



**HAL**  
open science

# Analytic and conformal scattering in general relativity

Jean-Philippe Nicolas

► **To cite this version:**

| Jean-Philippe Nicolas. Analytic and conformal scattering in general relativity. 2023. hal-04323480

**HAL Id: hal-04323480**

**<https://hal.science/hal-04323480v1>**

Preprint submitted on 5 Dec 2023

**HAL** is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

Public Domain

# Analytic and conformal scattering in general relativity

Jean-Philippe NICOLAS<sup>1</sup>

This paper is about two important trends of scattering theory in general relativity: time dependent spectral analytic scattering and conformal scattering. The former was initiated by Jonathan Dimock and Bernard Kay in the mid 1980's and is based on spectral and functional analysis. The latter was proposed by Roger Penrose in 1965 and then constructed for the first time by Gerard Friedlander in 1980 by putting together Penrose's conformal method and another analytic approach to scattering: the Lax-Phillips theory due to Peter Lax and Ralph Phillips. We shall review the history of the two approaches and explain their general principles. We shall also explore an important question: "can the tools of one approach be used to obtain a complete construction in the other?".

**Keywords.** Scattering theory, general relativity, black holes, conformal compactification, null infinity.

**Mathematics subject classification.** 35B40, 35L05, 35Q75, 83C57

## Contents

<b>1</b>	<b>Introduction</b>	<b>2</b>
<b>2</b>	<b>Spectral analytic time dependent scattering theory</b>	<b>4</b>
2.1	The one-space scattering . . . . .	5
2.1.1	Abstract principle . . . . .	5
2.1.2	The Weyl equation with short-range potential . . . . .	6
2.1.3	The basic tool: Cook's method . . . . .	8
2.1.4	The essential role of the Huygens principle . . . . .	8
2.1.5	A weak version of the Huygens principle . . . . .	9
2.2	The two-space scattering . . . . .	11
2.2.1	General principles . . . . .	11
2.2.2	The wave equation on the Schwarzschild metric . . . . .	12
<b>3</b>	<b>Another spectral analytic approach to scattering: the Lax-Phillips theory</b>	<b>15</b>
3.1	Finite dimensional case . . . . .	16
3.2	The wave equation . . . . .	16

---

<sup>1</sup>LMBA, UMR CNRS 6205, Université de Brest, 6 avenue Victor Le Gorgeu, 29200 Brest, France, jean-philippe.nicolas@univ-brest.fr

<b>4</b>	<b>Conformal scattering theory</b>	<b>18</b>
4.1	Main principle and general structure of the construction . . . . .	18
4.2	The wave equation on the Schwarzschild spacetime . . . . .	19
4.3	Strategy for a singular spacelike infinity . . . . .	23
4.4	Dealing with timelike infinities . . . . .	24
4.5	Freedom of choice of observer . . . . .	25
<b>5</b>	<b>Transversal constructions</b>	<b>26</b>

# 1 Introduction

A complete scattering theory provides precise information on the asymptotic behaviour of solutions to a field equation for large times and, in addition, proves that the knowledge of the asymptotic behaviour is equivalent to that of the solution itself. It entails the existence of a scattering operator, that maps the asymptotic behaviour in the past to the one in the future and is an isomorphism between well-chosen function spaces. We shall consider two important approaches to scattering theory in general relativity: spectral analytic time dependent scattering and conformal scattering. Our purpose is to present the essential features of each approach illustrated on simple examples, then to explore the links between them. We shall focus on linear massless fields on Ricci-flat spacetimes, in which case the conformal boundary is made up of null hypersurfaces.

Time dependent scattering theory in general relativity has been a very active research area for several decades now. It was initiated in the mid 1980's by the works of John Dimock and Bernard Kay on the scattering of scalar fields on the Schwarzschild metric [21, 22, 23, 24]. Then Alain Bachelot and his group developed systematically the analysis of scattering on the Schwarzschild metric, these efforts culminating in a rigorous proof of the Hawking effect in this framework [1, 2, 3, 4, 5, 6, 45, 46, 52]. In the early 2000's, the research moved towards the Kerr metric and the techniques used changed and got more involved, with Mourre theory and positive commutator methods, a detailed account of which can be found in the book by Yan Dereziński and Christian Gérard [20]. Mourre theory is inspired by quantum mechanics in Heisenberg's formalism; the idea is to construct, for the hamiltonian of the evolution equation under consideration, a so-called conjugate operator that resembles a position operator or a generator of dilations and is increasing along the evolution. This last property, derived locally in energy by means of pseudo-differential cut-offs, entails among other things a weak version of the Huygens principle (a minimal velocity estimate) that is essential for the construction of the scattering theory. There has been a very important activity in this direction, starting with works by Dietrich Häfner, at first on his own and then with the author, [33, 36, 34, 30, 31, 32, 16]. Recently, Nicolas Besset studied the scattering of charged scalar fields on charged backgrounds using resonance methods [13, 14, 15]. Moreover, the time dependent scattering of Dirac fields in the interior dynamical regions of black hole spacetimes such as Reissner-Nordström and Kerr has been investigated in the last few years by Dietrich Häfner, Mokdad Mokdad and the author [35] as well as Mokdad Mokdad with Rajai

Nasser and Milos Provcic [50, 51]. The main principle of time dependent scattering theory is to compare a physical evolution with a simplified one in order to show that a field obeying the physical evolution approaches asymptotically a solution to the simplified comparison equation. In adequate situations (stationary spacetimes for instance) this comparison can be performed using spectral analysis; this is the case of most of the works cited above and this is the trend that we shall describe in this paper.

There was also earlier work by Lax and Phillips, in particular their remarkable book published in 1967 simply entitled “scattering theory” [40], that was already an example of scattering in relativity, but in the flat case. The topic was a scattering theory for the wave equation in the presence of an obstacle. The approach was different from the trend we first mentioned. It is not a time dependent approach but uses a decomposition of the field on generalised eigenfunctions of the Hamiltonian of the wave equation. The part of their work that concerns the behaviour of the wave away from the obstacle is in itself very interesting and was to provide a major incentive for development of the first conformal scattering theory.

Conformal scattering was originally proposed for the full gravity by Roger Penrose in 1965 in his beautiful paper on the asymptotic behaviour of zero rest-mass fields [55]. The idea was to use the conformal method to set data for the Einstein equations on past null infinity and to evolve the solution right up to future null infinity. The first actual construction was due to Gerard Friedlander in 1980 [28], for the wave equation on a static asymptotically flat spacetime with a complete smooth compactification. There are no such solutions to the Einstein vacuum equations apart from Minkowski spacetime, but Friedlander was not assuming that his spacetime was Einstein. His motivation came from his earlier study of radiation fields [25, 26, 27] — that are the restrictions at the conformal boundary of the rescaled field and can be seen, equivalently, as the leading order of the asymptotic behaviour of the physical field along outgoing null geodesics — and from the Lax-Phillips theory in which the main object is a “translation representer” that is constructed by stationary spectral methods but is a posteriori re-interpreted as being exactly the radiation field. His idea was then to use the conformal technique to recover the translation representer without using spectral analysis. He assumed the spacetime to be static in order to ensure that the radiation field behaves as a translation representer (we shall see later on, that this behaviour requires merely stationarity). His idea was then extended to nonlinear equations by John Baez and his collaborators but still on static backgrounds [7, 8, 9, 10, 11]. The fundamental idea of conformal scattering is what was initially proposed by Penrose, that a complete scattering theory can be understood exactly as the well-posedness of the Goursat problem at the conformal boundary of the spacetime. This makes sense naturally for conformally invariant equations. For gravity, one would expect to use Helmut Friedrich’s conformal Einstein equations (see for example [29]). In 2004 Lionel Mason and the author [42] used Lars Hörmander’s simple resolution of the Goursat problem [37] based on energy estimates and weak convergence methods and adapted it to construct a conformal scattering theory for zero rest-mass fields of spin 0, 1/2 and 1 on asymptotically simple spacetimes with a smooth timelike infinity. Such spacetimes have an almost complete conformal compactification, the only singularity in the conformal boundary being at spacelike infinity, and they can be generically time dependent. This construction has been extended to a nonlinear Klein-Gordon equation by Jérémie Joudioux [38, 39], to Maxwell

fields on Schwarzschild-de Sitter-type spacetimes and to Dirac fields inside a Reissner-Nordström spacetime by Mokdad Mokdad [49] and to Maxwell potentials on asymptotically flat spacetimes by Grigalius Taujanskas and the author [54].

The paper is organised as follows. In Section 2 we present the principles of spectral analytic time dependent scattering, detailing the cases with one and two Hilbert spaces with explicit examples. Section 3 is devoted to a brief description of the Lax-Phillips theory in the case where there is no obstacle. Conformal scattering is then the object of Section 4 with again a description of the main ideas and some concrete examples. Finally, in Section 5, we explore the analogies of structure between the various approaches and describe some transversal constructions, i.e. methods that use the tools of one approach to infer the results of another.

## 2 Spectral analytic time dependent scattering theory

The principle of time dependent scattering theory is to describe the asymptotic behaviour of a field in a given asymptotic region as a solution to a simplified equation. Let us first consider the very simple example of the wave equation in 1 + 1 dimensions

$$\partial_t^2 \phi - \partial_x^2 \phi = 0, \quad (2.1)$$

as an evolution equation on the real line with respect to the time parameter  $t$ . In this case we have two asymptotic regions corresponding to  $x \rightarrow \pm\infty$  and we may wish to study the asymptotic behaviour of solutions in the distant past or future. We know from D'Alembert [18] that the solutions to (2.1) on  $\mathbb{R}_{(t,x)}^2$  are the functions (this extends trivially to distributions) of the form

$$\phi = F(x + t) + G(x - t). \quad (2.2)$$

The functions  $F(x + t)$  and  $G(x - t)$  satisfy the simpler equations

$$(\partial_t - \partial_x)(F(x + t)) = 0, \quad (2.3)$$

$$(\partial_t + \partial_x)(G(x - t)) = 0. \quad (2.4)$$

The asymptotic behaviour of (2.2) at  $-\infty$  as  $t \rightarrow +\infty$  and at  $+\infty$  as  $t \rightarrow -\infty$  is described by the function  $F(x + t)$  solution to the simplified equation (2.3). Similarly, the asymptotic behaviour of (2.2) at  $+\infty$  as  $t \rightarrow +\infty$  and at  $-\infty$  as  $t \rightarrow -\infty$  is described by the function  $G(x - t)$  solution to the simplified equation (2.4). The initial data for solutions to the simplified equations, i.e. the functions  $F(x)$  and  $G(x)$ , are called the scattering data for (2.2). More precisely,  $F$  is the future scattering datum at  $-\infty$  as well as the past scattering datum at  $+\infty$  and  $G$  is the future scattering datum at  $+\infty$  as well as the past scattering datum at  $-\infty$ . The knowledge of the solution (2.2) is equivalent to that of the scattering data  $F$  and  $G$ .

Although this is a completely trivial example, it shows clearly the basic features of a complete time-dependent scattering theory, that we shall recognise in the simplest situation where only one function space is involved.

## 2.1 The one-space scattering

We start by considering a time-dependent scattering theory obtained by comparing two operators that are self-adjoint on the same Hilbert space. In the next subsection, we explain the main ingredients of the theory in an abstract setting; this is classic material that can be found in many advanced functional analysis books, for instance M. Reed and B. Simon [57], Volumes 1 and 3. Then we move on to a concrete example of such a situation, for which we shall explore more precisely the methods used to construct the scattering theory.

### 2.1.1 Abstract principle

Let us consider a separable Hilbert space  $\mathcal{H}$ , two self-adjoint (usually unbounded) operators  $H_0$  and  $H$ , as well as the associated evolution equations

$$\frac{d\phi}{dt} + iH\phi = 0, \quad (2.5)$$

$$\frac{d\psi}{dt} + iH_0\psi = 0, \quad (2.6)$$

where the unknowns  $\phi$  and  $\psi$  are functions on  $\mathbb{R}$  with values in  $\mathcal{H}$ . The operator  $H$  will be referred to as the full dynamics and  $H_0$  as the simplified dynamics.

We denote by  $D(H)$  and  $D(H_0)$  the natural domains of the two operators, they are dense in  $\mathcal{H}$ . The propagators for both equations,  $e^{-itH}$  and  $e^{-itH_0}$ , are strongly continuous one-parameter groups of unitary operators on  $\mathcal{H}$ , which means that they satisfy the following properties (expressed here only for  $e^{-itH}$ )

1. for any  $t \in \mathbb{R}$   $e^{-itH}$  is a unitary operator on  $\mathcal{H}$ ,
2. for all  $t, s \in \mathbb{R}$ ,  $e^{-i(t+s)H} = e^{-itH} e^{-isH}$  and  $e^{-i0H}$  is the identity of  $\mathcal{H}$ ,
3. for any  $\phi_0 \in \mathcal{H}$ ,  $e^{-itH} \phi_0 \in \mathcal{C}^0(\mathbb{R}_t; \mathcal{H})$ ,
4. for any  $\phi_0 \in D(H)$ ,  $e^{-itH} \phi_0$  is differentiable on  $\mathbb{R}$  with values in  $D(H)$  and we have

$$\frac{d}{dt}(e^{-itH} \phi_0) = -iH e^{-itH} \phi_0 \quad \forall t \in \mathbb{R}.$$

The way one usually implements the comparison of the two dynamics for large times is to define the so-called wave operators.

**Definition 2.1.** *The direct and inverse wave operators, if they exist, are defined as the following strong limits of unitary operators*

$$\begin{aligned} W^\pm &:= s - \lim_{t \rightarrow \pm\infty} e^{itH} e^{-itH_0}, \\ \tilde{W}^\pm &:= s - \lim_{t \rightarrow \pm\infty} e^{itH_0} e^{-itH}. \end{aligned}$$

If in a certain sense the operators  $H$  and  $H_0$  are asymptotically close, then one can establish the following type of result

**Theorem 2.1.** *The direct and inverse wave operators exist, are isometries on  $\mathcal{H}$  and satisfy*

$$\tilde{W}^\pm = (W^\pm)^{-1} = (W^\pm)^*. \quad (2.7)$$

**Remark 2.1.** *Note that the framework we are presenting here is valid in a relatively simple case where the spectra of both  $H_0$  and  $H$  are purely absolutely continuous. When this is not the case, the definition of the wave operators requires to apply first a projector onto the absolutely continuous spectrum of the first operator.*

The existence of the direct and inverse wave operators is all there is to prove, their unitarity then follows automatically from the definition and the unitarity of the propagators. Equation (2.7) also follows directly since for  $t \in \mathbb{R}$

$$e^{itH} e^{-itH_0} e^{itH_0} e^{-itH} = \text{Id}_{\mathcal{H}}.$$

The existence of the inverse wave operators is usually referred to as ‘asymptotic completeness’.

### 2.1.2 The Weyl equation with short-range potential

We now move on to a slightly less simple case: the Weyl neutrino equation on Minkowski spacetime with a time-independent short-range skew potential

$$\partial_t \phi + \sigma^1 \partial_{x_1} \phi + \sigma^2 \partial_{x_2} \phi + \sigma^3 \partial_{x_3} \phi + iP(x)\phi = 0, \quad (2.8)$$

$$P \in C^\infty(\mathbb{R}^3; \mathcal{M}_2(\mathbb{C})), \quad {}^t \bar{P} = P, \quad P(x) = O(|x|^{-1-\varepsilon}), \quad \varepsilon > 0, \quad (2.9)$$

where  $x = (x_1, x_2, x_3)$  and

$$\sigma^1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \sigma^2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma^3 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix},$$

are the Pauli matrices<sup>2</sup>. We consider (2.8) as an evolution equation on  $\mathbb{R}^3$ . There is only one asymptotic region and we wish to compare the behaviour for large times of solutions to (2.8) with that of solutions to the free Weyl equation

$$\partial_t \psi + \sigma^1 \partial_{x_1} \psi + \sigma^2 \partial_{x_2} \psi + \sigma^3 \partial_{x_3} \psi = 0. \quad (2.10)$$

To do so, we can express the two equations in the form of Schrödinger equations: for (2.10), we get

$$\frac{\partial \psi}{\partial t} = -iH_0 \psi, \quad H_0 = -i\sigma^1 \partial_{x_1} \psi - i\sigma^2 \partial_{x_2} \psi - i\sigma^3 \partial_{x_3} \psi \quad (2.11)$$

---

<sup>2</sup>It would probably be much better to express the equation in 2-spinor form, but for our present purpose, it is more convenient to express it through a choice of spin-frame as an equation on a complex vector.

and for (2.8),

$$\frac{\partial \phi}{\partial t} = -iH\phi, \quad H = H_0 + P(x). \quad (2.12)$$

A usual and fairly natural way of expressing the property that solutions to (2.8) behave for large times as solutions to (2.10) is to show that direct and inverse wave operators can be constructed. The operators  $H$  and  $H_0$  are self-adjoint on  $L^2(\mathbb{R}^3; \mathbb{C}^2)$  with domain  $H^1(\mathbb{R}^3; \mathbb{C}^2)$ . Given data  $\phi_0 \in L^2(\mathbb{R}^3; \mathbb{C}^2)$ , (resp.  $\psi_0 \in L^2(\mathbb{R}^3; \mathbb{C}^2)$ ),  $\phi(t) := e^{-itH}\phi_0$  (resp.  $\psi(t) := e^{-itH_0}\psi_0$ ) is the unique solution to (2.8) (resp. (2.10)) such that  $\phi(0) = \phi_0$  (resp.  $\psi(0) = \psi_0$ ). We have the following theorem.

**Theorem 2.2.** *The direct and inverse wave operators*

$$W^\pm := s - \lim_{t \rightarrow \pm\infty} e^{itH} e^{-itH_0}, \quad (2.13)$$

$$\tilde{W}^\pm := s - \lim_{t \rightarrow \pm\infty} e^{itH_0} e^{-itH}, \quad (2.14)$$

are well-defined and are isometries of  $L^2(\mathbb{R}^3; \mathbb{C}^2)$ . Moreover, we have

$$\tilde{W}^\pm = (W^\pm)^* = (W^\pm)^{-1}. \quad (2.15)$$

The result is then easily interpreted as describing the asymptotic behaviour of solutions to (2.8). Indeed, for  $\phi_0 \in L^2(\mathbb{R}^3; \mathbb{C}^2)$ , put

$$\psi_0^\pm := \tilde{W}^\pm \phi_0. \quad (2.16)$$

Then we have

$$\lim_{t \rightarrow \pm\infty} \|e^{-itH}\phi_0 - e^{-itH_0}\psi_0^\pm\| = \lim_{t \rightarrow \pm\infty} \|e^{itH_0}e^{-itH}\phi_0 - \psi_0^\pm\| = \|\tilde{W}^\pm \phi_0 - \psi_0^\pm\| = 0, \quad (2.17)$$

where we have used the unitarity of  $e^{-itH_0}$  and its group properties. Equality (2.17) shows that solutions to (2.8) behave for large times as solutions to the free equation (2.10). The data  $\psi_0^\pm$  for the free equation defined by (2.16) are called the *past and future scattering data*.

**Definition 2.2.** *The scattering operator is defined by*

$$S := \tilde{W}^+ W^-. \quad (2.18)$$

*It summarises the whole evolution of the field by transforming the past scattering data into the future scattering data.*

The proof of Theorem 2.2 is entirely based on a simple argument called Cook's method. We shall now describe Cook's method and then explain how we can use it to prove the theorem.

### 2.1.3 The basic tool: Cook's method

Cook's method is quite systematically used in the construction of wave operators. It is based on a sufficient condition for the existence of a limit at infinity of a  $\mathcal{C}^1$  function on  $\mathbb{R}$ : the integrability of its derivative. We shall simply apply this condition in our case and derive a condition that corresponds to Cook's method. For the general result, see Theorem XI.4 from Reed and Simon's Volume 3 [57].

Let  $\psi_0 \in \mathcal{C}_0^\infty(\mathbb{R}^3; \mathbb{C}^2)$  which is a dense subspace of  $H^1(\mathbb{R}^3; \mathbb{C}^2) = D(H_0) = D(H)$  and of  $L^2(\mathbb{R}^3; \mathbb{C}^2)$ . We put  $\Psi(t) := e^{itH}e^{-itH_0}\psi_0$ . Then  $\Psi \in \mathcal{C}^k(\mathbb{R}_t; H^s(\mathbb{R}^3; \mathbb{C}^2))$  for all  $k, s \in \mathbb{N}$ , in particular

$$\Psi \in \mathcal{C}^1(\mathbb{R}_t; L^2(\mathbb{R}^3; \mathbb{C}^2))$$

and

$$\Psi'(t) = ie^{itH}(H - H_0)e^{-itH_0}\psi_0 = ie^{itH}P(x)e^{-itH_0}\psi_0. \quad (2.19)$$

Similarly, let  $\phi_0 \in \mathcal{C}_0^\infty(\mathbb{R}^3; \mathbb{C}^2)$  and put  $\Phi(t) := e^{itH_0}e^{-itH}\phi_0$ . Then

$$\Phi \in \mathcal{C}^1(\mathbb{R}_t; L^2(\mathbb{R}^3; \mathbb{C}^2))$$

and

$$\Phi'(t) = ie^{itH_0}(H_0 - H)e^{-itH}\phi_0 = -ie^{itH_0}P(x)e^{-itH}\phi_0. \quad (2.20)$$

Equations (2.19) and (2.20) show that if we can prove that  $P(x)e^{-itH_0}\psi_0$  and  $P(x)e^{-itH}\phi_0$  are in  $L^1(\mathbb{R}; L^2(\mathbb{R}^3; \mathbb{C}^2))$ , then we can infer the existence of the strong limits (2.13) and (2.14) for  $\psi_0, \phi_0 \in H^1(\mathbb{R}^3; \mathbb{C}^2)$ . This in turn entails that the direct and inverse wave operators are well defined using the boundedness of the propagators and the density of  $\mathcal{C}_0^\infty(\mathbb{R}^3; \mathbb{C}^2)$  in  $L^2(\mathbb{R}^3; \mathbb{C}^2)$ .

### 2.1.4 The essential role of the Huygens principle

We obtain first the integrability of  $\Psi(t)$  which will yield the existence of the direct wave operators. This is easier than the existence of the inverse wave operators because the free equation satisfies the Huygens principle. More precisely, given  $\psi_0 \in \mathcal{C}_0^\infty(\mathbb{R}^3; \mathbb{C}^2)$ , there exists  $R > 0$  such that  $\psi_0 \equiv 0$  in  $\{|x| \geq R\}$  and this entails that

$$e^{-itH_0}\psi_0 \equiv 0 \text{ for } |x| \leq |t| - R. \quad (2.21)$$

Consequently, for  $|t| > R$ , we have

$$\begin{aligned} \|P(x)e^{-itH_0}\psi_0\|_{L^2} &\leq \|P(x)\|_{L^\infty(\{|x| > |t| - R\})} \|e^{-itH_0}\psi_0\|_{L^2} \\ &= \|P(x)\|_{L^\infty(\{|x| > |t| - R\})} \|\psi_0\|_{L^2} \end{aligned}$$

and thanks to the assumptions on  $P$ ,  $\|P(x)\|_{L^\infty(\{|x| > |t| - R\})} = O(|t|^{-1-\varepsilon})$  and this proves that  $P(x)e^{-itH_0}\psi_0 \in L^1(\mathbb{R}_t; L^2(\mathbb{R}^3; \mathbb{C}^2))$ .

### 2.1.5 A weak version of the Huygens principle

Because of the presence of the potential  $P$ , Equation (2.8) does not satisfy the Huygens principle. In order to be able to transform the decay in space of  $P$  into the decay in time of  $P(x)e^{-itH}\phi_0$ , we must find an approximate version of the Huygens principle.

Trace-class perturbation techniques can be used to establish the existence of the inverse wave operators. The idea is essentially to understand the integrability on the spectral level, which is in a sense akin to a Fourier transform approach. The case of a potential  $V \in L^1(\mathbb{R}^3)$ , which is a little more restrictive than the situation we are considering here, is treated by such methods in the book by Baumgärtel and Wollenberg [12] Section 16.3.2. In the fully general short range case, a proof of the asymptotic completeness by the non-stationary phase method can be found in Thaller's monograph [58] Section 8.3.1. The drawback of both approaches is that they do not provide an intuitive geometrical understanding of a mechanism that replaces the Huygens principle. Mourre theory offers a different spectral approach to scattering that is much more technical but also much more flexible and powerful than, say, trace-class methods. One of its many advantages is an explicit version of a weak Huygens principle, the so-called minimal velocity estimate. In our case, we have the following result that is due to Daudé ([19] Proposition 4.1) and is valid for long-range potentials and in particular applies in the short-range case.

**Proposition 2.1.** *For any  $\delta \in ]0, 1[$  and any  $\chi \in C_0^\infty(\mathbb{R})$  such that  $\chi \equiv 0$  in a neighbourhood of 0, we have for all  $\phi \in \mathcal{H}$*

$$\int_1^{+\infty} \left\| \mathbf{1}_{[0, \delta[} \left( \frac{|x|}{t} \right) \chi(H) e^{-itH} \phi \right\|_{\mathcal{H}}^2 \frac{dt}{t} \leq C \|\phi\|_{\mathcal{H}}^2.$$

Note that the integral in time of the squared norm is an integral on the whole spacetime of a quadratic form in  $\psi$ . The cut-off function  $\mathbf{1}_{[0, \delta[} \left( \frac{|x|}{t} \right)$  simply restricts the integral to the interior of the narrow cone  $\left\{ \frac{|x|}{t} < \delta \right\}$  and the operator  $\chi(H)$  is a localisation in energy. The constant  $C$  depends only on  $\chi$  and on  $\delta$ . Since the function  $1/t$ , that is the measure density in the integral, is not integrable near infinity, the above estimate says in a weak sense that the integrand (without the density) tends to zero as  $t \rightarrow +\infty$ . In fact one can prove a stronger convergence, namely

$$s - \lim_{t \rightarrow +\infty} \mathbf{1}_{[0, 1 - \delta[} \left( \frac{|x|}{t} \right) \chi(H) e^{-itH} = 0$$

which is also proven by Daudé in the same paper (and in the same proposition). So although we do not have the strong Huygens principle, we see that at least locally in energy, the solution asymptotically vanishes within any narrow cone  $\left\{ \frac{|x|}{t} < \delta \right\}$ . This is enough to establish the existence of inverse wave operators using Cook's method on smooth and compactly supported data.

The proof of the above proposition is quite technically involved. However, in order to get an idea of the method, we can consider the simple case of a 1 + 1-dimensional version of the Weyl equation with a short-range potential, for which it is possible to obtain a minimal velocity

estimate using similar arguments, with neither energy cut-offs nor conjugate operator (in this case, a natural conjugate operator would be  $A = \gamma x$ ). The equation is

$$\partial_t \phi + iH\phi = 0, \quad H = -i\gamma \partial_x + V\mathbf{1}_2 \quad (2.22)$$

where

$$\gamma = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \mathbf{1}_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad V(x) = O(|x|^{-1-\alpha}), \quad \alpha > 0.$$

We consider a function  $f \in \mathcal{C}_0^\infty(\mathbb{R})$  such that, for  $0 < \varepsilon < 1$  given,  $\text{supp} f \subset [-1 + \varepsilon, 1 - \varepsilon]$ , and we define the function  $F$  by

$$F(t) = \int_{-\infty}^t (f(s))^2 ds.$$

Next, we introduce the observable

$$\Phi(t) := F\left(\gamma \frac{x}{t}\right).$$

Then

$$\frac{d}{dt} (e^{itH} \Phi(t) e^{-itH}) = e^{itH} D\Phi(t) e^{-itH}$$

where  $D\Phi(t)$  is the Heisenberg derivative of  $\Phi$ :

$$D\Phi(t) = \frac{d\Phi}{dt} + i[H, \Phi].$$

This can be expressed explicitly as follows

$$\begin{aligned} D\Phi(t) &= \frac{d\Phi}{dt} + \gamma \frac{d\Phi}{dx} \\ &= \gamma \frac{1}{t} \left(\gamma - \frac{x}{t}\right) f^2\left(\gamma \frac{x}{t}\right) \\ &= \frac{1}{t} \left(1 - \gamma \frac{x}{t}\right) f^2\left(\gamma \frac{x}{t}\right). \end{aligned}$$

Given the support of  $f$ , it follows that as a quadratic form,

$$D\Phi(t) \geq \frac{\varepsilon}{t} f^2\left(\gamma \frac{x}{t}\right).$$

Using this inequality, we obtain for  $T > 0$

$$\begin{aligned} \int_1^T \left\| f\left(\gamma \frac{x}{t}\right) e^{-itH} \phi \right\|^2 \frac{dt}{t} &= \int_1^T \left\langle f^2\left(\gamma \frac{x}{t}\right) e^{-itH} \phi, e^{-itH} \phi \right\rangle \frac{dt}{t} \\ &\leq \frac{1}{\varepsilon} \int_1^T \left\langle D\Phi(t) e^{-itH} \phi, e^{-itH} \phi \right\rangle dt \end{aligned}$$

and

$$\begin{aligned} \int_1^T \langle e^{itH} D\Phi(t) e^{-itH} \phi, \phi \rangle dt &= \frac{d}{dt} \int_1^T \langle e^{itH} \Phi(t) e^{-itH} \phi, \phi \rangle dt \\ &= (\langle e^{iTH} \Phi(T) e^{-iTH} \phi, \phi \rangle - \langle e^{iH} \Phi(1) e^{-iH} \phi, \phi \rangle). \end{aligned}$$

It follows that

$$\int_1^T \left\| f\left(\gamma \frac{x}{t}\right) e^{-itH} \phi \right\|^2 \frac{dt}{t} \leq \frac{2}{\varepsilon} \|\Phi\|_{L^\infty(\mathbb{R}; \mathcal{L}(\mathcal{H}))} \|\phi\|^2 \quad (2.23)$$

with

$$\|\Phi\|_{L^\infty(\mathbb{R}; \mathcal{L}(\mathcal{H}))} = \|F\|_{L^\infty} = \|f\|_{L^2}^2.$$

Taking the limit of (2.23) as  $T \rightarrow +\infty$  and using the fact that the bound depends only on the  $L^2$  norm of  $f$  and the localisation of its support, we obtain a minimal velocity estimate : for any  $\varepsilon \in ]0, 1[$  there exists  $C > 0$  such that for all  $\phi \in \mathcal{H}$ ,

$$\int_1^{+\infty} \left\| \mathbf{1}_{[-1+\varepsilon, 1-\varepsilon]} \left(\gamma \frac{x}{t}\right) e^{-itH} \phi \right\|^2 \frac{dt}{t} \leq C \|\phi\|^2. \quad (2.24)$$

## 2.2 The two-space scattering

In the previous paragraphs, we were working with two Hamiltonians that were self-adjoint on the same Hilbert space. This is far from being a general situation. Usually, the two Hamiltonians that we try to compare are self-adjoint on different Hilbert spaces and there may not even be a natural embedding of one space into the other. In this case, we need an identification operator that is a bounded operator from one space into the other that ideally realises an “approximate embedding”, by which we mean that at least for a large class of functions in the first space, the identification operator should transform them into elements of the second space with as little modification as possible, which, as we shall see on an example below, can still mean non trivial changes. We first formulate the general principles of two-Hilbert-space scattering, again this can be found in more details in many books (see for instance [57] Volumes 1 and 3), then describe explicitly the example of massless scalar fields on the Schwarzschild spacetime. We observe in particular that there may be several natural choices of comparison dynamics for a given equation and explore the consequences of these choices concerning the description of the asymptotic behaviour.

### 2.2.1 General principles

Let us consider two separable Hilbert spaces  $\mathcal{H}$  and  $\mathcal{H}_0$  and two (usually unbounded) operators  $H$  and  $H_0$  that are self-adjoint respectively on  $\mathcal{H}$  and  $\mathcal{H}_0$ . Let  $\mathcal{J} \in \mathcal{L}(\mathcal{H}_0; \mathcal{H})$  be an identification operator. We consider the same equations (2.5) and (2.6) as before and the associated propagators  $e^{-itH_0}$  and  $e^{-itH}$  that are now strongly continuous one parameter groups of

unitary operators on  $\mathcal{H}_0$  and  $\mathcal{H}$  respectively. We aim to define direct and inverse wave operators as follows

$$W^\pm := s - \lim_{t \rightarrow \pm\infty} e^{itH} \mathcal{J} e^{-itH_0}, \quad (2.25)$$

$$\tilde{W}^\pm := s - \lim_{t \rightarrow \pm\infty} e^{itH_0} \mathcal{J}^* e^{-itH}, \quad (2.26)$$

where the adjoint of  $\mathcal{J}$  is defined by

$$\langle \mathcal{J}\phi, \psi \rangle_{\mathcal{H}} = \langle \phi, \mathcal{J}^*\psi \rangle_{\mathcal{H}_0}, \quad \forall \phi \in \mathcal{H}_0, \quad \psi \in \mathcal{H}.$$

The interpretation of the asymptotic behaviour of solutions to the full equation will then depend of the nature of  $\mathcal{J}$ . We shall now consider an explicit example for which these abstract notions will become more concrete.

### 2.2.2 The wave equation on the Schwarzschild metric

The Schwarzschild metric in Schwarzschild coordinates  $(t, r, \omega)$  is expressed as

$$g = F(r)dt^2 - \frac{1}{F(r)}dr^2 - r^2d\omega^2, \quad F(r) = 1 - \frac{2M}{r}, \quad (2.27)$$

where  $d\omega^2$  is the Euclidean metric on the round 2-sphere. We work outside the black hole in the region  $\{r > 2M\}$ . The d'Alembertian operator associated with  $g$  has the form

$$\square_g = \frac{1}{F} \frac{\partial^2}{\partial t^2} - \frac{1}{r^2} \frac{\partial}{\partial r} r^2 F \frac{\partial}{\partial r} - \frac{1}{r^2} \Delta_{S^2}. \quad (2.28)$$

Introducing the Regge-Wheeler tortoise coordinate

$$r_* = r + 2M \log(r - 2M), \quad (2.29)$$

which satisfies

$$\frac{dr_*}{dr} = \frac{1}{F},$$

modifies the expression of the d'Alembertian into

$$\square_g = \frac{1}{F} \left( \frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial r_*^2} - \frac{2F}{r} \frac{\partial}{\partial r_*} \right) - \frac{1}{r^2} \Delta_{S^2}. \quad (2.30)$$

The first order term remaining in the d'Alembertian is not very pleasant in the sense that it has a slow fall-off at spacelike infinity, it is a long-range term as opposed to the short-range potential that we dealt with for the Weyl equation. However, a simple rescaling by  $r$  replaces it with a short range potential. Indeed, we have

$$r \square_g \frac{1}{r} = \frac{1}{F} \left( \frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial r_*^2} + \frac{FF'}{r} \right) - \frac{1}{r^2} \Delta_{S^2}, \quad (2.31)$$

where the left-hand side is to be understood as the composition of three operators, not as  $r$  times the d'Alembertian of  $1/r$ . Hence the wave equation outside a Schwarzschild black hole can be expressed on the unknown

$$\hat{\phi} = r\phi \quad (2.32)$$

as follows

$$\left( \frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial r_*^2} + F \left( \frac{-\Delta_{S^2}}{r^2} + \frac{F'}{r} \right) \right) \hat{\phi} = 0, \text{ on } \mathbb{R}_t \times \mathbb{R}_{r_*} \times S_\omega^2, \quad (2.33)$$

which is notably simpler than the equation satisfied by the unrescaled field  $\phi$ .

**Remark 2.2.** *This rescaling appears here as an innocent trick that allows to get rid of an annoying first order term in the equation, however the same rescaling will appear in the conformal scattering construction for the wave equation on the Schwarzschild metric as a consequence of the conformal compactification of the geometry. As we shall see shortly, this allows to compare the full equation with the wave equation on Minkowski spacetime in a simple manner and also to introduce an even simpler comparison dynamics that would not be accessible if the first order term remained.*

The Hamiltonian for Equation (2.33) is

$$H = i \begin{pmatrix} 0 & 1 \\ \partial_{r_*}^2 - F \left( \frac{-\Delta_{S^2}}{r^2} + \frac{F'}{r} \right) & 0 \end{pmatrix}. \quad (2.34)$$

There are two asymptotic regions, the horizon ( $r_* \rightarrow -\infty$ ) and infinity ( $r_* \rightarrow +\infty$ ), and we need to introduce a comparison dynamics for each. At the horizon, the function  $F$  falls-off exponentially fast in  $r_*$ . As a consequence a natural comparison dynamics in this region is given by

$$H_0 = i \begin{pmatrix} 0 & 1 \\ \partial_{r_*}^2 & 0 \end{pmatrix}. \quad (2.35)$$

At infinity, two approaches can be adopted. The first one consists in considering that the region is asymptotically flat and it is therefore natural to chose the Hamiltonian for the wave equation on Minkowski spacetime as a comparison dynamics. There is however a freedom in the manner in which we choose to glue Minkowski spacetime onto the Schwarzschild geometry in the neighbourhood of infinity; this can be expressed in a simple manner as a choice of coordinates on Minkowski spacetime in terms of the coordinates on Schwarzschild's spacetime. We choose to use the coordinates  $(t, r_*, \omega)$  as spherical coordinates in flat spacetime. We obtain the following comparison dynamics

$$H_1 = i \begin{pmatrix} 0 & 1 \\ \partial_{r_*}^2 + \frac{\Delta_{S^2}}{r_*^2} & 0 \end{pmatrix}. \quad (2.36)$$

Using the spherical symmetry, this can be understood as a short range perturbation of  $\partial_{r_*}^2$ . This is done by decomposing the field on the basis of spherical harmonics. For each given angular dependence the perturbation reduces to a short-range scalar potential. The eigenvalue  $l(l+1)$

of the spherical Laplacian appears as a factor in the potential, as a reminder of the fact that  $\Delta_{S^2}$  is not a bounded operator. The unitarity of the propagators allows to recombine the full solution and to infer the existence of the strong limit. Note that using  $r$  instead of  $r_*$ , however natural it may seem, would lead to an artificially long range perturbation that would make the scattering theory much more complicated and less meaningful. Another approach is to say that since the perturbation  $F(\frac{-\Delta_{S^2}}{r^2} + \frac{F'}{r})$  is short-range at infinity, we can choose  $H_0$  defined above in (2.35) as the comparison dynamics at infinity. Both the choices (2.35) and (2.36) are valid and they provide two different and in a sense inequivalent descriptions of the asymptotic behaviour of massless scalar fields on Minkowski spacetime. This is because the two Hamiltonians  $H_0$  and  $H_1$  are self-adjoint on two different Hilbert spaces, namely

$$\mathcal{H}_0 = \dot{H}^1(\mathbb{R}; L^2(S^2)) \times L^2(\mathbb{R} \times S^2), \quad (2.37)$$

$$\mathcal{H}_1 = \dot{H}^1(\mathbb{R}^3) \times L^2(\mathbb{R}^3), \quad (2.38)$$

that are the completion of  $\mathcal{C}_0^\infty(\mathbb{R} \times S^2)^2$  (resp.  $\mathcal{C}_0^\infty(\mathbb{R}^3)^2$ ) in the respective norms

$$\|(f, g)\|_{\mathcal{H}_0}^2 = \int_{\mathbb{R} \times S^2} (|\partial_{r_*} f(r_*, \omega)|^2 + |g(r_*, \omega)|^2) \, d\omega \, dr, \quad (2.39)$$

$$\|(f, g)\|_{\mathcal{H}_1}^2 = \int_{\mathbb{R}^3} \left( |\partial_{r_*} f(r_*, \omega)|^2 + \frac{1}{r_*^2} |\nabla_{S^2} f(r_*, \omega)|^2 + |g(r_*, \omega)|^2 \right) \, dr_* \, d^2\omega. \quad (2.40)$$

The corresponding wave operators take into account the existence of the two asymptotic regions and can be defined as follows for  $j = 0, 1$  depending on the comparison dynamics chosen at infinity

$$W^\pm := \left( s - \lim_{t \rightarrow \pm\infty} e^{itH} \overset{\leftarrow}{\mathcal{J}}_0 e^{-itH_0}, s - \lim_{t \rightarrow \pm\infty} e^{itH} \overset{\rightarrow}{\mathcal{J}}_j e^{-itH_j} \right), \quad (2.41)$$

$$\tilde{W}^\pm := s - \lim_{t \rightarrow \pm\infty} e^{itH_0} \overset{\leftarrow}{\mathcal{J}}_0^* e^{-itH} + s - \lim_{t \rightarrow \pm\infty} e^{itH_j} \overset{\rightarrow}{\mathcal{J}}_j^* e^{-itH}, \quad (2.42)$$

where the identification operators  $\mathcal{J}_0 \in \mathcal{L}(\mathcal{H}; \mathcal{H}_0)$  and  $\mathcal{J}_1 \in \mathcal{L}(\mathcal{H}; \mathcal{H}_1)$  are defined using cut-off functions

$$\begin{aligned} \overset{\leftarrow}{\chi} &\in \mathcal{C}_0^\infty(\mathbb{R}), \quad \overset{\leftarrow}{\chi}(x) \equiv 1 \text{ for } x \leq -R, \quad \overset{\leftarrow}{\chi}(x) \equiv 0 \text{ for } x \geq R, \\ \overset{\rightarrow}{\chi} &\in \mathcal{C}_0^\infty(\mathbb{R}), \quad \overset{\rightarrow}{\chi}(x) \equiv 1 \text{ for } x \geq R, \quad \overset{\rightarrow}{\chi}(x) \equiv 0 \text{ for } x \leq -R \end{aligned}$$

and the operators

$$\begin{aligned} \mu &:= \left( -\partial_{r_*}^2 + F\left(\frac{-\Delta_{S^2}}{r^2} + \frac{F'}{r}\right) \right)^{1/2}, \\ \mu_1 &:= \left( -\partial_{r_*}^2 - \frac{\Delta_{S^2}}{r_*^2} \right)^{1/2}, \\ \mu_0 &:= \left( -\partial_{r_*}^2 \right)^{1/2}, \end{aligned}$$

by

$$\overleftarrow{\mathcal{J}}_0 := \mu^{-1} \overleftarrow{\chi} \mu_0 \oplus \overleftarrow{\chi} = (\mu^{-1} \otimes 1) \overleftarrow{\chi} (\mu_0 \otimes 1), \quad (2.43)$$

$$\overrightarrow{\mathcal{J}}_j = \mu^{-1} \overrightarrow{\chi} \mu_j \oplus \overrightarrow{\chi} = (\mu^{-1} \otimes 1) \overrightarrow{\chi} (\mu_j \otimes 1). \quad (2.44)$$

The proof of the existence and completeness of the wave operators for the comparison dynamics  $H_1$  at infinity has been established by Dimock in [21]. The proof with the choice  $H_0$  is not explicitly present in the literature in the exact form given here but an equivalent result is proven by the author in [53].

The idea of the construction by Dimock is very clearly explained in his paper. One would naturally like to use mere cut-off functions as identification operators, but they are not bounded operators between the various energy spaces. However they are bounded at the level of the corresponding  $L^2$  spaces. So one composes  $\overleftarrow{\chi}$  or  $\overrightarrow{\chi}$  with the operators or their inverses

$$\begin{aligned} \mu \otimes 1 &\in \mathcal{L}(\mathcal{H}; L^2(\mathbb{R} \times S^2, dr_* d^2\omega)^2), \\ \mu_0 \otimes 1 &\in \mathcal{L}(\mathcal{H}_0; L^2(\mathbb{R} \times S^2, dr_* d^2\omega)^2), \\ \mu_1 \otimes 1 &\in \mathcal{L}(\mathcal{H}_1; L^2(\mathbb{R}^3, dr_* d^2\omega)^2), \end{aligned}$$

that are *isometries*. The existence of the wave operators then translates into the comparison between solutions to evolution equations associated to  $H$  and  $H_0$  or  $H_1$  seen through these isometries. Considering  $\Phi \in \mathcal{H}_0$  for instance and putting  $\Psi := s - \lim_{t \rightarrow +\infty} e^{itH} \overleftarrow{\mathcal{J}}_0 e^{-itH_0} \Phi$ , we have

$$\begin{aligned} 0 &= \lim_{t \rightarrow +\infty} \left\| e^{itH} (\mu^{-1} \otimes 1) \overleftarrow{\chi} (\mu_0 \otimes 1) e^{-itH_0} \Phi - \Psi \right\|_{\mathcal{H}} \\ &= \lim_{t \rightarrow +\infty} \left\| (\mu^{-1} \otimes 1) \overleftarrow{\chi} (\mu_0 \otimes 1) e^{-itH_0} \Phi - e^{-itH} \Psi \right\|_{\mathcal{H}} \\ &= \lim_{t \rightarrow +\infty} \left\| (\mu \otimes 1) (\mu^{-1} \otimes 1) \overleftarrow{\chi} (\mu_0 \otimes 1) e^{-itH_0} \Phi - (\mu \otimes 1) e^{-itH} \Psi \right\|_{L^2(\mathbb{R}^3; dr_* d^2\omega)} \\ &= \lim_{t \rightarrow +\infty} \left\| \overleftarrow{\chi} (\mu_0 \otimes 1) e^{-itH_0} \Phi - (\mu \otimes 1) e^{-itH} \Psi \right\|_{L^2(\mathbb{R}^3; dr_* d^2\omega)}. \end{aligned}$$

### 3 Another spectral analytic approach to scattering: the Lax-Phillips theory

The Lax-Phillips theory [40] is a spectral analytic scattering theory for the wave equation with an obstacle on Minkowski spacetime. We present it here in the trivial case where there is no obstacle. What we are interested in is the construction of the so-called “translation representer” that is the scattering data for the free wave equation. The main idea is inspired by a spectral construction for finite dimensional differential systems. We explain the finite dimensional case first, then move on to the wave equation on Minkowski spacetime.

### 3.1 Finite dimensional case

Consider the equation for a time-dependent vector in  $\mathbb{C}^n$  :

$$\partial_t V(t) = iAV(t)$$

where  $A$  is an  $n \times n$  hermitian matrix  $A$ . For the sake of simplicity, we assume that  $A$  has  $n$  distinct eigenvalues  $\sigma_1, \dots, \sigma_n$ . Let  $\{e_1, \dots, e_n\}$  be an orthonormal basis of eigenvectors of  $A$ . A vector  $V \in \mathbb{R}^n$  can be described as the function  $\tilde{V}$  from  $\mathbb{R}$  to itself that is zero everywhere except for

$$\tilde{V}(\sigma_i) := \langle V, e_i \rangle, \quad i = 1, 2, \dots, n.$$

The vector  $AV$  is then simply represented as the function  $\sigma\tilde{V}(\sigma)$ , i.e. the action of  $A$  is represented as the multiplication by the spectral parameter  $\sigma$ . Similarly, the unitary group  $e^{itA}$  is described as the multiplication by  $e^{it\sigma}$ . This is a *spectral representation* of the matrix  $A$  and its associated unitary group.

A Fourier transform in  $\sigma$  then gives naturally a *translation representation* of the group :

$$\mathcal{F}_\sigma(\widetilde{e^{itA}V})(r) = \mathcal{F}_\sigma(e^{it\sigma}\tilde{V})(r) = \hat{V}(r-t).$$

**Remark 3.1.** *Of course for it all to make sense, the Fourier transform must be understood on  $L^2(\mathbb{R}; d\mu)$  where  $\mu$  is the sum of Dirac measures on the  $\sigma'_j$ s.*

### 3.2 The wave equation

Now consider the wave equation on Minkowski spacetime :

$$\partial_t^2 \phi - \Delta \phi = 0. \tag{3.1}$$

As a Schrödinger-type equation, this takes the form

$$\partial_t U = iAU, \quad U := \begin{pmatrix} \phi \\ \partial_t \phi \end{pmatrix}, \quad A = -i \begin{pmatrix} 0 & 1 \\ \Delta & 0 \end{pmatrix}.$$

The operator  $A$  is self-adjoint on  $\mathcal{H} = \dot{H}^1(\mathbb{R}^3) \times L^2(\mathbb{R}^3)$ , completion of  $\mathcal{C}_0^\infty(\mathbb{R}^3) \times \mathcal{C}_0^\infty(\mathbb{R}^3)$  in the norm

$$\|U\|^2 := \int_{\mathbb{R}^3} (|\nabla u_1|^2 + |u_2|^2) d^3x, \quad U = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}.$$

In order to mimic the construction in the finite dimensional case, we look for the eigenvalues of  $A$ , i.e. for  $\sigma \in \mathbb{R}$  such that  $AU = \sigma U$ . This equation is equivalent to

$$\begin{cases} u_2 &= i\sigma u_1, \\ \Delta u_1 &= i\sigma u_2 = -\sigma^2 u_1. \end{cases} \tag{3.2}$$

This has no finite energy solution, however there are obvious solutions

$$e_{\sigma,\omega}(x) = \begin{pmatrix} e^{-i\sigma x \cdot \omega} \\ i\sigma e^{-i\sigma x \cdot \omega} \end{pmatrix}, \quad \omega \in S^2. \quad (3.3)$$

This means that  $H$  has no eigenvalues. In fact, its spectrum is purely absolutely continuous and for each spectral value  $\sigma \in \mathbb{R}$ , we have generalised eigenfunctions given by (3.3). So instead of a finite orthonormal basis of eigenvectors, we have a continuous family of generalised eigenfunctions. We now proceed exactly as in the finite dimensional case : consider  $U \in \mathcal{C}_0^\infty(\mathbb{R}^3) \times \mathcal{C}_0^\infty(\mathbb{R}^3)$ , we project it on a given generalised eigenfunction, in spite of the apparently obvious obstacle that  $e_{\sigma,\omega}$  is not an element of  $\mathcal{H}$ :

$$\begin{aligned} \tilde{U}(\sigma,\omega) &:= \frac{1}{(2\pi)^{3/2}} \langle U, e_{\sigma,\omega} \rangle_{\mathcal{H}} \\ &= \frac{1}{(2\pi)^{3/2}} \int_{\mathbb{R}^3} (\nabla u_1 \overline{\nabla e^{-i\sigma x \cdot \omega}} + u_2 \overline{i\sigma e^{-i\sigma x \cdot \omega}}) d^3x \\ &= \frac{1}{(2\pi)^{3/2}} \int_{\mathbb{R}^3} (u_1 \overline{(-\Delta e^{-i\sigma x \cdot \omega})} + u_2 \overline{i\sigma e^{-i\sigma x \cdot \omega}}) d^3x \\ &= \frac{1}{(2\pi)^{3/2}} \int_{\mathbb{R}^3} (\sigma^2 u_1 - i\sigma u_2) e^{i\sigma x \cdot \omega} d^3x \\ &= \sigma^2 \hat{u}_1(-\sigma\omega) - i\sigma \hat{u}_2(-\sigma\omega). \end{aligned}$$

It turns out that, although the intermediate calculations do not, the final formula extends to  $\mathcal{H}$  and the map that to  $U$  associates  $\tilde{U}$  extends as an isometry from  $\mathcal{H}$  onto  $L^2(\mathbb{R}_\sigma \times S^2)$ .

This provides a spectral representation of  $A$  and its propagator :

$$\widetilde{AU} = \sigma \tilde{U}, \quad \widetilde{e^{itA}U} = e^{it\sigma} \tilde{U}.$$

Just as in the finite dimensional case, we take the Fourier transform in  $\sigma$ . Denoting

$$\mathcal{R}U(s,\omega) := \mathcal{F}_\sigma(\tilde{U}(\cdot,\omega))(s), \quad (3.4)$$

we have

$$\mathcal{R}(e^{itA}U)(s,\omega) = (\mathcal{R}U)(s-t,\omega). \quad (3.5)$$

The representation (3.4) is of course also an isometry from  $\mathcal{H}$  onto  $L^2(\mathbb{R} \times S^2)$  and it provides, as shown by (3.5), a translation representation of the propagator for (3.1).

The translation representation  $\mathcal{R}$  can be expressed using the Radon transform of  $\mathbb{R}^3$ . Recall that the Radon transform is defined for a smooth compactly supported function  $f$  on  $\mathbb{R}^3$  by

$$Rf(s,\omega) = \int_{x \cdot \omega = s} f(x) d^2\sigma(x).$$

It has an inverse transform valid for smooth functions on  $\mathbb{R} \times S^2$

$$R^*\psi(x) = \int_{S^2} \psi(x \cdot \omega, \omega) d^2\omega.$$

It is easily seen that

$$\mathcal{R}U = \frac{1}{4\pi}(-\partial_s^2 Ru_1 + \partial_s Ru_2).$$

Using the inverse Radon transform, one can construct a converse map to  $\mathcal{R}$  given by

$$\mathcal{I}k = \frac{1}{2\pi}(R^*k, -R^*\partial_s k).$$

Lax and Phillips then use this converse map to establish the *asymptotic profile* property which is the geometric reinterpretation of the translation representer as the radiation field of the solution. More precisely, denoting  $k(s, \omega) = \mathcal{R}U(s, \omega)$ , we have

$$k(s, \omega) = - \lim_{r \rightarrow +\infty} r \partial_t \phi(r, (r+s)\omega). \quad (3.6)$$

## 4 Conformal scattering theory

### 4.1 Main principle and general structure of the construction

The fundamental idea of conformal scattering is that a complete scattering construction is equivalent to solving the Goursat problem at the conformal boundary. A simple and rigorous way of doing this was proposed by Hörmander in 1990 [37] for first order perturbations of the wave equation on spatially compact globally hyperbolic spacetimes. Given the finite propagation speed and the fact that first order symmetric hyperbolic systems also satisfy a perturbed second order wave equation, the results of [37] are valid for a large class of linear equations on fairly general globally hyperbolic spacetimes. Based on Hörmander’s method, the conformal scattering construction proposed by Mason and the author in 2004 [42] is done in three steps.

Consider a globally hyperbolic spacetime  $(\mathcal{M}, g)$  and its conformal compactification  $(\hat{\mathcal{M}}, \hat{g})$ ,  $\hat{g} = \Omega^2 g$ . We study a linear conformally invariant field equation on  $(\mathcal{M}, g)$ . Using its conformal invariance, we transpose it on  $(\hat{\mathcal{M}}, \hat{g})$  and perform the whole construction on the compactified spacetime. The important geometrical objects at this point are

- a Cauchy hypersurface  $\Sigma_0$  for  $(\mathcal{M}, g)$ ,
- the future and past components of the conformal boundary that we shall here simply denote by  $\mathcal{I}^\pm$  but which may contain also horizons when working outside a black hole.

*Step 1.* We construct so-called “trace operators”  $\mathbb{T}^\pm$  that map smooth and compactly supported initial data on  $\Sigma_0$  ( $\hat{\phi}_0 \in \mathcal{C}_0^\infty(\Sigma_0)$ ) to the restrictions  $\hat{\phi}^\pm$  at  $\mathcal{I}^\pm$  of the solution  $\hat{\phi}$  to the Cauchy problem for our equation. The quantities  $\hat{\phi}^\pm$  have been intensively studied in the literature, particularly by Friedlander [25, 26, 27] and are usually referred to as radiation fields. They represent the leading order of the asymptotic behaviour of the field along outgoing null geodesics. The existence of such restrictions or traces is ensured by the global hyperbolicity of the spacetime  $(\mathcal{M}, g)$ , the hyperbolicity of the equation and general results for hyperbolic equations due to Leray [41]. There are technical details at this

point associated with the fact that the conformal boundary is usually not closed and admits singularities. Using the finite propagation speed for the equation, we can excise the singularities and extend our mutilated compactified spacetime to a globally hyperbolic spacetime without singularities, on which Leray's results can be applied.

*Step 2.* We define natural Hilbert spaces  $\mathcal{H}$  on  $\Sigma_0$  and  $\mathcal{H}^\pm$  on  $\mathcal{I}^\pm$  and extend the trace operators  $\mathbb{T}^\pm$  as bounded operators from  $\mathcal{H}$  to  $\mathcal{H}^\pm$ . This is done by means of energy estimates. They are performed for solutions associated with smooth and compactly supported initial data. The basic energy fluxes on  $\Sigma_0$  and  $\mathcal{I}^\pm$  define non-negative quadratic forms with respect to which we complete  $\mathcal{C}_0^\infty(\Sigma_0)$  to obtain our Hilbert spaces. We then obtain estimates both ways between the norms on  $\Sigma_0$  and  $\mathcal{I}^\pm$

$$\|\hat{\phi}^\pm\|_{\mathcal{H}^\pm} \leq C\|\phi_0\|_{\mathcal{H}}, \quad (4.1)$$

$$\|\phi_0\|_{\mathcal{H}} \leq C\|\hat{\phi}^\pm\|_{\mathcal{H}^\pm}, \quad (4.2)$$

that are uniform on the considered family of solutions. These allow to extend uniquely our trace operators by density, we still denote them by  $\mathbb{T}^\pm$ . Moreover, because of the two converse estimates, the operators  $\mathbb{T}^\pm$  are one-to-one and with closed range.

*Step 3.* We prove that  $\mathbb{T}^\pm$  are isometries from  $\mathcal{H}$  onto  $\mathcal{H}^\pm$ . All that is left to prove at this stage is the surjectivity of  $\mathbb{T}^\pm$ . Since we already know that their range is closed, we merely need to establish that  $\text{Ran}(\mathbb{T}^\pm)$  is dense in  $\mathcal{H}^\pm$ . For this, we consider data  $\hat{\phi}^\pm$  in a dense subspace of  $\mathcal{H}^\pm$ , typically, smooth and compactly supported functions. For such data, following Hörmander, we construct a solution to the Goursat problem as a limit to solutions of the Cauchy problem for a modified equation with slowed down propagation (for which  $\mathcal{I}^\pm$  are now spacelike hypersurfaces).

The property that the operators  $\mathbb{T}^\pm$  exist and are isometries between Hilbert spaces  $\mathcal{H}$  on  $\Sigma_0$  and  $\mathcal{H}^\pm$  on  $\mathcal{I}^\pm$  is a precise statement of the well-posedness of the Goursat problem at  $\mathcal{I}^\pm$  with an explicit functional framework and therefore provides a complete scattering theory. The Cauchy hypersurface  $\Sigma_0$  can be considered as an intermediate technical tool that can then be dropped when considering the scattering operator

$$\mathbb{S} := \mathbb{T}^+ (\mathbb{T}^-)^{-1}. \quad (4.3)$$

This construction has the advantage of being insensitive to time dependence in general. However, when a singularity is present at timelike or even spacelike infinity, a complicated time dependence may make the proof much trickier.

We now give additional details about the construction on an explicit example.

## 4.2 The wave equation on the Schwarzschild spacetime

This case is the object of a paper by the author [53] to which we refer the reader for more detailed explanations. We work on the exterior ( $r > 2M$ ) of Schwarzschild's spacetime (2.27)

and we consider the wave equation

$$\square_g \phi = 0. \quad (4.4)$$

This is also the conformal wave equation since the Schwarzschild metric is Ricci flat. We then perform the following conformal compactification

$$\hat{g} := \Omega^2 g, \quad \Omega = R := \frac{1}{r}, \quad \text{Scal}_{\hat{g}} = 12MR. \quad (4.5)$$

We express the metric  $\hat{g}$  in three different coordinate systems: the Schwarzschild coordinates with the Regge-Wheeler radial variable  $(t, r_*, \omega)$ , the outgoing Eddington-Finkelstein coordinates with inverted radial variable  $(u = t - r_*, R = 1/r, \omega)$  and the incoming Eddington-Finkelstein coordinates with inverted radial variable  $(v = t + r_*, R, \omega)$ :

$$\hat{g} = \frac{r - 2M}{r^3} (dt^2 - dr_*^2) - d\omega^2, \quad (4.6)$$

$$= R^2(1 - 2MR)du^2 - 2dudR - d\omega^2, \quad (4.7)$$

$$= R^2(1 - 2MR)dv^2 + 2dv dR - d\omega^2. \quad (4.8)$$

The compactified Schwarzschild exterior acquires future and past null infinities

$$\mathcal{I}^+ \simeq \mathbb{R}_u \times \{R = 0\} \times S^2, \quad \mathcal{I}^- \simeq \mathbb{R}_v \times \{R = 0\} \times S^2, \quad (4.9)$$

as boundary. With our choice of conformal factor  $\Omega$ , spacelike infinity and timelike infinities remain at infinity and are not part of the boundary. They could be brought back to finite distance with a different choice of conformal factor, but they are conformal singularities and the rescaled metric would in any case not be regular there. The complete boundary is therefore made up of  $\mathcal{I}^\pm$  and of the future and past horizons  $\mathcal{H}^\pm$  as well as their crossover  $S_c$  (see Figure 1 for a Penrose diagram of the Schwarzschild exterior). An interesting feature of the conformal rescaling we have chosen is that the vector field  $\partial_t$  remains Killing for  $\hat{g}$ . Moreover, it is timelike and future oriented in the bulk of the compactified spacetime and extends to the boundary as a null vector field (as can be seen easily from (4.7)-(4.8) using the fact that in coordinates  $(u, R, \omega)$ ,  $\partial_t$  is  $\partial_u$  and in coordinates  $(v, R, \omega)$ , it is given by  $\partial_v$ ) that only vanishes at  $S_c$ .

The conformal wave equation on the compactified spacetime is

$$\square_{\hat{g}} \hat{\phi} + 2MR\hat{\phi} = 0. \quad (4.10)$$

The energy estimates described in step 2 in Subsection 4.1 are obtained by choosing first a stress-energy tensor for Equation (4.10). The usual stress-energy tensor for the wave equation is not conformally invariant and therefore does not yield a conserved stress-energy tensor for (4.10). It is easy in our case to choose simply the stress-energy tensor for the wave equation on the compactified spacetime

$$T_{ab} := \hat{\nabla}_a \hat{\phi} \hat{\nabla}_b \hat{\phi} - \frac{1}{2} \langle \hat{\nabla} \hat{\phi}, \hat{\nabla} \hat{\phi} \rangle_{\hat{g}} \hat{g}_{ab}.$$

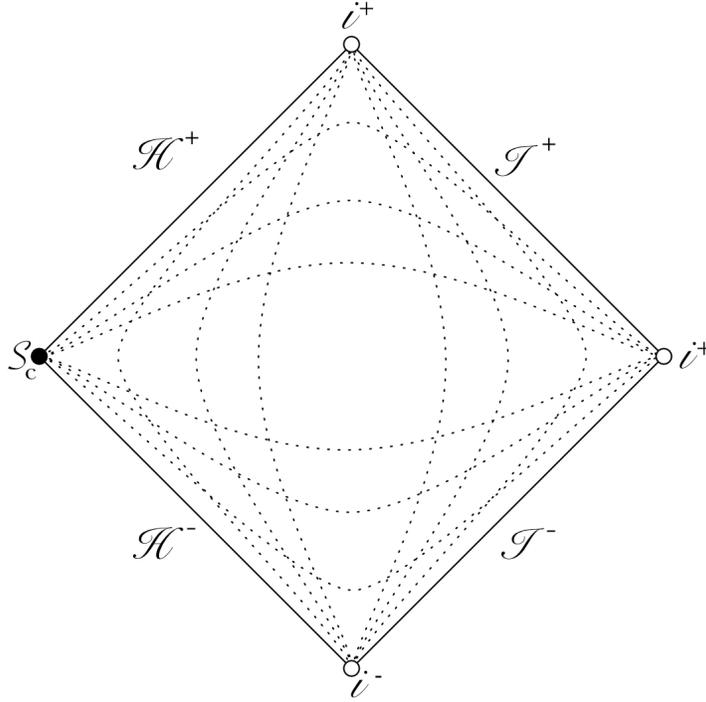


Figure 1: Penrose diagram of the exterior of a Schwarzschild black hole with the level hypersurfaces of  $t$  and  $r$  represented by dotted lines.

This is symmetric but not conserved since

$$\hat{\nabla}^a T_{ab} = \left( \square_{\hat{g}} \hat{\phi} \right) \hat{\nabla}_b \hat{\phi} = -2MR\hat{\phi}\hat{\nabla}_b \hat{\phi} \text{ when } \hat{\phi} \text{ satisfies (4.10).}$$

We also need to choose an observer, i.e. a timelike vector field, with which to contract the stress-energy tensor in order to obtain an energy current. A natural choice on the rescaled spacetime is  $\partial_t =: K$ . The corresponding energy current

$$\hat{J}_a = K^b \hat{T}_{ab} \tag{4.11}$$

is not conserved either,

$$\hat{\nabla}^a \hat{J}_a = -2MR\hat{\phi}\partial_t \hat{\phi}, \tag{4.12}$$

however the right-hand side of (4.12) is an exact divergence

$$V := MR\hat{\phi}^2 \partial_t, \quad 2MR\hat{\phi}\partial_t \hat{\phi} = \hat{\nabla}_a V^a.$$

As a consequence, the modified energy current

$$\tilde{J}^a := \hat{J}^a + V^a \tag{4.13}$$

is conserved. The associated energy flux across an oriented hypersurface is defined in terms of the Hodge dual of the 1-form  $\tilde{J}_a$ . We adopt the conventions of Penrose and Rindler [56] whereby for a 1-form  $\alpha_a$ ,

$$(*\alpha)_{abc} = e_{abcd}\alpha^d,$$

where  $e_{abcd}$  is the volume form for  $\hat{g}$ , which we also denote  $d\text{Vol}_{\hat{g}}$ . Given an oriented hypersurface  $S$ , the flux of  $\tilde{J}^a$  through  $S$  is then given by

$$\hat{\mathcal{E}}_S = -4 \int_S * \tilde{J}_a dx^a = \int_S \tilde{J}_a \hat{n}^a \hat{l} \lrcorner d\text{Vol}_{\hat{g}}, \quad (4.14)$$

where  $\hat{l}$  is a vector field transverse to  $S$  and compatible with the orientation of the hypersurface, and  $\hat{n}$  is the normal vector field to  $S$  such that  $\hat{g}(\hat{l}, \hat{n}) = 1$ . The energy fluxes across the Cauchy hypersurface  $\Sigma_0 := \{t = 0\}$ ,  $\mathcal{I}^+$  and  $\mathcal{H}^+$  are given by

$$\begin{aligned} \hat{\mathcal{E}}_{\Sigma_0}(\hat{\phi}) &= \frac{1}{2} \int_{\Sigma_0} \left( (\partial_t \hat{\phi})^2 + (\partial_{r_*} \hat{\phi})^2 + R^2 F |\nabla_{S^2} \hat{\phi}|^2 + 2MFR^3 \hat{\phi}^2 \right) dr_* d^2\omega, \\ \hat{\mathcal{E}}_{\mathcal{H}^+}(\hat{\phi}) &= \int_{\mathcal{H}^+} (\partial_v (\hat{\phi}|_{\mathcal{H}^+}))^2 dv d^2\omega, \\ \hat{\mathcal{E}}_{\mathcal{I}^+}(\hat{\phi}) &= \int_{\mathcal{I}^+} (\partial_u (\hat{\phi}|_{\mathcal{I}^+}))^2 du d^2\omega. \end{aligned}$$

Note that these are really all integrals on  $\mathbb{R} \times S^2$  but for the sake of clarity, we have preferred to indicate the geometrical localisation of the integration. We define the corresponding energy spaces on  $\Sigma_0$  and on the future boundary  $\mathcal{H}^+ \cup \mathcal{I}^+$  as follows:

- $\mathcal{H}$  is the completion of  $\mathcal{C}_0^\infty(\Sigma_0)^2$  in the norm defined by  $\hat{\mathcal{E}}_{\Sigma_0}$ ;
- $\mathcal{H}^+$  is the completion of  $\mathcal{C}_0^\infty(\mathcal{H}^+) \times \mathcal{C}_0^\infty(\mathcal{I}^+)$  in the norm

$$\|(f_1, f_2)\|_{\mathcal{H}^+}^2 = \hat{\mathcal{E}}_{\mathcal{H}^+}(f_1) + \hat{\mathcal{E}}_{\mathcal{I}^+}(f_2).$$

We have a similar construction of an energy space  $\mathcal{H}^-$  at the past boundary  $\mathcal{H}^- \cup \mathcal{I}^-$ .

Since the current  $\tilde{J}$  is conserved, we obtain the following energy identities for finite energy solutions

$$\hat{\mathcal{E}}_{\Sigma_0}(\hat{\phi}) = \hat{\mathcal{E}}_{\mathcal{H}^+}(\hat{\phi}) + \hat{\mathcal{E}}_{\mathcal{I}^+}(\hat{\phi}) = \hat{\mathcal{E}}_{\mathcal{H}^-}(\hat{\phi}) + \hat{\mathcal{E}}_{\mathcal{I}^-}(\hat{\phi}), \quad (4.15)$$

which, at the end of the three-step construction, will entail the following theorem

**Theorem 4.1.** *The trace operators  $\mathbb{T}^\pm$  that to smooth compactly supported initial data  $(\hat{\phi}_0, \hat{\phi}_1)$  for Equation (4.10) associate the restrictions  $(\hat{\phi}|_{\mathcal{H}^\pm}, \hat{\phi}|_{\mathcal{I}^\pm})$  of the rescaled solution at  $\mathcal{H}^\pm \cup \mathcal{I}^\pm$ , extend as isometries from  $\mathcal{H}$  onto  $\mathcal{H}^\pm$ .*

There are two difficulties in the proof of (4.15) that are related to the conformal singularities at  $i^0$  and  $i^\pm$  and that prevent us from applying directly the divergence Theorem. Such difficulties are not present when working on Minkowski spacetime for which the conformal boundary has no singularity; the singularity at  $i^0$  appears as soon as we work with a spacetime containing energy and the one at  $i^\pm$  is caused by the presence of the event horizon (future and past). We explain in the next subsections how to deal with them in the case of the wave equation. The constructions are described towards the future, the exact same thing can be done towards the past.

### 4.3 Strategy for a singular spacelike infinity

In a compactified spacetime where the only singularity is at spacelike infinity, the finite propagation speed is enough to deal with the difficulty. The idea is to work with a dense family of initial data that have compact support, typically smooth and compactly supported. For such data, the finite propagation speed will entail that the support of the solution remains away from spacelike infinity (see Figure 2). This means that we can use the divergence Theorem

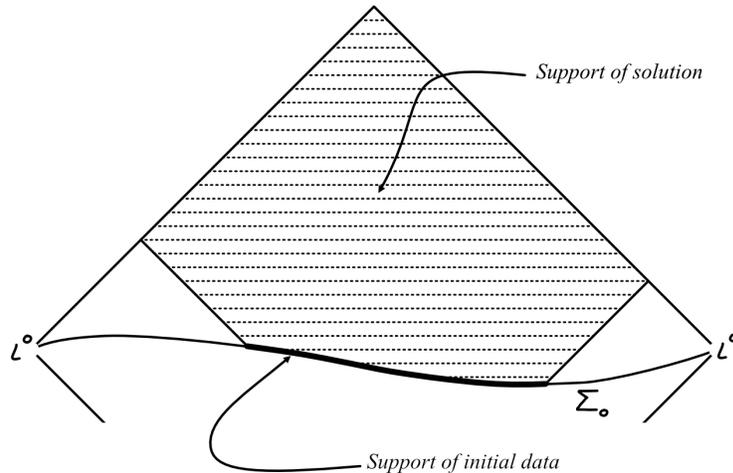


Figure 2: Avoiding the singularity at  $i^0$

to obtain energy estimates both ways between a Cauchy hypersurface and the future boundary  $\mathcal{I}^+$ . Moreover, provided we have some adequate approximate symmetry near  $i^0$  yielding an approximate conservation law, the constants in the estimates will be independent of the size of the support of the data; in [42], this is taken care of by assuming that the spacetime is diffeomorphic to the Schwarzschild metric in a neighbourhood of  $i^0$ . Then, by density of the family of smooth compactly supported functions in the energy space, the inequalities will be valid for all finite energy data, whether their support is compact or not. When timelike infinities are also singular, as is the case for the Schwarzschild metric, the strategy is still valid but must be used in combination with the method described in Subsection 4.4.

When solving the Goursat problem for smooth compactly supported scattering data on  $\mathcal{I}^+$  in step 3 in Subsection 4.1, the difficulty occurs again but this time, the support of the solution has no reason to remain away from  $i^0$ . One can still use the finite propagation speed but the method is a little more subtle in this case. We consider a spacelike hypersurface  $S$  that lies in the past of the support of the data on the future boundary, which may be made of  $\mathcal{I}^+$  and  $\mathcal{H}^+$ , see Figure 3. The solution to the Goursat problem can be obtained from Hörmander’s theorems

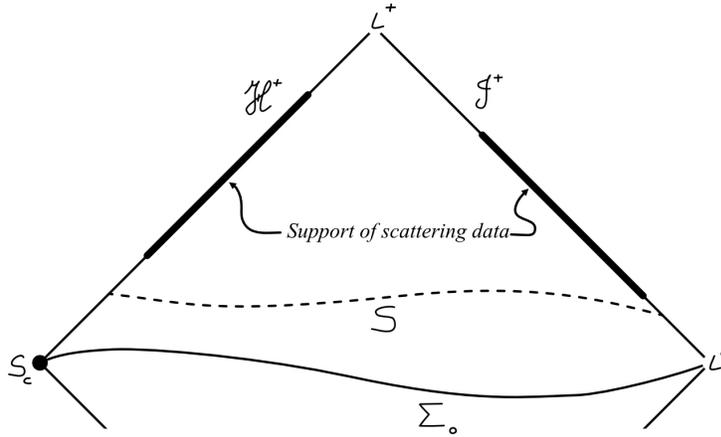


Figure 3: The choice of hypersurface  $S$  in the backwards Goursat problem.

[37] in the future of  $S$  and the restriction  $\hat{\phi}|_S$  of the solution  $\hat{\phi}$  to  $S$  has finite energy. Moreover,  $\hat{\phi}|_S$  vanishes at the intersection of  $S$  and the boundary. This implies, just like smooth compactly supported functions are dense in  $H_0^1$  of a bounded open set of  $\mathbb{R}^n$ , that  $\hat{\phi}|_S$  can be approached in energy norm by smooth compactly supported functions on  $S$ , where compactly supported means that they are supported away from the boundary. The same is true for the restriction of the time derivative of  $\hat{\phi}$  to  $S$  as it lies in  $L^2$  in which smooth compactly supported functions are dense. We can therefore approach the full data on  $S$  by a sequence of smooth compactly supported data for which we can solve the Cauchy problem towards the past between  $S$  and  $\Sigma_0$ . The solutions remain supported away from  $i_0$  and we obtain energy estimates both ways between  $S$  and  $\Sigma_0$  for them, with constants independent of the element of the sequence considered. Taking the limit shows that the restriction of  $\hat{\phi}$  to  $\Sigma_0$  has finite energy. In this construction, the hypersurface  $S$  plays the role of an intermediate technical tool for a given set of data.

#### 4.4 Dealing with timelike infinities

When timelike infinities are singular, unless we work in a spacetime for which the Huyghens principle is valid, solutions associated with compactly supported initial data will not remain supported away from  $i^\pm$  and finite propagation speed therefore cannot help us. In this case, we start by working away from timelike infinity. We consider a family of spacelike hypersurfaces

$\{S_T\}_{T \geq T_0}$  that are transverse to the future boundary and accumulate on  $i^+$  as  $T \rightarrow +\infty$ . We denote by  $\mathcal{H}_T^+$  and  $\mathcal{I}_T^+$  the parts of  $\mathcal{H}^+$  and  $\mathcal{I}^+$  located in the past of  $S_T$  (see Figure 4). Using the strategy explained in the previous subsection for dealing with the singularity at  $i_0$ ,

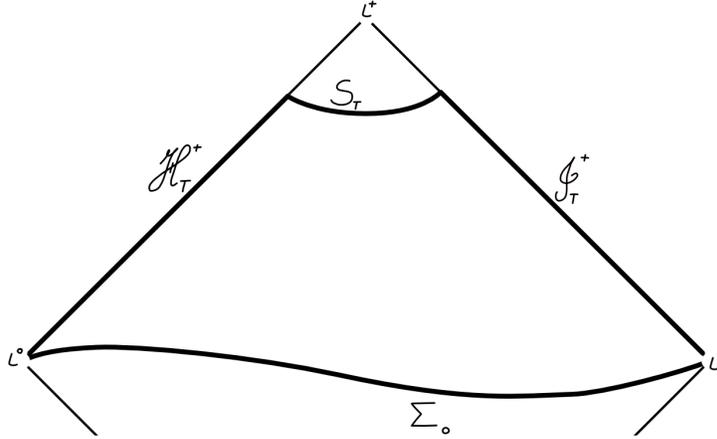


Figure 4: Dealing with a singular  $i^+$

we can obtain energy estimates both ways between  $\Sigma_0$  and the union  $\mathcal{H}_T^+ \cup S_T \cup \mathcal{I}_T^+$ . In the Schwarzschild case with the current  $\tilde{J}$ , these estimates will be identities since the current is conserved. Then we need to prove that when  $T \rightarrow +\infty$  the energy on  $S_T$  tends to 0. This is a global decay result on the physical spacetime in the future of a family of hyperboloids that are pushed further and further into the future. For the wave equation on the Schwarzschild metric, such results have been obtained by Dafermos and Rodnianski [17]. On the exterior of a Reissner-Nordström-de Sitter black hole for the Maxwell field, Mokdad established the decay estimates near  $i^\pm$  [48] and used them to infer the conformal scattering construction [47].

#### 4.5 Freedom of choice of observer

For the wave equation on the Schwarzschild spacetime, we have chosen the observer associated to the timelike Killing vector field  $\partial_t$ . Since it becomes null at the boundary, the norms on the energy spaces for scattering data give a fairly weak control, namely  $\dot{H}^1(\mathbb{R}; L^2(S^2))$ . We could choose an observer that is transverse to the boundary, this would give us a full  $\dot{H}^1$  control on the scattering data with some weights at timelike infinities and spacelike infinity depending on the behaviour of the vector field there. On Minkowski spacetime, the two types of choices can be made and two inequivalent scattering theories are obtained. First, one can work on the fully compactified spacetime that is embedded in the Einstein cylinder, using the conformal factor

$$\Omega = \frac{2}{\sqrt{1+(t+r)^2}\sqrt{1+(t-r)^2}}.$$

Then the conformal wave equation on the Einstein cylinder is the Klein-Gordon equation with mass 1 which has a conserved stress-energy tensor. Choosing the observer associated to the time translation along the Einstein cylinder gives a conserved energy current whose flux across  $\mathcal{S}$  and  $\{t = 0\}$  provides a full  $H^1$  control on both hypersurfaces. The control at the boundary is even stronger than  $\dot{H}^1$  because of the choice of stress-energy tensor, which is natural here since the scalar curvature does not vanish at  $\mathcal{S}$ . Alternatively, we can compactify Minkowski spacetime using the conformal factor  $\Omega = 1/r$  and choose the observer  $\partial_t$ . Then the compactified spacetime is Ricci flat and the conformal wave equation is the usual wave equation which also has a conserved stress-energy tensor. The conserved energy current we obtain gives an  $\dot{H}^1(\mathbb{R}; L^2(S^2))$  control on the boundary.

On Schwarzschild's spacetime, no conformal scattering theory has been constructed using an observer transverse to the boundary. However, the Morawetz vector field is such an observer and it has been used by L. Mason and the author in [43, 44] to study the peeling for various zero rest-mass field equations. It would be interesting to see whether the decay estimates of Dafermos and Rodnianski are enough to establish a complete scattering theory for the wave equation using the Morawetz vector field as an observer.

## 5 Transversal constructions

In spite of the fundamentally different approaches, there are many analogous structures that appear between the spectral analytic time dependent and conformal scattering constructions. First of all the wave operators and the trace operators seem to play similar roles, the trace operators corresponding to the inverse wave operators rather than the direct ones. Note that they also seem to enjoy analogous properties inherited from the symmetries of the background. For instance for the wave equation on the Schwarzschild metric, both are isometries between well-chosen Hilbert spaces. Also, there are freedoms of choices for both approaches that are of a different nature but entail in the complete theory remarkably close changes. For time-dependent scattering, the freedom in the choice of comparison dynamics is translated in the function space in which the scattering data live. For conformal scattering, it is the choice of observer that has an effect on the energy space of scattering data. Moreover, we have exhibited natural choices in both frameworks that entail exactly the same function spaces for the scattering data: for the wave equation on the Schwarzschild metric, the choice of  $H_0$  for a comparison dynamics and the choice of  $\partial_t$  as an observer produce the same function spaces  $\dot{H}^1(\mathbb{R} \times L^2(S^2)) \oplus \dot{H}^1(\mathbb{R} \times L^2(S^2))$  for the scattering data, whereas the choice of  $H_1$  and of an observer transverse to the boundary change the function space at infinity into  $\dot{H}^1(\mathbb{R} \times S^2)$  with possibly different asymptotic weights depending on the choice of behaviour of the observer near  $i^0$  and  $i^\pm$ . The two approaches seem to differ the most in the way they consider their scattering data; in the spectral construction, they live in a different space to the solutions and the link is made by sometimes obscure identification operators, whereas in the conformal picture, the scattering data are geometrically localised at the boundary of the spacetime and are the radiation fields. However, the function space of scattering data can often be made sense of geometrically as an asymptotic simplification of the

finite energy space for the complete equation. For the comparison dynamics  $H_0$  in the case of the wave equation on the Schwarzschild metric, the geometric interpretation is very natural, the propagator  $e^{-itH_0}$  plays the role of the principal null geodesic flow, the scattering data at infinity are exactly the restrictions at  $\mathcal{I}$  of the rescaled field, i.e. the radiation fields, and the inverse wave operators are naturally identified to the direct trace operators, modulo an identification of  $\mathcal{I}^\pm$  with the congruence of outgoing/incoming principal null geodesics.

These analogous structures raise a natural question: can one use the ingredients of one approach to infer the final result in the other? This is what we call a transversal construction. The theories themselves are of course not equivalent since their main ingredients, in order to be available, impose some constraints on the type of background and field equation considered and these constraints are different for each approach. For instance, the conformal method requires to work on manifolds that can be conformally compactified with a sufficiently smooth boundary and with field equations that are conformally invariant, or at least that change in a well controlled manner under a conformal rescaling. For the time-dependent method, whether it is based on spectral analysis or not, stationary frameworks are well adapted, but also time dependent situations with a strong and uniform time decay, see for example the time-dependent scattering theory for Dirac fields in the interior of a Reissner-Nordström black-hole established by D. Häfner, M. Mokdad and the author [35]. As for the Lax-Phillips theory, it has been constructed on Minkowski spacetime and its extensions obtained using the conformal method by Friedlander [28] and then J. Baez and its collaborators [7, 8, 9, 10, 11] are only in static situations. In spite of these differences, we would argue that the question at the beginning of the paragraph is a valid one, to which some partial answer has already been given. In a sense, conformal scattering itself is a transversal construction. It was initially constructed by Friedlander [28] after years of studying Penrose’s conformal method and radiation fields and also discovering in the Lax-Phillips theory that the translation representer and the radiation field are the same thing. His purpose was to reconstruct the whole Lax-Phillips theory using the conformal method to start from the radiation field instead of recovering it in the end. Several other transversal constructions have been obtained in the recent years.

1. **Conformal scattering entails time-dependent scattering.** This is a fairly general result established in [42] for asymptotically simple spacetimes and in [53] for the Schwarzschild metric. In the Schwarzschild case, we use the comparison dynamics  $H_0$  and the fact that the different connected components of the conformal boundary can be identified with the congruences of outgoing or incoming radial null geodesics (these happen to be the so-called principal null geodesics, that are generated by the double null eigenvectors of the Weyl tensor; for the Kerr metric, we have similar double null eigenvectors and associated geodesics with a non trivial twist). The restrictions of the rescaled field at the conformal boundary can then be understood as fields propagating in the spacetime according to this comparison dynamics. We recover a time dependent scattering theory that could also be constructed using spectral analysis. For asymptotically simple spacetimes with smooth timelike infinities, the idea is to construct a comparison dynamics that resembles  $H_0$  in the Schwarzschild case and can be understood as the flow of a congruence

that can be used to define the conformal boundary. There is a lot of flexibility in this construction and one recovers a time-dependent scattering — in the sense that we can define wave operators that compare two different dynamics — that is not based on spectral analytic methods. There are two natural possibilities to construct the congruences that will support the comparison dynamics. The first method consists in constructing a null congruence in the neighbourhood of the future boundary (resp. the past). We cannot in general extend it to the whole spacetime since caustics will occur, but outside a large world-tube that is compact in space in the physical spacetime, such a congruence can be constructed (see the first image in Figure 5). It can then be extended as a global causal congruence as shown in the second image in Figure 5. Alternatively, we can choose a global timelike congruence (see the third image in Figure 5) defined for instance by the integral lines of the gradient of a global time function. The two congruences shown in the second and third images can both be used to define a comparison dynamics; the parameter along the flow can for instance be defined by a global time function on the physical spacetime, whose level hypersurfaces accumulate at the boundary.

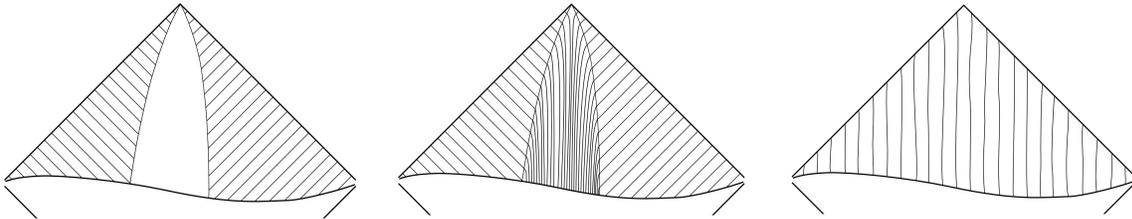


Figure 5: A null congruence near the future boundary, its extension as a global causal congruence and a global timelike congruence.

**2. In stationary situations, conformal scattering entails the Lax-Phillips theory.**

In such cases, we have a global timelike Killing vector field  $K$  in the bulk of the spacetime that becomes null at the conformal boundary. Choosing a time function  $t$  such that  $Kt = 1$ , we can consider the evolution by a time  $t$  of the rescaled field as an active action on the whole spacetime, including the field, of the flow  $\Phi_K(-t)$  of  $K$ . The resulting effect on the restriction at the conformal boundary of the rescaled field is a translation of time  $-t$  along the integral lines of  $K$  that are the null generators of the boundary. Hence, the radiation field is a translation representer, which is the fundamental object of the Lax-Phillips theory.

On the exterior of a Kerr black hole, that is only locally stationary, we have a Killing vector ( $\partial_t$  in Boyer-Lindquist coordinates) that is timelike near infinity but not near the horizon. In this case, the future (resp. past) radiation field at infinity will behave like a translation representer but it does not contain the whole information of the field. The remaining information consists of the restriction of the field on the future (resp. past)

horizon. There  $\partial_t$  is tangent to the horizon but spacelike because of the rotation. The scattering data at the horizon, under the evolution of the field, will undergo a translation of  $-t$  along the null generator of the horizon as well as a rotation at the rotation speed of the horizon.

3. **Time dependent scattering does not entail a conformal scattering theory in general.** This is because in general a time-dependent scattering is not geometrical, it relies on equations on one or two Hilbert spaces and there may not be a natural background spacetime let alone a conformal boundary. In some explicit situations, time-dependent scattering has been shown to entail a conformal scattering theory. This was done on a case by case basis by re-interpreting the direct wave operators as solving the Goursat problem at the conformal boundary, for instance by A. Bachelot for Maxwell fields on the Schwarzschild spacetime [1] and by D. Häfner and the author for Dirac fields outside a Kerr black hole [36].

The third point is worth investigating further. When the geometrical ingredients (a spacetime and its conformal compactification) are present, one may wonder whether a minimal velocity estimate may be enough to entail a conformal scattering theory. It would be extremely interesting if such estimates could provide alternatives to stronger uniform decay results that require a precise understanding of the local geometry of null geodesics, in particular the trapping sets and their nature. First steps must be done in well controlled situations with some explicit symmetries, but this is a fascinating question and a possible opening of a wide research area.

**Acknowledgments.** The author would like to thank the Royal Society for financial support on the occasion of the conference “At the interface of asymptotics, conformal methods and analysis in general relativity”.

## References

- [1] A. Bachelot, *Gravitational scattering of electromagnetic field by Schwarzschild black hole*, Annales de l’I.H.P. A **54** (1991), 261–320.
- [2] A. Bachelot, *Asymptotic completeness for the Klein-Gordon equation on the Schwarzschild metric*, Ann. Inst. Henri Poincaré, Physique Théorique **61** (1994), no. 4, 411–441.
- [3] A. Bachelot, *Quantum vacuum polarization at the black-hole horizon*, Ann. Inst. H. Poincaré Phys. Théor. **67** (1997), no. 2, 181–222.
- [4] A. Bachelot, *The Hawking effect*, Ann. Inst. H. Poincaré Phys. Théor. **70** (1999), no. 1, 41–99.
- [5] A. Bachelot, *Creation of fermions at the charged black-hole horizon*, Ann. Henri Poincaré **1** (2000), no. 6, 1043–1095.
- [6] A. Bachelot, A. Motet-Bachelot, *Les résonances d’un trou noir de Schwarzschild*, Ann. Inst. Henri Poincaré, Physique Théorique **59** (1993), no. 1, 3–68.

- [7] J.C. Baez, *Scattering and the geometry of the solution manifold of  $\square f + \lambda f^3 = 0$* , J. Funct. Anal. **83** (1989), 317–332.
- [8] J.C. Baez, *Scattering for the Yang-Mills equations*, Trans. Amer. Math. Soc. **315** (1989), 2, 823–832.
- [9] J.C. Baez, *Conserved quantities for the Yang-Mills equations*, Adv. Math. **82** (1990), 1, 126–131.
- [10] J.C. Baez, I.E. Segal, Zhou Z.F., *The global Goursat problem and scattering for nonlinear wave equations*, J. Funct. Anal. **93** (1990), 2, 239–269.
- [11] J.C. Baez, Zhou Z.F., *The global Goursat problem on  $\mathbb{R} \times S^1$* , J. Funct. Anal. **83** (1989), 364–382.
- [12] H. Baumgärtel, M. Wollenberg, *Mathematical scattering theory*, Operator theory, advances and applications Vol. 9, Springer Basel AG, 1983.
- [13] N. Besset, *Decay of the local energy for the charged Klein-Gordon equation in the exterior de Sitter-Reissner-Nordström spacetime*, Ann. Henri Poincaré **21** (2020), 8, 2433–2484.
- [14] N. Besset, *Scattering theory for the charged Klein-Gordon equation in the exterior De Sitter-Reissner-Nordström spacetime*, J. Geom. Anal. **31** (2021), 11, 10521–10585.
- [15] N. Besset, D. Häfner, *Existence of exponentially growing finite energy solutions for the charged Klein-Gordon equation on the de Sitter-Kerr-Newman metric*, J. Hyperbolic Differ. Equ. **18** (2021), 2, 293–310.
- [16] J. Borthwick, *Scattering theory for Dirac fields near an extreme Kerr-de Sitter black hole*, Ann. Inst. Fourier **73** (2023), 3, 919–997.
- [17] M. Dafermos, I. Rodnianski, *Lectures on black holes and linear waves*, Evolution equations, 97–205, Clay Math. Proc., 17, Amer. Math. Soc., Providence, RI, 2013, arXiv:0811.0354, 2008.
- [18] Jean Le Rond d’Alembert, *Recherches sur la courbe que forme une corde tendue mise en vibrations*, Histoire de l’Académie Royale des Sciences et Belles Lettres **3** (1747), 214–249.
- [19] T. Daudé, *Scattering theory for massless Dirac fields with long-range potentials*, J. Math. Pures Appl. **84** (2005), pp. 615–665.
- [20] J. Dereziński, C. Gérard, *Scattering theory of classical and quantum N-particle systems*, Springer, 1997.
- [21] J. Dimock, *Scattering for the wave equation on the Schwarzschild metric*, General Relativity and Gravitation, **17** (1985), 4, 353–369.

- [22] J. Dimock, B.S. Kay, *Scattering for massive scalar fields on Coulomb potentials and Schwarzschild metrics*, Classical Quantum Gravity **3** (1986), 71–80.
- [23] J. Dimock, B.S. Kay, *Classical and quantum scattering theory for linear scalar fields on the Schwarzschild metric. II*, J. Math. Phys. **27** (1986), 10, 2520–2525.
- [24] J. Dimock, B.S. Kay, *Classical and Quantum Scattering theory for linear scalar fields on the Schwarzschild metric I*, Ann. Phys. **175** (1987), 366–426.
- [25] F.G. Friedlander, *On the radiation field of pulse solutions of the wave equation*, Proc. Roy. Soc. Ser. A **269** (1962), 53–65.
- [26] F.G. Friedlander, *On the radiation field of pulse solutions of the wave equation II*, Proc. Roy. Soc. Ser. A **279** (1964), 386–394.
- [27] F.G. Friedlander, *On the radiation field of pulse solutions of the wave equation III*, Proc. Roy. Soc. Ser. A **299** (1967), 264–278.
- [28] F.G. Friedlander, *Radiation fields and hyperbolic scattering theory*, Math. Proc. Camb. Phil. Soc. **88** (1980), 483–515.
- [29] H. Friedrich, *Conformal Einstein evolution*, in: J. Frauendiener and H. Friedrich (eds) The Conformal Structure of Space-Time, Lecture Notes in Physics 604, Springer, 2002.
- [30] V. Georgescu, C. Gérard, D. Häfner, *Boundary values of resolvents of selfadjoint operators in Krein spaces*, J. Funct. Anal. **265** (2013), 12, 3245–3304.
- [31] V. Georgescu, C. Gérard, D. Häfner, *Resolvent and propagation estimates for Klein-Gordon equations with non-positive energy*, J. Spectr. Theory **5** (2015), 1, 113–192.
- [32] V. Georgescu, C. Gérard, D. Häfner, *Asymptotic completeness for superradiant Klein-Gordon equations and applications to the de Sitter-Kerr metric*, J. Eur. Math. Soc. (JEMS) **19** (2017), 8, 2371–2444.
- [33] D. Häfner, *Sur la théorie de la diffusion pour l'équation de Klein-Gordon dans la métrique de Kerr*, Dissertationes Mathematicae **421** (2003).
- [34] D. Häfner, *Creation of fermions by rotating charged black holes*, Mém. Soc. Math. Fr., Nouv. Sér. 117 (2009), 1–158.
- [35] D. Häfner, M. Mokdad, J.-P. Nicolas, *Scattering theory for Dirac fields inside a Reissner-Nordström-type black hole*, 2020, arXiv:2007.16139, J. Math. Phys. **62** (2021), 8, 081503, 15 pp.
- [36] D. Häfner, J.-P. Nicolas, *Scattering of massless Dirac fields by a slow Kerr black hole*, Reviews in Mathematical Physics **16** (2004), 1, 29–123.

- [37] L. Hörmander, *A remark on the characteristic Cauchy problem*, J. Funct. Anal. **93** (1990), 270-277.
- [38] J. Joudioux, *Conformal scattering for a nonlinear wave equation*, J. Hyperbolic Differ. Equ. **9** (2012), 1, 1–65.
- [39] J. Joudioux, *Hörmander’s method for the characteristic Cauchy problem and conformal scattering for a nonlinear wave equation*, Lett. Math. Phys. **110** (2020), 6, 1391–1423.
- [40] P.D. Lax, R.S. Phillips, *Scattering theory*, Academic Press 1967.
- [41] J. Leray, *Hyperbolic differential equations*, Princeton Institute for Advanced Studies 1953.
- [42] L. J. Mason, J.-P. Nicolas, *Conformal scattering and the Goursat problem*, Journal of Hyperbolic Differential Equations, **1** (2) (2004), 197–233.
- [43] L.J. Mason, J.-P. Nicolas, *Regularity at spacelike and null infinity*, J. Inst. Math. Jussieu, **8** (2009), 1, 179–208.
- [44] L.J. Mason, J.-P. Nicolas, *Peeling of Dirac and Maxwell fields on a Schwarzschild background*, J. Geom. Phys. **62** (2012), 867–889.
- [45] F. Melnyk, *Scattering on Reissner-Nordström metric for massive charged spin 1/2 fields*, Ann. H Poincaré **4**(5) (2003), 813–846.
- [46] F. Melnyk, *The Hawking effect for spin 1/2 fields*, Commun. Math. Phys. **244**(3) (2004) 483–525.
- [47] M. Mokdad, *Conformal scattering of Maxwell fields on Reissner-Nordström-de Sitter black hole spacetimes*, Ann. Inst. Fourier **69** (2019), 5, 2291–2329.
- [48] M. Mokdad, *Decay of Maxwell fields on Reissner-Nordström-de Sitter black holes*, Lett. Math. Phys. **110** (2020), 7, 1961–2018.
- [49] M. Mokdad, *Conformal scattering and the Goursat problem for Dirac fields in the interior of charged spherically symmetric black holes*, Rev. Math. Phys. **34** (2022), 1, Article ID 2150037, 16 pages.
- [50] M. Mokdad, R. Nasser, *On the scattering of waves inside charged spherically symmetric black holes*, Ann. Henri Poincaré **23** (2022), 9, 3191–3220.
- [51] M. Mokdad, M. Provcic, *Scattering of Dirac Fields in the Interior of Kerr-Newman(-Anti)-de Sitter Black Holes*, arXiv:2303.03835, 2023.
- [52] J.-P. Nicolas, *Scattering of linear Dirac fields by a spherically symmetric Black-Hole*, Ann. Inst. Henri Poincaré - Physique Théorique, **62** (1995), 2, pp. 145–179.

- [53] J.-P. Nicolas, *Conformal scattering on the Schwarzschild metric*, arXiv:1312.1386, Ann. Inst. Fourier (Grenoble) **66** (2016), 3, 1175–1216.
- [54] J.-P. Nicolas, G. Taujanskas, *Conformal scattering of Maxwell potentials*, arXiv:2211.14579, 2022.
- [55] R. Penrose, *Zero rest-mass fields including gravitation: asymptotic behavior*, Proc. Roy. Soc. **A284** (1965), p. 159–203.
- [56] R. Penrose and W. Rindler, *Spinors and space-time*, Vol. 1 and Vol. 2, Cambridge University Press, 1984 & 1986.
- [57] M. Reed and B. Simon, *Methods of modern mathematical physics*, Vols. 1-4, Academic Press, 1972-1979.
- [58] B. Thaller, *The Dirac equation*, Theoretical and Mathematical Physics, Springer, 1992.