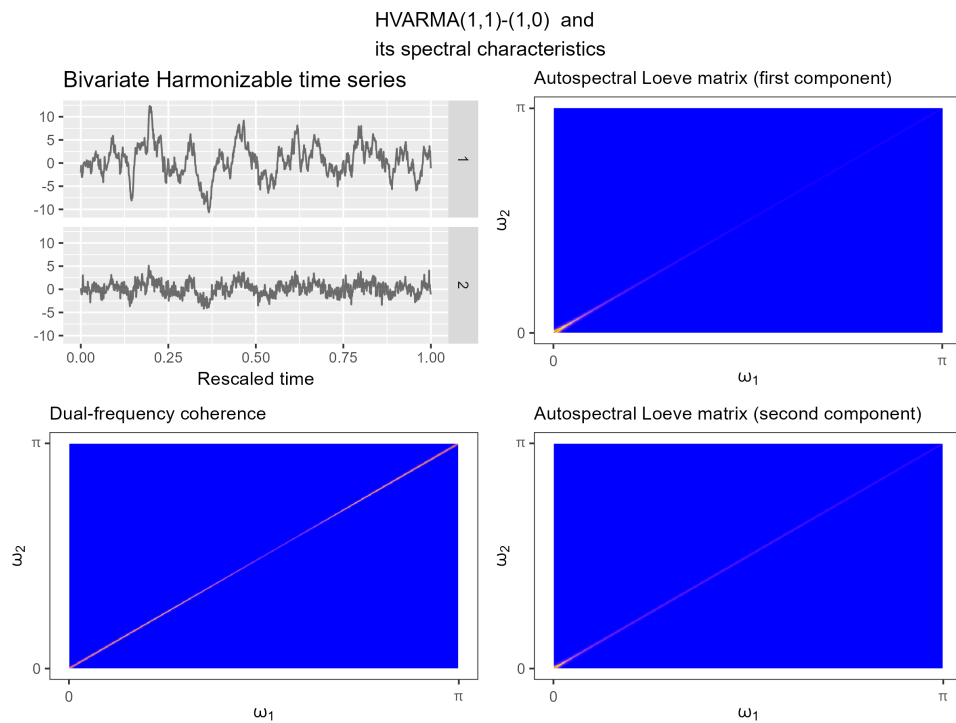


Figure 5: Realization of $HVARMA(1,1) - (1,0)$ (top left panel) together with the corresponding Loève (auto-)spectra (top right and bottom right panels) and dual-frequency coherence (bottom left panel). The values of the spectrum and coherence are displayed on a logarithmic scale, with colors ranging from blue through yellow to red to indicate larger values.

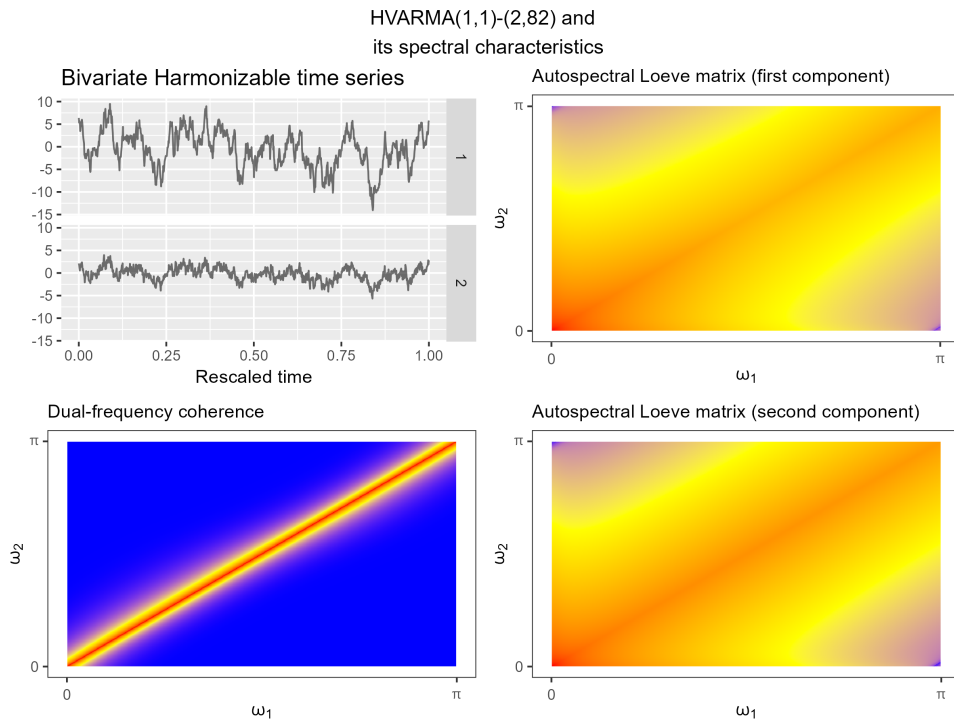


Figure 6: Realization of $HVARMA(1,1) - (2,82)$ (top left panel) together with the corresponding Loève (auto-)spectra (top right and bottom right panels) and dual-frequency coherence (bottom left panel). The values of the spectrum and coherence are displayed on a logarithmic scale, with colors ranging from blue through yellow to red to indicate larger values.

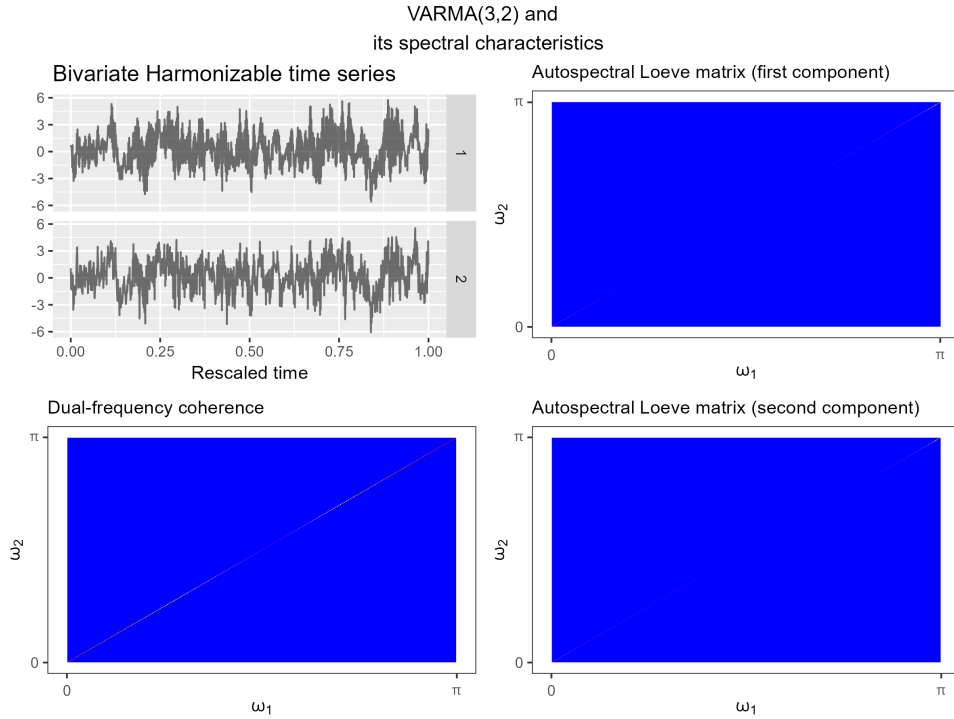


Figure 7: Realization of $VARMA(3,2)$ (top left panel) together with the corresponding Loève (auto-)spectra (top right and bottom right panels) and dual-frequency coherence (bottom left panel). The values of the spectrum and coherence are displayed on a logarithmic scale, with colors ranging from blue through yellow to red to indicate larger values.

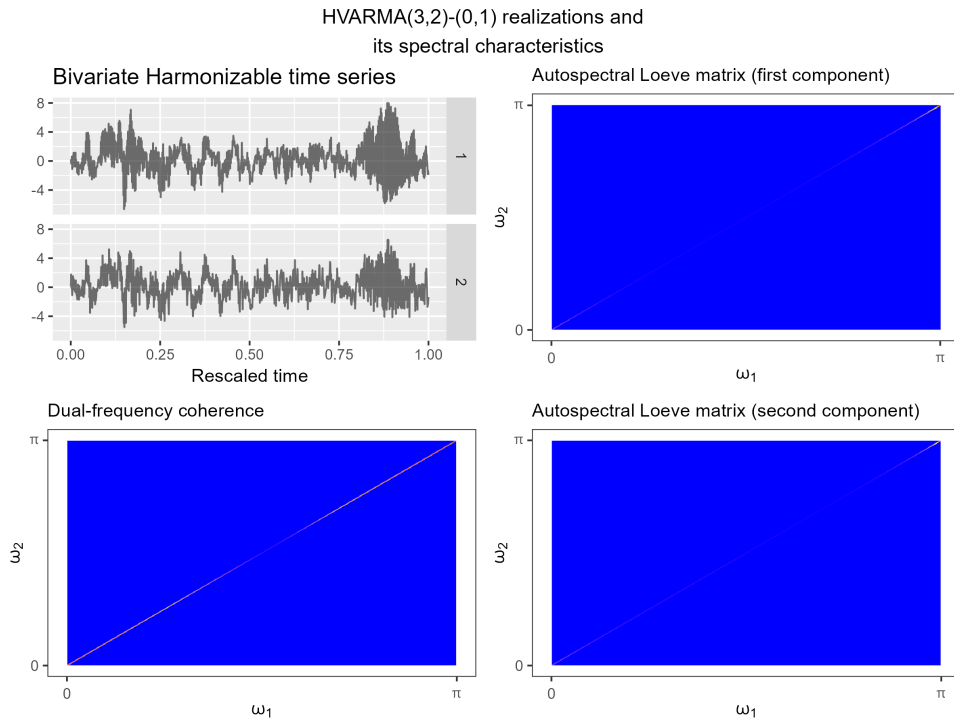


Figure 8: $HVARMA(3,2) - (0,1)$

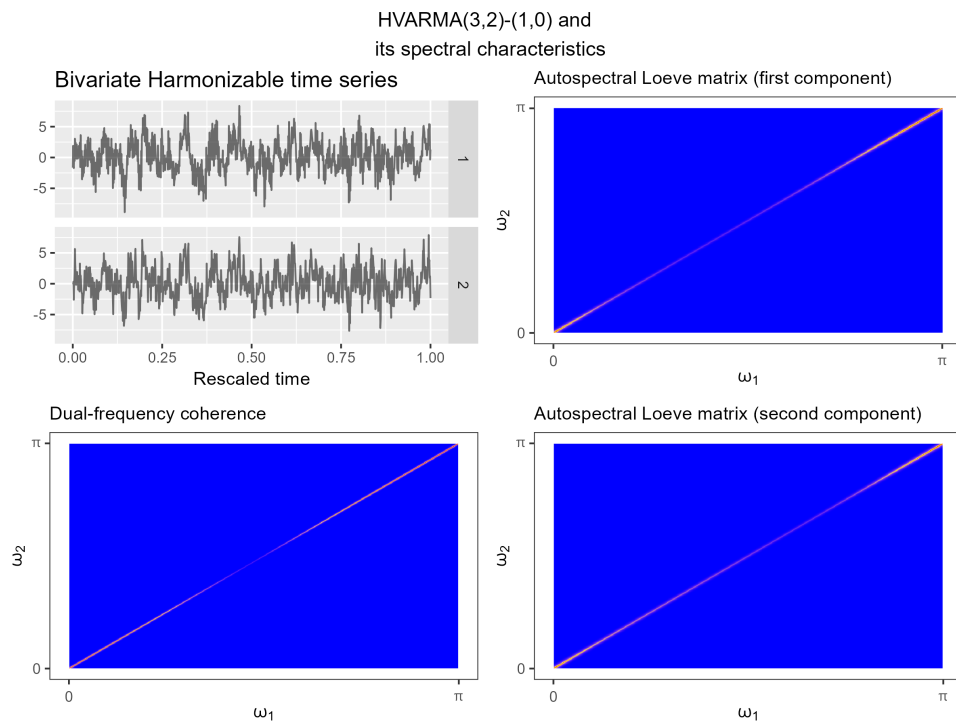


Figure 9: Realization of $HVARMA(3, 2) - (1, 0)$ (top left panel) together with the corresponding Loève (auto-)spectra (top right and bottom right panels) and dual-frequency coherence (bottom left panel). The values of the spectrum and coherence are displayed on a logarithmic scale, with colors ranging from blue through yellow to red to indicate larger values.

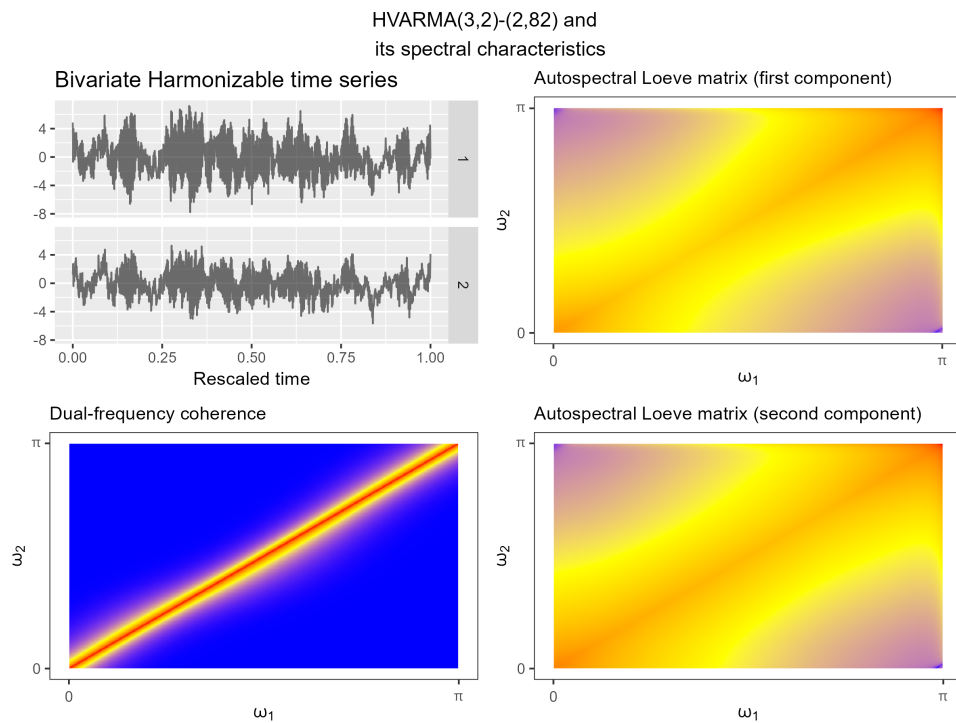


Figure 10: Realization of $HVARMA(3, 2) - (2, 82)$ (top left panel) together with the corresponding Loève (auto-)spectra (top right and bottom right panels) and dual-frequency coherence (bottom left panel). The values of the spectrum and coherence are displayed on a logarithmic scale, with colors ranging from blue through yellow to red to indicate larger values.

Cramèr's representation is that of a VARMA process of order (p, q) with correlated increments following a VARMA of order (p', q') . We exhibit the spectral characteristics of these $HVARMA(p, q) - (p', q')$ processes, and we show how to generate realizations from given models. Our results pave the way towards novel developments on estimation, prediction, model selection, etc., for harmonizable processes. Consequently, non-specialists will be able to model such complex processes in a relatively simple way (as it is currently possible, for example, for ARMA processes).

8 Annexe: Proofs

Proof of Lemma 3.1

Clearly, if there exists a d -variate white noise $\boldsymbol{\varepsilon} \stackrel{\text{def}}{=} (\varepsilon_n)_{n \in \mathbb{Z}}$ such that $\mathbf{Z}_n = \mathbf{S}_n \boldsymbol{\varepsilon}_n$, where $\sup_{n,j,k} |s_{n,j,k}| < \infty$, then the d -variate time series $\mathbf{Z} = (\mathbf{Z}_n)_{n \in \mathbb{Z}}$ satisfies the condition (Ha).

Conversely, suppose that $\mathbf{Z} = (\mathbf{Z}_n)_{n \in \mathbb{Z}}$ is a d -variate time series satisfying the condition (Ha). Let $d_n = \dim \mathcal{H}(\mathbf{Z}_n) \leq d$. When $d_n \neq 0$, there exists an orthonormal basis $\{\epsilon_{n,1}, \dots, \epsilon_{n,d_n}\}$ of $\mathcal{H}(\mathbf{Z}_n)$ (Schmidt method of orthonormalization). Since the Hilbert subspaces $\mathcal{H}(\mathbf{Z}_n)$, $n \in \mathbb{Z}$ are orthogonal, and we assume that the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ is sufficiently large, we can complete the orthonormal family $\{\epsilon_{n,j} : n \in \mathbb{Z}, j = 1, \dots, d_n\}$ to an orthonormal family $\{\epsilon_{n,j} : n \in \mathbb{Z}, j = 1, \dots, d\}$ of \mathcal{H} . Then the time series $(\boldsymbol{\varepsilon}_n)_{n \in \mathbb{Z}}$, where $\boldsymbol{\varepsilon}_n = (\epsilon_{n,1}, \dots, \epsilon_{n,d})^\top$ for $n \in \mathbb{Z}$, is a d -variate pure white noise, and the space $\mathcal{H}(\mathbf{Z}_n)$ is a subspace of the Hilbert space $\mathcal{H}(\boldsymbol{\varepsilon}_n)$. Hence, the proof of the lemma can be easily achieved using the assumption $\sup_{n,j} \mathbb{E}\{|Z_{n,j}|^2\} < \infty$.

Proof of Theorem 3.1

To prove the theorem, we use the notation of Lemma 3.1 and construct a bounded linear operator $\Xi : \mathcal{H}^d(\boldsymbol{\varepsilon}) \rightarrow \mathcal{H}^d(\mathbf{Z})$ such that $\mathbf{Z}_n = \Xi \boldsymbol{\varepsilon}_n$. Since the time series $(\boldsymbol{\varepsilon}_n)_{n \in \mathbb{Z}}$ is stationary, we get that the time series $(\mathbf{Z}_n)_{n \in \mathbb{Z}}$ is harmonizable with $\boldsymbol{\mu}_{\mathbf{Z}}(A) = \Xi \boldsymbol{\mu}_{\boldsymbol{\varepsilon}}(A)$ for $A \in \mathcal{B}(\mathbb{T})$ (Rao, 1982) (D.S, 1957 Theorem 3.2.19).

First, we prove that any d -variate harmonizable time series $(\mathbf{Z}_n)_{n \in \mathbb{Z}}$ whose stochastic spectral measure satisfies relation (1), satisfies the condition (Ha). Let $n \in \mathbb{Z}$ be fixed. Then $\{\epsilon_{n,j} : j = 1, \dots, d\}$ is an orthonormal basis of the Hilbert space $\mathcal{H}(\boldsymbol{\varepsilon}_n)$. Since $Z_{n,j} \in \mathcal{H}(\boldsymbol{\varepsilon}_n)$, we have that $Z_{n,j} = \mathbf{S}_{n,j} \boldsymbol{\varepsilon}_n$, where $(\mathbf{S}_{n,j})^\top = (s_{n,j,1}, \dots, s_{n,j,d})^\top \in \mathbb{C}^d$, for any $j = 1, \dots, d$. It follows that $\mathbf{Z}_n = \mathbf{S}_n \boldsymbol{\varepsilon}_n$, where \mathbf{S}_n is a $d \times d$ -matrix whose rows are the row vectors $\mathbf{S}_{n,j}$, $j = 1, \dots, d$.

Then, since the $\epsilon_{n,j}$, $n \in \mathbb{Z}, j = 1, \dots, d$, are linearly independent, we can define a linear operator Ξ_j from the linear space $\{\mathbf{A} \boldsymbol{\varepsilon}_n : n \in \mathbb{Z}, \mathbf{A} \in \mathbb{C}^{d \times d}\} \subset \mathcal{H}^d(\boldsymbol{\varepsilon})$ into $\mathcal{H}(\mathbf{Z})$ by $\Xi_j \boldsymbol{\varepsilon}_n = Z_{n,j}$.

Moreover, the $\epsilon_{n,j}$, $n \in \mathbb{Z}, j = 1, \dots, d$, are orthonormal in the Hilbert space $\mathcal{H}(\boldsymbol{\varepsilon})$, that is, uncorrelated random variables with zero mean and variance equal to 1. Thus, we have

$$\text{var}(Z_{n,j}) = \mathbb{E}(|Z_{n,j}|^2) = \|Z_{n,j}\|_{\mathcal{H}}^2 = \mathbf{S}_{n,j} \mathbf{S}_{n,j}^*.$$

The assumption (Ha) implies that $\sup_{n,j,k} |s_{n,j,k}| < \infty$. Consequently, the linear operator Ξ_j is bounded, and can be extended as a bounded linear operator from $\mathcal{H}^d(\boldsymbol{\varepsilon})$ to $\mathcal{H}(\mathbf{Z})$, also denoted Ξ_j . Finally, we can easily define a bounded linear operator Ξ from $\mathcal{H}^d(\boldsymbol{\varepsilon})$ to $\mathcal{H}^d(\mathbf{Z})$ such that $\mathbf{Z}_n = \Xi \boldsymbol{\varepsilon}_n$, for any $n \in \mathbb{Z}$. The bounded operators $\Xi_j : \mathcal{H}^d(\boldsymbol{\varepsilon}) \rightarrow \mathcal{H}(\mathbf{Z})$, $j = 1, \dots, d$ are the component operators of Ξ . Thus, the d -variate noise \mathbf{Z} is the image by a bounded linear operator Ξ of the d -variate stationary noise $\boldsymbol{\varepsilon}$, and from Dunford and Schwartz (1957, Theorem 3.2.19) (see also Corollary 4.3 in Rao, 1982), we get that \mathbf{Z} is harmonizable and its stochastic spectral measure $\boldsymbol{\mu}_{\mathbf{Z}}$ satisfies $\boldsymbol{\mu}_{\mathbf{Z}}(A) = \Xi(\boldsymbol{\mu}_{\boldsymbol{\varepsilon}}(A))$, $A \in \mathcal{B}(\mathbb{T})$.

Now we show that (1) holds. We know that $\mathcal{H}(\boldsymbol{\mu}_{\mathbf{Z}}) = \mathcal{H}(\mathbf{Z})$ is a Hilbert subspace of $\mathcal{H}(\boldsymbol{\varepsilon})$, and the family $\{\epsilon_{n,k} : n \in \mathbb{Z}, k = 1, \dots, d\}$ is an orthonormal basis of $\mathcal{H}(\boldsymbol{\varepsilon}) \subset L^2_{\mathbb{C}}(\mathbb{P})$. Therefore, for $j = 1, \dots, d$, the stochastic spectral measure μ_{Z_j} of the component time series $Z_j = (Z_{n,j})_{n \in \mathbb{Z}}$ can be decomposed as

$$\mu_{Z_j}(A) = \sum_{n \in \mathbb{Z}} \sum_{k=1}^d \mu_{Z_j, n, k}(A) \epsilon_{n, k}, \quad A \in \mathcal{B}(\mathbb{T}),$$

where the complex-valued measure $\mu_{Z_j, n, k}$ is defined on $\mathcal{B}(\mathbb{T})$ by $\mu_{Z_j, n, k}(A) \stackrel{\text{def}}{=} \langle \mu_{Z_j}(A) | \epsilon_{n, k} \rangle_{\mathcal{H}} = \mathbb{E}[\mu_{Z_j}(A) \overline{\epsilon_{n, k}}]$. Furthermore,

$$\langle Z_{m, j} | \epsilon_{n, k} \rangle_{\mathcal{H}} = \left\langle \int_{\mathbb{T}} e^{im\lambda} \mu_{Z_j}(d\lambda) \middle| \epsilon_{n, k} \right\rangle_{\mathcal{H}} = \int_{\mathbb{T}} e^{im\lambda} \mu_{Z_j, n, k}(d\lambda)$$

for any $m, n \in \mathbb{Z}$ and any $j, k = 1, \dots, d$. Besides, we know that $\langle Z_{m, j} | \epsilon_{n, k} \rangle_{\mathcal{H}} = 0$ for $m \neq n$ and $\langle Z_{m, j} | \epsilon_{n, k} \rangle_{\mathcal{H}} = s_{n, j, k}$. Thanks to the characterization of a complex-valued measure by its Fourier coefficients, we readily deduce that the measures $\mu_{Z_j, n, k}$, $n \in \mathbb{Z}$, $j, k = 1, \dots, d$ are absolutely continuous with respect to the Lebesgue measure on \mathbb{T} and

$$\mu_{Z_j, n, k}(A) = \frac{s_{n, j, k}}{2\pi} \left(\int_A e^{-in\lambda} d\lambda \right). \quad (25)$$

Then

$$\begin{aligned} \mu_{Z_j}(A) &= \frac{1}{2\pi} \sum_{n \in \mathbb{Z}} \sum_{k=1}^d \left(\int_A e^{-in\lambda} d\lambda \right) s_{n, j, k} \epsilon_{n, k} \\ &= \frac{1}{2\pi} \sum_{n \in \mathbb{Z}} \left(\int_A e^{-in\lambda} d\lambda \right) \mathbf{S}_{n, j} \boldsymbol{\epsilon}_n = \frac{1}{2\pi} \sum_{n \in \mathbb{Z}} \left(\int_A e^{-in\lambda} d\lambda \right) Z_{n, j} \end{aligned}$$

and consequently we get the relation (1).

Proof of Proposition 3.1

First, assume that $(\mathbf{Z}_n)_{n \in \mathbb{Z}}$ is strongly harmonizable. Then its spectral bimeasure $\mathbf{M}_{\mathbf{Z}}$ is extendible as a $\mathbb{C}^{d \times d}$ -valued measure on \mathbb{T}^2 .

$$\text{Cov}(\mathbf{Z}_n, \mathbf{Z}_{n+k}) = \iint_{\mathbb{T}^2} e^{i(n\lambda_1 - (n+k)\lambda_2)} \mathbf{M}_{\mathbf{Z}}(d\lambda_1, d\lambda_2) = \iint_{\mathbb{T}^2} e^{i(n(\lambda_1 - \lambda_2) - k\lambda_2)} \mathbf{M}_{\mathbf{Z}}(d\lambda_1, d\lambda_2).$$

We consider the transformation of \mathbb{T}^2 defined by $\mathcal{T} : (\lambda_1, \lambda_2) \mapsto (\lambda_1 - \lambda_2, \lambda_2)$, $\mathcal{T}^{-1} : (\lambda, \alpha) \mapsto (\lambda + \alpha, \alpha)$. Let $\mathbf{M}_{\mathbf{Z}}^{\mathcal{T}} = \mathbf{M}_{\mathbf{Z}} \circ \mathcal{T}^{-1}$ be the pushforward measure of $\mathbf{M}_{\mathbf{Z}}$ by the transformation \mathcal{T} and defined on \mathbb{T}^2 by $\mathbf{M}_{\mathbf{Z}}^{\mathcal{T}}(E) = \mathbf{M}_{\mathbf{Z}}(\mathcal{T}^{-1}(E))$. Then

$$\text{Cov}(\mathbf{Z}_n, \mathbf{Z}_{n+k}) = \iint_{\mathbb{T}^2} e^{i(n\lambda - k\alpha)} \mathbf{M}_{\mathbf{Z}}^{\mathcal{T}}(d\lambda, d\alpha).$$

We know that $\text{Cov}(\mathbf{Z}_n, \mathbf{Z}_{n+k}) = \mathbf{0}_{d \times d}$ for $k \neq 0$ and $\text{Var}(\mathbf{Z}_n) = \boldsymbol{\Sigma}_{\mathbf{Z}_n}^2$. Hence, considering the $\mathbb{C}^{d \times d}$ -valued measure $\mathbf{n}_{\mathbf{Z}}$ defined on \mathbb{T} by $\mathbf{n}_{\mathbf{Z}}(A) \stackrel{\text{def}}{=} \mathbf{M}_{\mathbf{Z}}^{\mathcal{T}}(A \times \mathbb{T})$, we can easily see that

$$\text{Cov}(\mathbf{Z}_n, \mathbf{Z}_{n+k}) = \frac{1}{2\pi} \iint_{\mathbb{T}^2} e^{i(n\lambda - k\alpha)} \mathbf{n}_{\mathbf{Z}}(d\lambda) d\alpha = \begin{cases} \int_{\mathbb{T}} e^{in\lambda} \mathbf{n}_{\mathbf{Z}}(d\lambda) & \text{for } k = 0 \\ 0 & \text{otherwise} \end{cases}$$

and by the characterization of a measure on \mathbb{T}^2 by its Fourier transform, we deduce that $\mathbf{M}_{\mathbf{Z}}^{\mathcal{T}}(d\lambda, d\alpha) = \frac{1}{2\pi} \mathbf{n}_{\mathbf{Z}}(d\lambda) d\alpha$, and the relation (3) is satisfied.

Now assume that $(\mathbf{Z}_n)_{n \in \mathbb{Z}}$ is such that there exists a measure $\mathbf{n}_{\mathbf{Z}}$ that satisfies relation (3). Then we can easily check that

$$\text{Cov}(\mathbf{Z}_n, \mathbf{Z}_{n+k}) = \frac{1}{2\pi} \iint_{\mathbb{T}^2} e^{i(n\lambda - k\alpha)} \mathbf{n}_{\mathbf{Z}}(d\lambda) d\alpha.$$

We denote $\mathbf{M}(d\lambda, d\alpha) \stackrel{\text{def}}{=} \frac{1}{2\pi} \mathbf{n}_{\mathbf{Z}}(d\lambda) d\alpha$. Then the measure which is image of the $\mathbb{C}^{d \times d}$ -valued measure \mathbf{M} on \mathbb{T}^2 by the transformation \mathcal{T}^{-1} is an extension of the spectral bimeasure $\mathbf{M}_{\mathbf{Z}}$ of the harmonizable time series \mathbf{Z} . Thus, the spectral bimeasure $\mathbf{M}_{\mathbf{Z}}$ is extendible to a measure and the time series \mathbf{Z} is strongly harmonizable. Furthermore, we obtain relations (5) and (4).

Proof of Proposition 3.2

Consider a d -variate pure white noise ε associated to the harmonizable time series \mathbf{Z} in the sense of Lemma 3.1.

First, assume that the d -variate stochastic measure $\mu_{\mathbf{Z}}$ admits a stochastic Radon-Nikodym density function with respect to the Lebesgue measure on \mathbb{T} , $\Delta_{\mathbf{Z}} : \mathbb{T} \rightarrow \mathcal{H}^d(\mathbf{Z})$. We denote $\Delta_{\mathbf{Z}} = (\Delta_{Z_1}, \dots, \Delta_{Z_d})$, $\Delta_{Z_j} : \mathbb{T} \rightarrow \mathcal{H}(\mathbf{Z})$, $j = 1, \dots, d$. Then

$$\mathbf{Z}_n = \int_{\mathbb{T}} e^{in\lambda} \mu_{\mathbf{Z}}(\lambda) = \int_{\mathbb{T}} e^{in\lambda} \Delta_{\mathbf{Z}}(\lambda) d\lambda$$

and

$$Z_{n,j} = \int_{\mathbb{T}} e^{in\lambda} \mu_{Z_j}(\lambda) = \int_{\mathbb{T}} e^{in\lambda} \Delta_{Z_j}(\lambda) d\lambda.$$

Moreover,

$$\langle Z_{n,j} | \epsilon_{m,k} \rangle_{\mathcal{H}} = \int_{\mathbb{T}} e^{in\lambda} \langle \Delta_{Z_j}(\lambda) | \epsilon_{m,k} \rangle_{\mathcal{H}} d\lambda.$$

Since ε is a d -variate pure white noise, using notation of Lemma 3.1, we have that $\langle Z_{n,j} | \epsilon_{m,k} \rangle_{\mathcal{H}} = s_{n,j,k} \mathbb{1}_{\{n=m\}}$. We get that $\langle \Delta_{Z_j}(\lambda) | \epsilon_{m,k} \rangle_{\mathcal{H}} = \frac{s_{n,j,k}}{2\pi} e^{-im\lambda}$ for all n, j, m and for almost every λ . From the facts that $\Delta_{Z_j}(\lambda) \in \mathcal{H}(\mathbf{Z}) \subset \mathcal{H}(\varepsilon)$, and $\{\epsilon_{m,k} : m \in \mathbb{Z}, k = 1, \dots, d\}$ is an orthonormal basis of $\mathcal{H}(\varepsilon)$, we obtain that

$$\Delta_{Z_j}(\lambda) = \frac{1}{2\pi} \sum_{m \in \mathbb{Z}} \sum_{k=1}^d s_{n,j,k} e^{-im\lambda} \epsilon_{m,k} \quad \text{and} \quad \sum_{n \in \mathbb{Z}} \sum_{k=1}^d |s_{n,j,k}|^2 < \infty.$$

Hence, the relations (7) and (6) are satisfied.

Conversely, assume that the relation (6) is satisfied. Then the function $\Delta_{\mathbf{Z}} = (\Delta_{Z_1}, \dots, \Delta_{Z_d})$ is well-defined on \mathbb{T} by relation (7). Note that the convergence is in $\mathcal{H}(\mathbf{Z}) \subset L_{\mathbb{C}}^2(\mathbb{P})$. Let $j = 1, \dots, d$ be fixed. In the following, we use the notation of the proof of Theorem 3.1. From the relation (25), the function $\lambda \mapsto \mathbb{E}[\Delta_{Z_j}(\lambda) \overline{\Delta_{Z_j}(\lambda)}] = \frac{s_{n,j,k}}{2\pi} e^{-in\lambda}$ is the Radon-Nikodym density function of the complex measure $\mu_{Z_j, n, k}$ with respect to the Lebesgue measure on \mathbb{T} . Since $\mathcal{H}(\mathbf{Z})$ and $\mathcal{H}(\varepsilon)$ are separable Hilbert spaces, the Lebesgue measure on \mathbb{T} is bounded and

$$\|\Delta_{Z_j}(\lambda)\|_{\mathcal{H}}^2 = \mathbb{E}(\Delta_{Z_j}(\lambda) \overline{\Delta_{Z_j}(\lambda)}) = \frac{1}{4\pi^2} \sum_{n \in \mathbb{Z}} \sum_{k=1}^d |s_{n,j,k}|^2 < \infty,$$

and we can prove that the function $D_{Z_k} : \mathbb{T} \rightarrow \mathcal{H}(\mathbf{Z})$ is Bochner integrable (strongly integrable) (Dunford & Schwartz, 1957, Chap III, Th. III.2.21), and

$$\mu_{Z_j}(A) = \int_A \Delta_{Z_j}(\lambda) d\lambda, \quad A \in \mathcal{B}(\mathbb{T}).$$

This ends the proof.

Proof of Theorem 4.1

Let $(\varepsilon)_{n \in \mathbb{Z}}$ be a d -variate pure white noise associated to the harmonizable noise $(\mathbf{Z}_n)_{n \in \mathbb{Z}}$ in Lemma 3.1. It is known that the time series $(\mathbf{Y}_n)_{n \in \mathbb{Z}}$ of the form

$$\mathbf{Y}_n = \sum_{m \in \mathbb{Z}} \mathbf{A}_{n-m} \varepsilon_m \tag{26}$$

is well-defined (the convergence is in $H^d(\varepsilon)$), and it is stationary.

Let $n \in \mathbb{Z}$ be fixed. Since the family $\{\varepsilon_{n,k} : n \in \mathbb{Z}, k = 1, \dots, d\}$ is an orthonormal basis of $\mathcal{H}(\varepsilon)$, we have

$$\left\| \sum_{m=N_1}^{N_2} \mathbf{A}_{n-m} \varepsilon_m \right\|_{\mathcal{H}^d(\varepsilon)}^2 = \sum_{m=N_1}^{N_2} \text{tr}(\mathbf{A}_{n-m} \mathbf{A}_{n-m}^*)$$

for any $-\infty < N_1 < N_2 < \infty$. Then the series (26) is a Cauchy series in the Hilbert space $\mathcal{H}^d(\varepsilon)$. Thus, it is convergent in $\mathcal{H}^d(\varepsilon)$ and its sum \mathbf{Y}_n is well-defined. The stationarity of the d -variate time series $(\mathbf{Y}_n)_{n \in \mathbb{Z}}$ is evident.

Now consider the bounded linear operator $\Xi : \mathcal{H}^d(\varepsilon) \rightarrow \mathcal{H}^d(\mathbf{Z})$ such that $\Xi \varepsilon_n = \mathbf{Z}_n$ defined in the proof of Theorem 3.1. The operator Ξ applied to the moving average series (26) entails the convergence in $\mathcal{H}^d(\mathbf{Z})$ of the moving average series (9)

$$\sum_m \mathbf{A}_{n-m} \Xi(\varepsilon_m) = \sum_m \mathbf{A}_{n-m} \mathbf{Z}_m.$$

Thus, the d -variate time series $(\mathbf{X}_n)_{n \in \mathbb{Z}} = (\Xi \mathbf{Y}_n)_{n \in \mathbb{Z}}$ is well-defined. Moreover, as the image of the stationary time series $(\mathbf{Y}_n)_{n \in \mathbb{Z}}$ by the bounded linear operator Ξ , it is harmonizable. Furthermore, the stochastic spectral measure $\boldsymbol{\mu}_{\mathbf{X}}$ of \mathbf{X} coincides with $\Xi \boldsymbol{\mu}_{\mathbf{Y}}$. Since the topological group \mathbb{T} is compact and $\widehat{\mathbf{A}} \in L^2_{\mathbb{C}^d \times d}(\mathbb{T})$, we have $\widehat{\mathbf{A}} \in L^1_{\mathbb{C}^d \times d}(\mathbb{T})$ and the inversion formula gives

$$\mathbf{A}_n = \frac{1}{2\pi} \int_{\mathbb{T}} e^{-in\lambda} \widehat{\mathbf{A}}(\lambda) d\lambda.$$

Hence,

$$\mathbf{X}_n = \sum_{m \in \mathbb{Z}} \mathbf{A}_{n-m} \mathbf{Z}_m = \frac{1}{2\pi} \sum_{m \in \mathbb{Z}} \left(\int_{\mathbb{T}} e^{i(n-m)\lambda} \widehat{\mathbf{A}}(-\lambda) d\lambda \right) \mathbf{Z}_m.$$

Thus, the stochastic spectral measure $\boldsymbol{\mu}_{\mathbf{Z}}$ satisfies

$$\boldsymbol{\mu}_{\mathbf{X}}(A) = \frac{1}{2\pi} \sum_{m \in \mathbb{Z}} \left(\int_A e^{-im\lambda} \widehat{\mathbf{A}}(-\lambda) d\lambda \right) \mathbf{Z}_m,$$

and the convergence is in \mathcal{H}^d .

Then, using the inequality (2), we can easily get that the d -variate function $\widehat{\mathbf{A}}$ is $\boldsymbol{\mu}_{\mathbf{Z}}$ -integrable and the equality (10) is satisfied.

8.1 Proof of Proposition 5.1

We notice that $\widehat{\phi}_n = \phi + D_n^{-1} N_n$ where

$$D_n \stackrel{\text{def}}{=} \sum_{k=1}^n X_{k-1}^2 \quad \text{and} \quad N_n \stackrel{\text{def}}{=} \sum_{k=1}^n Z_k X_{k-1}.$$

1) *Asymptotic behaviour of D_n .* The sequence $\{D_n\}$ is a nondecreasing and nonnegative. Here we show that $D_\infty \stackrel{\text{def}}{=} \lim_{n \rightarrow \infty} \uparrow D_n = \infty$ a.e.

First, we assure that $|\phi| < 1$. Since $Z_n \perp \mathcal{F}_{n-1}$, $n \geq 1$, we can write

$$\text{var}(D_n) = \sum_{k=1}^n \phi^{4(k-2)} \text{var}((S_{k-1} + X_0)^2) + 2 \sum_{k_1=2}^n \sum_{k_2=1}^{k_1-1} \phi^{2(k_1+k_2-2)} \text{var}((S_{k_2-1} + X_0)^2)$$

and

$$\text{var}(S_{k-1}^2) = \sum_{j=1}^{k-1} \phi^{-4j} \text{var}(Z_j^2) + 4 \sum_{j_1=2}^{k-1} \sum_{j_2=1}^{j_1-1} \phi^{-2(j_1+j_2)} \sigma_{j_1}^2 \sigma_{j_2}^2.$$

Then, we deduce that $n^{-1}(D_n - \mathbf{E}(D_n))$ converges to 0 in quadratic mean. Furthermore, applying Borel Cantelli lemma, we verify that the convergence a.e. is also available. Since

$$\mathbf{E}(D_n) = \sum_{k=2}^n \sum_{j=1}^{k-1} \phi^{2(k-j-1)} \sigma_j^2 + \sum_{k=1}^n \phi^{2(k-1)} \times \mathbf{E}(X_0^2)$$

we obtain

$$\liminf_{n \rightarrow \infty} n^{-1} \mathbf{E}(D_n) = (1 - \phi^2)^{-1} \liminf_{n \rightarrow \infty} \mu_n > 0.$$

Hence $D_\infty = \infty$ a.e. in this case.

Next, assume that $|\phi| = 1$. The central limit theorem for independent random variables (see, e.g., Loève, 1978, sec. 21.1) entails that $s_n^{-1} X_n$ converges in distribution to the standard normal law $\mathcal{N}(0, 1)$. Here $s_n^2 \stackrel{\text{def}}{=} \sum_{k=1}^n \sigma_k^2$. We deduce that

$$\liminf_{n \rightarrow \infty} \mathbf{P}(D_n > s_n^2 \epsilon) \geq \lim_{n \rightarrow \infty} \mathbf{P}(X_n^2 > s_n^2 \epsilon) = 1 - F_2(\epsilon)$$

for any $\epsilon > 0$, where $F_2(\epsilon)$ is the value of the distribution function of the $\chi^2(1)$ distribution. Since s_n^2 converges to ∞ , we deduce that D_n converges in probability to ∞ as $n \rightarrow \infty$. Hence $D_\infty = \infty$ a.e.

Finally, assume that $|\phi| > 1$. In this case, we know that

$$\lim_{n \rightarrow \infty} \phi^{-2n} X_n^2 = (S_\infty + X_0)^2 \quad \text{a.e.}$$

If $\mathbf{P}(S_\infty + X_0 = 0) = 0$ we readily deduce that $\lim_n X_n^2 = \infty$ and $D_\infty = \infty$ a.e.

2) *Asymptotic behaviour of N_n .* For each $\phi \in \mathbb{R}$, we have just seen that $D_\infty \stackrel{\text{def}}{=} \lim_n \uparrow D_n = \infty$ a.e. Then $\mathbf{P}(\cup_n \{D_n > 0\}) = 1$. Since $Z_n \perp\!\!\!\perp \mathcal{F}_{n-1}$, $n \geq 1$, the time series $\{N_n\}$ is a martingale with respect to the filtration $\{\mathcal{F}_n\}$ and the strong law of large numbers for martingales (see Theorem 2.18 in Hall & Heyde, 1980) entails that $\lim_n D_n^{-1} N_n = 0$ a.e. on the set

$$\{D_\infty = \infty\} \cap \{D_n > 0\} \cap \left\{ \sum_{k=n}^{\infty} D_k^{-2} \mathbf{E}(Z_k X_{k-1})^2 \mid \mathcal{F}_{k-1} < \infty \right\}$$

for any $n \geq 1$. However

$$\sum_{k=n}^{\infty} D_k^{-2} \mathbf{E}(X_{k-1}^2 Z_k^2 \mid \mathcal{F}_{k-1}) = \sum_{k=n}^{\infty} D_k^{-2} X_{k-1}^2 \sigma_k^2 \leq \sum_{k=n}^{\infty} D_k^{-2} (D_k - D_{k-1}) \sup_k \sigma_k^2.$$

Since $0 < D_n \leq D_k \leq D_{k+1}$, for any $n \leq k$, we easily verify that

$$0 \leq \sum_{k=n}^{\infty} D_k^{-2} (D_k - D_{k-1}) \leq 2D_n^{-1} < \infty.$$

Then we readily achieve the proof of Proposition 5.1.

8.2 Proof of Proposition 5.2

First, following the proof of Proposition 5.1 when $|\phi| < 1$, under the hypotheses of Proposition 5.2, we obtain that

$$\lim_{n \rightarrow \infty} n^{-1} D_n = (1 - \phi^2)^{-1} \mu > 0 \quad \text{a.e.}$$

Next, the conditional quadratic variation $\langle N \rangle$ of the square-integrable martingale $\{N_n\}$ is equal to

$$\langle N \rangle_n \stackrel{\text{def}}{=} \sum_{k=1}^{n-1} \mathbb{E}((N_k - N_{k-1})^2 \mid \mathcal{F}_{k-1}) = \sum_{k=1}^{n-1} X_{k-1}^2 \sigma_k^2.$$

To apply the central limit theorem for martingales (see, e.g., Dacunha-Castelle & Duflo, 1993, Theorem 2.8.41), it suffices to show that the two following conditions are fulfilled:

(C1) : $n^{-1}\langle N \rangle_n$ converges in probability to some positive value γ_ϕ ,

(C2) : Lindeberg condition is satisfied, that is, for each $\epsilon > 0$:

$$\lim_{n \rightarrow \infty} n^{-1} \sum_{k=1}^n \mathbb{E}((X_{k-1}^2 Z_k^2 \mathbb{I}_{\{\sqrt{n}|X_{k-1}Z_k| > \epsilon\}} \mid \mathcal{F}_{k-1})) = 0 \quad \text{in probability.}$$

Thanks to the independence condition, $Z_k \perp\!\!\!\perp \mathcal{F}_{k-1}$, $k \geq 1$, we can write

$$\mathbb{E}(\langle N \rangle_n) = \sum_{k=1}^{n-1} \phi^{2(k-1)} \sum_{j=1}^{n-k} \sigma_j^2 \sigma_{j+k}^2 + \mathbb{E}(X_0^2) \sum_{k=1}^n \phi^{2(k-1)} \sigma_k^2$$

and $\text{var}(\langle N \rangle_n) \leq \text{var}(D_n) \times \sup_k \sigma_k^4$. Then we deduce that the condition (C1) is satisfied with $\gamma_\phi = \sum_k \phi^{2(k-1)} \eta_k < \infty$.

Now we study the condition (C2). Since $0 \leq \mathbb{I}_{\{\sqrt{n}|X_{k-1}Z_k| > \epsilon\}} \leq \epsilon^{-1} \sqrt{n} |X_{k-1}Z_k|$ and $Z_k \perp\!\!\!\perp \mathcal{F}_{k-1}$, we have

$$\begin{aligned} 0 \leq \mathbb{E}(X_{k-1}^2 Z_k^2 \mathbb{I}_{\{a_n^{1/2}|X_{k-1}Z_k| > \epsilon\}} \mid \mathcal{F}_{k-1}) &\leq \epsilon^{-1} \sqrt{n} \mathbb{E}(|X_{k-1}Z_k|^3 \mid \mathcal{F}_{k-1}) \\ &= \epsilon^{-1} \sqrt{n} |X_{k-1}|^3 \mathbb{E}(|Z_k|^3). \end{aligned}$$

We know that $\mathbb{E}(|X_{k-1}|^3) \leq \mathbb{E}(X_{k-1}^4)^{3/4}$. From the independence condition, we have

$$\begin{aligned} \mathbb{E}(X_{k-1}^4)^{3/4} &\leq 8\phi^{3(k-1)} \left(\mathbb{E}(|S_{k-1}|^4)^{3/4} + \mathbb{E}(|X_0|^4)^{3/4} \right. \\ &\quad \left. + 3\mathbb{E}(|S_{k-1}|^4)^{1/2} \mathbb{E}(|X_0|^4)^{1/4} + 3\mathbb{E}(|S_{k-1}|^4)^{1/4} \mathbb{E}(|X_0|^4)^{1/2} \right) \end{aligned}$$

for any $k > 1$. Besides,

$$\begin{aligned} \mathbb{E}(|S_{k-1}|^4) &= \sum_{j=1}^{k-1} \phi^{-4j} \mathbb{E}(Z_j^4) + 6 \sum_{j_1=2}^{k-1} \sum_{j_2=1}^{j_1-1} \phi^{-2(j_1+j_2)} \sigma_{j_1}^2 \sigma_{j_2}^2 \\ &\leq \sup_j \mathbb{E}(Z_j^4) \sum_{j=1}^{k-1} \phi^{-4j} + 6 \sup_j \sigma_j^4 \sum_{j_1=2}^{k-1} \sum_{j_2=1}^{j_1-1} \phi^{-2(j_1+j_2)}. \end{aligned}$$

Then we obtain that $\mathbb{E}(|X_{k-1}|^3) = \mathcal{O}(1)$. We deduce that the condition (C2) (Lindeberg condition) is satisfied. We can apply the central limit theorem for martingales: the time series $\{n^{-1/2}N_n\}$ converges in distribution to $\mathcal{N}(0, \gamma_\phi)$. Then we readily achieve the proof of Proposition 5.2.

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