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Spectral characteristics of Harmonizable VARMA processes

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Abstract

Harmonizable processes form a wide class of nonstationary processes, which admit a convenient Fourier analysis and have spectral distributions characterized by correlated components. They are proved to be useful in many fields of application, for example, recently they were successfully applied in the analysis of replicated ElectroEncephaloGram signals for studying the brain connectivity. In this paper, we introduce a parametric form for these harmonizable processes, namely Harmonizable Vector AutoRegressive and Moving Average models (HVARMA). In the same spirit as of standard VARMA models, they are derived as a unique solution of a difference equation based on a properly defined concept of harmonizable noise. We exhibit their spectral characteristics and provide a way to generate realizations from a given HVARMA model.

keywords: Harmonizable processes, Loève spectrum, time series, VARMA models.

1 Introduction

Stationary processes are widely used because of the wide set of available tools for their analysis, both in the time and frequency domains. However, there are many problems for which stationarity is an unacceptable limitation and for which it is of interest to maintain a convenient and easy-to-interpret spectral analysis (e.g., econometrics, telecommunication, vibroacoustics, mechanics). Harmonizable processes are a natural extension of stationary processes with a spectral (frequency) distribution, whose components are correlated. Their covariance function is bivariate and admits a two-dimensional Fourier decomposition (Loève spectrum). Hence, the study of this class of time series is possible through Fourier analysis methods.

Harmonizable processes have been studied in the literature over the past 70 years. We refer the reader in particular to the works of Loève (1948-1963) (see Cramér, 1961; Loève, 1978; Rozanov, 1959). Numerous applications of the concept of a harmonizable process have been developed in signal theory, in communications, in mechanics, etc, (see, e.g., Gardner et al., 2006; Napolitano, 2019; Serpedin et al., 2005; Setoodeh & Haykin, 2017). That is to say, in fields where filtering problems are common, and frequency domain

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methods are highly regarded by practitioners for describing or inferring the properties of noisy signals. Recent applications of harmonizable time series allowed to uncover new research directions for EEG data (Aston et al., 2023; Gorrostieta et al., 2019). In these papers, the authors developed a nonparametric procedure for estimating the Loève spectrum (and some generalizations such as the spatial time-localized Loève spectrum, Aston et al. (see 2023)).

Despite more than 70-years of history of harmonizable processes and their usefulness, there are no parametric models for such processes, depriving researchers and practitioners from a whole scope of powerful methods for estimation, prediction, data generation. . . It is worth noting that there are few publications in the literature on linear harmonizable time series. One can cite Mehlman (1991, 1992), where the structure of multivariate harmonizable time series is analyzed and the theory of moving average representation is presented along with a study of the prediction problem.

Taking the above into account, the contribution of this paper is threefold: first, we fill in the gap in the literature by introducing VARMA-type model for harmonizable processes; second, we exhibit their spectral characteristics; and third, we address the issue of generating non-trivial realizations of harmonizable time series with a given Loève spectrum.

In what follows, we deal with linear multivariate harmonizable time series defined as moving average of harmonizable noise. In contrast to the theory presented in Mehlman (1991, 1992), we assume that the harmonizable noise is uncorrelated, but its variance is not constant over time. Thus, the harmonizable noise can be considered as innovations for harmonizable models. For this purpose, hereafter we describe the spectrum of multivariate linear harmonizable time series and, in particular, the Vector Autoregressive Moving Average with harmonizable innovation denoted therein as HVARMA models. Our approach follows some ideas from Brockwell and Davis (1991) by putting attention on the filtering problem, and on the transfer of the Loève spectrum of the harmonizable noise to the Loève spectrum of the resulting moving average time series.

This paper is organized as follows. In Section 2 a brief description of the spectral representation of multivariate harmonizable time series is presented. In Section 3, the notion of multivariate harmonizable noise (or innovation) is introduced and its spectrum characterized. This is the basic building block of linear models. Additionally, necessary and sufficient conditions for strong harmonizability and for the existence of a stochastic spectral density are given. Section 4 is devoted to defining multivariate linear harmonizable time series as moving average of harmonizable noise. Section 5 deals with multivariate harmonizable autoregressive moving average (VARMA) models. Section 6 details a proper methodology for generating realizations of harmonizable time series from known spectral characteristics. Finally, a short discussion is provided in Section 7. All proofs are deferred to Section 8.

2 Background

In this section, we introduce the basic notations and recall some definitions and properties of harmonizable processes. For more details, we refer the reader to Loève (1978), Mehlman (1991), Rao (1982), and Rozanov (1959).

2.1 Notation

Let d be a fixed non-zero integer. From now on, \mathbb{C}^d denotes the set of d -dimensional complex column vectors and $\mathbb{T} \stackrel{\text{def}}{=} \mathbb{R}/2\pi$ represented by $(-\pi, \pi]$ is the topological dual group of \mathbb{Z} . In this paper, we assume that the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ is sufficiently large so that there exists a sequence of uncorrelated

random d -dimensional vectors with a variance matrix equal to the identity matrix of size $d \times d$ denoted as $\mathbb{I}_{d \times d}$, and which are not correlated with the d -dimensional times series $(\mathbf{Z}_n)_{n \in \mathbb{Z}}$ studied in the following. Otherwise, one can always replace any probability space $(\Omega, \mathcal{F}, \mathbb{P})$ by an enlarged probability space with this property. For the construction of such an enlargement, we refer the reader to Loève (1978).

Hereafter, we consider only complex-valued random variables defined on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$, that have zero mean and are mean square integrable. For simplicity, in what follows, when we write "time series" we mean second order time series with zero mean, i.e., a sequence square integrable random vectors with zero mean (or variables) on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$. We denote by $\mathcal{H} = L^2_{\mathbb{C}}(\mathbb{P})$ the complex Hilbert space of square integrable complex-valued random variables with scalar product $\langle x|y \rangle_{\mathcal{H}} = \mathbb{E}(x\bar{y})$. The inflated space $\mathcal{H}^d \stackrel{\text{def}}{=} \bigoplus_1^d \mathcal{H} = L^2_{\mathbb{C}^d}(\mathbb{P})$ is a Hilbert space with the scalar product $\langle \mathbf{x}|\mathbf{y} \rangle_{\mathcal{H}^d} \stackrel{\text{def}}{=} \mathbb{E}(\mathbf{x}^\top \bar{\mathbf{y}})$. If \mathbf{x} and \mathbf{y} are two zero mean random vectors in $L^2_{\mathbb{C}^d}(\mathbb{P})$ then the covariance between \mathbf{x} and \mathbf{y} is the $d \times d$ -matrix $\text{Cov}(\mathbf{x}, \mathbf{y}) = \mathbb{E}(\mathbf{x}\mathbf{y}^*) = (\text{cov}(x_i, y_j))_{1 \leq i, j \leq d}$, where $\mathbf{x} = (x_1, \dots, x_d)^\top$ and $\mathbf{y} = (y_1, \dots, y_d)^\top \in \mathcal{H}^d$.

Here and in the following, the superscript \top denotes the transpose operator, and the superscript $'^*$ is the adjoint operator, i.e., the conjugate transpose operator.

Let $\mathbf{X} = (\mathbf{X}_n)_{n \in \mathbb{Z}}$ be a d -variate time series and let $\mathcal{H}(\mathbf{X}_n)$ be the Hilbert subspace of \mathcal{H} generated by $X_{n,1}, \dots, X_{n,d}$, i.e., $\mathcal{H}(\mathbf{X}_n) \stackrel{\text{def}}{=} \{\mathbf{a}^\top \mathbf{X}_n : \mathbf{a} \in \mathbb{C}^d\}$, where $\mathbf{X}_n = (X_{n,1}, \dots, X_{n,d})^\top$. Notice that $\dim \mathcal{H}(\mathbf{X}_n) \leq d$. Moreover, let $\mathcal{H}^d(\mathbf{X}_n)$ be the Hilbert subspace of \mathcal{H}^d generated by \mathbf{X}_n , i.e., $\mathcal{H}^d(\mathbf{X}_n) \stackrel{\text{def}}{=} \{\mathbf{A}\mathbf{X}_n : \mathbf{A} \in \mathbb{C}^{d \times d}\}$, $\mathcal{H}^d(\mathbf{X}_n) = \mathcal{H}(\mathbf{X}_n) \oplus \dots \oplus \mathcal{H}(\mathbf{X}_n)$ and $\dim \mathcal{H}^d(\mathbf{X}_n) = d \times \dim \mathcal{H}(\mathbf{X}_n) \leq d^2$. In the following, we also consider $\mathcal{H}(\mathbf{X})$ a Hilbert subspace of \mathcal{H} generated by the Hilbert subspaces $\mathcal{H}(\mathbf{X}_n)$, $n \in \mathbb{Z}$, and $\mathcal{H}^d(\mathbf{X})$ the Hilbert subspace of \mathcal{H}^d generated by the subspaces $\mathcal{H}^d(\mathbf{X}_n)$, $n \in \mathbb{Z}$. If we replace \mathbf{X} by a d -variate white noise $\boldsymbol{\varepsilon} = (\boldsymbol{\varepsilon}_n)_{n \in \mathbb{Z}}$, then the Hilbert subspaces $\mathcal{H}(\boldsymbol{\varepsilon}_n)$, $n \in \mathbb{Z}$, are orthogonal with the same dimension equal to the rank of the covariance matrix $\boldsymbol{\Sigma}_{\boldsymbol{\varepsilon}}^2 = \text{Var}(\boldsymbol{\varepsilon}_n)$. If, in addition, the d -variate white noise is pure (see Definition 3.1), then the family $\{\epsilon_{n,1}, \dots, \epsilon_{n,d}\}$ is the orthonormal basis of $\mathcal{H}(\boldsymbol{\varepsilon}_n)$. Thus, $\dim \mathcal{H}(\boldsymbol{\varepsilon}_n) = d$ and $\dim \mathcal{H}^d(\boldsymbol{\varepsilon}_n) = d^2$.

2.2 Multivariate harmonizable time series

Definition 2.1 A d -variate time series $\mathbf{X} = (\mathbf{X}_n)_{n \in \mathbb{Z}}$ is called harmonizable when there exists a (σ -additive) vector measure $\boldsymbol{\mu}_{\mathbf{X}} : \mathcal{B}(\mathbb{T}) \rightarrow \mathcal{H}^d$, such that

$$\mathbf{X}_n = \int_{\mathbb{T}} e^{in\lambda} \boldsymbol{\mu}_{\mathbf{X}}(d\lambda).$$

The measure $\boldsymbol{\mu}_{\mathbf{X}}$ is unique, it is called stochastic spectral measure of the time series \mathbf{X} . The integral with respect to the vector measure $\boldsymbol{\mu}_{\mathbf{X}}$ is in the sense of Dunford and Schwartz (see chapter IV, section 10, Dunford & Schwartz, 1957).

In the following, for a harmonizable time series \mathbf{X} , we will consider the Hilbert subspace $\mathcal{H}(\boldsymbol{\mu}_{\mathbf{X}})$ of \mathcal{H} generated by $\mu_{\mathbf{X},j}(A)$, $j = 1, \dots, d$, $A \in \mathcal{B}(\mathbb{T})$. Then $\mathcal{H}(\boldsymbol{\mu}_{\mathbf{X}}) = \mathcal{H}(\mathbf{X})$ and $\mathcal{H}^d(\boldsymbol{\mu}_{\mathbf{X}}) = \mathcal{H}^d(\mathbf{X})$ (Rao, 1982).

A harmonizable time series can also be characterized by a dependence structure in the frequency domain. More specifically, we have the following result.

Proposition 2.1 (Rao, 1982; Rozanov, 1959) A d -variate time series \mathbf{X} is harmonizable if and only if there exists a positive semi-definite bimeasure $\mathbf{M}_{\mathbf{X}} : \mathcal{B}(\mathbb{T}) \times \mathcal{B}(\mathbb{T}) \rightarrow \mathbb{C}^{d \times d}$ (the set function $\mathbf{M}_{\mathbf{X}}$ is σ -additive with respect to each of its components) such that,

$$\mathbb{E}(\mathbf{X}_m \mathbf{X}_n^*) = \int \int_{\mathbb{T}^2} e^{i(m\lambda_1 - n\lambda_2)} \mathbf{M}_{\mathbf{X}}(d\lambda_1, d\lambda_2).$$

The bimeasure $\mathbf{M}_{\mathbf{X}}$ is unique, and it is called the spectral bimeasure of the time series \mathbf{X} . The above integral with respect to the bimeasure $\mathbf{M}_{\mathbf{X}}$ is a Morse-Transue integral (Rao, 1982; Rozanov, 1959). Furthermore,

$$\mathbf{M}_{\mathbf{X}}(A, B) = \text{Cov}(\boldsymbol{\mu}_{\mathbf{X}}(A), \boldsymbol{\mu}_{\mathbf{X}}(B)) = \text{E}(\boldsymbol{\mu}_{\mathbf{X}}(A)\boldsymbol{\mu}_{\mathbf{X}}^*(B))$$

for any A and $B \in \mathcal{B}(\mathbb{T})$.

Remark 2.1 • The bimeasure $\mathbf{M}_{\mathbf{X}}$ is not necessarily extendible as a $\mathbb{C}^{d \times d}$ -valued measure on \mathbb{T}^2 .

When the bimeasure $\mathbf{M}_{\mathbf{X}}$ is extendible as a $\mathbb{C}^{d \times d}$ -valued measure on \mathbb{T}^2 , the time series is called strongly harmonizable (Loève harmonizable), otherwise it is called weakly harmonizable.

- Stationary (wide-sense or second order) times series are strongly harmonizable. A harmonizable time series is stationary if and only if its stochastic spectral measure is orthogonally scattered, i.e., its spectral bimeasure is concentrated on the main diagonal of \mathbb{T}^2 : $\mathbf{M}_{\mathbf{X}}(A, B) = \text{E}(\boldsymbol{\mu}_{\mathbf{X}}(A)\boldsymbol{\mu}_{\mathbf{X}}^*(B)) = \mathbf{0}_{d \times d}$, when $A \cap B = \emptyset$, $A, B \in \mathcal{B}(\mathbb{T})$. Then the spectral measure $\mathbf{m}_{\mathbf{X}}$ on \mathbb{T} of the stationary time series \mathbf{X} satisfies $\mathbf{m}_{\mathbf{X}}(A) = \mathbf{M}_{\mathbf{X}}(A, \mathbb{T}) = \mathbf{M}_{\mathbf{X}}^*(\mathbb{T}, A)$ for any $A \in \mathcal{B}(\mathbb{T})$.

3 Multivariate harmonizable noise or innovation

For the sake of clarity, we first recall the definition of a white noise $\boldsymbol{\varepsilon} \stackrel{\text{def}}{=} (\boldsymbol{\varepsilon}_n)_{n \in \mathbb{Z}}$. Then, we state a necessary and sufficient condition for a time series $\mathbf{Z} \stackrel{\text{def}}{=} (\mathbf{Z}_n)_{n \in \mathbb{Z}}$ to be a harmonizable noise, and we characterize its stochastic spectral measure. Finally, we establish necessary and sufficient conditions for \mathbf{Z} to be strongly harmonizable, and to admit a stochastic spectral density.

Definition 3.1 (White noise) A d -variate time series $\boldsymbol{\varepsilon} \stackrel{\text{def}}{=} (\boldsymbol{\varepsilon}_n)_{n \in \mathbb{Z}}$ is a white noise when it has zero mean and $\text{Cov}(\boldsymbol{\varepsilon}_m, \boldsymbol{\varepsilon}_n) = \text{E}(\boldsymbol{\varepsilon}_m \boldsymbol{\varepsilon}_n^*) = \boldsymbol{\Sigma}_{\boldsymbol{\varepsilon}}^2 \mathbf{1}_{\{m=n\}}$ for some hermitian positive semi-definite $d \times d$ -matrix $\boldsymbol{\Sigma}_{\boldsymbol{\varepsilon}}^2$. A d -variate white noise is a stationary time series. In the following, a d -variate white noise is said to be a d -variate pure white noise when $\boldsymbol{\Sigma}_{\boldsymbol{\varepsilon}}^2 = \mathbb{I}_{d \times d}$ (identity $d \times d$ -matrix).

We consider the following condition for a d -variate time series $\mathbf{Z} \stackrel{\text{def}}{=} (\mathbf{Z}_n)_{n \in \mathbb{Z}}$, whose covariance matrix is time-varying.

- (Ha)** $\text{E}(\mathbf{Z}_n) = \mathbf{0}_d$ and $\text{Cov}(\mathbf{Z}_m, \mathbf{Z}_n) = \text{E}(\mathbf{Z}_m \mathbf{Z}_n^*) = \boldsymbol{\Sigma}_{\mathbf{Z}_n}^2 \mathbf{1}_{\{m=n\}}$,
where the components of the covariance matrices $\boldsymbol{\Sigma}_{\mathbf{Z}_n}^2$, $n \in \mathbb{Z}$, are uniformly bounded, that is,
 $\sup_{n,j} \text{E}(|Z_{n,j}|^2) < \infty$.

Below we establish a fundamental building block for our results, stating a necessary and sufficient condition for the time series to satisfy **(Ha)**.

Lemma 3.1 (Fundamental Lemma) A d -variate time series $\mathbf{Z} \stackrel{\text{def}}{=} (\mathbf{Z}_n)_{n \in \mathbb{Z}}$ fulfills condition **(Ha)** if and only if there exists a d -variate white noise $\boldsymbol{\varepsilon} \stackrel{\text{def}}{=} (\boldsymbol{\varepsilon}_n)_{n \in \mathbb{Z}}$ and a family of $d \times d$ -matrices $\mathbf{S}_n = (s_{n,j,k})_{j,k=1,\dots,d}$ such that $\sup_{n,j,k} |s_{n,j,k}| < \infty$ and $\mathbf{Z}_n = \mathbf{S}_n \boldsymbol{\varepsilon}_n$, $n \in \mathbb{Z}$.

Remark 3.1 (i) In Lemma 3.1 the white noise $(\boldsymbol{\varepsilon}_n)_{n \in \mathbb{Z}}$ is not unique and it can be selected as a d -variate pure white noise.

- (ii) $\text{Var}(\mathbf{Z}_n) = \mathbf{S}_n \boldsymbol{\Sigma}_{\boldsymbol{\varepsilon}}^2 \mathbf{S}_n^*$. In the case of a pure white noise, we have $\text{Var}(\mathbf{Z}_n) = \mathbf{S}_n \mathbf{S}_n^*$ and $\text{E}(\mathbf{Z}_n^{\top} \overline{\mathbf{Z}_n}) = \text{tr}(\mathbf{S}_n \mathbf{S}_n^*)$.

Below we state the harmonizability of the time series $(\mathbf{Z}_n)_{n \in \mathbb{Z}}$ satisfying the condition **(Ha)**. The proof of the following theorem relies on the construction of a bounded linear operator $\Xi : \mathcal{H}^d(\boldsymbol{\varepsilon}) \rightarrow \mathcal{H}^d(\mathbf{Z})$ such that $\mathbf{Z}_n = \Xi \boldsymbol{\varepsilon}_n$, where the time series $(\boldsymbol{\varepsilon}_n)_{n \in \mathbb{Z}}$ is stationary. For more details, we refer the reader to Section 8, proof of Theorem 3.1.

Theorem 3.1 (Harmonizable noise or innovation) *A d -variate time series $(\mathbf{Z}_n)_{n \in \mathbb{Z}}$ fulfilling condition (Ha) is harmonizable. Its stochastic spectral measure $\boldsymbol{\mu}_{\mathbf{Z}}$ defined on $\mathcal{B}(\mathbb{T})$ satisfies*

$$\boldsymbol{\mu}_{\mathbf{Z}}(A) = \frac{1}{2\pi} \sum_{n \in \mathbb{Z}} \left(\int_A e^{-in\lambda} d\lambda \right) \mathbf{Z}_n, \quad (1)$$

where the convergence is in $\mathcal{H}^d(\mathbf{Z})$. Conversely, any d -variate harmonizable time series $(\mathbf{Z}_n)_{n \in \mathbb{Z}}$ whose stochastic spectral measure satisfies relation (1), satisfies the condition **(Ha)**. In what follows, such a time series will be referred to as d -variate harmonizable noise.

Remark 3.2 (i) Note that the spectral bimeasure $\mathbf{M}_{\mathbf{Z}}$ of $(\mathbf{Z}_n)_{n \in \mathbb{Z}}$ satisfies

$$\mathbf{M}_{\mathbf{Z}}(A, B) = \frac{1}{4\pi^2} \sum_{n \in \mathbb{Z}} \left(\int_A \int_B e^{-in(\lambda_1 - \lambda_2)} d\lambda_1 d\lambda_2 \right) \boldsymbol{\Sigma}_{\mathbf{Z}_n}^2,$$

where the convergence of the series is in \mathbb{R} .

(ii) If $\boldsymbol{\varepsilon} \stackrel{\text{def}}{=} (\boldsymbol{\varepsilon}_n)_{n \in \mathbb{Z}}$ is d -variate white noise, then it is harmonizable noise and it is stationary. The stochastic spectral measure $\boldsymbol{\mu}_{\boldsymbol{\varepsilon}}$ is orthogonally scattered in the sense that $\mathbb{E}[\boldsymbol{\mu}_{\boldsymbol{\varepsilon}}(A)\boldsymbol{\mu}_{\boldsymbol{\varepsilon}}^*(B)] = \mathbf{0}_{d \times d}$ when $A \cap B = \emptyset$. It admits a spectral measure $\mathbf{m}_{\boldsymbol{\varepsilon}}$ defined on $\mathcal{B}(\mathbb{T})$ which satisfies

$$\mathbf{m}_{\boldsymbol{\varepsilon}}(A \cap B) = \mathbf{M}_{\boldsymbol{\varepsilon}}(A, B) = \mathbb{E}[\boldsymbol{\mu}_{\boldsymbol{\varepsilon}}(A)\boldsymbol{\mu}_{\boldsymbol{\varepsilon}}^*(B)], \quad A, B \in \mathcal{B}(\mathbb{T}).$$

(iii) When $\boldsymbol{\varepsilon} \stackrel{\text{def}}{=} (\boldsymbol{\varepsilon}_n)_{n \in \mathbb{Z}}$ is d -variate pure white noise, its spectral measure $\mathbf{m}_{\boldsymbol{\varepsilon}}$ has a density function with respect to Lebesgue measure on \mathbb{T} , which coincides with the $\mathbb{C}^{d \times d}$ -valued constant function $\frac{1}{2\pi} \mathbb{I}_{d \times d}$. Every function $f \in L^2_{\mathbb{C}}(\mathbb{T})$ is $\boldsymbol{\mu}_{\boldsymbol{\varepsilon}}$ -integrable and

$$\left\| \int_{\mathbb{T}} f(\lambda) \boldsymbol{\mu}_{\boldsymbol{\varepsilon}}(d\lambda) \right\|_{\mathcal{H}^d}^2 = \text{tr} \left(\mathbb{E} \left[\left(\int_{\mathbb{T}} f(\lambda) \boldsymbol{\mu}_{\boldsymbol{\varepsilon}}(d\lambda) \right) \left(\int_{\mathbb{T}} f(\lambda) \boldsymbol{\mu}_{\boldsymbol{\varepsilon}}(d\lambda) \right)^* \right] \right) = \frac{d}{2\pi} \int_{\mathbb{T}} |f(\lambda)|^2 d\lambda.$$

(iv) Consider a d -variate harmonizable noise \mathbf{Z} and a d -variate pure white noise $\boldsymbol{\varepsilon}$ associated to \mathbf{Z} in Lemma 3.1. Applying the previous remark (iii) and Dunford and Schwartz (Theorem 3.2.19 1957), every function $f \in L^2(\mathbb{T})$ is integrable with respect to $\boldsymbol{\mu}_{\mathbf{Z}} = \Xi \boldsymbol{\mu}_{\boldsymbol{\varepsilon}}$, where Ξ is the bounded linear operator defined in the proof of Theorem 3.1. Furthermore,

$$\left\| \int_{\mathbb{T}} f(\lambda) \boldsymbol{\mu}_{\mathbf{Z}}(d\lambda) \right\|_{\mathcal{H}^d}^2 \leq \sup_{n,j,k} |s_{n,j,k}|^2 \left\| \int_{\mathbb{T}} f(\lambda) \boldsymbol{\mu}_{\boldsymbol{\varepsilon}}(d\lambda) \right\|_{\mathcal{H}^d}^2 = \frac{\zeta^2 d}{2\pi} \int_{\mathbb{T}} |f(\lambda)|^2 d\lambda. \quad (2)$$

where $\zeta^2 = \sup_{n,j,k} |s_{n,j,k}|^2$. Consequently, the stochastic spectral measure $\boldsymbol{\mu}_{\mathbf{Z}}$ is absolutely continuous with respect to Lebesgue measure on \mathbb{T} . Nevertheless, this does not imply that there exists a function $\boldsymbol{\Delta}_{\mathbf{Z}} : \mathbb{T} \rightarrow \mathcal{H}^d$ that is integrable in some sense (Pettis or Bochner, Dunford and Schwartz (see 1957)) and such that

$$\boldsymbol{\mu}_{\mathbf{Z}}(A) = \int_A \boldsymbol{\Delta}_{\mathbf{Z}}(\lambda) d\lambda.$$

Neither that the spectral bimeasure $\mathbf{M}_{\mathbf{Z}}$ is extendible to a $\mathbb{C}^{d \times d}$ -valued measure on \mathbb{T}^2 (see Proposition 3.1).

Before presenting other theoretical results, we give some elementary examples of harmonizable noises.

Example 3.1 Let $\boldsymbol{\varepsilon} \stackrel{\text{def}}{=} (\boldsymbol{\varepsilon}_n)_{n \in \mathbb{Z}}$ be a d -variate white noise. Consider the following time series:

- (i) $\mathbf{Z}_n^{(1)} \stackrel{\text{def}}{=} \boldsymbol{\varepsilon}_n$ for any $n \in \mathbb{Z}$.
- (ii) $\mathbf{Z}_n^{(2)} \stackrel{\text{def}}{=} \boldsymbol{\varepsilon}_n$ for $n_1 \leq n \leq n_2$, and $\mathbf{Z}_n^{(2)} \stackrel{\text{def}}{=} 0$ otherwise, $n_1 < n_2$ being fixed in \mathbb{Z} (Rao, 1982).
- (iii) $\mathbf{Z}_n^{(3)} \stackrel{\text{def}}{=} \boldsymbol{\varepsilon}_n$ for $n \geq 0$, and $\mathbf{Z}_n^{(3)} \stackrel{\text{def}}{=} 0$ otherwise (Rao, 1982).
- (iv) Let $d = 2$ and $\mathbf{S}_n^{(4)} \stackrel{\text{def}}{=} \begin{pmatrix} 1 & 0 \\ 0 & 1 + (-1)^n \end{pmatrix}$, $\mathbf{S}_n^{(5)} \stackrel{\text{def}}{=} \begin{pmatrix} 1 & 0 \\ 0 & 1 \wedge |n|^{-1} \end{pmatrix}$. Set $\mathbf{Z}_n^{(i)} \stackrel{\text{def}}{=} \mathbf{S}_n^{(i)} \boldsymbol{\varepsilon}_n$, for $i = 4, 5$.
- (v) Let $\mathbf{X} \stackrel{\text{def}}{=} (\mathbf{X}_n)_{n \in \mathbb{Z}}$ be a $L_{\mathbb{C}^d}^2(\mathbb{P})$ -bounded time series. Consider the innovation series $\mathbf{Z}^{(6)} = (\mathbf{Z}_n^{(6)})_{n \in \mathbb{Z}}$ defined by $\mathbf{Z}_{n,j}^{(6)} \stackrel{\text{def}}{=} X_{n,j} - P_{n-1} X_{n,j}$, where $P_{n-1} X_{n,j}$ is the one-step prediction of $X_{n,j}$, i.e., the orthogonal projection of $X_{n,j}$ onto the sub-Hilbert space $\mathcal{H}_{n-1}(\mathbf{X})$ of $\mathcal{H}(\mathbf{X}) \subset L_{\mathbb{C}}^2(\mathbb{P})$ generated by $\{X_{p,k} : p \leq n-1, k = 1, \dots, d\}$.

Then the series $\mathbf{Z}^{(i)} = (\mathbf{Z}_n^{(i)})_{n \in \mathbb{Z}}$, $i = 1, \dots, 6$ are d -variate harmonizable noises.

Next, we state a condition for the d -variate time series $(\mathbf{Z}_n)_{n \in \mathbb{Z}}$ to be strongly harmonizable.

Proposition 3.1 (Strong harmonizability) *A d -variate harmonizable noise $(\mathbf{Z}_n)_{n \in \mathbb{Z}}$ is strongly harmonizable if and only if*

$$\boldsymbol{\Sigma}_{\mathbf{Z}_n}^2 = \int_{\mathbb{T}} e^{in\lambda} \mathbf{n}_{\mathbf{Z}}(d\lambda), \quad (3)$$

where $\mathbf{n}_{\mathbf{Z}}$ is some $\mathbb{C}^{d \times d}$ -valued measure on \mathbb{T} . In this case, the measure $\mathbf{n}_{\mathbf{Z}}$ is unique, for every $A \in \mathcal{B}(\mathbb{T})$

$$\mathbf{n}_{\mathbf{Z}}(A) = \iint_{\mathbb{T}^2} \mathbb{I}_A(\lambda_1 + \lambda_2) \mathbf{M}_{\mathbf{Z}}(d\lambda_1, d\lambda_2), \quad (4)$$

and for every $E \in \mathcal{B}(\mathbb{T}^2)$ we have

$$\mathbf{M}_{\mathbf{Z}}(E) = \frac{1}{2\pi} \iint_{\mathbb{T}^2} \mathbb{I}_E(\lambda - \alpha, \alpha) \mathbf{n}_{\mathbf{Z}}(d\lambda) d\alpha. \quad (5)$$

\mathbb{I}_E is the indicator function of the set E , that is, $\mathbb{I}_E(\lambda) = 1$ if $\lambda \in E$ and $\mathbb{I}_E(\lambda) = 0$ otherwise.

Remark 3.3 (i) The measure $\mathbf{n}_{\mathbf{Z}}$ in Proposition 3.1 satisfies

$$\mathbf{n}_{\mathbf{Z}}^*(A) = \mathbf{n}_{\mathbf{Z}}(-A) = \mathbf{n}_{\mathbf{Z}}(A), \quad A \in \mathcal{B}(\mathbb{T}).$$

Indeed

$$\begin{aligned} \mathbf{M}_{\mathbf{Z}}^*(d\lambda_1, d\lambda_2) &= \mathbf{M}_{\mathbf{Z}}(d\lambda_2, d\lambda_1) \implies \\ \mathbf{n}_{\mathbf{Z}}^*(A) &= \iint_{\mathbb{T}^2} \mathbb{I}_A(\lambda_1 + \lambda_2) \mathbf{M}_{\mathbf{Z}}^*(d\lambda_1, d\lambda_2) = \iint_{\mathbb{T}^2} \mathbb{I}_A(\lambda_1 + \lambda_2) \mathbf{M}_{\mathbf{Z}}(d\lambda_2, d\lambda_1) = \mathbf{n}_{\mathbf{Z}}(A) \end{aligned}$$

and

$$\begin{aligned} 0 \leq \mathbf{v} \boldsymbol{\Sigma}_{\mathbf{Z}_n}^2 \mathbf{v}^* &= (\mathbf{v} \boldsymbol{\Sigma}_{\mathbf{Z}_n}^2 \mathbf{v}^*)^* \implies \\ \int_{\mathbb{T}} e^{in\lambda} (\mathbf{v} \mathbf{n}_{\mathbf{Z}}(d\lambda) \mathbf{v}^*) &= \int_{\mathbb{T}} e^{-in\lambda} (\mathbf{v} \mathbf{n}_{\mathbf{Z}}(d\lambda) \mathbf{v}^*)^* = \int_{\mathbb{T}} e^{-in\lambda} (\mathbf{v} \mathbf{n}_{\mathbf{Z}}^*(d\lambda) \mathbf{v}^*) = \int_{\mathbb{T}} e^{in\lambda} (\mathbf{v} \mathbf{n}_{\mathbf{Z}}^*(-d\lambda) \mathbf{v}^*), \quad n \in \mathbb{Z}. \end{aligned}$$

- (ii) A white noise $(\boldsymbol{\varepsilon}_n)_{n \in \mathbb{Z}}$ is strongly harmonizable, and the measure $\mathbf{n}_{\boldsymbol{\varepsilon}}$ coincides with $\boldsymbol{\Sigma}_{\boldsymbol{\varepsilon}}^2 \delta_0$, where δ_0 is the Dirac measure at 0.

(iii) Following the notations of Examples 3.1, the d -variate time series $\mathbf{Z}^{(3)}$ is weakly harmonizable and not strongly harmonizable (Rao (1982, Sec.2 p.301) and Helson and Lowdenslager (1958, p.183)), while $\mathbf{Z}^{(1)}$ is a white noise. Applying Proposition 3.1, we establish that the d -variate time series $\mathbf{Z}^{(2)}$, $\mathbf{Z}^{(4)}$ and $\mathbf{Z}^{(5)}$ are strongly harmonizable noises.

– For $\mathbf{Z}^{(2)}$, let $f_{\mathbf{Z}^{(2)}}(\lambda) \stackrel{\text{def}}{=} \sum_{k=n_1}^{n_2} e^{i\lambda k}$, then $\text{Var}(\mathbf{Z}_n^{(2)}) = \frac{1}{2\pi} \int_{\mathbb{T}} e^{-i\lambda n} f_{\mathbf{Z}^{(2)}}(\lambda) d\lambda$ and hence $\mathbf{n}_{\mathbf{Z}^{(2)}}(d\lambda) = \frac{1}{2\pi} f_{\mathbf{Z}^{(2)}}(\lambda) d\lambda$.

– For $\mathbf{Z}^{(4)}$, notice that

$$(1 + (-1)^n)^2 = (1 + \cos(\pi n))^2 = 2 + 2\cos(\pi n) = 2 + e^{i\pi n} + e^{-i\pi n}.$$

Then we obtain that

$$\mathbf{n}_{\mathbf{Z}^{(4)}} = \Sigma_{\varepsilon}^2 \begin{pmatrix} \delta_0 \\ 2\delta_0 + \delta_{\pi} + \delta_{-\pi} \end{pmatrix}.$$

– For $\mathbf{Z}^{(5)}$ we have

$$(1 \wedge |n|^{-1})^2 = 1 \wedge |n|^{-2}$$

Consider the function $f_{\mathbf{Z}^{(5)}}(\lambda) \stackrel{\text{def}}{=} 1 + 2 \sum_{n=1}^{\infty} \frac{1}{n^2} \cos(\lambda n)$ and the real valued measure defined on \mathbb{T} by $\mathbf{n}_{\mathbf{Z}^{(5)}}(A) \stackrel{\text{def}}{=} \frac{1}{2\pi} \int_A f_{\mathbf{Z}^{(5)}}(\lambda) d\lambda$. Then we obtain that

$$\mathbf{n}_{\mathbf{Z}^{(5)}} = \Sigma_{\varepsilon}^2 \begin{pmatrix} \delta_0 \\ \mathbf{n}_{\mathbf{Z}^{(5)}} \end{pmatrix}.$$

(iv) For any integrable function $\mathbf{G} : \mathbb{T} \rightarrow \mathbb{C}^{d \times d}$, $\mathbf{G} \in L^1_{\mathbb{C}^{d \times d}}(\mathbb{T})$, the ('convolution') function

$$\mathbf{D}(\lambda) \stackrel{\text{def}}{=} \int_{\mathbb{T}} \mathbf{G}(\alpha) \mathbf{G}^*(\alpha + \lambda) d\alpha.$$

The $d \times d$ -matrix $\int_{\mathbb{T}} e^{in\lambda} \mathbf{D}(\lambda) d\lambda$ is positive semi-definite for any n . It is therefore the covariance matrix of a zero mean d -variate random vector \mathbf{Z}_n : $\Sigma_{\mathbf{Z}_n}^2 = \int_{\mathbb{T}} e^{in\lambda} \mathbf{D}(\lambda) d\lambda$. Moreover, we can enlarge the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ in such a way that we can select the zero mean d -valued random vectors \mathbf{Z}_n , $n \in \mathbb{Z}$, so that they are uncorrelated: $\text{Cov}(\mathbf{Z}_m, \mathbf{Z}_n) = \Sigma_n^2 \mathbb{1}_{\{m=n\}}$. In this case, the d -variate time series $\mathbf{Z} = (\mathbf{Z}_n)_{n \in \mathbb{Z}}$ is a strongly harmonizable noise, so that $\mathbf{n}_{\mathbf{Z}}(d\lambda) = \mathbf{D}(\lambda) d\lambda$.

In the following proposition based on a general result of Szökefalvi-Nagy and Foias (1970, pp.183–184) (see also Niemi, 1978), we obtain a characterization for the stochastic measure of a d -variate harmonizable noise admitting a stochastic Radon-Nikodym density function with respect to the Lebesgue measure, i.e., for which there exists a strongly measurable Lebesgue-integrable function $\Delta_{\mathbf{Z}} : \mathbb{T} \rightarrow \mathcal{H}^d(\mathbf{Z})$ such that

$$\mu_{\mathbf{Z}}(A) = \int_A \Delta_{\mathbf{Z}}(\lambda) d\lambda.$$

Proposition 3.2 (Stochastic spectral density) *The stochastic spectral measure $\mu_{\mathbf{Z}}$ of a d -variate harmonizable noise \mathbf{Z} admits a density function with respect to the Lebesgue measure on \mathbb{T} , if and only if the total energy of the d -variate time series \mathbf{Z} is finite:*

$$\sum_n \text{tr} \Sigma_{\mathbf{Z}_n}^2 = \sum_n \text{tr} \text{Var}(\mathbf{Z}_n) = \sum_n \mathbb{E}(\mathbf{Z}_n^{\top} \mathbf{Z}_n) = \sum_n \sum_j \sigma_{\mathbf{Z}_n, j}^2 < \infty. \quad (6)$$

In this case, the stochastic spectral density function $\Delta_{\mathbf{Z}}$ on \mathbb{T} satisfies

$$\Delta_{\mathbf{Z}}(\lambda) = \frac{1}{2\pi} \sum_{n \in \mathbb{Z}} e^{-in\lambda} \mathbf{Z}_n, \quad (7)$$

where the convergence of series (7) is in \mathcal{H}^d .

Moreover, the d -variate harmonizable noise \mathbf{Z} is strongly harmonizable. The spectral bimeasure $\mathbf{M}_{\mathbf{Z}}$ is extendible to a $\mathbb{C}^{d \times d}$ -valued measure on \mathbb{T}^2 with density function

$$\mathbf{F}_{\mathbf{Z}}(\lambda_1, \lambda_2) = \mathbb{E}(\Delta_{\mathbf{Z}}(\lambda_1) \Delta_{\mathbf{Z}}^*(\lambda_2)) = \frac{1}{4\pi^2} \sum_{n \in \mathbb{Z}} e^{-in(\lambda_1 - \lambda_2)} \Sigma_{\mathbf{Z}_n}^2.$$

Thus, $\mathbf{n}_{\mathbf{Z}}(d\lambda) = \mathbf{D}_{\mathbf{Z}}(\lambda) d\lambda$, where

$$\mathbf{D}_{\mathbf{Z}}(\lambda) = \frac{1}{4\pi^2} \sum_{n \in \mathbb{Z}} e^{-in\lambda} \Sigma_{\mathbf{Z}_n}^2. \quad (8)$$

Remark 3.4 • The stochastic spectral density function $\Delta_{\mathbf{Z}}$ is stationary with respect to the frequencies: $\mathbf{F}_{\mathbf{Z}}(\lambda_1, \lambda_2) = \mathbb{E}(\Delta_{\mathbf{Z}}(\lambda_1) \Delta_{\mathbf{Z}}^*(\lambda_2)) = \mathbf{D}_{\mathbf{Z}}(\lambda_1 - \lambda_2)$.

- A (non-zero) white noise $\varepsilon = (\varepsilon_n)_{n \in \mathbb{Z}}$ has no stochastic spectral density with respect to the Lebesgue measure on \mathbb{T} . Indeed, condition (6) is not satisfied, because $\Sigma_{\varepsilon_n}^2$ is constant and different from the zero $d \times d$ -matrix $\mathbf{0}_{d \times d}$. Recall, however, that the non-zero white noise ε has a spectral density with respect to the Lebesgue measure on \mathbb{T} which is constant and equal to $\frac{1}{2\pi} \mathbb{I}_{d \times d}$.

4 Multivariate harmonizable linear model

In this section, we show that multivariate time series defined as moving average of a harmonizable noise is harmonizable. This is a key result that allows us to introduce harmonizable ARMA time series in the next section.

Let $\mathbf{Z} = (\mathbf{Z}_n)_{n \in \mathbb{Z}}$ be a d -variate harmonizable noise and $\mathbf{A} \stackrel{\text{def}}{=} (\mathbf{A}_n)_{n \in \mathbb{Z}}$ be a family of complex $d \times d$ -matrices such that $\sum_n \text{tr}(\mathbf{A}_n \mathbf{A}_n^*) < \infty$. Then the d -variate time series $\mathbf{X} = (\mathbf{X}_n)_{n \in \mathbb{Z}}$ is well-defined by

$$\mathbf{X}_n = \sum_{m \in \mathbb{Z}} \mathbf{A}_{n-m} \mathbf{Z}_m = \sum_{m \in \mathbb{Z}} \mathbf{A}_m \mathbf{Z}_{n-m}, \quad (9)$$

where the convergence is in $\mathcal{H}(\mathbf{Z}) \subset L_{\mathbb{C}^d}^2(\mathbb{P})$ (see Theorem 4.1 below, for stationary case see Brockwell and Davis (1991)). Notice that the convergence in the sum (9) is almost sure when $\sum_n |\mathbf{A}_n| < \infty$. The d -variate time series \mathbf{X} is the transformation of the d -variate harmonizable noise \mathbf{Z} by a time invariant linear filter and

$$\text{Cov}(\mathbf{X}_n, \mathbf{X}_m) = \sum_{j \in \mathbb{Z}} \mathbf{A}_{n-j} \Sigma_{\mathbf{Z}_j}^2 \mathbf{A}_{m-j}^*.$$

Thanks to Plancherel theorem (Rudin, 1967, see Theorem 1.6.1) the Fourier transform $\widehat{\mathbf{A}} : \mathbb{T} \rightarrow \mathbb{C}^{d \times d}$ of the sequence $\mathbf{A} = (\mathbf{A}_n)_{n \in \mathbb{Z}}$ is well-defined, and is square integrable on \mathbb{T} , i.e., the $\mathbb{C}^{d \times d}$ -valued function $\lambda \mapsto \widehat{\mathbf{A}}(\lambda) \widehat{\mathbf{A}}^*(\lambda)$ is integrable with respect to the Lebesgue measure on \mathbb{T} . Furthermore, since the topological group \mathbb{T} is compact, the Fourier transform $\widehat{\mathbf{A}}$ is integrable and

$$\mathbf{A}_n = \frac{1}{2\pi} \int_{\mathbb{T}} \widehat{\mathbf{A}}(\lambda) e^{-in\lambda} d\lambda.$$

When the sequence \mathbf{A} is absolutely summable ($\sum_n |\mathbf{A}_n| < \infty$), the Fourier transform $\widehat{\mathbf{A}}$ is equal to

$$\widehat{\mathbf{A}}(\lambda) = \sum_{n \in \mathbb{Z}} \mathbf{A}_n e^{in\lambda}.$$

In the following theorem, we state that the time series $(\mathbf{X}_n)_{n \in \mathbb{Z}}$ given by (9) is harmonizable.

Theorem 4.1 *Let $\mathbf{Z} \stackrel{\text{def}}{=} (\mathbf{Z}_n)_{n \in \mathbb{Z}}$ be a d -variate harmonizable noise, and let $\mathbf{A} \stackrel{\text{def}}{=} (\mathbf{A}_n)_{n \in \mathbb{Z}}$ be a family of complex $d \times d$ -matrices such that $\sum_n \text{tr}(\mathbf{A}_n \mathbf{A}_n^*) < \infty$. Then the d -variate time series $\mathbf{X} \stackrel{\text{def}}{=} (\mathbf{X}_n)_{n \in \mathbb{Z}}$ defined by the moving average series (9) is well-defined in $\mathcal{H}^d(\mathbf{Z})$ and it is harmonizable. Moreover, the Fourier transform $\widehat{\mathbf{A}}$ of \mathbf{A} is $\mu_{\mathbf{Z}}$ -integrable and the stochastic spectral measure $\mu_{\mathbf{X}}$ of \mathbf{X} satisfies*

$$\mu_{\mathbf{X}}(A) = \frac{1}{2\pi} \sum_{n \in \mathbb{Z}} \left(\int_A e^{-in\lambda} \widehat{\mathbf{A}}(-\lambda) d\lambda \right) \mathbf{Z}_n = \int_A \widehat{\mathbf{A}}(-\lambda) \mu_{\mathbf{Z}}(d\lambda), \quad (10)$$

where $A \in \mathcal{B}(\mathbb{T})$. Therefore, the spectral bimeasure of \mathbf{X} can be expressed by

$$\begin{aligned} \mathbf{M}_{\mathbf{X}}(A, B) &= \frac{1}{4\pi^2} \sum_{n \in \mathbb{Z}} \left(\int_A e^{-in\lambda_1} \widehat{\mathbf{A}}(-\lambda_1) d\lambda_1 \right) \Sigma_{\mathbf{Z}_n}^2 \left(\int_B e^{in\lambda_2} \widehat{\mathbf{A}}^*(-\lambda_2) d\lambda_2 \right) \\ &= \int_A \int_B \widehat{\mathbf{A}}(-\lambda_1) \mathbf{M}_{\mathbf{Z}}(d\lambda_1, d\lambda_2) \widehat{\mathbf{A}}^*(-\lambda_2), \end{aligned} \quad (11)$$

where $A, B \in \mathcal{B}(\mathbb{T})$. The last integral is in the sense of an integral with respect to a bimeasure (see (Rao, 1982), see also (Rozanov, 1959)).

Remark 4.1 (i) If $(\mathbf{Z}_n)_{n \in \mathbb{Z}}$ is strongly harmonizable, then $(\mathbf{X}_n)_{n \in \mathbb{Z}}$ is strongly harmonizable. The spectral bimeasure $\mathbf{M}_{\mathbf{X}}$ is extendible to a $\mathbb{C}^{d \times d}$ -valued measure on \mathbb{T}^2 and the integral in equality (11) is the Lebesgue integral with respect to the $\mathbb{C}^{d \times d}$ -valued measure $\mathbf{M}_{\mathbf{Z}}$.

Indeed, since $\sum_n \text{tr}(\mathbf{A}_n \mathbf{A}_n^*) < \infty$, Proposition 3.1 implies that

$$\text{Cov}(\mathbf{X}_n, \mathbf{X}_m) = \sum_{j \in \mathbb{Z}} \int_{\mathbb{T}} e^{ij\lambda} (\mathbf{A}_{n-j} \mathbf{n}_{\mathbf{Z}}(d\lambda) \mathbf{A}_{m-j}^*)$$

Applying the Parseval's equality to each of the components of $\text{Cov}(\mathbf{X}_n, \mathbf{X}_m)$ we obtain that

$$\text{Cov}(\mathbf{X}_n, \mathbf{X}_m) = \frac{1}{2\pi} \int_{\mathbb{T}} \left(\int_{\mathbb{T}} e^{i(n(\alpha+\lambda)-m\lambda)} \widehat{\mathbf{A}}(\alpha-\lambda) \mathbf{n}_{\mathbf{Z}}(d\alpha) \widehat{\mathbf{A}}^*(-\lambda) \right) d\lambda,$$

and the spectral bimeasure $\mathbf{M}_{\mathbf{X}}$ is extendible to a vector valued measure on \mathbb{T}^2 , and

$$\mathbf{M}_{\mathbf{X}}(A \times B) = \frac{1}{2\pi} \int_B \left(\int_A \widehat{\mathbf{A}}(\lambda_1 - 2\lambda_2) \mathbf{n}_{\mathbf{Z}}(d\lambda_1 - \lambda_2) \widehat{\mathbf{A}}^*(-\lambda_2) \right) d\lambda_2.$$

(ii) If $(\mathbf{Z}_n)_{n \in \mathbb{Z}}$ has a stochastic spectral density $\Delta_{\mathbf{Z}}$ (Proposition 3.2), then $(\mathbf{X}_n)_{n \in \mathbb{Z}}$ has a stochastic spectral density which coincides with $\Delta_{\mathbf{X}}(\lambda) = \widehat{\mathbf{A}}(-\lambda) \Delta_{\mathbf{Z}}(\lambda)$:

$$\mu_{\mathbf{X}}(A) = \int_A \widehat{\mathbf{A}}(-\lambda) \Delta_{\mathbf{Z}}(\lambda) d\lambda.$$

Thus the spectral bimeasure $\mathbf{M}_{\mathbf{X}}$ is extendible to a measure on \mathbb{T}^2 with a density function

$$\mathbf{F}_{\mathbf{X}}(\lambda_1, \lambda_2) = \widehat{\mathbf{A}}(-\lambda_1) \mathbf{D}_{\mathbf{Z}}(\lambda_1 - \lambda_2) \widehat{\mathbf{A}}^*(-\lambda_2)$$

and

$$\begin{aligned} \text{Cov}(\mathbf{X}_n, \mathbf{X}_m) &= \int_{\mathbb{T}} \left(\sum_{n \in \mathbb{Z}} e^{ij\lambda} \mathbf{A}_{n-j} \mathbf{D}_{\mathbf{Z}}(\lambda) \mathbf{A}_{m-j}^* \right) d\lambda \\ &= \int_{\mathbb{T}^2} e^{-i(n\lambda_1 - m\lambda_2)} \widehat{\mathbf{A}}(-\lambda_1) \mathbf{D}_{\mathbf{Z}}(\lambda_1 - \lambda_2) \widehat{\mathbf{A}}^*(-\lambda_2) d\lambda_1 d\lambda_2. \end{aligned}$$

(iii) If $(\mathbf{Z}_n)_{n \in \mathbb{Z}}$ has a discrete stochastic spectral measure, then $(\mathbf{X}_n)_{n \in \mathbb{Z}}$ has a discrete stochastic spectral measure: $\boldsymbol{\mu}_{\mathbf{X}}(\lambda) = \widehat{\mathbf{A}}(-\lambda)\boldsymbol{\mu}_{\mathbf{Z}}(\lambda)$ and $\mathbf{M}_{\mathbf{X}}(\lambda_1, \lambda_2) = \widehat{\mathbf{A}}(-\lambda_1)\mathbf{M}_{\mathbf{Z}}(\lambda_1, \lambda_2)\widehat{\mathbf{A}}^*(-\lambda_2)$.

5 Harmonizable VARMA model

Our chosen definition for harmonizable noise enables us to build harmonizable VARMA (HVARMA) models following the ideas of Brockwell and Davis (1991, sect. 3.1). Below, we introduce HVARMA models and give some of their properties. This is the starting point of many possible studies on their estimation, prediction, etc., but also the key results that will allow us to propose a method for generating realizations of HVARMA time series later in the paper.

5.1 Definition and spectrum

Consider a d -variate white noise $\boldsymbol{\varepsilon}$ associated to the d -variate noise \mathbf{Z} as in Lemma 3.1: $\mathbf{Z}_n = \mathbf{S}_n \boldsymbol{\varepsilon}_n = \Xi(\boldsymbol{\varepsilon}_n)$, with $\sup_{n,j,k} |s_{n,j,k}| < \infty$. Let \mathbf{Y} be a solution of the difference equation

$$\mathbf{Y}_n + \boldsymbol{\phi}_1 \mathbf{Y}_{n-1} + \cdots + \boldsymbol{\phi}_p \mathbf{Y}_{n-p} = \boldsymbol{\varepsilon}_n + \boldsymbol{\theta}_1 \boldsymbol{\varepsilon}_{n-1} + \cdots + \boldsymbol{\theta}_q \boldsymbol{\varepsilon}_{n-q}. \quad (12)$$

The operator Ξ , introduced in the proof of Theorem 3.1, which is a bounded operator on $\mathcal{H}^d(\boldsymbol{\varepsilon})$, can be applied to relation (12).

Thus,

$$\Xi(\mathbf{Y}_n) + \boldsymbol{\phi}_1 \Xi(\mathbf{Y}_{n-1}) + \cdots + \boldsymbol{\phi}_p \Xi(\mathbf{Y}_{n-p}) = \mathbf{Z}_n + \boldsymbol{\theta}_1 \mathbf{Z}_{n-1} + \cdots + \boldsymbol{\theta}_q \mathbf{Z}_{n-q}.$$

Consequently, the d -variate time series $(\mathbf{X}_n)_{n \in \mathbb{Z}} = (\Xi(\mathbf{Y}_n))_{n \in \mathbb{Z}}$ is well-defined, and is the solution of the VARMA equation:

$$\mathbf{X}_n + \boldsymbol{\phi}_1 \mathbf{X}_{n-1} + \cdots + \boldsymbol{\phi}_p \mathbf{X}_{n-p} = \mathbf{Z}_n + \boldsymbol{\theta}_1 \mathbf{Z}_{n-1} + \cdots + \boldsymbol{\theta}_q \mathbf{Z}_{n-q}, \quad (13)$$

where $\mathbf{Z} = (\mathbf{Z}_n)_{n \in \mathbb{Z}}$ is a d -variate harmonizable noise and $\boldsymbol{\phi}_1, \dots, \boldsymbol{\phi}_p, \boldsymbol{\theta}_1, \dots, \boldsymbol{\theta}_q \in \mathbb{C}^{d \times d}$. We define the polynomial matrices $\boldsymbol{\Phi}(z) = \mathbb{I}_{d \times d} + \boldsymbol{\phi}_1 z + \cdots + \boldsymbol{\phi}_p z^p$ and $\boldsymbol{\Theta}(z) = \mathbb{I}_{d \times d} + \boldsymbol{\theta}_1 z + \cdots + \boldsymbol{\theta}_q z^q$ associated to the difference equation (13).

From now on, we assume that $\det \boldsymbol{\Phi}(e^{i\lambda}) \neq 0$ for any λ . Then the difference equation (13) admits a unique d -variate harmonizable solution.

$$\mathbf{X}_n = \int_{\mathbb{T}} e^{in\lambda} \boldsymbol{\Phi}(e^{-i\lambda})^{-1} \boldsymbol{\Theta}(e^{-i\lambda}) \boldsymbol{\mu}_{\mathbf{Z}}(d\lambda) = \sum_{j \in \mathbb{Z}} \mathbf{A}_j \mathbf{Z}_{n-j}, \quad (14)$$

where the matrix \mathbf{A}_j is the coefficient of z^j in the Laurent expansion

$$\boldsymbol{\Phi}(z)^{-1} \boldsymbol{\Theta}(z) = \sum_{j \in \mathbb{Z}} \mathbf{A}_j z^j.$$

Since $\det \boldsymbol{\Phi}(e^{i\lambda}) \neq 0$ for any λ , the Laurent series is absolutely convergent in some ring (annulus) $r^{-1} < z < r$ for some $r > 1$, and thus $\sum_{n \in \mathbb{Z}} |\mathbf{A}_n| < \infty$ (see, for e.g., Melnyk, 2023) and $\widehat{\mathbf{A}}(\lambda) = \boldsymbol{\Phi}(e^{-i\lambda})^{-1} \boldsymbol{\Theta}(e^{-i\lambda})$. The relation (10) implies that the stochastic spectral measure $\boldsymbol{\mu}_{\mathbf{X}}$ verifies

$$\boldsymbol{\mu}_{\mathbf{X}}(A) = \int_A \boldsymbol{\Phi}(e^{-i\lambda})^{-1} \boldsymbol{\Theta}(e^{-i\lambda}) \boldsymbol{\mu}_{\mathbf{Z}}(d\lambda) = \frac{1}{2\pi} \sum_{n \in \mathbb{Z}} \left(\int_A e^{-in\lambda} \boldsymbol{\Phi}(e^{-i\lambda})^{-1} \boldsymbol{\Theta}(e^{-i\lambda}) d\lambda \right) \mathbf{Z}_n.$$

Furthermore,

$$\begin{aligned} \text{Cov}(\mathbf{X}_n, \mathbf{X}_m) &= \sum_{j \in \mathbb{Z}} \mathbf{A}_{n-j} \Sigma_{\mathbf{Z}_j}^2 \mathbf{A}_{m-j}^* \\ &= \iint_{\mathbb{T}^2} e^{i(n\lambda_1 - m\lambda_2)} (\Phi(e^{-i\lambda_1})^{-1} \Theta(e^{-i\lambda_1})) \mathbf{M}_{\mathbf{Z}}(d\lambda_1, d\lambda_2) (\Theta^*(e^{-i\lambda_2}) \Phi^*(e^{-i\lambda_2})^{-1}) \end{aligned}$$

and the spectral bimeasure can be expressed as

$$\begin{aligned} \mathbf{M}_{\mathbf{X}}(A, B) &= \iint_{A \times B} (\Phi(e^{-i\lambda_1})^{-1} \Theta(e^{-i\lambda_1})) \mathbf{M}_{\mathbf{Z}}(d\lambda_1, d\lambda_2) (\Theta^*(e^{-i\lambda_2}) \Phi^*(e^{-i\lambda_2})^{-1}) \\ &= \frac{1}{4\pi^2} \sum_{n \in \mathbb{Z}} \left(\int_A e^{-in\lambda} \Phi(e^{-i\lambda})^{-1} \Theta(e^{-i\lambda}) d\lambda \right) \Sigma_{\mathbf{Z}_n}^2 \left(\int_B e^{in\lambda} \Theta^*(e^{-i\lambda}) \Phi^*(e^{-i\lambda})^{-1} d\lambda \right). \end{aligned} \quad (15)$$

In the remark below we give some important properties of HVARMA models.

Remark 5.1 (i) If $(\mathbf{Z}_n)_{n \in \mathbb{Z}}$ is a strongly harmonizable noise, then $(\mathbf{X}_n)_{n \in \mathbb{Z}}$ is strongly harmonizable.

(ii) If the harmonizable noise $(\mathbf{Z}_n)_{n \in \mathbb{Z}}$ has a stochastic spectral density $\Delta_{\mathbf{Z}}$, then $(\mathbf{X}_n)_{n \in \mathbb{Z}}$ has a stochastic spectral density $\Delta_{\mathbf{X}}(\lambda) = \Phi(e^{-i\lambda})^{-1} \Theta(e^{-i\lambda}) \Delta_{\mathbf{Z}}(\lambda)$, and

$$\begin{aligned} \text{Cov}(\mathbf{X}_n, \mathbf{X}_m) &= \frac{1}{4\pi^2} \iint_{\mathbb{T}^2} e^{i(n\lambda_1 - m\lambda_2)} \Phi(e^{-i\lambda_1})^{-1} \Theta(e^{-i\lambda_1}) \mathbf{D}_{\mathbf{Z}}(\lambda_1 - \lambda_2) \Theta^*(e^{-i\lambda_2}) \Phi^*(e^{-i\lambda_2})^{-1} d\lambda_1 d\lambda_2. \end{aligned}$$

(iii) Causality (Brockwell & Davis, 1991, Theorem 3.1.1) :

$$\det \Phi(z) \neq 0 \quad \text{for } |z| \leq 1 \quad \implies \quad \mathbf{X}_n = \sum_{j=-\infty}^n \mathbf{A}_{n-j} \mathbf{Z}_j = \sum_{j=0}^{\infty} \mathbf{A}_j \mathbf{Z}_{n-j}.$$

(iv) Invertibility (Brockwell & Davis, 1991, Theorem 3.1.2) :

$$\det \Theta(z) \neq 0 \quad \text{for } |z| \leq 1 \quad \implies \quad \mathbf{Z}_n = \sum_{j=-\infty}^n \mathbf{\Pi}_{n-j} \mathbf{X}_j = \sum_{j=0}^{\infty} \mathbf{\Pi}_j \mathbf{X}_{n-j}.$$

Here $\Theta(z)^{-1} \Phi(z) = \sum_{j \in \mathbb{Z}} \mathbf{\Pi}_j z^j$.

(v) In (iii) and (iv), we have equivalence whenever the polynomial functions $\det \Phi(\cdot)$ and $\det \Theta(\cdot)$ have no common zeroes.

6 Simulation harmonizable time series with a given Loève spectrum

In this section, we give tools to generate finite time sample realizations of a HVARMA process with known Loève spectrum, while ensuring they provide a correct approximation of the associated time series defined on \mathbb{Z} .

We first show, in Section 6.1, how to construct a periodic harmonizable time series on \mathbb{Z} from its discrete Loève spectrum. However, this method does not permit to get a harmonizable noise, which is the basic building block of our HVARMA model. In Section 6.2 we describe how to overcome this difficulty by defining harmonizable noise and time series directly on the finite group \mathbb{Z}_N . Next, results of Section 6.3

ensure that the restriction of harmonizable noise and time series on \mathbb{Z} to \mathbb{Z}_N provides a proper approximation. Consequently, we consider in Section 6.4 the extension by zeros of harmonizable noise and time series defined on the finite group \mathbb{Z}_N to (the full space) \mathbb{Z} . We prove that in general this does not produce a harmonizable time series, but a harmonizable noise on \mathbb{Z} , and a linear harmonizable time series. On that basis, we can construct a HVARMA time series on \mathbb{Z} as described in Section 6.5. Finally, we provide some numerical examples of realizations from HVARMA process in Section 6.6.

Hereafter, N is a fixed integer, $N \geq 1$. We denote as $\mathbb{Z}_N \stackrel{\text{def}}{=} \mathbb{Z}/(2N+1)$ the finite group of integer numbers modulo $2N+1$ that we represent as $\{-N, \dots, N\}$. Its dual group \mathbb{T}_N can be represented by $\{\omega_j : j \in \mathbb{Z}_N\}$, where $\omega_j \stackrel{\text{def}}{=} \frac{2\pi j}{2N+1}$.

6.1 Construction of a periodic harmonizable time series

We consider a d -variate time series $(\mathbf{X}_n)_{n \in \mathbb{Z}}$ defined as:

$$\mathbf{X}_n \stackrel{\text{def}}{=} \sum_{j=-N}^N e^{in\omega_j} \boldsymbol{\mu}_j,$$

where the random vectors (called increments) $\boldsymbol{\mu}_j$ are extracted from a d -variate zero mean stationary time series $(\boldsymbol{\mu}_n)_{n \in \mathbb{Z}}$ with covariance function

$$\mathbf{K}_\boldsymbol{\mu}(r) \stackrel{\text{def}}{=} \text{Cov}[\boldsymbol{\mu}_{j+r}, \boldsymbol{\mu}_j].$$

$(\mathbf{X}_n)_{n \in \mathbb{Z}}$ is periodic with period $2N+1$, zero mean, strongly harmonizable with a stochastic spectral measure $\boldsymbol{\mu}_\mathbf{X}$ which has a discrete support at frequencies ω_j , $j = -N, \dots, N$, $\boldsymbol{\mu}_\mathbf{X}(\{\omega_j\}) = \boldsymbol{\mu}_j$. Furthermore, its spectral bimeasure $\mathbf{M}_\mathbf{X}$ is a discrete measure on \mathbb{T}^2

$$\mathbf{M}_\mathbf{X}(\{(\omega_{j_1}, \omega_{j_2})\}) = \text{Cov}[\boldsymbol{\mu}_\mathbf{X}(\omega_{j_1}), \boldsymbol{\mu}_\mathbf{X}(\omega_{j_2})] = \text{Cov}[\boldsymbol{\mu}_{j_1}, \boldsymbol{\mu}_{j_2}] = \mathbf{K}_\boldsymbol{\mu}(j_1 - j_2).$$

Additionally,

$$\text{Cov}[\mathbf{X}_{n+k}, \mathbf{X}_n] = \sum_{j_1=-N}^N \sum_{r=j_1-N}^{j_1+N} e^{ik\omega_{j_1}} e^{in\omega_r} \mathbf{K}_\boldsymbol{\mu}(r).$$

Assume that $\mathbf{K}_\boldsymbol{\mu}$ is periodic with period $2N+1$.

Then we have

$$\text{Cov}[\mathbf{X}_{n+k}, \mathbf{X}_n] = \sum_{j_1=-N}^N \sum_{r=-N}^N e^{ik\omega_{j_1}} e^{in\omega_r} \mathbf{K}_\boldsymbol{\mu}(r).$$

Therefore, $\text{Cov}[\mathbf{X}_{n+k}, \mathbf{X}_n] = \mathbf{0}_{d \times d}$ for $k \neq 0 \pmod{2N+1}$. While, for $k = 0 \pmod{2N+1}$, we have

$$\text{Cov}[\mathbf{X}_{n+k}, \mathbf{X}_n] = \text{Var}[\mathbf{X}_n] = (2N+1) \sum_{r=-N}^N e^{in\omega_r} \mathbf{K}_\boldsymbol{\mu}(r).$$

Hence, \mathbf{X} cannot be a non-identically null harmonizable noise. In other words, due to its periodicity, the process \mathbf{X} cannot be a harmonizable noise.

6.2 Time series indexed by a finite group

Similarly to Sections 2.2 and 3, we can consider harmonizable time series and harmonizable noise defined on \mathbb{Z}_N by replacing \mathbb{Z} by \mathbb{Z}_N and \mathbb{T} by \mathbb{T}_N in Definitions 2.1 and 3.1 and condition (Ha). In this case Proposition 2.1, Lemma 3.1 as well as Theorem 3.1 are valid. Additionally, since the group \mathbb{Z}_N and its dual group \mathbb{T}_N are finite, any zero mean second order time series $\boldsymbol{\chi} = (\boldsymbol{\chi}_n)_{n \in \mathbb{Z}_N}$ is strongly harmonizable with a discrete stochastic spectral measure:

$$\boldsymbol{\mu}_{\boldsymbol{\chi}}(\omega_j) = \frac{1}{2N+1} \sum_{n \in \mathbb{Z}_N} e^{-in\omega_j} \boldsymbol{\chi}_n = \frac{1}{2N+1} \sum_{n \in \mathbb{Z}_N} e^{-in\omega_1} \boldsymbol{\chi}_n, \quad (16)$$

and a discrete spectral bimeasure

$$\mathbf{M}_{\boldsymbol{\chi}}(\omega_{j_1}, \omega_{j_2}) = \frac{1}{(2N+1)^2} \sum_{n_1 \in \mathbb{Z}_N} \sum_{n_2 \in \mathbb{Z}_N} e^{-i(n_1 j_1 - n_2 j_2)\omega_1} \mathbf{E}(\boldsymbol{\chi}_{n_1} \boldsymbol{\chi}_{n_2}^*).$$

Hence,

$$\boldsymbol{\chi}_n = \sum_{j \in \mathbb{Z}_N} e^{inj\omega_1} \boldsymbol{\mu}_{\boldsymbol{\chi}}(\omega_j),$$

and

$$\text{Cov}(\boldsymbol{\chi}_{n_1}, \boldsymbol{\chi}_{n_2}) = \sum_{j_1 \in \mathbb{Z}_N} \sum_{j_2 \in \mathbb{Z}_N} e^{-i(n_1 j_1 - n_2 j_2)\omega_1} \mathbf{M}_{\boldsymbol{\chi}}(\omega_{j_1}, \omega_{j_2}).$$

Notice that any (second order) time series $\boldsymbol{\chi}$ can be extended periodically to a strongly harmonizable time series $\tilde{\boldsymbol{\chi}}$ on \mathbb{Z} with the discrete stochastic spectral measure $\boldsymbol{\mu}_{\tilde{\boldsymbol{\chi}}}(\omega_j) = \boldsymbol{\mu}_{\boldsymbol{\chi}}(\omega_j)$ for $j = -N, \dots, N$, and $\boldsymbol{\mu}_{\tilde{\boldsymbol{\chi}}}(\omega) = \mathbf{0}_d$ otherwise.

6.2.1 Harmonizable noise on \mathbb{Z}_N

Let $\boldsymbol{\mu} = (\boldsymbol{\mu}_j)_{j \in \mathbb{Z}_N}$ be a zero mean stationary d -variate time series on \mathbb{Z}_N with covariance matrix function $\mathbf{K}_{\boldsymbol{\mu}}(r) \stackrel{\text{def}}{=} \text{Cov}(\boldsymbol{\mu}_{j+r}, \boldsymbol{\mu}_j) = \mathbf{E}(\boldsymbol{\mu}_{j+r} \boldsymbol{\mu}_j^*)$. An example can be found in Section 6.2.3. Then, the time series $\boldsymbol{\zeta} = (\boldsymbol{\zeta}_n)_{n \in \mathbb{Z}_N}$ defined by

$$\boldsymbol{\zeta}_n \stackrel{\text{def}}{=} \sum_{j \in \mathbb{Z}_N} e^{in\omega_j} \boldsymbol{\mu}_j, \quad (17)$$

is the (strongly) harmonizable noise on \mathbb{Z}_N (see Theorem 3.1) with the stochastic spectral measure $\boldsymbol{\mu}_{\boldsymbol{\zeta}}$ defined on \mathbb{T}_N by $\boldsymbol{\mu}_{\boldsymbol{\zeta}}(\omega_j) = \boldsymbol{\mu}_j$, and

$$\boldsymbol{\Sigma}_{\boldsymbol{\zeta}_n}^2 \stackrel{\text{def}}{=} \text{Var}(\boldsymbol{\zeta}_n) = \sum_{j \in \mathbb{Z}_N} e^{in\omega_j} \mathbf{n}_{\boldsymbol{\zeta}}(\omega_j) = (2N+1) \sum_{j \in \mathbb{Z}_N} e^{in\omega_j} \mathbf{K}_{\boldsymbol{\mu}}(j) = \int_{\mathbb{T}_N} e^{in\omega} \mathbf{n}_{\boldsymbol{\zeta}}(d\omega),$$

where $\mathbf{n}_{\boldsymbol{\zeta}}(\omega_j) = (2N+1) \mathbf{K}_{\boldsymbol{\mu}}(j)$ (see relation (3)). Notice that the time series $\boldsymbol{\zeta}$ is a real-valued vector if and only if $\bar{\boldsymbol{\mu}}_j = \boldsymbol{\mu}_{-j}$, $j \in \mathbb{Z}_N$.

Conversely, if $\boldsymbol{\zeta} = (\boldsymbol{\zeta}_n)_{n \in \mathbb{Z}_N}$ is any harmonizable noise on \mathbb{Z}_N , then the time series $\boldsymbol{\mu} = (\boldsymbol{\mu}_j)_{j \in \mathbb{Z}_N}$ defined by

$$\boldsymbol{\mu}_j \stackrel{\text{def}}{=} \boldsymbol{\mu}_{\boldsymbol{\zeta}}(\omega_j) = \frac{1}{2N+1} \sum_{n \in \mathbb{Z}_N} e^{-in\omega_j} \boldsymbol{\zeta}_n$$

is (second order) stationary on \mathbb{Z}_N with the covariance function

$$\mathbf{K}_{\boldsymbol{\mu}_{\boldsymbol{\zeta}}}(j) = \frac{1}{2N+1} \mathbf{n}_{\boldsymbol{\zeta}}(\omega_j) = \frac{1}{2N+1} \sum_{n \in \mathbb{Z}_N} e^{-in\omega_j} \text{Var}(\boldsymbol{\zeta}_n) = \frac{1}{2N+1} \sum_{n \in \mathbb{Z}_N} e^{-in\omega_j} \boldsymbol{\Sigma}_{\boldsymbol{\zeta}_n}^2. \quad (18)$$

6.2.2 Moving average on \mathbb{Z}_N

If $\zeta = (\zeta_n)_{n \in \mathbb{Z}_N}$ is a harmonizable noise on \mathbb{Z}_N and $(\Gamma_n)_{n \in \mathbb{Z}_N}$ is a family of complex $d \times d$ -matrices, then the times series $\chi = (\chi_n)_{n \in \mathbb{Z}_N}$ defined by

$$\chi_n \stackrel{\text{def}}{=} \sum_{m \in \mathbb{Z}_N} \Gamma_{n-m} \zeta_m \quad (19)$$

is (strongly) harmonizable on \mathbb{Z}_N with stochastic spectral measure and spectral bimeasure

$$\mu_\chi(\omega_j) = \widehat{\Gamma}_{-j} \mu_\zeta(\omega_j) \quad \text{and} \quad M_\chi(\omega_{j_1}, \omega_{j_2}) = \widehat{\Gamma}_{-j_1} \mathbf{K}_{\mu_\zeta}(j_1 - j_2) \widehat{\Gamma}_{-j_2}^*$$

where

$$\widehat{\Gamma}_j = \sum_{k \in \mathbb{Z}_N} \Gamma_k e^{ik\omega_j}.$$

Notice that in formula (19), Γ_n is periodically extended to \mathbb{Z} by setting $\Gamma_{n+kN} = \Gamma_n$.

6.2.3 Moving average with band-limited spectrum

We consider a white noise $(\varepsilon_n)_{n \in \mathbb{Z}_N}$ on \mathbb{Z}_N and a family $(\vartheta_n)_{n \in \mathbb{Z}_N}$ of complex $d \times d$ -matrices. Then the moving average

$$\mu_j = \sum_{k \in \mathbb{Z}_N} \vartheta_{j-k} \varepsilon_k$$

defines a stationary times series $\mu = (\mu_j)_{j \in \mathbb{Z}_N}$ with covariance function

$$\mathbf{K}_\mu(r) = \mathbb{E}(\mu_{j+r} \mu_j^*) = \sum_{k \in \mathbb{Z}_N} \vartheta_{k+r} \Sigma_\varepsilon^2 \vartheta_k^*.$$

The harmonizable noise ζ defined by relation (6.5) satisfies the equality $\zeta_n = \widehat{\theta}_n \widehat{\varepsilon}_n$, where

$$\widehat{\theta}_n \stackrel{\text{def}}{=} \sum_{k \in \mathbb{Z}_N} e^{in\omega_k} \theta_k \quad \text{and} \quad \widehat{\varepsilon}_n \stackrel{\text{def}}{=} \sum_{k \in \mathbb{Z}_N} e^{in\omega_k} \varepsilon_k.$$

Notice that $\widehat{\varepsilon}$ is a white noise with variance $\Sigma_\varepsilon^2 = (2N+1)\Sigma_\varepsilon^2$.

Whenever $\vartheta_k = \mathbf{0}_d$ for $k = \pm q, \dots, \pm N \bmod (2N+1)$, for some $0 < q \leq N/2$ then $\mathbf{K}_\mu(r) = \mathbf{0}_{d \times d}$ for $r = \pm 2q, \dots, \pm N \bmod (2N+1)$ and the harmonizable noise ζ has a band-limited (Loève) spectrum: $n_\zeta(\omega_j) = \mathbf{0}_{d \times d}$ for $j = \pm 2q, \dots, \pm N \bmod (2N+1)$. In addition, the harmonizable moving average time series χ defined by relation (19) has a band-limited (Loève) spectrum: $M_\chi(\omega_{j_1}, \omega_{j_2}) = \mathbf{0}_{d \times d}$ for $j_1 - j_2 = \pm 2q, \dots, \pm N \bmod (2N+1)$.

6.2.4 Autoregressive time series on \mathbb{Z}_N

First, notice that the definition of an autoregressive time series with uncorrelated innovations on \mathbb{Z}_N need a constraining condition to justify its existence. Indeed, consider the autoregressive time series $\mu_n = \Phi \mu_{n-1} + \varepsilon_n$, $n \in \mathbb{Z}_N$, where $\{\varepsilon_n\}_{n \in \mathbb{Z}_N}$ is a d -variate white noise and Φ is a complex $d \times d$ -matrix. To justify the existence of such a time series $\{\mu_n\}_{n \in \mathbb{Z}_N}$, we need to assume the constraint : $\mu_{-N} = \Phi \mu_N + \varepsilon_{-N}$. That is, the $d \times d$ -matrix $\mathbb{I}_{d \times d} - \Phi^{2N+1}$ is invertible and

$$\mu_{-N} = (\mathbb{I}_{d \times d} - \Phi^{2N+1})^{-1} (\varepsilon_{-N} + \Phi^{2N} \varepsilon_{-N+1} + \dots + \Phi^{2N-k} \varepsilon_{-N+k+1} + \dots + \Phi \varepsilon_N).$$

In this case,

$$\mu_j = (\mathbb{I}_{d \times d} - \Phi^{2N+1})^{-1} (\varepsilon_j + \Phi^{2N} \varepsilon_{j+1} + \dots + \Phi^{2N-k} \varepsilon_{j+k+1} + \dots + \Phi \varepsilon_{j+2N})$$

for $j = -N, \dots, N$ and where $\varepsilon_{k+N} = \varepsilon_{k-N-1}$ for $k = 1, \dots, 2N$. Then, we can derive the expression for $K_{\boldsymbol{\mu}}(r)$ in terms of $\boldsymbol{\Phi}$ and r . For instance

$$K_{\boldsymbol{\mu}}(0) = (\mathbb{I}_{d \times d} - \boldsymbol{\Phi}^{2N+1})^{-1} (\mathbb{I} + (\boldsymbol{\Phi}\boldsymbol{\Phi}^*)^{2N} + \dots + (\boldsymbol{\Phi}\boldsymbol{\Phi}^*)^{2N-k} + \dots + \boldsymbol{\Phi}\boldsymbol{\Phi}^*) (\mathbb{I}_{d \times d} - \boldsymbol{\Phi}^{*2N+1})^{-1}.$$

6.3 Restriction to \mathbb{Z}_N of a harmonizable process on \mathbb{Z}

In this section, we study the link between the spectrum of the time series under analysis and the spectrum of the observation interpreted as a time series on a finite group. Let $\mathbf{X} = (\mathbf{X}_n)_{n \in \mathbb{Z}}$ be a harmonizable times series on \mathbb{Z} . Then the times series $\boldsymbol{\chi} = (\boldsymbol{\chi}_n)_{n \in \mathbb{Z}_N}$ defined by $\boldsymbol{\chi}_n = \mathbf{X}_n$ for $n = -N, \dots, N$, is a strongly harmonizable on \mathbb{Z}_N with

$$\boldsymbol{\mu}_{\boldsymbol{\chi}}(\omega_j) = \frac{1}{2N+1} \sum_{n \in \mathbb{Z}_N} e^{-in\omega_j} \boldsymbol{\chi}_n = \frac{1}{2N+1} \sum_{n=-N}^N e^{-in\omega_j} \mathbf{X}_n = \int_{\mathbb{T}} \frac{W_N(\lambda - \omega_j)}{2N+1} \boldsymbol{\mu}_{\mathbf{X}}(d\lambda)$$

and

$$\mathbf{M}_{\boldsymbol{\chi}}(\omega_{j_1}, \omega_{j_2}) = \iint_{\mathbb{T} \times \mathbb{T}} \frac{W_N(\lambda_1 - \omega_{j_1}) W_N(-\lambda_2 + \omega_{j_2})}{(2N+1)^2} \mathbf{M}_{\mathbf{X}}(d\lambda_1, d\lambda_2),$$

where

$$W_N(\lambda) = \sum_{n=-N}^N e^{in\lambda} = 1 + 2 \sum_{n=1}^N \cos(n\lambda).$$

Notice that $W_N(0) = 2N+1$ and for $\lambda \neq 0 \pmod{2\pi}$ we have $|W_N(\lambda)|^2 = \frac{\sin^2((2N+1)\lambda/2)}{\sin^2(\lambda/2)}$. Hence, $\boldsymbol{\mu}_{\boldsymbol{\chi}}(\omega_j) \approx \boldsymbol{\mu}_{\mathbf{X}}(\omega_j)$ (in the sense of the L^2 -norm) and $\mathbf{M}_{\boldsymbol{\chi}}(\omega_{j_1}, \omega_{j_2}) \approx \mathbf{M}_{\mathbf{X}}(\omega_{j_1}, \omega_{j_2})$ for large N (Rozanov (see Theorem 2.2 1959) and Rao (1982)). Moreover, if $\lambda_1 < \lambda'_1$ and $\boldsymbol{\mu}_{\mathbf{X}}(\{\lambda_1\}) = \boldsymbol{\mu}_{\mathbf{X}}(\{\lambda'_1\}) = 0$, we have

$$\lim_{N \rightarrow \infty} \frac{1}{2\pi} \sum_{k=-N}^N \frac{e^{-i\lambda'_1 k} - e^{-i\lambda_1 k}}{-ik} \mathbf{X}_k = \boldsymbol{\mu}_{\mathbf{X}}([\lambda_1, \lambda'_1]) \quad \text{in } L^2\text{-norm.}$$

If $\lambda_1 < \lambda'_1$, $\lambda_2 < \lambda'_2$ and $\boldsymbol{\mu}_{\mathbf{X}}(\{\lambda_1\}) = \boldsymbol{\mu}_{\mathbf{X}}(\{\lambda'_1\}) = \boldsymbol{\mu}_{\mathbf{X}}(\{\lambda_2\}) = \boldsymbol{\mu}_{\mathbf{X}}(\{\lambda'_2\}) = 0$, then we have

$$\lim_{N_1, N_2 \rightarrow \infty} \frac{1}{4\pi^2} \sum_{k_1=-N_1}^{N_1} \sum_{k_2=-N_2}^{N_2} \frac{e^{-i\lambda'_1 k_1} - e^{-i\lambda_1 k_1}}{-ik_1} \times \frac{e^{i\lambda'_2 k_2} - e^{i\lambda_2 k_2}}{ik_2} \mathbb{E}[\mathbf{X}_{k_1} \mathbf{X}_{k_2}^*] = \mathbf{M}_{\mathbf{X}}([\lambda_1, \lambda'_1] \times [\lambda_2, \lambda'_2]).$$

Remark 6.1 If \mathbf{Z} is a harmonizable noise on \mathbb{Z} , then the time series $\boldsymbol{\zeta}$ defined on \mathbb{Z}_N by $\boldsymbol{\zeta}_n = \mathbf{Z}_n$ for $n = -N, \dots, N$, is a harmonizable noise on \mathbb{Z}_N with

$$\boldsymbol{\mu}_{\boldsymbol{\zeta}}(\omega_j) = \frac{1}{2N+1} \sum_{n=-N}^N e^{-in\omega_j} \mathbf{Z}_n, \quad \mathbf{M}_{\boldsymbol{\zeta}}(\omega_{j_1}, \omega_{j_2}) = \frac{1}{(2N+1)^2} \sum_{n=-N}^N e^{-in(\omega_{j_1} - \omega_{j_2})} \boldsymbol{\Sigma}_{\mathbf{Z}_n}^2$$

and

$$\mathbf{n}_{\boldsymbol{\zeta}}(\omega_j) = \frac{1}{(2N+1)^2} \sum_{n=-N}^N e^{-in\omega_j} \boldsymbol{\Sigma}_{\mathbf{Z}_n}^2.$$

6.4 Extension to \mathbb{Z} of a harmonizable process defined on \mathbb{Z}_N

Since we cannot obtain a non-null harmonizable noise on \mathbb{Z} from periodic extension, we discuss hereafter how to obtain a finite time sample, from the extension by zeros. We show that we can obtain harmonizable noise on \mathbb{Z} and characterize its moving average form.

Let χ be a harmonizable time series on \mathbb{Z}_N , and let $\tilde{\mathbf{X}}$ be the time series on \mathbb{Z} defined by $\tilde{\mathbf{X}}_n = \chi_n$ for $n = -N, \dots, N$, and $\tilde{\mathbf{X}}_n = \mathbf{0}_d$ otherwise. Then, in general, we cannot say whether the time series $\tilde{\mathbf{X}}$ is harmonizable or not. Nevertheless, for a harmonizable noise $\zeta = (\zeta_n)_{n \in \mathbb{Z}_N}$, the time series $\tilde{\mathbf{Z}} = (\tilde{\mathbf{Z}}_n)_{n \in \mathbb{Z}}$ defined by $\tilde{\mathbf{Z}}_n = \zeta_n$ for $n = -N, \dots, N$, and $\tilde{\mathbf{Z}}_n = \mathbf{0}_d$ otherwise, is a harmonizable noise on \mathbb{Z} . Indeed, condition (Ha) is fulfilled by $\tilde{\mathbf{Z}}$.

Furthermore, from Proposition 3.2, the stochastic spectral measure $\mu_{\tilde{\mathbf{Z}}}$ admits a density function with respect to the Lebesgue measure on \mathbb{T} :

$$\Delta_{\tilde{\mathbf{Z}}}(\lambda) = \frac{1}{2\pi} \sum_{n=-N}^N e^{-in\lambda} \zeta_n = \frac{1}{2\pi} \sum_{j \in \mathbb{Z}_N} W_N(\omega_j - \lambda) \mu_{\zeta}(\omega_j),$$

and the measure $\mathbf{n}_{\tilde{\mathbf{Z}}}$ admits a density function

$$D_{\tilde{\mathbf{Z}}}(\lambda) = \frac{1}{4\pi^2} \sum_{n=-N}^N e^{-in\lambda} \text{Var}(\zeta_n) = \frac{2N+1}{4\pi^2} \sum_{j \in \mathbb{Z}_N} W_N(\omega_j - \lambda) \mathbf{K} \mu_{\zeta}(j).$$

6.4.1 Linear harmonizable time series on \mathbb{Z} with innovation $\tilde{\mathbf{Z}}$

Let $(\mathbf{A}_n)_{n \in \mathbb{Z}}$ be a family of matrices such that $\sum_n \text{tr}(\mathbf{A}_n \mathbf{A}_n^*) < \infty$. Define the time series \mathbf{X} on \mathbb{Z} by

$$\mathbf{X}_n \stackrel{\text{def}}{=} \sum_{m=-N}^N \mathbf{A}_{n-m} \tilde{\mathbf{Z}}_m.$$

Then the time series \mathbf{X} is strongly harmonizable with the stochastic spectral density

$$\begin{aligned} \Delta_{\mathbf{X}}(\lambda) &= \hat{\mathbf{A}}(-\lambda) \Delta_{\tilde{\mathbf{Z}}}(\lambda) = \frac{1}{2\pi} \hat{\mathbf{A}}(-\lambda) \sum_{n=-N}^N e^{-in\lambda} \zeta_n \\ &= \frac{1}{2\pi} \hat{\mathbf{A}}(-\lambda) \sum_{j \in \mathbb{Z}_N} W_N(\omega_j - \lambda) \mu_{\zeta}(\omega_j). \end{aligned}$$

Its spectral bimeasure admits a density function (see Remark (3.4))

$$\begin{aligned} \mathbf{F}_{\mathbf{X}}(\lambda_1, \lambda_2) &= \hat{\mathbf{A}}(-\lambda_1) \mathbf{F}_{\tilde{\mathbf{Z}}}(\lambda_1, \lambda_2) \hat{\mathbf{A}}^*(-\lambda_2) = \hat{\mathbf{A}}(-\lambda_1) D_{\tilde{\mathbf{Z}}}(\lambda_1 - \lambda_2) \hat{\mathbf{A}}^*(-\lambda_2) \\ &= \frac{2N+1}{4\pi^2} \sum_{j \in \mathbb{Z}_N} W_N(\omega_j - \lambda_1 + \lambda_2) \hat{\mathbf{A}}(-\lambda_1) \mathbf{K} \mu_{\zeta}(j) \hat{\mathbf{A}}^*(-\lambda_2). \end{aligned} \quad (20)$$

Now consider the "restriction" χ on \mathbb{Z}_N of the time series \mathbf{X} defined by $\chi_n = \mathbf{X}_n$ for $n = -N, \dots, N$. From Section 6.3, we know that χ is a harmonizable time series in \mathbb{Z}_N with the stochastic spectral measure

$$\mu_{\chi}(\omega_j) = \int_{\mathbb{T}} \frac{W_N(\lambda - \omega_j)}{2N+1} \mu_{\mathbf{X}}(d\lambda) = \frac{1}{2\pi(2N+1)} \sum_{k \in \mathbb{Z}_N} \left(\int_{\mathbb{T}} W_N(\lambda - \omega_j) W_N(\omega_k - \lambda) \hat{\mathbf{A}}(-\lambda) d\lambda \right) \mu_{\zeta}(\omega_k)$$

and spectral bimeasure

$$\begin{aligned} \mathbf{M}_{\chi}(\omega_{j_1}, \omega_{j_2}) &= \iint_{\mathbb{T} \times \mathbb{T}} \frac{W_N(\lambda_1 - \omega_{j_1}) W_N(-\lambda_2 + \omega_{j_2})}{(2N+1)^2} \mathbf{M}_{\mathbf{X}}(d\lambda_1, d\lambda_2) \\ &= \iint_{\mathbb{T} \times \mathbb{T}} \frac{W_N(\lambda_1 - \omega_{j_1}) W_N(\lambda_2 - \omega_{j_2})}{(2N+1)^2} \hat{\mathbf{A}}(-\lambda_1) D_{\tilde{\mathbf{Z}}}(\lambda_1 - \lambda_2) \hat{\mathbf{A}}^*(-\lambda_2) d\lambda_1 d\lambda_2 \\ &= \frac{1}{4\pi^2(2N+1)} \sum_{j \in \mathbb{Z}_N} \iint_{\mathbb{T} \times \mathbb{T}} W_N(\lambda_1 - \omega_{j_1}) W_N(\lambda_2 - \omega_{j_2}) W_N(\omega_j - \lambda_1 + \lambda_2) \times \\ &\quad \times \hat{\mathbf{A}}(-\lambda_1) \mathbf{K} \mu_{\zeta}(j) \hat{\mathbf{A}}^*(-\lambda_2) d\lambda_1 d\lambda_2. \end{aligned} \quad (21)$$

Recall that $W_N(0) = 2N+1$ and for $\lambda \neq 0 \pmod{2\pi}$ we have $W_N(\lambda)^2 = \frac{\sin^2((2N+1)\lambda/2)}{\sin^2(\lambda/2)}$.

6.4.2 Band-limited harmonizable time series

We consider a white noise $(\varepsilon_n)_{n \in \mathbb{Z}_N}$ on \mathbb{Z}_N and a family $(\vartheta_n)_{n \in \mathbb{Z}_N}$ of complex $d \times d$ -matrices. Then the moving average

$$\boldsymbol{\mu}_j = \sum_{k \in \mathbb{Z}_N} \vartheta_{j-k} \varepsilon_k$$

defines a stationary times series $\boldsymbol{\mu} = (\boldsymbol{\mu}_j)_{j \in \mathbb{Z}_N}$ with covariance function

$$\mathbf{K}_{\boldsymbol{\mu}}(r) = \mathbb{E}(\boldsymbol{\mu}_{j+r} \boldsymbol{\mu}_j^*) = \sum_{k \in \mathbb{Z}_N} \vartheta_{k+r} \boldsymbol{\Sigma}_{\varepsilon}^2 \vartheta_k^*.$$

Whenever $\vartheta_k = \mathbf{0}_d$ for $k = \pm q, \dots, \pm N$ for some $0 < q \leq N/2$, then $\mathbf{K}_{\boldsymbol{\mu}}(r) = \mathbf{0}_{d \times d}$ for $r = \pm 2q, \dots, \pm N$, and the harmonizable noise $\boldsymbol{\zeta}$ defined by relation (6.5) has a band-limited (Loève) spectrum: $\mathbf{n}_{\boldsymbol{\zeta}}(\omega_j) = \mathbf{0}_{d \times d}$ for $j = \pm 2q, \dots, \pm N$.

Hence, when $\widehat{\mathbf{A}}$ is *smooth enough*, relations (20) and (21) imply that

$$\mathbf{F}_{\mathbf{X}}(\lambda_1, \lambda_2)/(2N+1)^2 \quad \text{and} \quad \mathbf{M}_{\mathbf{X}}(\omega_{j_1}, \omega_{j_2})/(2N+1)^2$$

are close to 0 for sufficiently large N , and (λ_1, λ_2) , $(\omega_{j_1}, \omega_{j_2})$ are 'far' from the main diagonal.

6.5 Harmonizable VARMA time series on \mathbb{Z}_N

We are now ready to define HVARMA models. First, let us consider the following difference equation:

$$\mathbf{X}_n + \phi_1 \mathbf{X}_{n-1} + \dots + \phi_p \mathbf{X}_{n-p} = \tilde{\mathbf{Z}}_n + \theta_1 \tilde{\mathbf{Z}}_{n-1} + \dots + \theta_q \tilde{\mathbf{Z}}_{n-q}, \quad (22)$$

where $\{\tilde{\mathbf{Z}}_n\}$ is extension by zeros of a harmonizable noise on \mathbb{Z}_N . Then the results of the previous section apply with $\widehat{\mathbf{A}}(\lambda) = \boldsymbol{\Phi}(e^{i\lambda})^{-1} \boldsymbol{\Theta}(e^{i\lambda})$.

Then, the generation of HVARMA time series is as follows:

- Generate a realization from the (incremental) process $\boldsymbol{\mu} = (\boldsymbol{\mu}_j)_{j \in \mathbb{Z}_N}$ according to a zero mean VARMA process of order (p', q') defined on \mathbb{Z}_N , denoted as $\widetilde{\text{VARMA}}(p', q')$, with matrix polynomials $\tilde{\boldsymbol{\Phi}}(\cdot)$, $\tilde{\boldsymbol{\Theta}}(\cdot)$. Then, the time series $\boldsymbol{\zeta} = (\boldsymbol{\zeta}_n)_{n \in \mathbb{Z}_N}$ defined by

$$\boldsymbol{\zeta}_n \stackrel{\text{def}}{=} \sum_{j \in \mathbb{Z}_N} e^{in\omega_j} \boldsymbol{\mu}_j, \quad \omega_j = \frac{2\pi j}{2N+1},$$

is a (strongly) harmonizable noise on \mathbb{Z}_N .

- Generate a time series realization on \mathbb{Z}_N as $\mathcal{X}_n = \sum_{j \in \mathbb{Z}_N} e^{i\omega_j n} \widehat{\mathbf{A}}(\omega_j) \boldsymbol{\mu}_j$, where $\widehat{\mathbf{A}}(\lambda)$ defines a VARMA process of order (p, q) , denoted as $\text{VARMA}(p, q)$.

We denote the resulting process as $H\text{VARMA}(p, q) - (p', q')$, where (p, q) is the order of the VARMA process (see the equation 22) and (p', q') is the order of VARMA incremental process.

6.6 Some examples

In this section, we consider the following four examples of bivariate HARMA time series generated using the previously described procedure.

- a) $HVARMA(1, 1) - (1, 1)$

$$\tilde{\Phi} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}; \tilde{\Theta} = \begin{pmatrix} 0.9 & 0 \\ 0 & 0.9 \end{pmatrix}; \tilde{\Sigma} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix};$$

$$\Phi = \begin{pmatrix} 0.9 & 0 \\ 0 & 0 \end{pmatrix}; \Theta = \begin{pmatrix} 0 & 0 \\ 0 & 0.9 \end{pmatrix}; \Sigma = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix};$$

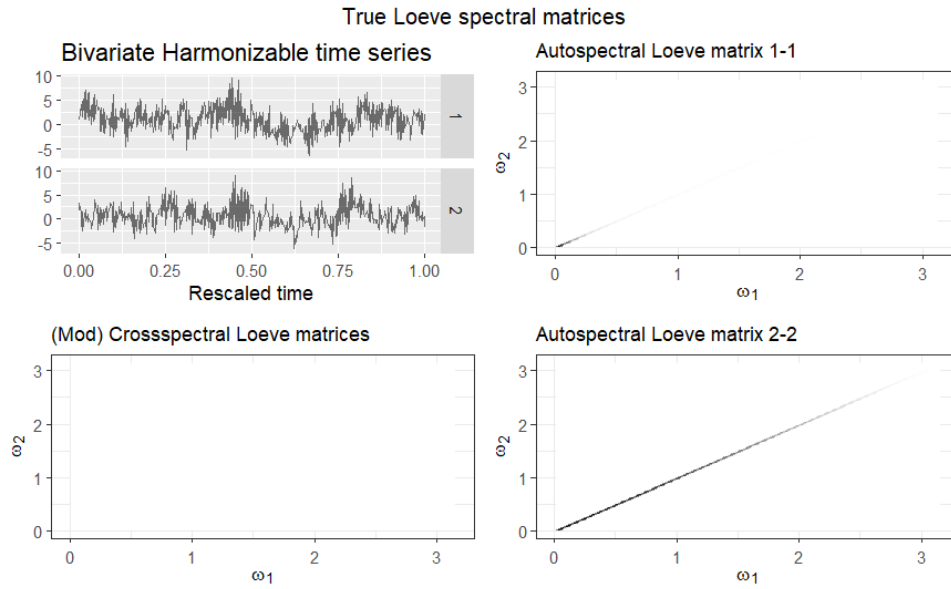


Figure 1: Example of a simple $HVARMA(1, 1) - (1, 1)$

- b) $HVARMA(1, 1) - (1, 1)$

$$\tilde{\Phi} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}; \tilde{\Theta} = \begin{pmatrix} -0.9 & 0 \\ 0 & -0.9 \end{pmatrix}; \tilde{\Sigma} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix};$$

$$\Phi = \begin{pmatrix} -0.9 & 0 \\ 0 & 0 \end{pmatrix}; \Theta = \begin{pmatrix} 0 & 0 \\ 0 & -0.9 \end{pmatrix}; \Sigma = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix};$$

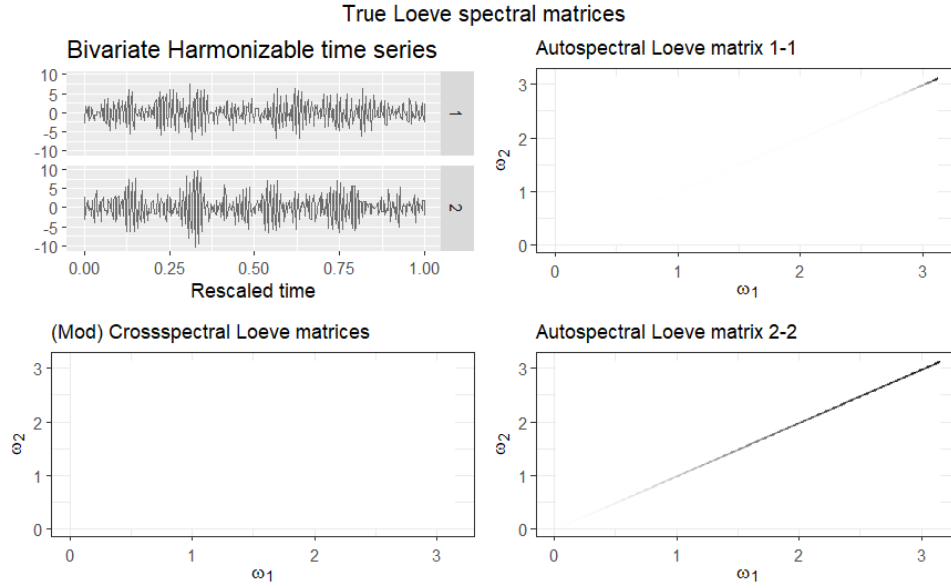


Figure 2: Example of a simple $HVARMA(1, 1) - (1, 1)$

- c) Complex process $HVARMA(2, 2) - (1, 2)$.

$$\tilde{\Phi} = \begin{pmatrix} -0.9 & 0.5 \\ 0.5 & 0 \end{pmatrix}; (\tilde{\Theta}_1, \tilde{\Theta}_2) = \left(\begin{pmatrix} 0.2 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0.1 & 0 \\ 0 & 0 \end{pmatrix} \right); \tilde{\Sigma} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

$$(\Phi_1, \Phi_2) = \left(\begin{pmatrix} -0.9 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0.5 & 0 \\ 0 & 0 \end{pmatrix} \right); (\Theta_1, \Theta_2) = \left(\begin{pmatrix} 0.4 & 0 \\ 0 & 0.9 \end{pmatrix}, \begin{pmatrix} -0.2 & 0 \\ 0 & 0.4 \end{pmatrix} \right); \Sigma = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

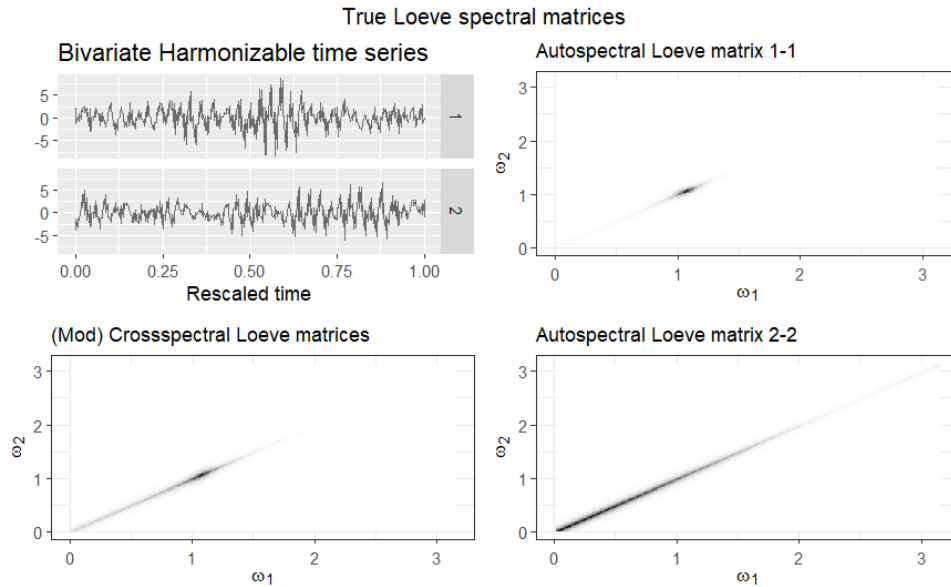


Figure 3: Example of a $HVARMA(2, 2) - (1, 2)$

- d) A $HVARMA(1, 1) - (2, 82)$ process. Due to the high order of the incremental process, we do not provide the values of the parameters.

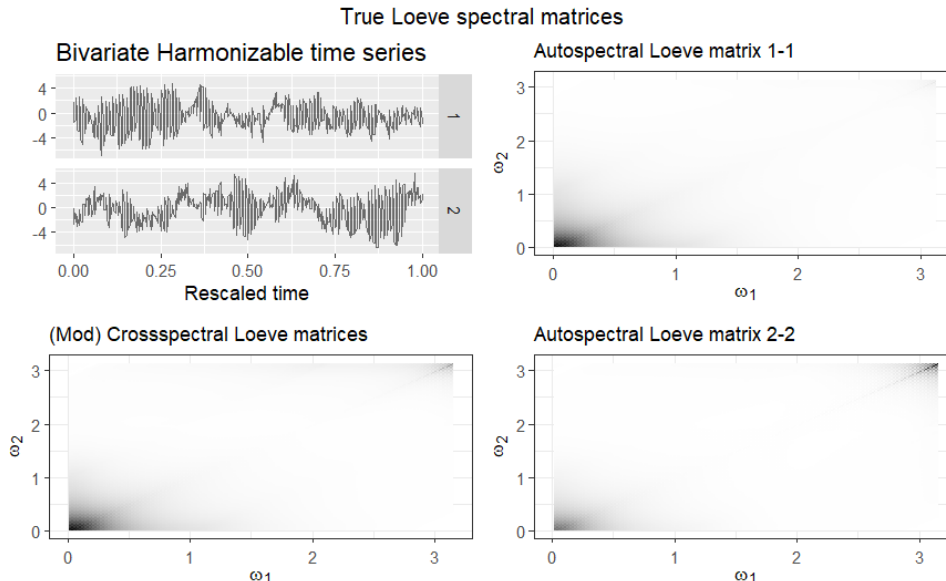


Figure 4: Example of a complicated $HVARMA(1, 1) - (2, 82)$

Example realizations of these processes are shown in Figures 1-4 (see top left panels). In addition, the corresponding Loève spectrum is given in each figure in the top left and bottom right panels. Finally, in the bottom left panels is presented the modulus of the cross-Loève spectrum. The first examples a) and b) in Figures 1,2 are $HVARMA(1, 1) - (1, 1)$. These time series are independent of each other and both harmonizable with simple incremental processes. Hence, the cross-Loève spectrum is the null spectrum. Figures 3,4 illustrate the much more complicated examples c) and d). Those involve higher order VARMA polynomials and cross-dependencies in the incremental process. They demonstrate the variety of temporal-spectral characteristics that can be modelled using such parametric class.

7 Summary

Many nonstationary models assume that stationarity holds locally, e.g., piecewise stationary, locally stationary processes. ARMA models with time-varying coefficients (Dahlhaus, 2012) provide a very powerful and convenient parametric class that is flexible enough to describe many real life signals. Nevertheless, those models do not allow for capturing many complex dependencies that characterize some signals. In this paper, we describe a novel parametric model for a wide class of nonstationary time series that present correlation between their frequency components. The proposed Harmonizable VARMA process is obtained as a solution of a difference equation based on a harmonizable noise. We show that its Cramèr's type spectral representation is the one of a VARMA process of order (p, q) with correlated increments following a VARMA of order (p', q') . We exhibit the spectral characteristics of these $HVARMA(p, q) - (p', q')$ processes, and we show how to generate realizations from given models. Our results pave the way towards novel developments on estimation, prediction, model selection, etc., for harmonizable processes. Consequently, non-specialists will be able to model such complex processes in a relatively simple way (as it is currently possible, for example, for ARMA processes).

8 Annexe: Proofs

8.1 Proof of Lemma 3.1

Clearly, if there exists a d -variate white noise $\boldsymbol{\varepsilon} \stackrel{\text{def}}{=} (\varepsilon_n)_{n \in \mathbb{Z}}$ such that $\mathbf{Z}_n = \mathbf{S}_n \boldsymbol{\varepsilon}_n$, where $\sup_{n,j,k} |s_{n,j,k}| < \infty$, then the d -variate time series $\mathbf{Z} = (\mathbf{Z}_n)_{n \in \mathbb{Z}}$ satisfies the condition (Ha).

Conversely, suppose that $\mathbf{Z} = (\mathbf{Z}_n)_{n \in \mathbb{Z}}$ is a d -variate time series satisfying the condition (Ha). Let $d_n = \dim \mathcal{H}(\mathbf{Z}_n) \leq d$. When $d_n \neq 0$, there exists an orthonormal basis $\{\epsilon_{n,1}, \dots, \epsilon_{n,d_n}\}$ of $\mathcal{H}(\mathbf{Z}_n)$ (Schmidt method of orthonormalization). Since the Hilbert subspaces $\mathcal{H}(\mathbf{Z}_n)$, $n \in \mathbb{Z}$ are orthogonal, and we assume that the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ is sufficiently large, we can complete the orthonormal family $\{\epsilon_{n,j} : n \in \mathbb{Z}, j = 1, \dots, d_n\}$ to an orthonormal family $\{\epsilon_{n,j} : n \in \mathbb{Z}, j = 1, \dots, d\}$ of \mathcal{H} . Then the time series $(\boldsymbol{\varepsilon}_n)_{n \in \mathbb{Z}}$, where $\boldsymbol{\varepsilon}_n = (\epsilon_{n,1}, \dots, \epsilon_{n,d})^\top$ for $n \in \mathbb{Z}$, is a d -variate pure white noise, and the space $\mathcal{H}(\mathbf{Z}_n)$ is a subspace of the Hilbert space $\mathcal{H}(\boldsymbol{\varepsilon}_n)$. Hence, the proof of the lemma can be easily achieved using the assumption $\sup_{n,j} \mathbb{E}\{|Z_{n,j}|^2\} < \infty$.

8.2 Proof of Theorem 3.1

To prove the theorem, we use the notation of Lemma 3.1 and construct a bounded linear operator $\Xi : \mathcal{H}^d(\boldsymbol{\varepsilon}) \rightarrow \mathcal{H}^d(\mathbf{Z})$ such that $\mathbf{Z}_n = \Xi \boldsymbol{\varepsilon}_n$. Since the time series $(\boldsymbol{\varepsilon}_n)_{n \in \mathbb{Z}}$ is stationary, we get that the time series $(\mathbf{Z}_n)_{n \in \mathbb{Z}}$ is harmonizable with $\boldsymbol{\mu}_{\mathbf{Z}}(A) = \Xi \boldsymbol{\mu}_{\boldsymbol{\varepsilon}}(A)$ for $A \in \mathcal{B}(\mathbb{T})$ (Rao, 1982) (D.S, 1957 Theorem 3.2.19).

First, we prove that any d -variate harmonizable time series $(\mathbf{Z}_n)_{n \in \mathbb{Z}}$ whose stochastic spectral measure satisfies relation (1), satisfies the condition (Ha). Let $n \in \mathbb{Z}$ be fixed. Then $\{\epsilon_{n,j} : j = 1, \dots, d\}$ is an orthonormal basis of the Hilbert space $\mathcal{H}(\boldsymbol{\varepsilon}_n)$. Since $Z_{n,j} \in \mathcal{H}(\boldsymbol{\varepsilon}_n)$, we have that $Z_{n,j} = \mathbf{S}_{n,j} \boldsymbol{\varepsilon}_n$, where $(\mathbf{S}_{n,j})^\top = (s_{n,j,1}, \dots, s_{n,j,d})^\top \in \mathbb{C}^d$, for any $j = 1, \dots, d$. It follows that $\mathbf{Z}_n = \mathbf{S}_n \boldsymbol{\varepsilon}_n$, where \mathbf{S}_n is a $d \times d$ -matrix whose rows are the row vectors $\mathbf{S}_{n,j}$, $j = 1, \dots, d$.

Then, since the $\epsilon_{n,j}$, $n \in \mathbb{Z}, j = 1, \dots, d$, are linearly independent, we can define a linear operator Ξ_j from the linear space $\{\mathbf{A} \boldsymbol{\varepsilon}_n : n \in \mathbb{Z}, \mathbf{A} \in \mathbb{C}^{d \times d}\} \subset \mathcal{H}^d(\boldsymbol{\varepsilon})$ into $\mathcal{H}(\mathbf{Z})$ by $\Xi_j \boldsymbol{\varepsilon}_n = Z_{n,j}$.

Moreover, the $\epsilon_{n,j}$, $n \in \mathbb{Z}, j = 1, \dots, d$, are orthonormal in the Hilbert space $\mathcal{H}(\boldsymbol{\varepsilon})$, that is, uncorrelated random variables with zero mean and variance equal to 1. Thus, we have

$$\text{var}(Z_{n,j}) = \mathbb{E}(|Z_{n,j}|^2) = \|Z_{n,j}\|_{\mathcal{H}}^2 = \mathbf{S}_{n,j} \mathbf{S}_{n,j}^*.$$

The assumption (Ha) implies that $\sup_{n,j,k} |s_{n,j,k}| < \infty$. Consequently, the linear operator Ξ_j is bounded, and can be extended as a bounded linear operator from $\mathcal{H}^d(\boldsymbol{\varepsilon})$ to $\mathcal{H}(\mathbf{Z})$, also denoted Ξ_j . Finally, we can easily define a bounded linear operator Ξ from $\mathcal{H}^d(\boldsymbol{\varepsilon})$ to $\mathcal{H}^d(\mathbf{Z})$ such that $\mathbf{Z}_n = \Xi \boldsymbol{\varepsilon}_n$, for any $n \in \mathbb{Z}$. The bounded operators $\Xi_j : \mathcal{H}^d(\boldsymbol{\varepsilon}) \rightarrow \mathcal{H}(\mathbf{Z})$, $j = 1, \dots, d$ are the component operators of Ξ . Thus, the d -variate noise \mathbf{Z} is the image by a bounded linear operator Ξ of the d -variate stationary noise $\boldsymbol{\varepsilon}$, and from Dunford and Schwartz (1957, Theorem 3.2.19) (see also Corollary 4.3 Rao, 1982), we get that \mathbf{Z} is a harmonizable and its stochastic spectral measure $\boldsymbol{\mu}_{\mathbf{Z}}$ satisfies $\boldsymbol{\mu}_{\mathbf{Z}}(A) = \Xi(\boldsymbol{\mu}_{\boldsymbol{\varepsilon}}(A))$, $A \in \mathcal{B}(\mathbb{T})$.

Now we show that (1) holds. Let $j = 1, \dots, d$ be fixed. We know that $\mathcal{H}(\boldsymbol{\mu}_{\mathbf{Z}}) = \mathcal{H}(\mathbf{Z})$ is a Hilbert subspace of $\mathcal{H}(\boldsymbol{\varepsilon})$, and the family $\{\epsilon_{n,k} : n \in \mathbb{Z}, k = 1, \dots, d\}$ is an orthonormal basis of $\mathcal{H}(\boldsymbol{\varepsilon}) \subset L_{\mathbb{C}}^2(\mathbb{P})$. Therefore, the stochastic spectral measure μ_{Z_j} of the component time series $Z_j = (Z_{n,j})_{n \in \mathbb{Z}}$ can be decomposed as

$$\mu_{Z_j}(A) = \sum_{n \in \mathbb{Z}} \sum_{k=1}^d \mu_{Z_j, n, k}(A) \epsilon_{n, k}, \quad A \in \mathcal{B}(\mathbb{T}),$$

where the complex-valued measure $\mu_{Z_j, n, k}$ is defined on $\mathcal{B}(\mathbb{T})$ by $\mu_{Z_j, n, k}(A) \stackrel{\text{def}}{=} \langle \mu_{Z_j}(A) | \epsilon_{n, k} \rangle_{\mathcal{H}} = \mathbb{E}[\mu_{Z_j}(A) \overline{\epsilon_{n, k}}]$. Besides, the law of large numbers for harmonizable time series (Rozanov, 1959, Theorem 2.2) applied to $Z_j = (Z_{m, j})_{m \in \mathbb{Z}}$ implies that

$$\lim_{N \rightarrow \infty} \frac{1}{N} \sum_{m=1}^N e^{-ik\lambda} Z_{m, j} = \mu_{Z_j}(\{\lambda\}),$$

for any $\lambda \in \mathbb{T}$. The convergence is in $\mathcal{H}(\mathbf{Z}) \subset \mathcal{H}(\boldsymbol{\varepsilon})$. Thus,

$$\lim_{N \rightarrow \infty} \frac{1}{N} \sum_{m=1}^N e^{-im\lambda} \langle Z_{m, j} | \epsilon_{n, k} \rangle_{\mathcal{H}} = \lim_{N \rightarrow \infty} \frac{1}{N} \sum_{m=1}^N e^{-im\lambda} \mathbb{E}[Z_{m, j} \overline{\epsilon_{n, k}}] = \mu_{Z_j, n, k}(\{\lambda\}),$$

for all n and k . However, by choice of the family $\{\epsilon_{n, k} : n \in \mathbb{Z}, k = 1, \dots, d\}$ (Lemma 3.1)

$$\frac{1}{N} \sum_{m=1}^N e^{-im\lambda} \langle Z_{m, j} | \epsilon_{n, k} \rangle_{\mathcal{H}} = \frac{1}{N} \sum_{m=1}^N e^{-im\lambda} \langle \mathbf{S}_{m, j} \boldsymbol{\varepsilon}_m | \epsilon_{n, k} \rangle_{\mathcal{H}} = \frac{1}{N} e^{-in\lambda} s_{n, j, k} \mathbb{I}_{\{n=1, \dots, N\}},$$

which converges to 0 as $N \rightarrow \infty$ for all n and k . Consequently, $\mu_{Z_j}(\{\lambda\}) = 0$ for any j , and $\boldsymbol{\mu}_{\mathbf{Z}}(\{\lambda\}) = 0$ for any λ .

Using the Lévy's inversion formula (Rozanov, 1959, Theorem 3.2), we get that

$$\lim_{n \rightarrow \infty} \frac{1}{2\pi} \sum_{m=-N}^N \left(\int_{\lambda_1}^{\lambda_2} e^{-im\lambda} d\lambda \right) Z_{m, j} = \lim_{N \rightarrow \infty} \frac{1}{2\pi} \sum_{m=-N}^N \frac{e^{-im\lambda_2} - e^{-im\lambda_1}}{-im} Z_{m, j} = \mu_{Z_j}([\lambda_1, \lambda_2]),$$

for all $\lambda_1 < \lambda_2 \in \mathbb{T}$. The convergence is in $\mathcal{H}(\mathbf{Z}) \subset \mathcal{H}(\boldsymbol{\varepsilon})$. We conclude that

$$\lim_{N \rightarrow \infty} \frac{1}{2\pi} \sum_{m=-N}^N \left(\int_{\lambda_1}^{\lambda_2} e^{-im\lambda} d\lambda \right) \langle Z_{m, j} | \epsilon_{n, k} \rangle_{\mathcal{H}} = \mu_{Z_j, n, k}([\lambda_1, \lambda_2]),$$

for all n and k . However,

$$\begin{aligned} \frac{1}{2\pi} \sum_{m=-N}^N \left(\int_{\lambda_1}^{\lambda_2} e^{-im\lambda} d\lambda \right) \langle Z_{m, j} | \epsilon_{n, k} \rangle_{\mathcal{H}} &= \frac{1}{2\pi} \sum_{m=-N}^N \sum_{l=1}^d \left(\int_{\lambda_1}^{\lambda_2} e^{-im\lambda} d\lambda \right) s_{m, j, l} \langle \epsilon_{m, l} | \epsilon_{n, k} \rangle_{\mathcal{H}} \\ &= \frac{s_{n, j, k}}{2\pi} \left(\int_{\lambda_1}^{\lambda_2} e^{-in\lambda} d\lambda \right) \mathbb{I}_{\{-N < n \leq N\}}. \end{aligned}$$

Thus,

$$\mu_{Z_j, n, k}([\lambda_1, \lambda_2]) = \frac{s_{n, j, k}}{2\pi} \left(\int_{\lambda_1}^{\lambda_2} e^{-in\lambda} d\lambda \right),$$

and the complex-valued measures $\mu_{Z_j, n, k}$, $n \in \mathbb{Z}$, $j, k = 1, \dots, d$ are absolutely continuous with respect to the Lebesgue measure on \mathbb{T} . More precisely,

$$\mu_{Z_j, n, k}(A) = \frac{s_{n, j, k}}{2\pi} \left(\int_A e^{-in\lambda} d\lambda \right). \quad (23)$$

Then

$$\begin{aligned} \mu_{Z_j}(A) &= \frac{1}{2\pi} \sum_{n \in \mathbb{Z}} \sum_{k=1}^d \left(\int_A e^{-in\lambda} d\lambda \right) s_{n, j, k} \epsilon_{n, k} \\ &= \frac{1}{2\pi} \sum_{n \in \mathbb{Z}} \left(\int_A e^{-in\lambda} d\lambda \right) \mathbf{S}_{n, j} \boldsymbol{\varepsilon}_n = \frac{1}{2\pi} \sum_{n \in \mathbb{Z}} \left(\int_A e^{-in\lambda} d\lambda \right) Z_{n, j} \end{aligned}$$

and consequently we get the relation (1).

8.3 Proof of Proposition 3.1

First, assume that $(\mathbf{Z}_n)_{n \in \mathbb{Z}}$ is strongly harmonizable. Then its spectral bimeasure $\mathbf{M}_{\mathbf{Z}}$ is extendible as a $\mathbb{C}^{d \times d}$ -valued measure on \mathbb{T}^2 .

$$\text{Cov}(\mathbf{Z}_n, \mathbf{Z}_{n+k}) = \iint_{\mathbb{T}^2} e^{i(n\lambda_1 - (n+k)\lambda_2)} \mathbf{M}_{\mathbf{Z}}(d\lambda_1, d\lambda_2) = \iint_{\mathbb{T}^2} e^{i(n(\lambda_1 - \lambda_2) - k\lambda_2)} \mathbf{M}_{\mathbf{Z}}(d\lambda_1, d\lambda_2).$$

We consider the transformation of \mathbb{T}^2 defined by $\mathcal{T} : (\lambda_1, \lambda_2) \mapsto (\lambda_1 - \lambda_2, \lambda_2)$, $\mathcal{T}^{-1} : (\lambda, \alpha) \mapsto (\lambda + \alpha, \alpha)$. Let $\mathbf{M}_{\mathbf{Z}}^{\mathcal{T}} = \mathbf{M}_{\mathbf{Z}} \circ \mathcal{T}^{-1}$ be the pushforward measure of $\mathbf{M}_{\mathbf{Z}}$ by the transformation \mathcal{T} and defined on \mathbb{T}^2 by $\mathbf{M}_{\mathbf{Z}}^{\mathcal{T}}(E) = \mathbf{M}_{\mathbf{Z}}(\mathcal{T}^{-1}(E))$. Then

$$\text{Cov}[\mathbf{Z}_n, \mathbf{Z}_{n+k}] = \iint_{\mathbb{T}^2} e^{i(n\lambda - k\alpha)} \mathbf{M}_{\mathbf{Z}}^{\mathcal{T}}(d\lambda, d\alpha).$$

We know that $\text{Cov}(\mathbf{Z}_n, \mathbf{Z}_{n+k}) = \mathbf{0}_{d \times d}$ for $k \neq 0$ and $\text{Var}[\mathbf{Z}_n] = \Sigma_{\mathbf{Z}_n}^2$. Hence, considering the $\mathbb{C}^{d \times d}$ -valued measure $\mathbf{n}_{\mathbf{Z}}$ defined on \mathbb{T} by $\mathbf{n}_{\mathbf{Z}}(A) \stackrel{\text{def}}{=} \mathbf{M}_{\mathbf{Z}}^{\mathcal{T}}(A \times \mathbb{T})$, we can easily see that

$$\text{Cov}(\mathbf{Z}_n, \mathbf{Z}_{n+k}) = \frac{1}{2\pi} \iint_{\mathbb{T}^2} e^{i(n\lambda - k\alpha)} \mathbf{n}_{\mathbf{Z}}(d\lambda) d\alpha = \begin{cases} \int_{\mathbb{T}} e^{in\lambda} \mathbf{n}_{\mathbf{Z}}(d\lambda) & \text{for } k = 0 \\ 0 & \text{otherwise} \end{cases}$$

and by the characterization of a measure on \mathbb{T}^2 by its Fourier transform, we deduce that $\mathbf{M}_{\mathbf{Z}}^{\mathcal{T}}(d\lambda, d\alpha) = \frac{1}{2\pi} \mathbf{n}_{\mathbf{Z}}(d\lambda) d\alpha$, and the relation (3) is satisfied.

Now assume that $(\mathbf{Z}_n)_{n \in \mathbb{Z}}$ is such that there exists a measure $\mathbf{n}_{\mathbf{Z}}$ that satisfies relation (3). Then we can easily check that

$$\text{Cov}(\mathbf{Z}_n, \mathbf{Z}_{n+k}) = \frac{1}{2\pi} \iint_{\mathbb{T}^2} e^{i(n\lambda - k\alpha)} \mathbf{n}_{\mathbf{Z}}(d\lambda) d\alpha.$$

We denote $\mathbf{M}(d\lambda, d\alpha) \stackrel{\text{def}}{=} \frac{1}{2\pi} \mathbf{n}_{\mathbf{Z}}(d\lambda) d\alpha$. Then the measure which is image of the $\mathbb{C}^{d \times d}$ -valued measure \mathbf{M} on \mathbb{T}^2 by the transformation \mathcal{T}^{-1} is an extension of the spectral bimeasure $\mathbf{M}_{\mathbf{Z}}$ of the harmonizable time series \mathbf{Z} . Thus, the spectral bimeasure $\mathbf{M}_{\mathbf{Z}}$ is extendible to a measure and the time series \mathbf{Z} is strongly harmonizable. Furthermore, we obtain relations (5) and (4).

8.4 Proof of Proposition 3.2

Consider a d -variate pure white noise ε associated to the harmonizable time series \mathbf{Z} in the sense of Lemma 3.1.

First, assume that the d -variate stochastic measure $\mu_{\mathbf{Z}}$ admits a stochastic Radon-Nikodym density function with respect to the Lebesgue measure on \mathbb{T} , $\Delta_{\mathbf{Z}} : \mathbb{T} \rightarrow \mathcal{H}^d(\mathbf{Z})$. We denote $\Delta_{\mathbf{Z}} = (\Delta_{Z_1}, \dots, \Delta_{Z_d})$, $\Delta_{Z_j} : \mathbb{T} \rightarrow \mathcal{H}(\mathbf{Z})$, $j = 1, \dots, d$. Then

$$\mathbf{Z}_n = \int_{\mathbb{T}} e^{in\lambda} \mu_{\mathbf{Z}}(\lambda) = \int_{\mathbb{T}} e^{in\lambda} \Delta_{\mathbf{Z}}(\lambda) d\lambda$$

and

$$Z_{n,j} = \int_{\mathbb{T}} e^{in\lambda} \mu_{Z_j}(\lambda) = \int_{\mathbb{T}} e^{in\lambda} \Delta_{Z_j}(\lambda) d\lambda.$$

Moreover,

$$\langle Z_{n,j} | \epsilon_{m,k} \rangle_{\mathcal{H}} = \int_{\mathbb{T}} e^{in\lambda} \langle \Delta_{Z_j}(\lambda) | \epsilon_{m,k} \rangle_{\mathcal{H}} d\lambda.$$

Since ε is a d -variate pure white noise, using notation of Lemma 3.1, we have that $\langle Z_{n,j} | \epsilon_{m,k} \rangle_{\mathcal{H}} = s_{n,j,k} \mathbb{1}_{\{n=m\}}$. We get that $\langle \Delta_{Z_j}(\lambda) | \epsilon_{m,k} \rangle_{\mathcal{H}} = \frac{s_{n,j,k}}{2\pi} e^{-im\lambda}$ for all n, j, m and for almost every λ . From the

facts that $\Delta_{Z_j}(\lambda) \in \mathcal{H}(\mathbf{Z}) \subset \mathcal{H}(\varepsilon)$, and $\{\epsilon_{m,k} : m \in \mathbb{Z}, k = 1, \dots, d\}$ is an orthonormal basis of $\mathcal{H}(\varepsilon)$, we obtain that

$$\Delta_{Z_j}(\lambda) = \frac{1}{2\pi} \sum_{m \in \mathbb{Z}} \sum_{k=1}^d s_{n,j,k} e^{-im\lambda} \epsilon_{m,k} \quad \text{and} \quad \sum_{n \in \mathbb{Z}} \sum_{k=1}^d |s_{n,j,k}|^2 < \infty.$$

Hence, the relations (7) and (6) are satisfied.

Conversely, assume that the relation (6) is satisfied. Then the function $\mathbf{\Delta}_Z = (\Delta_{Z_1}, \dots, \Delta_{Z_d})$ is well-defined on \mathbb{T} by relation (7). Note that the convergence is in $\mathcal{H}(\mathbf{Z}) \subset L^2_{\mathbb{C}}(\mathbb{P})$. Let $j = 1, \dots, d$ be fixed. In the following, we use the notation of the proof of Theorem 3.1. From the relation (23), the function $\lambda \mapsto \mathbb{E}[\Delta_{Z_j}(\lambda) \overline{\epsilon_{n,k}}] = \frac{s_{n,j,k}}{2\pi} e^{-in\lambda}$ is the Radon-Nikodym density function of the complex measure $\mu_{Z_j, n, k}$ with respect to the Lebesgue measure on \mathbb{T} . Since $\mathcal{H}(\mathbf{Z})$ and $\mathcal{H}(\varepsilon)$ are separable Hilbert spaces, the Lebesgue measure on \mathbb{T} is bounded and

$$\|\Delta_{Z_j}(\lambda)\|_{\mathcal{H}}^2 = \mathbb{E}(\Delta_{Z_j}(\lambda) \overline{\Delta_{Z_j}(\lambda)}) = \frac{1}{4\pi^2} \sum_{n \in \mathbb{Z}} \sum_{k=1}^d |s_{n,j,k}|^2 < \infty,$$

and we can prove that the function $D_{Z_k} : \mathbb{T} \rightarrow \mathcal{H}(\mathbf{Z})$ is Bochner integrable (strongly integrable) (Dunford & Schwartz, 1957, Chap III, Th. III.2.21), and

$$\mu_{Z_j}(A) = \int_A \Delta_{Z_j}(\lambda) d\lambda, \quad A \in \mathcal{B}(\mathbb{T}).$$

This ends the proof.

8.5 Proof of Theorem 4.1

Let $(\varepsilon)_{n \in \mathbb{Z}}$ be a d -variate pure white noise associated to the harmonizable noise $(\mathbf{Z}_n)_{n \in \mathbb{Z}}$ in Lemma 3.1. It is known that the time series $(\mathbf{Y}_n)_{n \in \mathbb{Z}}$ of the form

$$\mathbf{Y}_n = \sum_{m \in \mathbb{Z}} \mathbf{A}_{n-m} \varepsilon_m \tag{24}$$

is well-defined (the convergence is in $H^d(\varepsilon)$), and it is stationary. Let $n \in \mathbb{Z}$ be fixed. Since the family $\{\epsilon_{n,k} : n \in \mathbb{Z}, k = 1, \dots, d\}$ is an orthonormal basis of $\mathcal{H}(\varepsilon)$, we have

$$\left\| \sum_{m=N_1}^{N_2} \mathbf{A}_{n-m} \varepsilon_m \right\|_{\mathcal{H}^d(\varepsilon)}^2 = \sum_{m=N_1}^{N_2} \text{tr}(\mathbf{A}_{n-m} \mathbf{A}_{n-m}^*)$$

for any $-\infty < N_1 < N_2 < \infty$. Then the series (24) is a Cauchy series in the Hilbert space $\mathcal{H}^d(\varepsilon)$. Thus, it is convergent in $\mathcal{H}^d(\varepsilon)$ and its sum \mathbf{Y}_n is well-defined. The stationarity of the d -variate time series $(\mathbf{Y}_n)_{n \in \mathbb{Z}}$ is evident. Now consider the bounded linear operator $\Xi : \mathcal{H}^d(\varepsilon) \rightarrow \mathcal{H}^d(\mathbf{Z})$ such that $\Xi \varepsilon_n = \mathbf{Z}_n$ defined in the proof of Theorem 3.1. The operator Ξ applied to the moving average series (24) entails the convergence in $\mathcal{H}^d(\mathbf{Z})$ of the moving average series (9)

$$\sum_m \mathbf{A}_{n-m} \Xi(\varepsilon_m) = \sum_m \mathbf{A}_{n-m} \mathbf{Z}_m.$$

Thus, the d -variate times series $(\mathbf{X}_n)_{n \in \mathbb{Z}} = (\Xi \mathbf{Y}_n)_{n \in \mathbb{Z}}$ is well-defined. Moreover, as the image of the stationary times series $(\mathbf{Y}_n)_{n \in \mathbb{Z}}$ by the bounded linear operator Ξ , it is harmonizable. Furthermore, the

stochastic spectral measure $\mu_{\mathbf{X}}$ of \mathbf{X} coincides with $\Xi\mu_{\mathbf{Y}}$. Since the topological group \mathbb{T} is compact and $\widehat{\mathbf{A}} \in L^2_{\mathbb{C}^{d \times d}}(\mathbb{T})$, we have $\widehat{\mathbf{A}} \in L^1_{\mathbb{C}^{d \times d}}(\mathbb{T})$ and the inversion formula gives

$$\mathbf{A}_n = \frac{1}{2\pi} \int_{\mathbb{T}} e^{-in\lambda} \widehat{\mathbf{A}}(\lambda) d\lambda.$$

Hence,

$$\mathbf{X}_n = \sum_{m \in \mathbb{Z}} \mathbf{A}_{n-m} \mathbf{Z}_m = \frac{1}{2\pi} \sum_{m \in \mathbb{Z}} \left(\int_{\mathbb{T}} e^{i(n-m)\lambda} \widehat{\mathbf{A}}(-\lambda) d\lambda \right) \mathbf{Z}_m.$$

Thus, the stochastic spectral measure $\mu_{\mathbf{Z}}$ satisfies

$$\mu_{\mathbf{X}}(A) = \frac{1}{2\pi} \sum_{m \in \mathbb{Z}} \left(\int_A e^{-im\lambda} \widehat{\mathbf{A}}(-\lambda) d\lambda \right) \mathbf{Z}_m,$$

and the convergence is in \mathcal{H}^d .

Then, using the inequality (2), we can easily get that the d -variate function $\widehat{\mathbf{A}}$ is $\mu_{\mathbf{Z}}$ -integrable and the equality (10) is satisfied.

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