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Stabilisation of the complex double integrator by means of a saturated linear feedback

Yacine Chitour ^{*†}

April 14, 2021

To E. D. Sontag, for his 70th birthday.

Consider the saturated complex double integrator, i.e., the linear control system $\dot{x} = Ax + B\sigma(u)$, where $x \in \mathbb{R}^4$, $u \in \mathbb{R}$, $B \in \mathbb{R}^4$, the 4×4 matrix A is not diagonalizable and admits a non zero purely imaginary eigenvalue of multiplicity two, the pair (A, B) is controllable and $\sigma : \mathbb{R} \rightarrow \mathbb{R}$ is a saturation function. We prove that there exists a linear feedback $u = K^T x$ such that the resulting closed loop system given by $\dot{x} = Ax + B\sigma(K^T x)$ is globally asymptotically stable with respect to the origin.

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1 Introduction

In this paper, we address the issue of stabilizing a finite dimensional linear control system by means of a saturated control. That is, one has

$$(\Sigma) \quad \dot{x} = Ax + B\sigma(u), \quad x \in \mathbb{R}^n, \quad u \in \mathbb{R}^m, \tag{1}$$

where n, m are positive integers, and A and B are $n \times n$ and $n \times m$ matrices respectively with real entries. Here $\sigma = (\sigma_i)_{1 \leq i \leq m}$, where each $\sigma_i : \mathbb{R} \rightarrow \mathbb{R}$ is a saturation function, i.e., any locally Lipschitz function so that there exist positive real numbers a_1, b_1, a_2, b_2 where $s_{a_1, b_1}(x) \leq \sigma_i(\xi) \leq s_{a_2, b_2}(x)$ for $\xi \in \mathbb{R}$, with $s_{a, b} : \mathbb{R} \rightarrow \mathbb{R}$ is the function defined for any positive real numbers a, b by $s_{a, b}(x) = \frac{a x}{\max(b, |x|)}$. For instance, \arctan , \tanh or the standard saturation function $s_{1,1}$ are typical examples of saturation functions. We refer to [4] and [11] as standard references on the study of these systems in control theory. For the rest of the paper, we will assume that the pair (A, B) is controllable.

The basic issue consists in finding a continuous feedback law $u = k(x)$ such that the closed system associated with (Σ) and $k(\cdot)$ and equal to $\dot{x} = Ax + B\sigma(k(x))$ is globally asymptotically stable (GAS for short) with respect to the origin. It has been shown (in [8] for instance) that a necessary condition for the existence of such a feedback $k(\cdot)$ is that the real part of any eigenvalue of A is non negative. Note also that optimal control can furnish a stabilizing feedback, which is discontinuous in general. It is not difficult to see that the above mentioned stabilization issue gets not so easy in case where A admits non trivial Jordan blocks associated with purely imaginary eigenvalues. One can first try to seek linear feedbacks, i.e., $k(x) = K^T x$ with K an $m \times n$ matrix. However, it has been established in [3] that, if A is a Jordan block of order 3, then (Σ) cannot be stabilized by a linear feedback law, a result which has been extended in [10] to the case where A is any Jordan block of order $n \geq 3$. One had therefore to rely on non linear feedback laws $u = k(x)$ and it is in [12] that the stabilization issue was solved for Jordan block of order $n \geq 3$ and scalar input (i.e., $m = 1$) by using the celebrated feedback referred to as “nested saturations”. Such a feedback has been also used in [9] to handle the general case described by (1). As a matter of fact, the solution given in that reference relies on a (partial) solution of a more general problem related to (Σ) , that is its L_p -stabilization. Recall that, once a stabilizing feedback $u = k(x)$ has been determined for (Σ) , one wants to understand its robustness properties and for that purpose, one considers the input-output map $\phi_{k,p} : d \mapsto x_d$, where the disturbance d belongs to $L_p(\mathbb{R}_+, \mathbb{R}^m)$ for some $p \in [1, \infty]$, and x_d is the (unique) solution of $\dot{x} = Ax + B\sigma(k(x) + d)$ starting at the origin at $t = 0$. If $\phi_{k,p}$ takes values in $L_p(\mathbb{R}_+, \mathbb{R}^n)$, then the feedback $k(\cdot)$ is said to be L_p -stabilizing and it has finite (L_p) gain if $\phi_{k,p}$ is a bounded (non linear) operator. In case A is neutrally stable, (Σ) is stabilizable by a linear feedback law, which turns out to

have finite gain for every $p \in [1, \infty]$, cf. [5] while detailed results have been given in [1] for the double integrator relatively to L_p -stabilization of several feedback laws. In the general case described by (1), the situation is more complicated since the input-output map $\phi_{k,p}$ associated with the nested saturation feedback is not L_p -stable in general. The first general solution of a feedback law for (Σ) with finite L_p gain has been given in [7] inspired by a solution given in [6] for the stabilization issue and based on high and low gain techniques. Note though that the feedbacks provided by [7] are implicit enough to render their use for practical issues rather difficult and therefore a much simpler solution, based on sliding mode ideas, has been provided in [2] for a feedback law for (Σ) with finite L_p gain for A equal to any Jordan block of order $n \geq 3$ and scalar input.

One of the issues left open in that long string of research consists in determining conditions for the existence (or non existence) of linear stabilizing feedbacks for (Σ) if the state dimension n is larger than two. In particular, the first case not covered by existing results deals with the so called “complex double integrator” (*CDI* for short), i.e., one considers (Σ) in the special case $n = 4$, $m = 1$ and A not diagonalizable with two non zero purely imaginary eigenvalues. It means that

$$A \text{ is similar to } A_\omega := \begin{pmatrix} \omega A_0 & I_2 \\ 0_2 & \omega A_0 \end{pmatrix}, \quad (2)$$

where $\omega > 0$, I_2 and 0_2 are the 2×2 identity and zero matrices respectively and

$$A_0 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

In the present paper, we bring a positive answer to the stabilization issue associated with *CDI* by means of a linear feedback, in the case where the saturation function is further assumed to be odd, non decreasing and with a derivative non increasing on \mathbb{R}_+ . The main idea consists in embedding *CDI* into a continuous family of linear control systems with saturated control $(T_\varepsilon)_{\varepsilon>0}$ so that *CDI* = T_1 and the stabilization by means of a linear feedback of *CDI* is equivalent to that of T_ε for any $\varepsilon > 0$. Then, in a first step, one characterizes a limit system T_0 for $(T_\varepsilon)_{\varepsilon>0}$, as ε tends to zero, which is GAS with respect to the origin and also a strict Lyapunov function V associated with T_0 . It is worth noticing that T_0 is a linear control system with saturated control with a radial saturation, cf. (16). The second and more complicated step consists in establishing that T_ε is GAS with respect to the origin, for ε small enough. This is done by considering T_ε as a perturbation of T_0 and by proceeding at non trivial estimates of the variations of V along trajectories of T_ε .

We close this introduction by proposing a conjecture regarding the stabilization issue associated with (Σ) by means of a linear feedback under the condition that (A, B) is controllable. We claim that (Σ) is stabilizable by means of a linear feedback if and only if the purely imaginary eigenvalues of A do not admit any Jordan block of order larger than or equal to three.

2 Notations and statements of the main result

If $x \in \mathbb{R}$, let $E(x)$ be its integer part. When ε tends to $x_0 \in \mathbb{R} \cup \infty$, the notation $g(\varepsilon) = O(f(\varepsilon))$ means that there exists $C_0 > 0$ independent of ε such that $|g(\varepsilon)| \leq C_0|f(\varepsilon)|$ as ε tends to x_0 and the notation $g(\varepsilon) = o(f(\varepsilon))$ means that $|g(\varepsilon)| \leq C(\varepsilon)|f(\varepsilon)|$ with $C(\varepsilon) > 0$ tending to zero as ε tends to x_0 .

If $f : \mathbb{R} \rightarrow \mathbb{R}$ is a function and $t_1 \leq t_2$ two times, we use $\Delta f \Big|_{t_1}^{t_2}$ to denote $f(t_2) - f(t_1)$.

For $n, m \in \mathbb{N}^*$, let $M_{n,m}(\mathbb{R})$ (resp. $M_{n,m}(\mathbb{C})$) be the set of $n \times m$ matrix with real (resp. complex) entries and, if $n = m$, we simply use $M_n(\mathbb{R})$ (resp. $M_n(\mathbb{C})$). We use (e_1, e_2) , $I_2 \in M_2(\mathbb{R})$ and $J_2 \in M_2(\mathbb{R})$ denote the canonical basis of \mathbb{R}^2 , the identity matrix of \mathbb{R}^2 and the 2-dimensional real *Jordan block*, i.e., $J_2 e_i = e_{i-1}$, for $1 \leq i \leq 2$ with the convention that $e_0 = 0$. We also consider $J_2^c \in M_4(\mathbb{R})$ the complex Jordan block defined as $J_2^c = J_2 \otimes I_2$.

For $\omega \geq 0$, we define the matrix $J_2(\omega)$ as follows

$$J_2(0) = J_2, \quad J_2(\omega) = \omega I_2 \otimes A_0 + J_2^c, \quad \text{for } \omega > 0. \quad (3)$$

For $\varepsilon > 0$, let D_ε be the 4-dimensional diagonal matrix defined by

$$D_\varepsilon = \text{diag}(\varepsilon^2, \varepsilon^2, \varepsilon, \varepsilon). \quad (4)$$

For $\theta \in S^1$, we use R_θ to denote the rotation of \mathbb{R}^2 of angle θ , i.e. the matrix

$$R_\theta = \begin{pmatrix} c_\theta & -s_\theta \\ s_\theta & c_\theta \end{pmatrix}, \quad \text{where } c_\theta := \cos(\theta), \quad s_\theta := \sin(\theta).$$

We use A_0 to denote $R_{\pi/2}$.

If $x \in \mathbb{R}^2$, we use x^\perp to denote $A_0 x$, the orthogonal of x . If in addition $x \neq 0$, then $x/\|x\| \in S^1$ and we use $\theta_x \in [0, 2\pi)$ the corresponding angle. In particular, $x = \|x\| R_{\theta_x} e_1 = -\|x\| R_{\theta_x} e_2^\perp$.

Definition 1 (Saturation function). *A function $\sigma : \mathbb{R} \rightarrow \mathbb{R}$ is called a scalar saturation function if it verifies the following:*

(s1) σ is an odd and globally Lipschitz function;

(s2) $\sigma(\xi)\xi > 0$ for every non zero $\xi \in \mathbb{R}$, and

$$\lim_{\xi \rightarrow +\infty} \sigma(\xi) = \sigma_\infty > 0, \quad \lim_{\xi \rightarrow 0} \frac{\sigma(\xi)}{\xi} = \sigma'(0) > 0;$$

(s3) σ is non decreasing and σ' is non increasing on \mathbb{R}_+ .

Examples of saturation functions are arctan, tanh and the standard saturation function defined by $\sigma_s(\xi) = \frac{\xi}{\max(1, |\xi|)}$. Note that Item (s3) is not usually considered in the standard definition of saturation function.

Remark 2. As easy consequences of the definition, the following holds true:

(c1) For $\xi \in \mathbb{R}$, consider

$$\Sigma(\xi) = \int_0^\xi \sigma(v)dv. \quad (5)$$

Then Σ is an even, positive definite function tending linearly to infinity as $|\xi|$ tends to infinity;

(c2) for every non zero ξ , one has that $\sigma'(\xi) \leq \sigma(\xi)/\xi$ and $\xi \mapsto \sigma(\xi)/\xi$ is an even function, differentiable on \mathbb{R}^* and decreasing over \mathbb{R}_+^* ;

(c3) σ' is continuous at $\xi = 0$ and there exists $\xi_0 > 0$ such that $\sigma'(\xi) \geq \sigma'(0)/2$ for $\xi \in [-\xi_0, \xi_0]$.

A proof of the above items is given in Appendix.

Definition 3 (Stabilizing linear feedback). *Given a linear control system with input subject to saturation $(\Sigma) : \dot{x} = Ax + B\sigma(u)$ with $x, B \in \mathbb{R}^4$, $u \in \mathbb{R}$, $A \in M_4(\mathbb{R})$ and $\sigma : \mathbb{R} \rightarrow \mathbb{R}$ a saturation function. A vector $K \in \mathbb{R}^4$ is called a stabilizing linear feedback for (Σ) if the closed loop system $\dot{x} = Ax + B\sigma(K^T x)$ is globally asymptotically stable (GAS) with respect to the origin.*

In this paper, we prove the following result.

Theorem 4. *Let (CDI) be the saturated complex double integrator, that is the control system given by*

$$(CDI) \quad \dot{x} = J_2(\omega)x - b\sigma(u), \quad (6)$$

where $x \in \mathbb{R}^4$, $u \in \mathbb{R}$, $\sigma : \mathbb{R} \rightarrow \mathbb{R}$ is a saturation function, $\omega > 0$ and

$$b = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}, \quad b_i \in \mathbb{R}^2 \text{ for } i = 1, 2,$$

with $(J_2(\omega), b)$ is controllable. Then, (CDI) admits a stabilizing linear feedback.

3 Proof of Theorem 4

We start the argument by first providing a normal form for (CDI). Since $(J_2(\omega), b)$ is controllable, then b_2 must be a non zero vector of \mathbb{R}^2 . Then one gets the following.

Proposition 5. *The control system (CDI) defined in (6) can be brought, up to a linear change of variable and a time rescaling, to the form*

$$(CDI)_1 \quad \begin{cases} \dot{x}_1 &= 2\pi A_0 x_1 + x_2, \\ \dot{x}_2 &= 2\pi A_0 x_2 - e_2 \sigma(u), \end{cases} \quad (7)$$

where σ is a saturation function with $\sigma_\infty = \sigma'(0) = 1$.

Proof. If $b_1 \neq 0$, pick $\alpha > 0$ and a rotation U_1 so that $b_2 = \alpha U_1 b_1$. Perform first the linear change of variable given by $(\alpha U_1 x_1 - x_2, \alpha U_1 x_2)$ and then the linear change of variable given by $\beta U_2(x_1, x_2)$, with $\beta \alpha U_1 U_2 b_2 = e_2$. One gets that (Σ) has been brought to the form

$$\begin{cases} \dot{x}_1 &= \omega A_0 x_1 + x_2, \\ \dot{x}_2 &= \omega A_0 x_2 - e_2 \sigma(u). \end{cases}$$

Next consider $X_1(t) = \lambda x_1(2\pi t/\omega)$ and $X_2(t) = \lambda x_2(2\pi t/\omega)/\omega$ and $\sigma(k_1 u)/k_2$ to conclude for appropriate choices of $\lambda, k_1, k_2 > 0$. □

One has to determine a stabilizing linear feedback $K \in \mathbb{R}^4$ for $(CDI)_1$, i.e., that there exists $K \in \mathbb{R}^4$ such that the closed loop system defined by

$$(S_1) \quad \begin{cases} \dot{x}_1 &= 2\pi A_0 x_1 + x_2, \\ \dot{x}_2 &= 2\pi A_0 x_2 - e_2 \sigma(K^T x), \end{cases} \quad (8)$$

is GAS with respect to the origin. For that purpose, we imbed (S_1) into a family of dynamical systems $(S_\varepsilon)_{\varepsilon > 0}$ defined as follows. For $\varepsilon > 0$, the curves $t \mapsto x_\varepsilon(t) = D_\varepsilon x(t/\varepsilon)$, where $t \mapsto x(t)$ is any trajectory of (S_1) and D_ε has been defined in (4), are exactly the trajectories of the dynamical system (S_ε) given by

$$(S_\varepsilon) \quad \begin{cases} \dot{x}_1 &= \frac{2\pi A_0}{\varepsilon} x_1 + x_2, \\ \dot{x}_2 &= \frac{2\pi A_0}{\varepsilon} x_2 - e_2 \sigma(K_\varepsilon^T x), \end{cases} \quad x = (x_1, x_2) \in \mathbb{R}^4, \quad K_\varepsilon = D_\varepsilon^{-1} K. \quad (9)$$

The following lemma is immediate.

Lemma 6. *There exists a stabilizing linear feedback $K_1 \in \mathbb{R}^4$ rendering (S_1) GAS with respect to the origin if and only if, for every $\varepsilon > 0$, there exists a stabilizing linear feedback $K_\varepsilon \in \mathbb{R}^4$ rendering (S_ε) GAS with respect to the origin.*

The rest of the section is devoted to an argument for the next proposition.

Proposition 7. *There exists $\varepsilon_0 > 0$ such that, for every $\varepsilon \in (0, \varepsilon_0)$, there exists a stabilizing linear feedback $K_\varepsilon \in \mathbb{R}^4$ rendering (S_ε) GAS with respect to the origin.*

Proposition 7, together with Lemma 6, achieves the stabilisation objective for $(CDI)_1$, i.e., Theorem 4 holds true.

3.1 Limiting behavior for (T_ε) as $\varepsilon \rightarrow 0$.

Clearly, understanding the asymptotic behaviour of (S_ε) for any fixed value of $\varepsilon > 0$ is as difficult as fixing $\varepsilon = 1$. The strategy we follow is made of two steps. In the first one, we let ε tend to zero or infinity and expect to characterize a limit system which is GAS with respect to the origin. Then, in a second step, considering (S_ε) (for ε small or large enough) as a perturbation of the limit system, we aim at extending the GAS property of the limit system to neighboring (S_ε) 's.

As ε tends to infinity, it is not difficult to see that a limit system exists (by simply cancelling the terms in $2\pi A_0/\varepsilon$), but the latter “contains” a double integrator and hence it is unstable with respect to the origin for any choice of linear feedback K . In that case, we cannot even complete the first step of our strategy. As ε tends to zero, the term $2\pi A_0/\varepsilon$ blows up but the flow associated with this linear term corresponds to a rotation and thus remains uniformly bounded. Relying on a variation of constant formula, one obtains a family $(T_\varepsilon)_{\varepsilon>0}$ of dynamical systems on \mathbb{R}^4 which admits a limit (T_0) as ε tends to zero in a sense precised below.

One passes from $(S_\varepsilon)_{\varepsilon>0}$ to $(T_\varepsilon)_{\varepsilon>0}$ using the time-varying linear change of variable $Y_\varepsilon(t) = R_{-2\pi t/\varepsilon}x(t)$. Setting

$$b_\varepsilon(t) = R_{-2\pi t/\varepsilon}e_2, \quad (10)$$

and choosing

$$K_\varepsilon = \begin{pmatrix} e_2 \\ e_2 \end{pmatrix}, \quad (11)$$

an easy computation yields that $Y_\varepsilon = (y_1, y_2)$ is a trajectory of

$$\begin{cases} \dot{y}_1 &= y_2, \\ \dot{y}_2 &= -b_\varepsilon \sigma(b_\varepsilon^T(y_1 + y_2)), \end{cases}$$

where we have dropped the time dependence in b_ε for notational simplicity. We finally define $z = y_1 + y_2$ and $y = y_2$ to get the following one-parameter family $(T_\varepsilon)_{\varepsilon>0}$ of time-varying dynamical systems on \mathbb{R}^4 given by

$$(T_\varepsilon) \quad \begin{cases} \dot{z} &= y - b_\varepsilon \sigma(b_\varepsilon^T z), \\ \dot{y} &= -b_\varepsilon \sigma(b_\varepsilon^T z). \end{cases} \quad (12)$$

It is immediate to see that Proposition 7 holds true if, for $\varepsilon > 0$ small enough, (T_ε) is GAS with respect to the origin (with the definition of GAS uniformly with respect to time in the case of non autonomous ODEs).

We have the following lemma which is the key step to identify the limit system (T_0) .

Lemma 8. *Assume that σ is a saturation function as defined in Definition 1. Let S be the modified saturation function associated with σ as defined in Appendix. Then, the family of time-varying vector fields on \mathbb{R}^2 , $(f_\varepsilon(t, \cdot))_{t \geq 0}$, defined by*

$$f_\varepsilon(t, z) = b_\varepsilon \sigma(b_\varepsilon^T z), \quad (t, z) \in \mathbb{R}_+ \times \mathbb{R}^2, \quad (13)$$

converges, as ε tends to zero, to the vector field $f : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ given by

$$f(z) = \begin{cases} S(\|z\|) \frac{z}{\|z\|} & \text{if } z \neq 0, \\ 0 & \text{if } z = 0, \end{cases} \quad (14)$$

for the weak-* topology of $L^\infty(\mathbb{R}_+, \mathbb{R}^2)$, i.e., for every $z \in \mathbb{R}^2$ and $g \in L^1(\mathbb{R}_+, \mathbb{R}^2)$,

$$\lim_{\varepsilon \rightarrow 0} \int_0^\infty f_\varepsilon^T(t, z) g(t) dt = f^T(z) \int_0^\infty g(t) dt,$$

and the above convergence is uniform with respect to $z \in \mathbb{R}^2$.

Proof. It is enough to show that for every $0 \leq a < c$, one has $\lim_{\varepsilon \rightarrow 0} I_\varepsilon = f(z)$, where

$$I_\varepsilon = \frac{1}{c-a} \int_a^c f_\varepsilon(t, z) dt, \quad (15)$$

and that the convergence is uniform with respect to $z \in \mathbb{R}^2$. For $z = 0$, the result is true with no limit involved.

Hence we suppose in the sequel that $z \neq 0$. Since $z = -\|z\| R_{\theta_z} e_2^\perp$, one has that

$$b_\varepsilon^T z = -\|z\| e_2^T R_{-2\pi t/\varepsilon - \pi/2} R_{\theta_z} R_{\pi/2} e_2 = \|z\| s_{\theta_z + 2\pi t/\varepsilon}.$$

Hence one has that

$$I_\varepsilon = \frac{1}{c-a} \int_a^c \sigma(\|z\| s_{\theta_z + 2\pi t/\varepsilon}) b_\varepsilon dt.$$

After performing the change of time $v = \theta_z + 2\pi t/\varepsilon$, one gets that $I_\varepsilon = \frac{1}{c-a} R_{\theta_z} J_\varepsilon$ where

$$J_\varepsilon = \frac{\varepsilon}{2\pi} \int_{\theta_z + 2\pi a/\varepsilon}^{\theta_z + 2\pi c/\varepsilon} \sigma(\|z\| s_v) \begin{pmatrix} s_v \\ c_v \end{pmatrix} dv.$$

Set $k = E\left(\frac{2\pi(c-a)}{\varepsilon}\right)$. Then

$$J_\varepsilon = O(\varepsilon) + \frac{\varepsilon}{2\pi} \int_{\theta_z + 2\pi a/\varepsilon}^{\theta_z + 2\pi a/\varepsilon + k} \sigma(\|z\| s_v) \begin{pmatrix} s_v \\ c_v \end{pmatrix} dv = O(\varepsilon) + k\varepsilon \int_0^1 \sigma(\|z\| s_{2\pi v}) \begin{pmatrix} s_{2\pi v} \\ c_{2\pi v} \end{pmatrix} dv,$$

where the last equality holds since $v \rightarrow \sigma(\|z\| s_{2\pi v}) \begin{pmatrix} s_{2\pi v} \\ c_{2\pi v} \end{pmatrix}$ is 1-periodic. Moreover the terms $O(\varepsilon)$ do not depend on z . It is then immediate to compute that

$$\int_0^1 \sigma(\|z\| s_{2\pi v}) \begin{pmatrix} s_{2\pi v} \\ c_{2\pi v} \end{pmatrix} dv = S(\|z\|) \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

Since $R_{\theta_z} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \frac{z}{\|z\|}$, the lemma is proved. □

According to the previous lemma, the one-parameter family of time-varying dynamical systems $(T_\varepsilon)_{\varepsilon > 0}$ converges for the weak-* topology of $L^\infty(\mathbb{R}_+, \mathbb{R}^4)$ to the dynamical system (T_0) defined on \mathbb{R}^4 by

$$(T_0) \quad \begin{cases} \dot{z} &= y - f(z), \\ \dot{y} &= -f(z), \end{cases} \quad (16)$$

where the vector field f on \mathbb{R}^2 has been defined in (14). To study (T_0) , we need the following lemma.

Lemma 9. *Let $f : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ be the vector field defined in (14). Then f is bounded, of class C^1 and, for every $(z, y) \in \mathbb{R}^4$, one has*

$$y^T \left(f(z + y) - f(z) \right) \geq 0, \quad (17)$$

with equality if and only if $y = 0$.

Proof. From Proposition 20, we have that f is bounded and, since S is of class C^1 and $\xi \rightarrow S(\xi)/\xi$ is decreasing, f is differentiable everywhere, C^1 outside the origin and $df(0) = S'(0)I_2$. Indeed, for $z \neq 0$, one has that

$$df(z) = S'(\|z\|) \frac{zz^T}{\|z\|^2} + \frac{S(\|z\|)}{\|z\|} \left(I_2 - \frac{zz^T}{\|z\|^2} \right). \quad (18)$$

Note that, since $z \in \mathbb{R}^2$, one has that $I_2 - \frac{zz^T}{\|z\|^2} = \frac{z^\perp (z^\perp)^T}{\|z\|^2}$. Clearly $df(z)$ is bounded and continuous at $z = 0$. Moreover, since both S' and $\xi \mapsto S(\xi)/\xi$ are positive functions, then $df(z)$ is symmetric positive definite for every $z \in \mathbb{R}^2$.

For every $(z, y) \in \mathbb{R}^4$, one has

$$y^T \left(f(z + y) - f(z) \right) = \int_0^1 y^T df(z + sy) y ds, \quad (19)$$

which is clearly non negative, and strictly positive if $y \neq 0$ since $z \mapsto df(z)$ is everywhere positive definite. \square

As a consequence of Lemma 9, we have the following proposition, which describes the asymptotic behaviour of trajectories of (T_0) .

Proposition 10. *Trajectories of (T_0) given in (16) are defined for all non negative times. Moreover, consider the function $V_0 : \mathbb{R}^4 \rightarrow \mathbb{R}_+$ given by*

$$V_0(z, y) = \|y\|^2 + \int_0^{\|z\|} S(\xi) d\xi + \int_0^{\|z-y\|} S(\xi) d\xi. \quad (20)$$

Then V_0 is a C^1 , positive definite and radially unbounded function which is a strict Lyapunov function along trajectories of (T_0) . As a consequence, (T_0) is GAS with respect to the origin.

Proof. The vector field on \mathbb{R}^4 defining (T_0) is C^1 , thanks to Lemma 9, and, since its growth at infinity is linear, trajectories of (T_0) are defined for all non negative times. Properties of V_0 are immediate and we next check that V_0 is a strict Lyapunov function for (T_0) . Indeed, if we use \dot{V}_0 to denote the time derivative of V_0 along non trivial trajectories of (T_0) , one gets that

$$\begin{aligned} \dot{V}_0 &= -S(\|z\|)^2 - y^T \left(f(z) - f(z - y) \right) = -S(\|z\|)^2 - \int_0^1 y^T df(g(s)) y ds \\ &= -S(\|y\|)^2 - \int_0^1 \left[S'(\|g(s)\|) \frac{(y^T g(s))^2}{\|g(s)\|^2} + \frac{S(\|g(s)\|)}{\|g(s)\|} \frac{(y^T g(s)^\perp)^2}{\|g(s)\|^2} \right] ds, \end{aligned} \quad (21)$$

where $g(s) = z - (1 - s)y$ for $s \in [0, 1]$. One gets the conclusion by using Lemma 9. \square

Remark 11. Note that (T_0) is locally exponentially stable at the origin since the linearized system associated with (T_0) at the origin is defined by the Hurwitz matrix

$$\begin{pmatrix} -S'(0)I_2 & I_2 \\ -S'(0)I_2 & 0 \end{pmatrix} = \begin{pmatrix} -S'(0) & 1 \\ -S'(0) & 0 \end{pmatrix} \otimes I_2.$$

Remark 12. Recall that the double integrator (DI) is the linear control system defined on \mathbb{R}^2 by $\dot{x} = J_2x + e_2u$. For any feedback $u = -\sigma(k^T x)$ where $k \in \mathbb{R}^2$ has positive coordinates and σ is a saturation function, the closed loop system $\dot{x} = J_2x - e_2\sigma(k^T x)$ is GAS with respect to the origin. After a linear change of variable and time, such a system can be brought to the form corresponding to (T_0) namely

$$(DI) \quad \begin{cases} \dot{z} &= y - \sigma(z), \\ \dot{y} &= -\sigma(z), \end{cases} \quad (22)$$

with $(z, y) \in \mathbb{R}^2$. It has been proved in [13] that the radially unbounded positive definite function $V : \mathbb{R}^2 \rightarrow \mathbb{R}_+$ given by

$$V(z, y) = y^2 + \int_0^z \sigma(\xi)d\xi + \int_0^{z-y} \sigma(\xi)d\xi,$$

is a strict Lyapunov function for (DI) . It is immediate to see that V_0 is a simple adaptation of V to (T_0) .

Remark 13. Let $F_2 : \mathbb{R}^4 \rightarrow \mathbb{R}^4$ be the vector field on \mathbb{R}^4 defining (T_0) . It is rather immediate to see that, for every $n \geq 1$, one can define a vector field F_n on \mathbb{R}^{2n} where $F_n(z, y)$ is defined exactly as $F_2(z, y)$, now with z and y vectors in \mathbb{R}^n . (For $n = 1$, $z/\|z\|$ must be understood as the sign of $z \in \mathbb{R}$.) Then the conclusions of Proposition 10 extend verbatim to F_n with the same Lyapunov function V_0 now defined on \mathbb{R}^{2n} .

3.2 Study of (T_ε) for ε small enough.

By characterizing (T_0) , we have achieved the first step of the strategy devised to prove Proposition 7. We next turn to the second step and for that purpose we will analyse the variations of V_0 along trajectories of (T_ε) for ε small enough.

The time derivative \dot{V}_0 of V_0 along non trivial trajectories of (T_ε) is given by

$$\dot{V}_0 = -S(\|z\|) \left(b_\varepsilon^T \frac{z}{\|z\|} \right) \sigma(b_\varepsilon^T z) \quad (23)$$

$$- y^T \left(f(z) - f(z - y) \right) \quad (24)$$

$$+ 2y^T \left(f(z) - b_\varepsilon^T \sigma(b_\varepsilon^T z) \right). \quad (25)$$

Clearly the two first terms (23) and (24) are non positive and one must handle the effect of the third one (25). As a matter of fact, if $b_\varepsilon^T z = 0$ and $z = y \neq 0$, then $\dot{V}_0 = S(\|z\|)\|z\|$,

which is positive and unbounded over \mathbb{R}^2 . Then V_0 cannot be a Lyapunov function for (T_ε) for $\varepsilon > 0$ since clearly \dot{V}_0 (the time derivative of V_0 along non trivial trajectories of (T_ε)) can clearly be positive. To circumvent this problem, we will evaluate variations of V_0 on appropriate time intervals when $\|(z, y)\|$ is large.

Remark 14. One could have also written \dot{V}_0 as

$$\dot{V}_0 = -S(\|z\|)^2 - y^T \left(f(z) - f(z - y) \right) + (2y - f(z))^T \left(f(z) - b_\varepsilon^T \sigma(b_\varepsilon^T z) \right),$$

with a more handleable first term since it is ε -free. However, it introduces an extra quantity in the third term, which turns out to be not so easy to deal with.

We aim at establishing the following key technical proposition.

Proposition 15. *There exists $\varepsilon_0 > 0$, $R, C_1 > 0$ and $\rho \in (0, 1)$, such that, for every $\varepsilon \in (0, \varepsilon_0)$, $(z_0, y_0) \in \mathbb{R}^4$ with $V(z_0, y_0) \geq R$, there exists $T(z_0, y_0)$ such that*

$$\rho \max(1, \|y_0\|) \leq T(z_0, y_0) \leq 2\rho \max(1, \|y_0\|), \quad (26)$$

for which

$$\Delta V_0 \Big|_0^{T(z_0, y_0)} \leq -C_1 T(z_0, y_0), \quad (27)$$

along every trajectory of (T_ε) starting at (z_0, y_0) .

Proof. The several constants will be fixed along the argument but typically ε_0 and ρ will be small compared to one while R will be large compared to one. Let us stress that ρ , R and ε_0 will be eventually modified in the argument (typically by decreasing ρ and ε_0 and increasing R) but these choices remain “universal”, i.e., only depending on ε_0 and thus independent of $\varepsilon < \varepsilon_0$. We will also use the symbol C_R to denote positive constants that only depend on R and σ .

We fix $(z_0, y_0) \in \mathbb{R}^4$ with $V(z_0, y_0) \geq R$ and simply use T to denote $T(z_0, y_0)$. Note that $\dot{V}_0 \geq -1 - 3\sqrt{V_0}$. In particular, as long as $V_0 \geq 1$, one has that $(\sqrt{V_0}) \geq -2$ and hence $\sqrt{V_0} \geq \sqrt{V_0(0)}(1 - 4\rho\pi)$ on $[0, T\pi]$. In particular, $\sqrt{V_0} \geq R^{1/2}/2$ on $[0, T\pi]$.

There are two key quantities to estimate, namely

$$L_\varepsilon = - \int_0^T S(\|z\|) \left(b_\varepsilon^T \frac{z}{\|z\|} \right) \sigma(b_\varepsilon^T z) dt, \quad (28)$$

and

$$K_\varepsilon = K_\varepsilon^1 + K_\varepsilon^2, \quad (29)$$

where

$$K_\varepsilon^1 = - \int_0^T y^T \left(f(z) - f(z - y) \right) dt, \quad K_\varepsilon^2 = 2 \int_0^T y^T \left(f(z) - b_\varepsilon^T \sigma(b_\varepsilon^T z) \right) dt. \quad (30)$$

Assume that

$$L_\varepsilon \leq -3C_1 T, \quad (31)$$

and

$$K_\varepsilon \leq (C_1 + C_2\varepsilon)T, \quad (32)$$

for some positive constants C_1, C_2 independent of ε small enough. Clearly the above two inequalities yield (27). \square

We are now left to establish (31) and (32). This is the purpose of the next two lemmas.

Lemma 16. *With the above notations, there exists a positive constant such that (31) holds true.*

Proof. We distinguish two cases.

(L1) For every $t \in [0, T]$, one has $\|y(t)\| \leq \frac{\pi\|z(t)\|}{\varepsilon}$;

(L2) there exists $\bar{t} \in [0, T]$ such that $\|y(\bar{t})\| > \frac{\pi\|z(\bar{t})\|}{\varepsilon}$.

Assume that (L1) holds true. Then $z(t) \neq 0$ for every $t \in [0, T]$ and $\theta_z(t)$ is well defined and absolutely continuous. Moreover

$$\dot{\theta}_z = \frac{(z^\perp)}{\|z\|} \frac{d}{dt} \left(\frac{z}{\|z\|} \right) = \frac{(z^\perp)^T \dot{z}}{\|z\|^2}.$$

Taking into account the estimate in (L1), one gets that $|\dot{\theta}_z| \leq 4\pi/3\varepsilon$ on $[0, T]$.

In the case where $\|y_0\| \leq R^{1/2}/3$, then $\|y\| \leq R^{1/2}/2$ and $\|z\| \geq R/2 > 1$ on $[0, T]$ for R universal constant large enough. Assume now that $\|y_0\| > R^{1/2}/3$. It is immediate to see that

$$(1 - 2\rho)\|y_0\| \leq \|y(t)\| \leq (1 + 2\rho)\|y_0\|, \quad t \in [0, T]. \quad (33)$$

On the other hand, let

$$E_z := \{t \in [0, T] \mid \|z(t)\| < 1\}. \quad (34)$$

If E_z is not empty, let $\tilde{t} \in E_z$. From the dynamics and (33), one gets that

$$z(t) = z(\tilde{t}) + (t - \tilde{t}) \left(y(\tilde{t}) + O(1) \right), \quad t \in [0, T]$$

where $\|O(1)\|$ can be chosen smaller than one, thanks to Proposition 5. This implies that

$$\|z(t)\| \geq \left(|t - \tilde{t}| - \varepsilon \right) (\|y(\tilde{t})\| - 1). \quad (35)$$

From that, it is easy to deduce that E_z is contained in an subinterval of $[0, T\pi]$ of length smaller than $2/\|y_0\|$ and hence there exists a subinterval I_L of $[0, T]$ of length at least $T/2$ such that for $t \in I_L$,

- $\|z(t)\| \geq 1$,
- $|\dot{\theta}_z| \leq 4\pi/3\varepsilon$.

Then one gets,

$$L_\varepsilon \leq - \int_{I_L} S(\|z\|) \left(b_\varepsilon^T \frac{z}{\|z\|} \right) \sigma(b_\varepsilon^T z) dt \leq -S(1) \int_{I_L} s_{2\pi t/\varepsilon + \theta_z} \sigma(s_{2\pi t/\varepsilon + \theta_z}) dt, \quad (36)$$

since both S and σ are increasing. We now perform the change of time $\tau(t) = 2\pi t/\varepsilon + \theta_z$. Since $2\pi/3\varepsilon \leq \dot{\tau} \leq 5\pi/3\varepsilon$ on I_L , $t \mapsto \tau(t)$ realises an increasing bijection between I_L and an interval \tilde{I}_L with $2\pi|I_L|/3\varepsilon \leq |\tilde{I}_L| \leq 4\pi|I_L|\varepsilon$. One deduces from (36) the following

$$L_\varepsilon \leq - \frac{3\varepsilon S(1)}{10\pi} \int_{\tilde{I}_L} s_\tau \sigma(s_\tau) d\tau. \quad (37)$$

Since $\tau \mapsto s_\tau \sigma(s_\tau)$ is π -periodic, it is easy to see that $\int_{\tilde{I}_L} s_\tau \sigma(s_\tau) d\tau \geq S(1)T\pi/6\varepsilon + O(1)$, which implies that $L_\varepsilon \leq -4C_1T + T\varepsilon O(1)$ for some universal constant C_1 . Then (31) holds if (L1) holds true.

We now assume that (L2) holds true. In particular we have that $\|y(\bar{t})\| \geq R^{1/2}/2$ and both (33) and (35) hold true. It is immediate to see that, outside an interval $I_{bad} \subset [0, T]$ of length at most $4\pi\varepsilon$ and containing \bar{t} , one has $\|y(t)\| \leq \frac{\pi\|z(t)\|}{\varepsilon}$. We can therefore select a subinterval of $[0, T]$ of length at least $T/2$ on which the previous inequality holds true on it. We are back to (L1) and that concludes the proof of (31). \square

Lemma 17. *With the above notations, (32) holds true.*

Proof. In the sequel, we will use the notation $O(\cdot)$ only when the involved bounds do not depend on ε . We first perform the change of time $s = t/\varepsilon$ and rewrite $K_\varepsilon^1, K_\varepsilon^2$ defined in (32) as

$$K_\varepsilon^1 = -\varepsilon \int_0^{T/\varepsilon} y(\varepsilon s)^T \left(f(z(\varepsilon s)) - f(z(\varepsilon s) - y(\varepsilon s)) \right) ds \quad (38)$$

and

$$K_\varepsilon^2 = 2\varepsilon \int_0^{T/\varepsilon} y(\varepsilon s)^T \left(f(z(\varepsilon s)) - b_\varepsilon^T \frac{z(\varepsilon s)}{\|z(\varepsilon s)\|} \right) ds. \quad (39)$$

We start by several trivial remarks. With our choice of T and since $\dot{y} = O(1)$, then clearly $\|y\| = O(\max(1, \|y_0\|))$. We can therefore always assume that T/ε is an integer since otherwise the error made in (39) is $\varepsilon O(\max(1, \|y_0\|)) = \varepsilon O(T)$ and hence negligible if we establish (31).

We can then set $T/\varepsilon = k$. We now decompose the integral terms in K_ε^1 and K_ε^2 according to

$$\int_0^k \cdots = \sum_{j=0}^{k-1} \int_j^{j+1} \cdots,$$

and then perform the change of times $s = j + v$ in each interval $[j, (j+1)]$. We deduce from (38) and (39) that, for $0 \leq j \leq k-1$, one has

$$K_\varepsilon^1 = -\varepsilon \sum_{j=0}^{k-1} K_{\varepsilon,j}^1, \quad K_\varepsilon^2 = 2\varepsilon \sum_{j=0}^{k-1} K_{\varepsilon,j}^2, \quad (40)$$

where

$$K_{\varepsilon,j}^1 = \int_0^1 y^T \left(f(z) - f(z-y) \right) dv, \quad K_{\varepsilon,j}^2 = \int_0^1 y^T \left(f(z) - b_1 \sigma(b_1^T z) \right) dv, \quad (41)$$

where the argument of both z, y is equal to $j\pi\varepsilon + \pi\varepsilon v$.

We need the following notations,

$$z_j = z(\varepsilon j), \quad y_j = y(\varepsilon j), \quad z_j(v) = z_j + \varepsilon v y_j, \quad 0 \leq j \leq k-1, \quad v \in [0, 1]. \quad (42)$$

We also have the following estimates, easily deduced from (42),

$$y(\varepsilon j + \varepsilon v) = y_j + \varepsilon O(1)v, \quad z(\varepsilon j + \varepsilon v) = z_j(v) + \varepsilon O(1)v, \quad (43)$$

where $\|O(1)\| \leq 1$.

We next consider, for $0 \leq j \leq k-1$, the quantities \tilde{K}_ε^1 and \tilde{K}_ε^2 obtained as K_ε^1 and K_ε^2 in (40) but, instead of $K_{\varepsilon,j}^1$ and $K_{\varepsilon,j}^2$, we use the integrals

$$\tilde{K}_{\varepsilon,j}^1 = \int_0^1 y_j^T \left(f(z) - f(z-y) \right) dv, \quad \tilde{K}_{\varepsilon,j}^2 = \int_0^1 y_j^T \left(f(z) - b_1 \sigma(b_1^T z) \right) dv, \quad (44)$$

where still the argument of z is equal to $j\pi\varepsilon + \pi\varepsilon v$. From (43) and the fact that f and σ are bounded, one gets that, for $0 \leq j \leq k-1$,

$$K_{\varepsilon,j}^1 = \tilde{K}_{\varepsilon,j}^1 + O(1)\varepsilon, \quad K_{\varepsilon,j}^2 = \tilde{K}_{\varepsilon,j}^2 + O(1)\varepsilon. \quad (45)$$

One deduces that, for $i = 1, 2$,

$$K_\varepsilon^i = \tilde{K}_\varepsilon^i + \varepsilon^2 k O(1) = \tilde{K}_\varepsilon^i + \varepsilon T \pi O(1). \quad (46)$$

Setting $\tilde{K}_\varepsilon = \tilde{K}_\varepsilon^1 + \tilde{K}_\varepsilon^2$, one deduces from the previous equation that the argument amounts to prove the estimate (32) for \tilde{K}_ε .

We claim that, for $0 \leq j \leq k-1$, one has that

$$- \int_0^1 y_j^T b_1 \left(\sigma(b_1^T z) - \sigma(b_1^T z_j) \right) dv \leq O(1)\varepsilon. \quad (47)$$

Observe first that one gets from (43)

$$b_1^T z = b_1^T z_j + \varepsilon v \left(b_1^T y_j + O(1) \right). \quad (48)$$

To get the claim, one can see that

$$y_j^T b_1 \left(\sigma(b_1^T z) - \sigma(b_1^T z_j) \right) \geq 0,$$

as soon as $b_1^T z_j (b_1^T z - b_1^T z_j) > 0$ since σ is increasing. By (48), the previous inequality does not hold true only if $|y_j^T b_{2\pi v}| = O(1)$, in which case,

$$|y_j^T b_1 \left(\sigma(b_1^T z) - \sigma(b_1^T z_j) \right)| = \varepsilon O(1).$$

This concludes the argument of the claim (47).

Noticing that

$$\int_0^1 y_j^T b_1 \sigma(b_1^T z_j) dv = y_j^T f(z_j),$$

and using (47), one deduces that in the estimate $\tilde{K}_{\varepsilon,j}^2$, one can replace $\sigma(b_1^T z)$ by $f(z_j)$. We are therefore left to show that the following quantity

$$\varepsilon \sum_{j=0}^{k-1} \int_0^1 y_j^T \left(f(z) + f(z-y) - 2f(z_j) \right) dv \quad (49)$$

satisfies the estimate (32).

Notice that, for $0 \leq j \leq k-1$,

$$f(z) + f(z-y) - 2f(z_j) = f(z) - f(z_j) + f(z-y) - f(z_j) = \varepsilon \left(y_j + O(1) \right).$$

One deduces that if $\|y_0\| \leq R^{1/2}$, then $\|y_j\| = O(1)$ and $\tilde{K}_{\varepsilon,j}^i \leq \varepsilon O(1)$ for $i = 1, 2$ and $0 \leq j \leq k-1$, which yields the desired estimate for $K_\varepsilon = \varepsilon TO(R)$.

We can hence assume that $\|y_0\| \geq R^{1/2}$ and then, $\|y_0\|(1-\rho) \leq \|y_j\| \leq \|y_0\|(1+\rho)$ for $0 \leq j \leq k-1$.

Similarly to (47), we claim that, for $0 \leq j \leq k-1$, one has

$$\int_0^1 y_j^T \left(f(z) + f(z-y) - f(z_j(1)) - f(z_j(1) - y_j) \right) dv \leq O(1)\varepsilon. \quad (50)$$

Indeed, we get from (43) that

$$z = z_j(1) - \varepsilon \left((1-v)y_j + vO(1) \right), \quad z-y = z_j(1) - y_j - \varepsilon \left((1-v)y_j + vO(1) \right).$$

For $0 \leq j \leq k-1$ and, as long as $(1-v)\|y_j\| > vO(1)$, one deduces from (17) that

$$y_j^T \left(f(z) - f(z_j(1)) \right) \leq 0, \quad y_j^T \left(f(z-y) - f(z_j(1) - y_j) \right) \leq 0.$$

The inequality $(1-v)\|y_j\| \leq vO(1)$ occurs for v close to 1 and on a subinterval of length $O(1)/\|y_j\|$. Using on that subinterval that f is globally Lipschitz, one derives (50).

From (49) and (50), the argument of Lemma 17 reduces to prove that the quantity M_ε defined by

$$M_\varepsilon = \varepsilon \sum_{j=0}^{k-1} M_{\varepsilon,j}, \quad M_{\varepsilon,j} = y_j^T \left(f(z_j(1)) + f(z_j(1) - y_j) - 2f(z_j) \right), \quad 0 \leq j \leq k-1, \quad (51)$$

satisfies the estimate (32).

For $0 \leq j \leq k-1$, set $x_j(v) = z_j - vy_j$ for $v \in [0, 1]$. Notice that

$$z_j = z_j(0) = x_j(0), \quad z_j - y_j = z_j(0) - y_j = x_j(1),$$

and then one can rewrite (51) as

$$M_{\varepsilon,j} = y_j^T \left(f(z_j(1)) - f(z_j(0)) + f(z_j(1) - y_j) - f(z_j(0) - y_j) + f(x_j(1)) - f(x_j(0)) \right). \quad (52)$$

By using (19) in the previous equality, one has for every $0 \leq j \leq k-1$ that

$$M_{\varepsilon,j} = \int_0^1 \left(M_{\varepsilon,j}^1(v) + M_{\varepsilon,j}^2(v) \right) dv, \quad (53)$$

where

$$\begin{aligned} M_{\varepsilon,j}^1(v) &= \varepsilon \left[S'(\|z_j(v)\|) \frac{(y_j^T z_j(v))^2}{\|z_j(v)\|^2} + S'(\|z_j(v) - y_j\|) \frac{(y_j^T (z_j(v) - y_j))^2}{\|z_j(v) - y_j\|^2} \right] \\ &\quad - S'(\|x_j(v)\|) \frac{(y_j^T x_j(v))^2}{\|x_j(v)\|^2} \end{aligned} \quad (54)$$

and

$$\begin{aligned} M_{\varepsilon,j}^2(v) &= \varepsilon \left[\frac{S(\|z_j(v)\|)}{\|z_j(v)\|} \frac{(y_j^T z_j(v)^\perp)^2}{\|z_j(v)\|^2} + \frac{S(\|z_j(v) - y_j\|)}{\|z_j(v) - y_j\|} \frac{(y_j^T (z_j(v) - y_j)^\perp)^2}{\|z_j(v) - y_j\|^2} \right] \\ &\quad - \frac{S(\|x_j(v)\|)}{\|x_j(v)\|} \frac{(y_j^T x_j(v)^\perp)^2}{\|x_j(v)\|^2}. \end{aligned} \quad (55)$$

Moreover note that, for every $0 \leq j \leq k-1$ and $v \in [0, 1]$, one has

$$y_j^T z_j(v)^\perp = y_j^T (z_j(v) - y_j)^\perp = y_j^T x_j(v)^\perp = y_j^T z_j. \quad (56)$$

To obtain the required estimate, we subdivide the discussion into two cases and consider a constant C_* large with respect to one, which will be fixed later.

Case 1. For every $t \in [0, T]$, one has that $\|z - y\| \leq C_* \|z\|/2$. We will prove that $M_{\varepsilon,j}^1(v) + M_{\varepsilon,j}^2(v) < 0$ for every $0 \leq j \leq k-1$ and $v \in [0, 1]$.

As a consequence of the case assumption, one gets, for every $0 \leq j \leq k-1$ and $v \in [0, 1]$ that

$$\|x_j(v)\| \leq C_* \|z_j(v)\|, \quad \|x_j(v)\| \leq C_* \|z_j(v) - y_j\|. \quad (57)$$

Using Item (S2) in Proposition 20, one has, for every $0 \leq j \leq k-1$ and $v \in [0, 1]$, that

$$\frac{S(\|z_j(v)\|)}{\|z_j(v)\|} \leq C_* \frac{S(\|x_j(v)\|)}{\|x_j(v)\|}, \quad \frac{S(\|z_j(v) - y_j\|)}{\|z_j(v) - y_j\|} \leq C_* \frac{S(\|x_j(v)\|)}{\|x_j(v)\|}. \quad (58)$$

By taking into account (56), one has that

$$\frac{(y_j^T Z)^2}{\|y_j\|^2 \|Z\|^2} \leq C_*^2 \frac{(y_j^T x_j(v)^\perp)^2}{\|y_j\|^2 \|x_j(v)\|^2},$$

where

$$Z \in \{z_j(v)^\perp, (z_j(v) - y_j)^\perp\}.$$

Then, one deduces from the previous inequalities and (58) that, for every $0 \leq j \leq k-1$ and $v \in [0, 1]$, one has that

$$M_{\varepsilon,j}^2(v) \leq (\varepsilon C_*^3 - 1) \frac{S(\|x_j(v)\|) (y_j^T x_j(v)^\perp)^2}{\|x_j(v)\| \|x_j(v)\|^2} \leq 0, \quad (59)$$

where the last inequality is obtained for ε small enough.

To handle $M_{\varepsilon,j}^1(v)$, first notice that, for every $0 \leq j \leq k-1$ and $v \in [0, 1]$, one can deduce from the case assumption and Item (S3) in Proposition 20 that

$$S'(\|z_j(v)\|) \leq \frac{C_*^3}{C_0} S'(\|x_j(v)\|), \quad S'(\|z_j(v) - y_j\|) \leq \frac{C_*^3}{C_0} S'(\|x_j(v)\|). \quad (60)$$

In the case where

$$\frac{(y_j^T x_j(v))^2}{\|y_j\|^2 \|x_j(v)\|^2} \geq 1/\sqrt{2}, \quad (61)$$

one deduces that

$$M_{\varepsilon,j}^1(v) \leq \left(\frac{4\varepsilon C_*^3}{C_0} - 1 \right) \frac{S(\|x_j(v)\|) (y_j^T x_j(v)^\perp)^2}{\|x_j(v)\| \|x_j(v)\|^2} \leq 0, \quad (62)$$

where the last inequality is obtained for ε small enough. One finally gets from (59) and (62) that $M_{\varepsilon,j}^1(v) + M_{\varepsilon,j}^2(v) \leq 0$. If (61) does not hold then

$$\frac{(y_j^T x_j(v)^\perp)^2}{\|y_j\|^2 \|x_j(v)\|^2} \geq 1/\sqrt{2}. \quad (63)$$

In that case,

$$M_{\varepsilon,j}^1(v) \leq \varepsilon \frac{2\varepsilon C_*^3}{C_0} \frac{S(\|x_j(v)\|) (y_j^T x_j(v)^\perp)^2}{\|x_j(v)\| \|x_j(v)\|^2} \leq \frac{4\varepsilon C_*^3}{C_0} \frac{S(\|x_j(v)\|) (y_j^T x_j(v)^\perp)^2}{\|x_j(v)\| \|x_j(v)\|^2}.$$

Adding the above inequality with (59) yields that

$$M_{\varepsilon,j}^1(v) + M_{\varepsilon,j}^2(v) \leq \left(\varepsilon C_*^3 (2/C_0 + 1) - 1 \right) \frac{S(\|x_j(v)\|) (y_j^T x_j(v)^\perp)^2}{\|x_j(v)\| \|x_j(v)\|^2} \leq 0,$$

where the last inequality is obtained for ε small enough. The argument for Case 1-Lemma 17 is complete.

Case 2. There exists $\bar{t} \in [0, T]$ such that $\|z(\bar{t}) - y(\bar{t})\| \leq C_* \|z(\bar{t})\|/2$. One deduces at once that

$$\|z(\bar{t})\| \leq \frac{1}{\frac{C_*}{2} - 1} \|y(\bar{t})\|.$$

For C_* universal constant large enough with respect to one, we have that $\|y(\bar{t})\| \geq R^{1/2}/2$ and we can easily rewrite (33) as

$$(1 - 4\rho)\|y(\bar{t})\| \leq \|y(t)\| \leq (1 + 4\rho)\|y(\bar{t})\|, \quad t \in [0, T]. \quad (64)$$

By computations similar to those leading to (35), one gets that there exists a subinterval I_{bad} of $[0, T]$ of length at most $4/C_*$ such that $\|z(t) - y(t)\| \leq C_* \|z(t)\|$ for $t \in [0, T] \setminus I_{bad}$. We can therefore subdivide $[0, T]$ in at most three disjoint subintervals, I_1, I_2 and I_{bad} such that, if one writes $M_\varepsilon = M_{\varepsilon,1} + M_{\varepsilon,bad} + M_{\varepsilon,2}$ according to the subdivision $[0, T] = I_1 \cup I_{bad} \cup I_2$, then both $M_{\varepsilon,1}$ and $M_{\varepsilon,2}$ are negative since we can apply to each of them Case 1 and one has the direct estimate

$$M_{\varepsilon,bad} \leq 4|I_{bad}|\|y(\bar{t})\| \leq \frac{16}{C_*}\|y_0\| \leq \frac{32}{\rho C_*}T.$$

By choosing C_* large enough with respect to ρ and C_1 , one finally obtains (32). \square

3.3 Proof of Proposition 7

This will be obtained in three steps, with the help of Proposition 15. The first step is an easy consequence of Proposition 15.

Lemma 18. *Consider the constants ε_0 and R defined in Proposition 15. Then, for every $\varepsilon \in (0, \varepsilon_0)$ and every $(z_0, y_0) \in \mathbb{R}^4$, there exists a time $T_1(z_0, y_0)$ such that*

$$V_0(z(t), y(t)) \leq 2R, \quad t \geq T_1(z_0, y_0), \quad (65)$$

where (z, y) denotes the trajectory of (T_ε) starting at (z_0, y_0) .

Proof. First, notice that the inequality $V_0(z, y) \geq M$ for M large implies that either $\|y\| \geq \sqrt{M/2}$ or $\|z\| \geq M/2$.

We now start the argument of the proposition. Fix $\varepsilon \in (0, \varepsilon_0)$ and $(z_0, y_0) \in \mathbb{R}^4$ and consider the trajectory (z, y) of (T_ε) starting at (z_0, y_0) . Clearly, an immediate argument by contradiction using Proposition 15 yields that there exists a time $T_1 \geq 0$ such that $V_0(z(T_1), y(T_1)) \leq R$. One can show the conclusion by taking $T_1 = T_1(z_0, y_0)$. Indeed, if it is not possible, then by a obvious continuity argument there exists $T_2 > T'_1 \geq T_1$ such that

$$\frac{3R}{2} = V_0(z(T'_1), y(T'_1)) \leq V_0(z(t), y(t)) \leq 2R = V_0(z(T'_1), y(T'_1)), \quad T'_1 \leq t \leq T_2.$$

Since $\Delta V_0 \Big|_{T'_1}^{T_2} = R/2$ and $\|y(t)\| \leq 2R^{1/2}$ on $[T'_1, T_2]$, one deduces that $T_2 - T'_1 \geq 1/2$.

Applying Proposition 15 from T'_1 immediately yields that there exists $t_1 \in [T'_1, T_2]$ such that $\Delta V_0 \Big|_{T'_1}^{t_1} < 0$ which is a contradiction. \square

The second step to complete the proof of Proposition 7 consists in improving Estimate (27) and get a more precise one with the additional information that trajectories are now universally bounded (i.e., independently of ε) thanks to Lemma 18. We get the following.

Lemma 19. *With the above notations, there exist ε_0 and $C_R > 0$ such that, for every $\varepsilon \in (0, \varepsilon_0)$ and every $(z_0, y_0) \in \mathbb{R}^4$, there exists a time $T_2 := T_2(z_0, y_0)$ for which, for every $T \geq \rho$ with $1/2 \leq T \leq 2$ and T/ε integer, one has*

$$\Delta V_0 \Big|_{T_2}^{T_2+T} \leq -C_R \int_{T_2}^{T_2+T} [\|y(t)\|^2 + (b_\varepsilon^T z)^2] dt. \quad (66)$$

Proof. To proceed, one looks back at the argument of Proposition 15. We can suppose with no loss of generality, in the argument of Proposition 15 that $\|z(t)\| \leq 2R^2$ and $\|y(t)\| \leq 2R$ for every $t \in [0, T]$ where now $T \geq \rho$ is arbitrary. Moreover we will now use the following obvious estimates: there exists two positive constants C_R^1 and C_R^2 depending only on R such that, for $V_0(z, y) \leq 2R^2$ and $t \in [0, T]$ it holds

$$\|\dot{z}\| + \|\dot{y}\| \leq C_R^2 (\|y\| + \|z\|), \quad C_R^1 \|(z, y)\|^2 \leq V_0(z, y) \leq C_R^2 \|(z, y)\|^2, \quad |\dot{V}_0| \leq C_R^2 V_0. \quad (67)$$

We now follow again the proof of Proposition 15 with the objective of providing better estimates of all the $\varepsilon TO(1)$ that have appeared. We start by choosing T so that $1/2 \leq T \leq 2$ and T/ε integer. In that way, we have eliminated the error term occurring when one performs the the change of times $s = j + v$ in each interval $[j, j + 1]$, for $0 \leq j \leq k - 1$ to pass from (38) and (39) to (44).

The next error terms to handle are those occurring in (45) and then in (46). Those occurring in (45) can now be replaced by

$$C_R \varepsilon \left(\int_0^1 (b_1^T z)^2 dv \right)^{1/2} \left(\int_0^1 \|y\|^2 dv \right)^{1/2},$$

by using systematically Cauchy-Schwarz inequality and the global Lipschitz character of f . By plugging the factor ε inside the integrals, coming back to the time scale $t \in [0, T]$ and then summing up with respect to $0 \leq j \leq k - 1$, we can bound the error term in (46) as

$$2C_R \varepsilon \sum_{j=0}^{k-1} \left(\int_{Tj/k}^{T(j+1)/k} (b_\varepsilon^T z)^2 dt \right)^{1/2} \left(\int_{Tj/k}^{T(j+1)/k} \|y\|^2 dt \right)^{1/2},$$

where C_R is a positive constant only depending on R . This is then trivially smaller than

$$C_R \varepsilon \left(\int_0^T (b_\varepsilon^T z)^2 dt + \int_0^T \|y\|^2 dt \right). \quad (68)$$

All the other error terms $\varepsilon TO(1)$ can be bounded in a similar way together with the fact that there exists some positive constant C_R only depending on R , for every $0 \leq j \leq k - 1$,

$$\|y_i\| \leq C_R \left(\int_0^1 \|y\|^2 dv \right)^{1/2}, \quad \|z_i\|^2 \leq C_R \left(\int_0^1 (b_1^T z)^2 dv + \int_0^1 \|y\|^2 dv \right).$$

This follows simply from the left part of (67).

On the other hand, it is immediate that one can improve the estimates in (54) and (55) to derive that $M_{\varepsilon,j}^1(v) + M_{\varepsilon,j}^1(v)$ are upper bounded by $-C_R \|y_j\|^2$, which implies that for $0 \leq j \leq k-1$, one has

$$M_{\varepsilon,j} \leq -C_R \|y_j\|^2,$$

for some positive constant C_R only depending on R . Hence the quantity M_ε defined in (51) is upper bounded as

$$M_\varepsilon \leq -C_R \varepsilon \sum_{j=0}^{k-1} \|y_j\|^2.$$

After coming back to the time scale $t \in [0, T]$, one easily recognizes that the right-hand side of the above inequality is a Riemann sum of the function $t \mapsto \|y(t)\|^2$. Since it has a derivative bounded by some positive constant only depending on R , one gets that

$$M_\varepsilon \leq -C_R \int_0^T \|y\|^2 dt.$$

Gathering all the above estimates and eventually diminishing ε_0 finally yields that

$$K_\varepsilon \leq C_R^1 \varepsilon \int_0^T (b_\varepsilon^T z)^2 dt - C_R^2 \int_0^T \|y\|^2 dt, \quad (69)$$

for some positive constants C_R^1, C_R^2 only depending on R .

On the other hand, one has by using (65) that there exists a positive constant $C_R > 0$ such that

$$L_\varepsilon = - \int_0^T \frac{S(\|z\|)}{\|z\|} \frac{\sigma(b_\varepsilon^T z)}{b_\varepsilon^T z} (b_\varepsilon^T z)^2 dt \leq -C_R \int_0^T (b_\varepsilon^T z)^2 dt. \quad (70)$$

By collecting (69) and (70), we deduce (66). □

The final step of the argument takes advantage of the previous estimate. We obtain from the above that the L^2 -norm of $b_\varepsilon^T z$ over \mathbb{R}_+ is finite and, since the time derivative of $b_\varepsilon^T z$ is bounded (with a constant depending on ε), we deduce that $b_\varepsilon^T z$ tends to zero at t tends to infinity by Barbalat Lemma. Recall now that $b_\varepsilon^T z = K_\varepsilon^T x$ as the latter term appears in (S_ε) defined in (9) with the choice of K_ε made in (10). Then we have that $K_\varepsilon^T x$ tends to zero at t tends to infinity. One can therefore rewrite (S_ε) as $\dot{x} = A_\varepsilon x + f(t)b$, where

$$A_\varepsilon = J_2\left(\frac{2\pi}{\varepsilon}\right) - bb^T, \quad b = \begin{pmatrix} 0 \\ e_2 \end{pmatrix}, \quad f(t) = K_\varepsilon^T x - \sigma(K_\varepsilon^T x). \quad (71)$$

Since A_ε is Hurwitz and f tends to zero at t tends to infinity, one concludes that any trajectory x of (S_ε) converges to zero as t tends to infinity. Since $f(t)$ is actually a $o(\|x\|)$, as $\|x\|$ tends to zero, one also gets that (S_ε) is locally exponentially stable with respect to the origin.

The proof of Proposition 7 is complete.

4 Appendix

In this section, we collect technical results used throughout the paper.

We start by providing an argument for the items in Remark 2. Item (c1) is immediate. For Item (c2), it is enough to prove the statements for $\xi > 0$ and conclude by continuity. The first part of that item follows from the fact that σ' is decreasing over \mathbb{R}_+ and the inequality

$$\sigma(\xi) = \int_0^\xi \sigma'(v)dv \geq \xi\sigma'(\xi), \quad \forall \xi \geq 0. \quad (72)$$

The second statement is a consequence of the following equality

$$\left(\frac{\sigma(\xi)}{\xi}\right)' = \frac{\xi\sigma'(\xi) - \sigma(\xi)}{\xi^2}, \quad \forall \xi \neq 0,$$

and (72). As for Item (c3), notice that for every $\xi \neq 0$,

$$\sigma'(\xi) \leq 2\frac{\sigma(\xi)}{\xi} - \frac{\sigma(\xi/2)}{\xi/2} = \frac{\sigma(\xi) - \sigma(\xi/2)}{\xi/2} = \int_{\xi/2}^\xi \sigma'(s) ds \leq \sigma'(\xi/2).$$

By letting ξ tend to zero, one gets that

$$\limsup_{\xi \rightarrow 0} \sigma'(\xi) \leq \sigma'(0) \leq \liminf_{\xi \rightarrow 0} \sigma'(\xi),$$

hence the last part of Item (c3).

We now prove the following results on the modified saturation function.

Proposition 20 (Modified saturation function associated with a saturation function σ). *The modified saturation function $S : \mathbb{R} \rightarrow \mathbb{R}$ associated with the saturation function σ is the function defined on \mathbb{R} by*

$$S(\xi) = \int_0^1 s_{2\pi v} \sigma(\xi s_{2\pi v}) dv = \frac{2}{\pi} \int_0^{\pi/2} s_v \sigma(\xi s_v) dv. \quad (73)$$

Then S has the following properties.

(S1) *One has the following expressions for S' :*

$$S'(\xi) = \frac{2}{\pi} \int_0^{\pi/2} \sigma'(\xi s_v) s_v^2 dv, \quad (74)$$

and, for $\xi \neq 0$,

$$S'(\xi) = \frac{2}{\pi} \int_0^{\pi/2} \frac{\sigma(\xi) - \sigma(\xi s_v)}{\xi(1 - s_v)} h(v) dv, \quad (75)$$

where $h : [0, \pi/2] \rightarrow \mathbb{R}_+$ is the continuous function defined by $g(v) = \frac{1-s_v}{c_v^2} s_v (1+c_v^2)$ for $v \in [0, \pi/2)$ and $h(\pi/2) = 1/2$. As a consequence, S is a saturation function of class C^1 with $S_\infty = \sigma_\infty/2$, $S'(0) = \sigma'(0)/2$ and $S' > 0$;

(S2) there exists $C_2 > 0$ such that, for every $\xi \in \mathbb{R}$ and $M \geq 1$, one has

$$S'(M\xi) \geq \frac{C_2}{M^3} S'(\xi). \quad \xi \in \mathbb{R}. \quad (76)$$

Proof. The definition of S shows that it is positive and bounded. Eq. (74) is immediate, which implies that $S'(0) = \sigma'(0)/2$, S' is bounded, positive and non increasing. Moreover, (75) implies that S' is continuous for $\xi \neq 0$. It remains to show the continuity of S' at $\xi = 0$. For that purpose, note that from (74) one has

$$S'(\xi) - S'(0) = \frac{2}{\pi} \int_0^{\pi/2} \left(\sigma'(\xi s_v) - \sigma'(0) \right) dv.$$

Continuity of σ' at $\xi = 0$ immediately implies continuity of S' at $\xi = 0$.

As for Item (S2), one can assume $\xi > 0$ with no loss of generality. From (74) and the facts that σ' is decreasing and $2v/\pi \leq s_v \leq v$ for $v \in [0, 2\pi]$, one deduces that there exists two universal constants $C_1, C_2 > 0$ such that,

$$\frac{C_1}{\xi^3} \int_0^\xi \sigma'(v)v^2 dv \leq \frac{C_1}{\xi^3} \int_0^{\pi\xi/2} \sigma'(v)v^2 dv \leq S'(\xi) \leq \frac{C_2}{\xi^3} \int_0^\xi \sigma'(v)v^2 dv, \quad \forall \xi > 0.$$

For $\xi > 0$, set $H(\xi) = \int_0^\xi \sigma'(v)v^2 dv$, which is an increasing function. For $M \geq 1$, one gets

$$M^3 \xi^3 S'(M\xi) \geq C_1 H(M\xi) \geq C_1 H(\xi) \geq \frac{C_1}{C_2} \xi^3 S'(\xi),$$

from which the conclusion follows. □

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