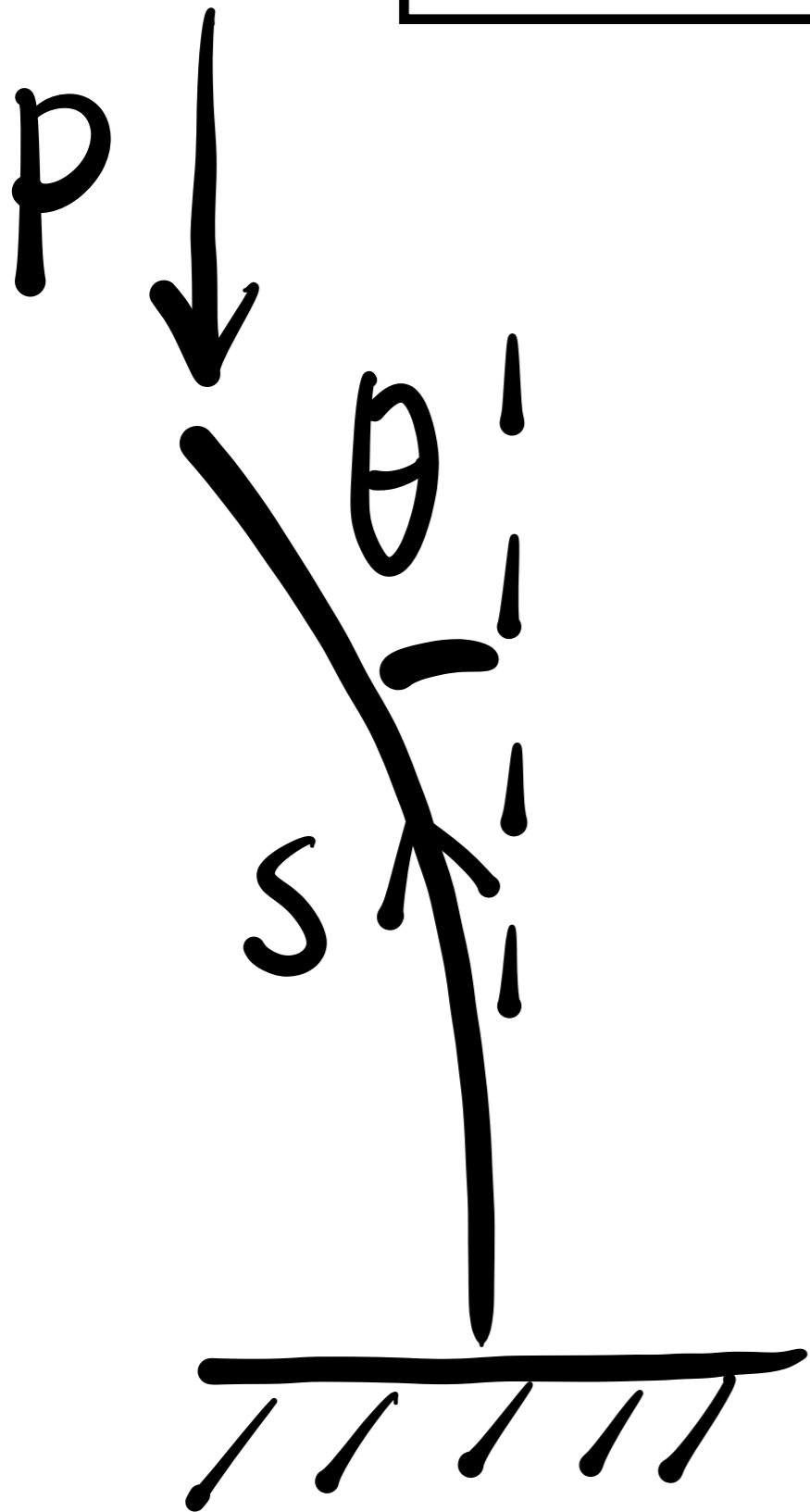


# Elastic beams as dynamical systems

sebastien neukirch  
d'alembert institute for mechanics  
sorbonne university & CNRS  
Paris, France

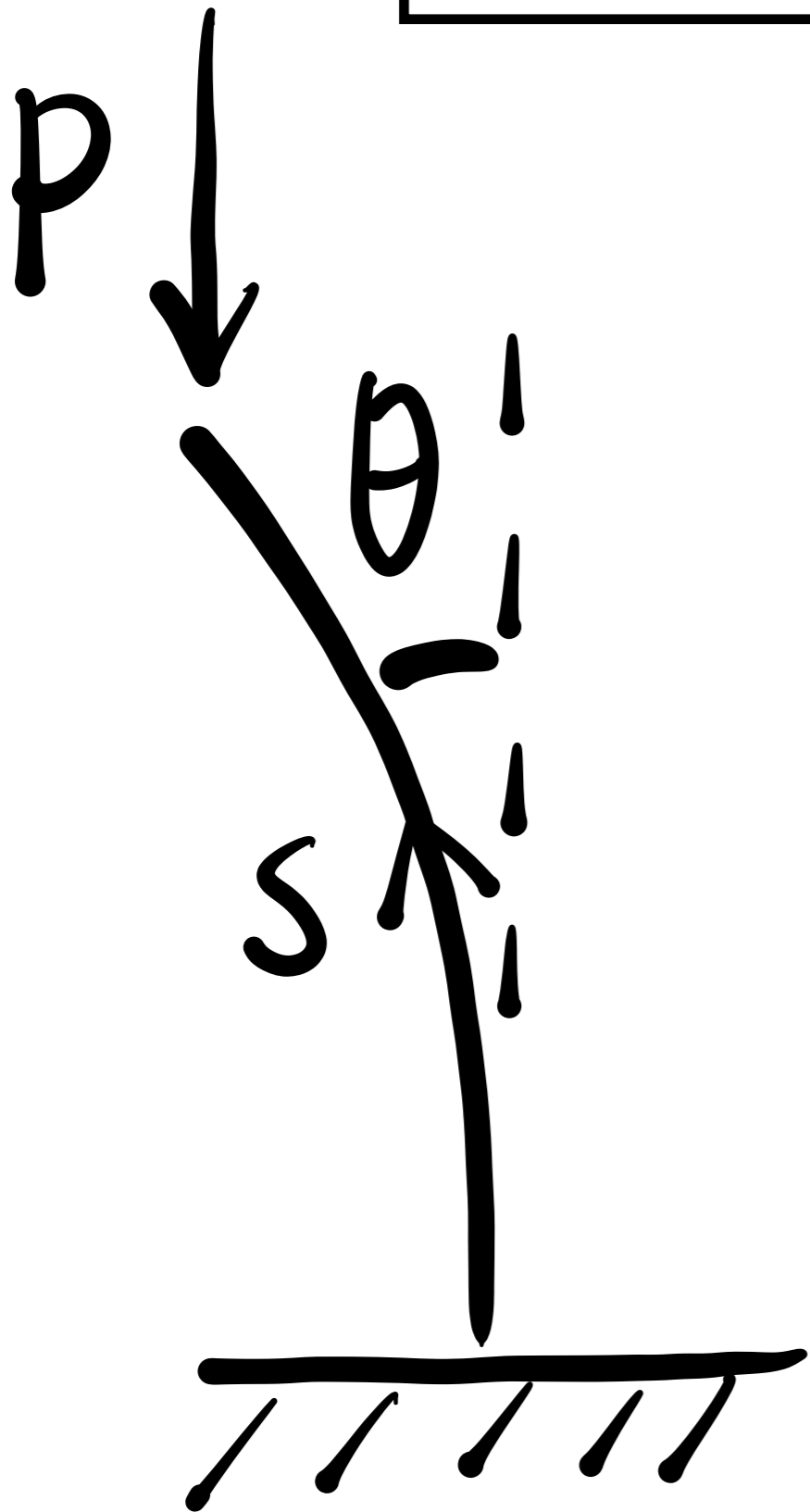
# Buckling of a beam



$$\theta'' + P \sin \theta(s) = 0$$

$$(\quad)' \equiv \frac{d}{ds} \quad (EI = 1)$$

# Buckling of a beam

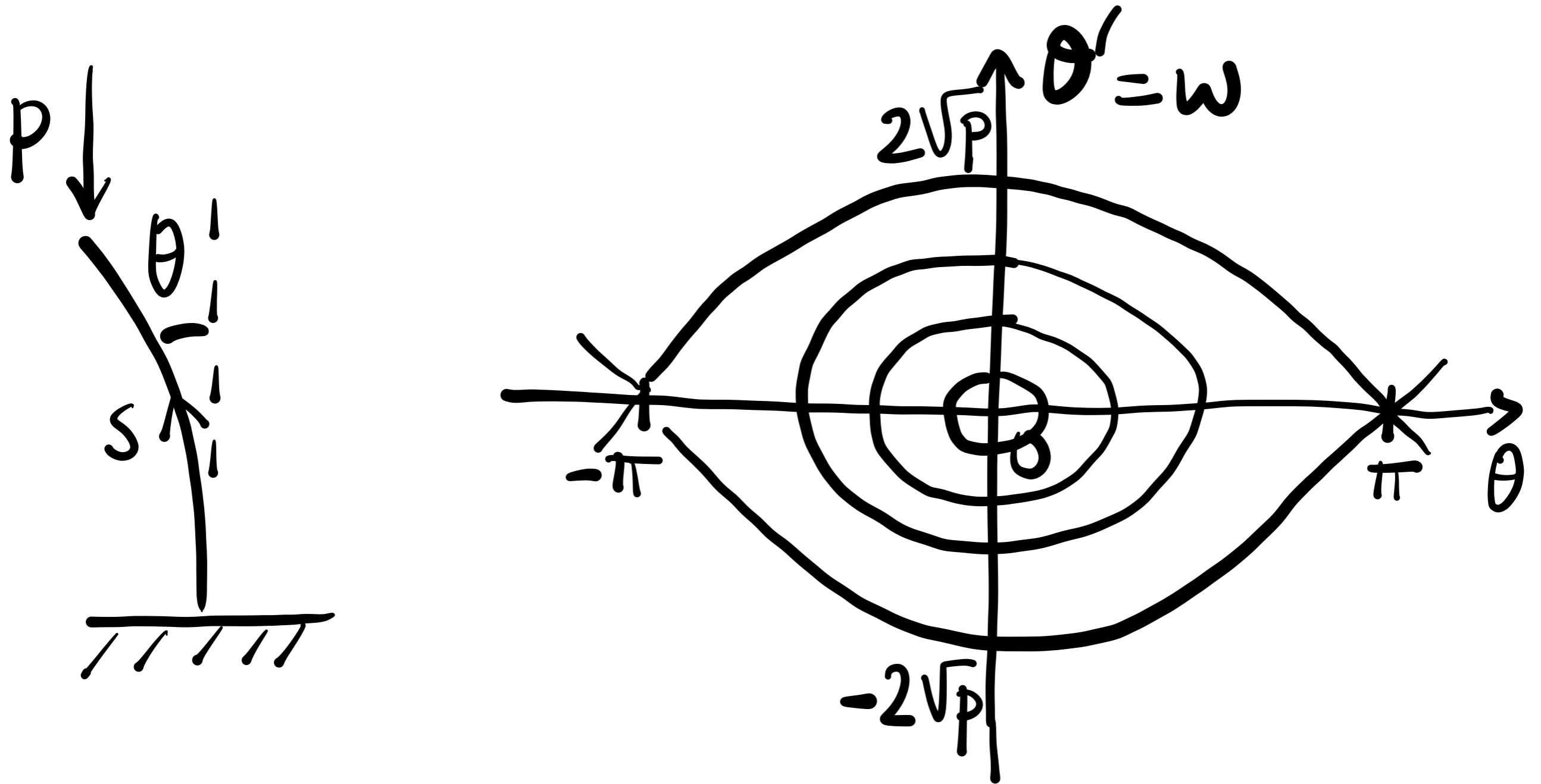


$$\theta'' + P \sin \theta(s) = 0$$

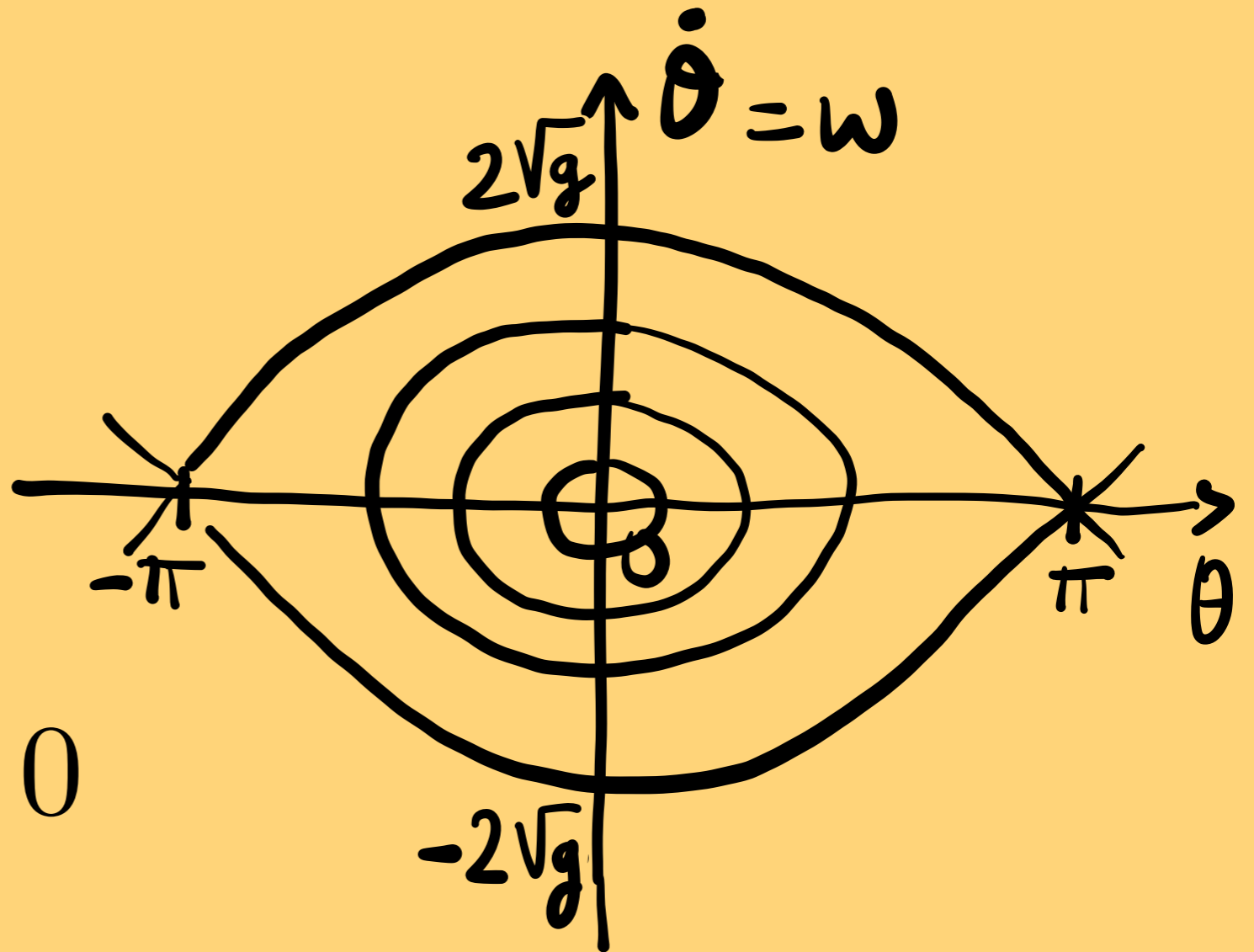
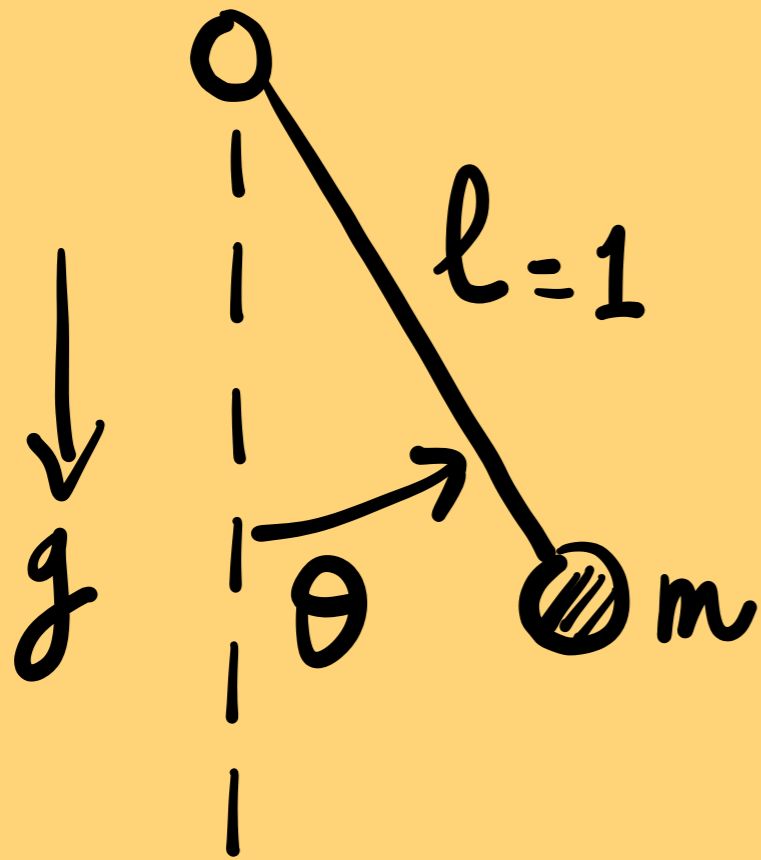


$$\begin{cases} \theta' = \omega \\ \omega' = -P \sin \theta \end{cases}$$

# Buckling of a beam



# Oscillations of a pendulum

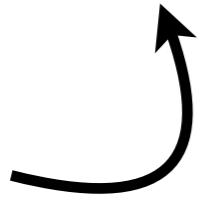


$$\ddot{\theta} + g \sin \theta(t) = 0$$

$$(\dot{\quad}) \equiv \frac{d}{dt}$$

# Kirchhoff static-dynamic analogy

(1824-1887)



but a limited analogy ...

**Elastica**

**(BVP) Boundary  
Value  
Problem**

**Pendulum**

**(IVP) Initial  
Value  
Problem**

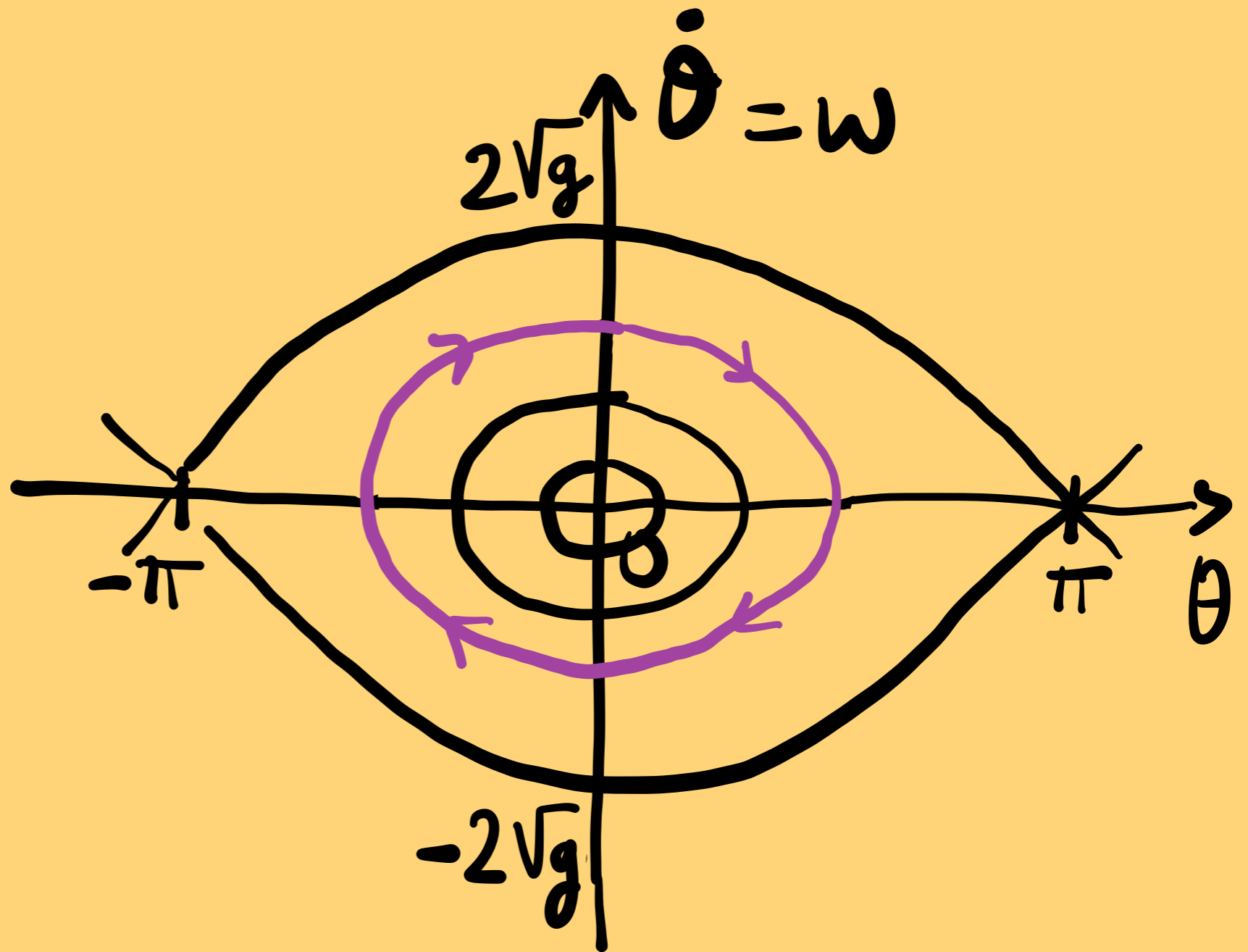
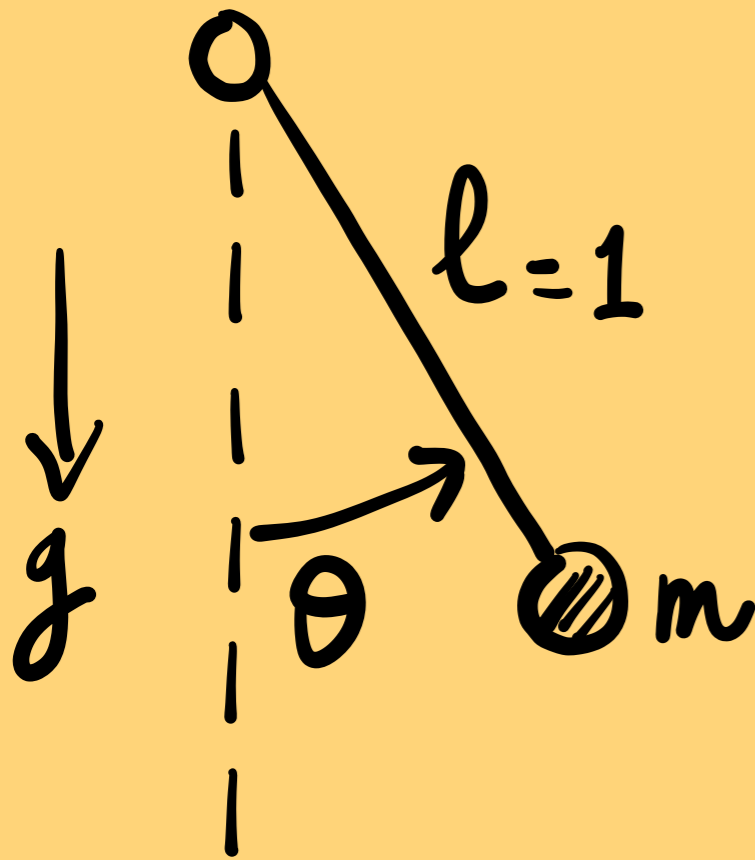
*G. Kirchhoff 1877, A. Love 1927*

*see e.g.*

*Nizette+Goriely J Math Phys 2001*

*Roman+Gay+Clanet arXiv:2006.02742*

# Oscillations of a pendulum



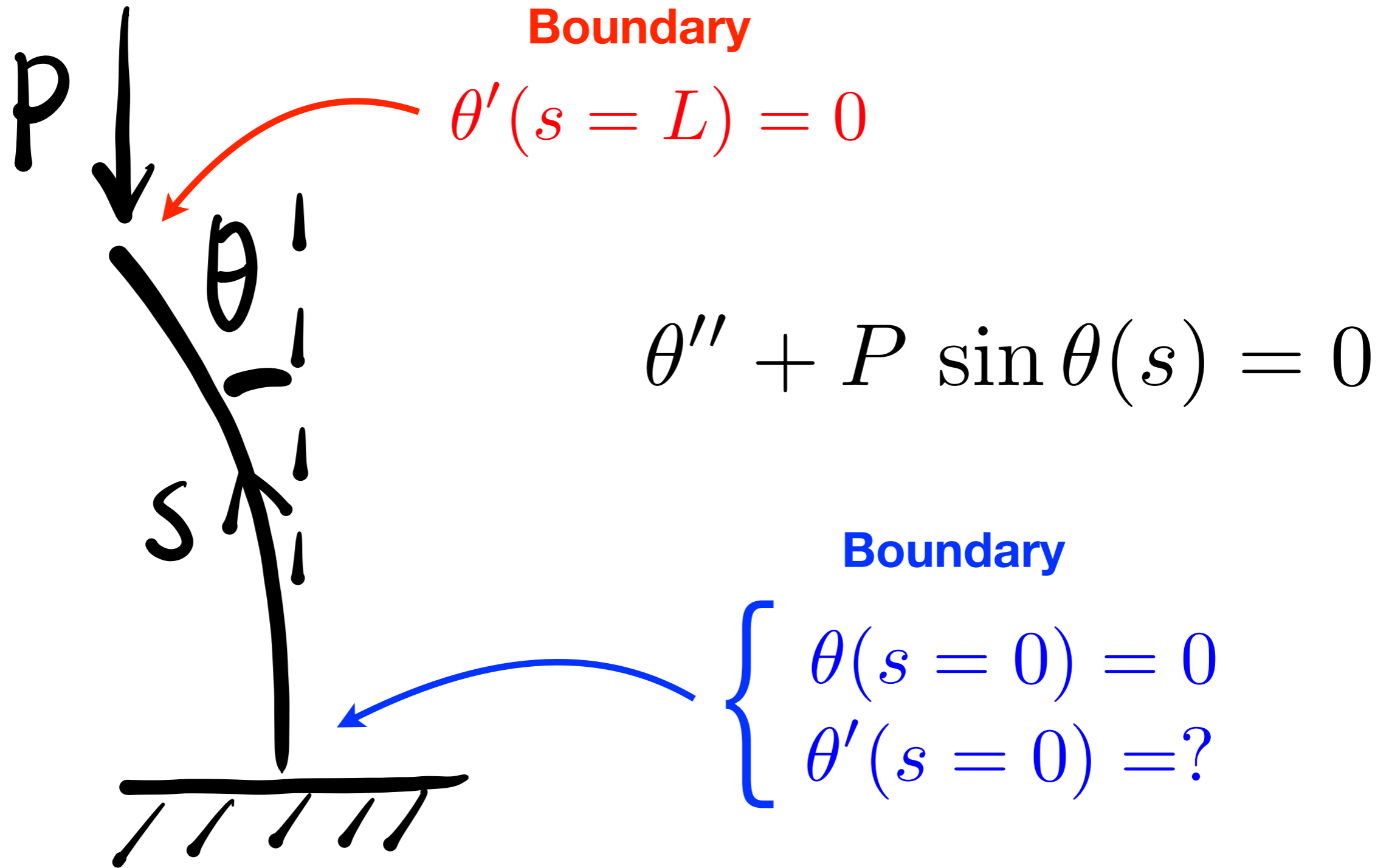
initial-time values

$$\theta(t=0) = 0.2$$

$$\dot{\theta}(t=0) = 0.3$$

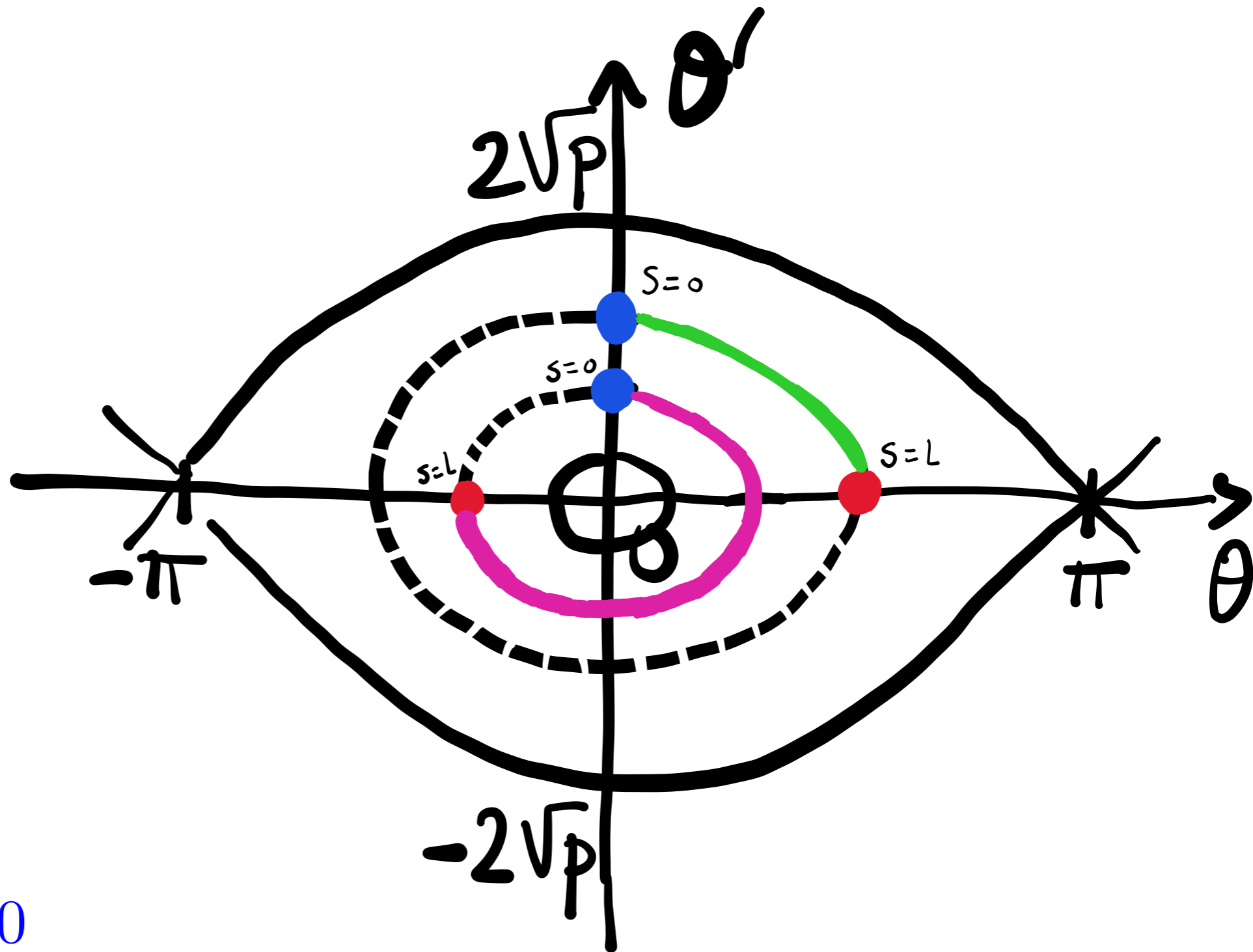
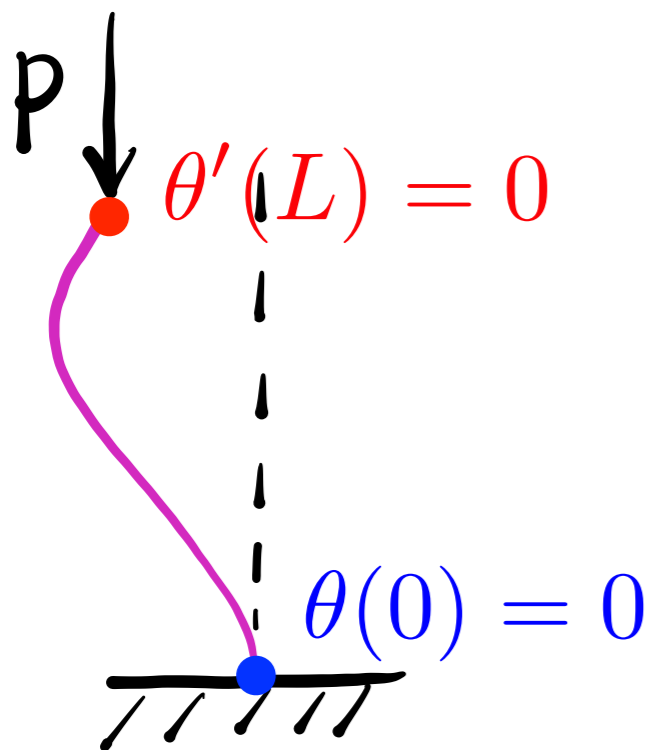
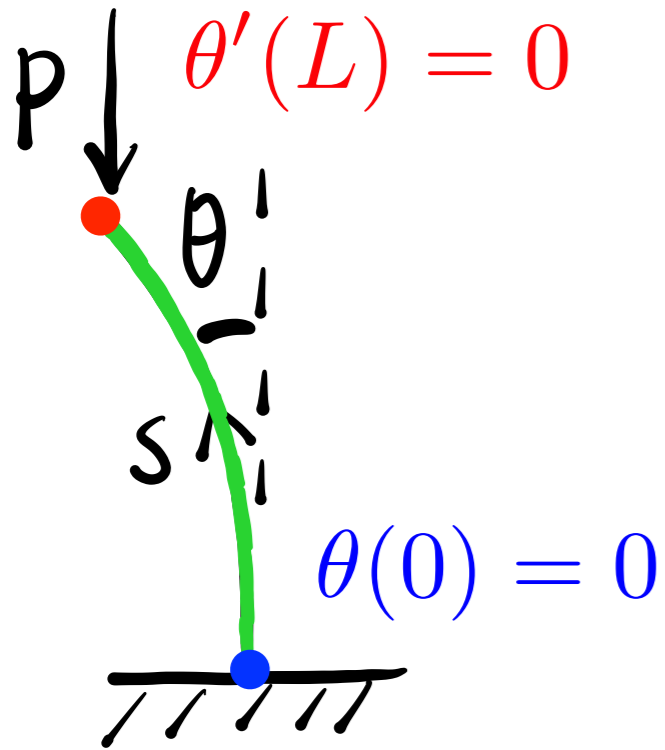
Cauchy problem : solution is unique

# Buckling of a beam





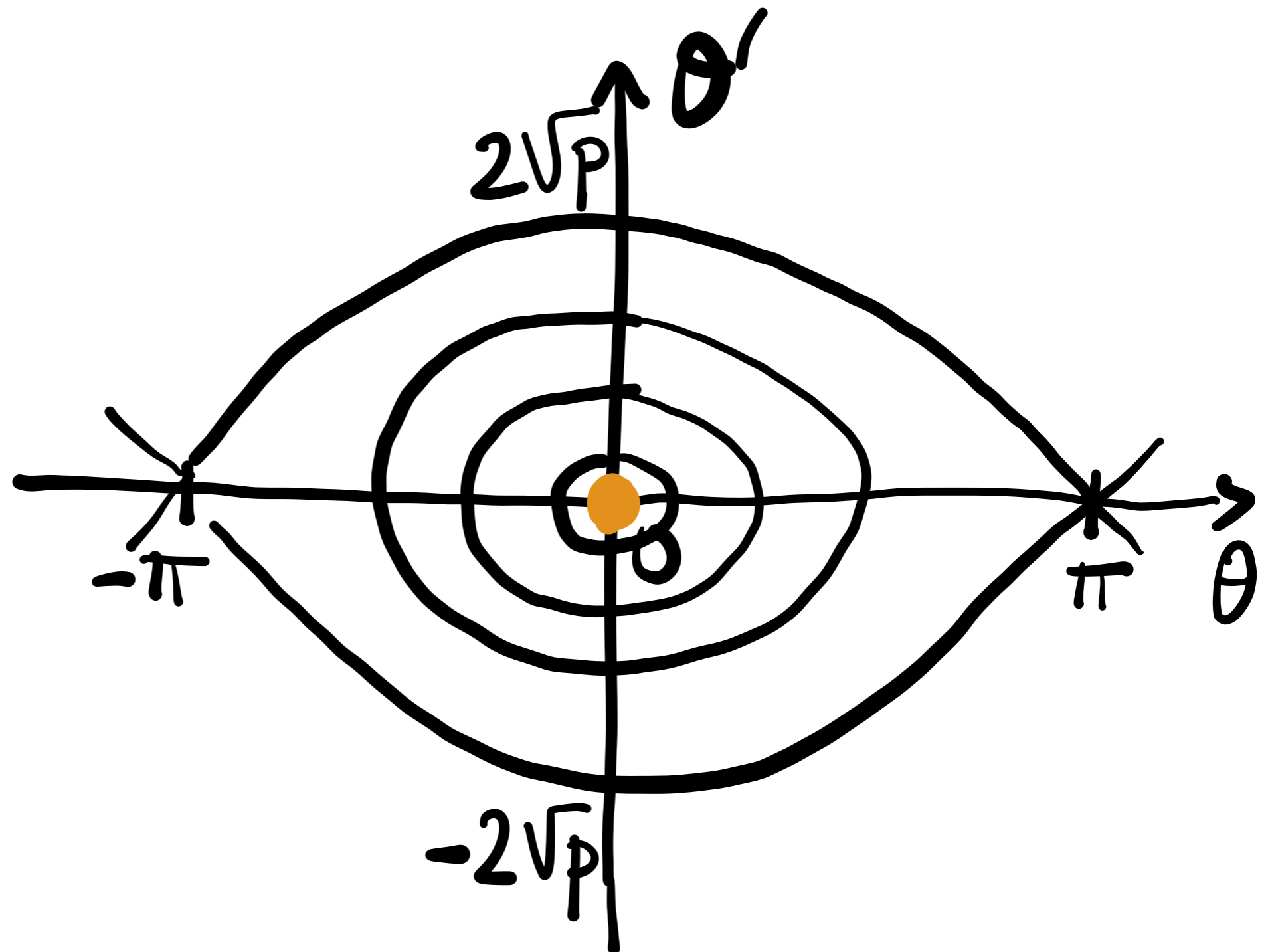
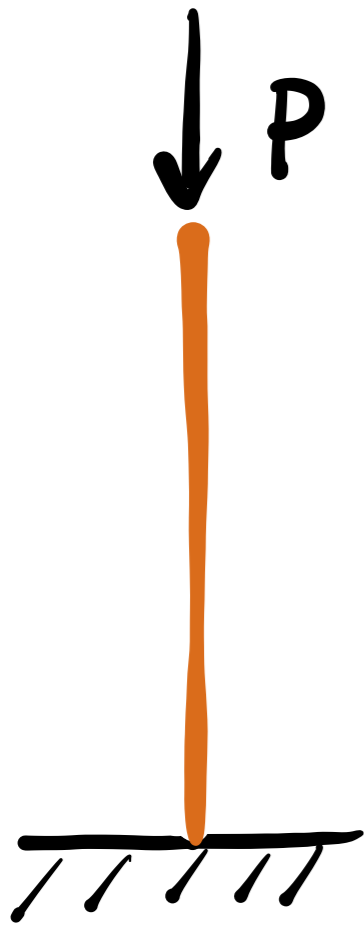
# Buckling of a beam



generally, there are multiple solutions

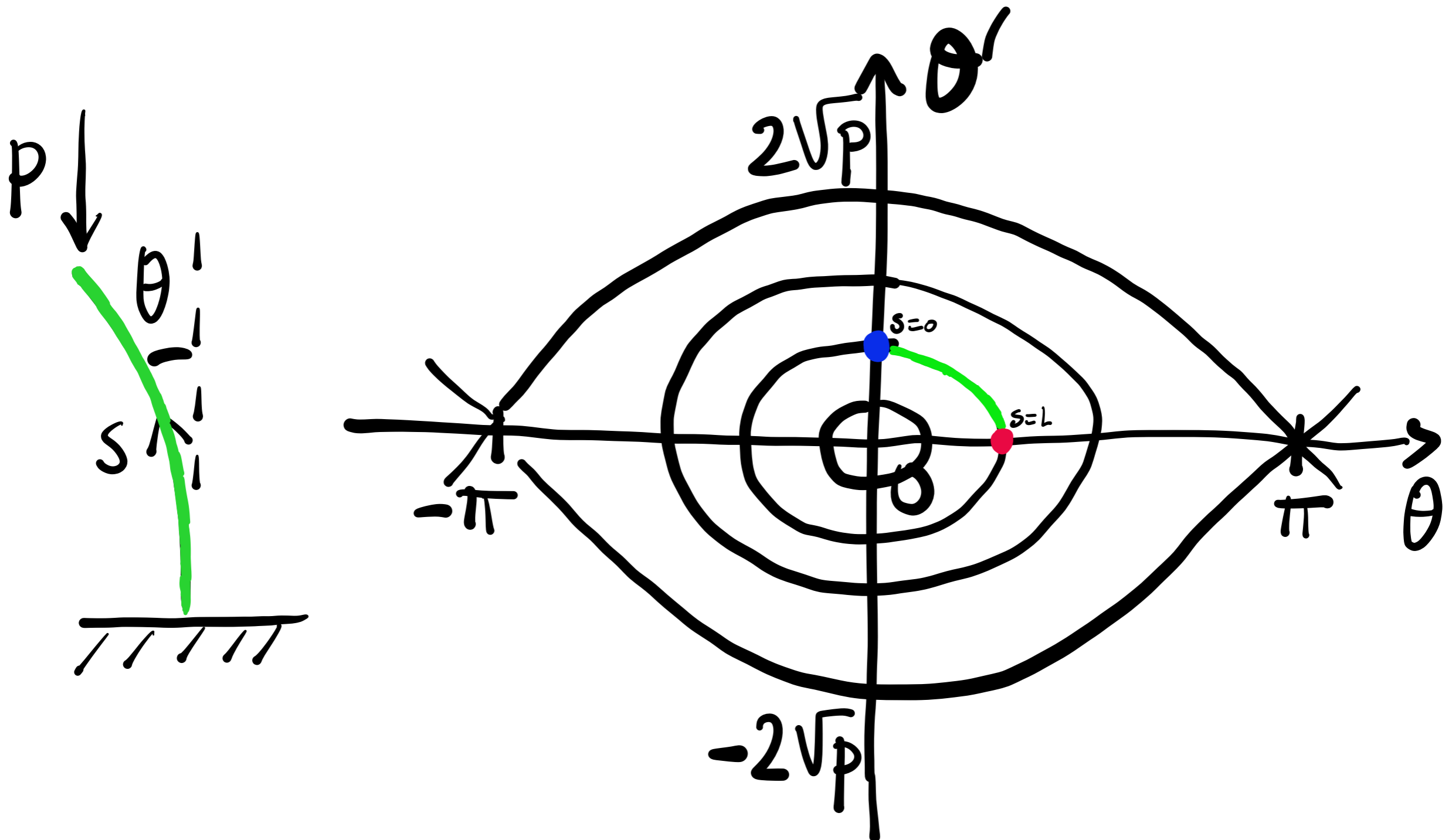
# Buckling of a beam

*There is always the trivial solution*



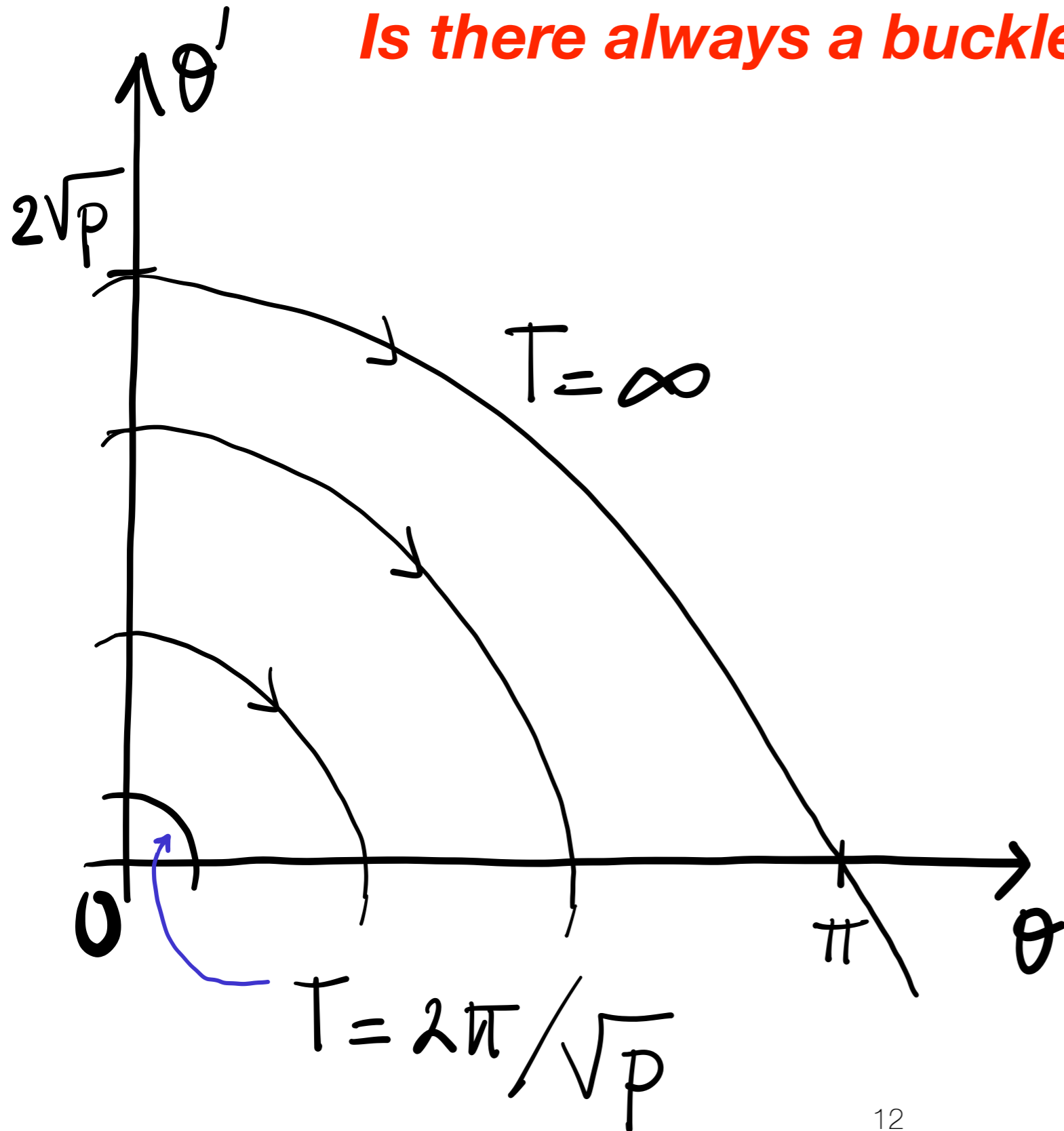
# Buckling of a beam

*Is there always a buckled solution?*



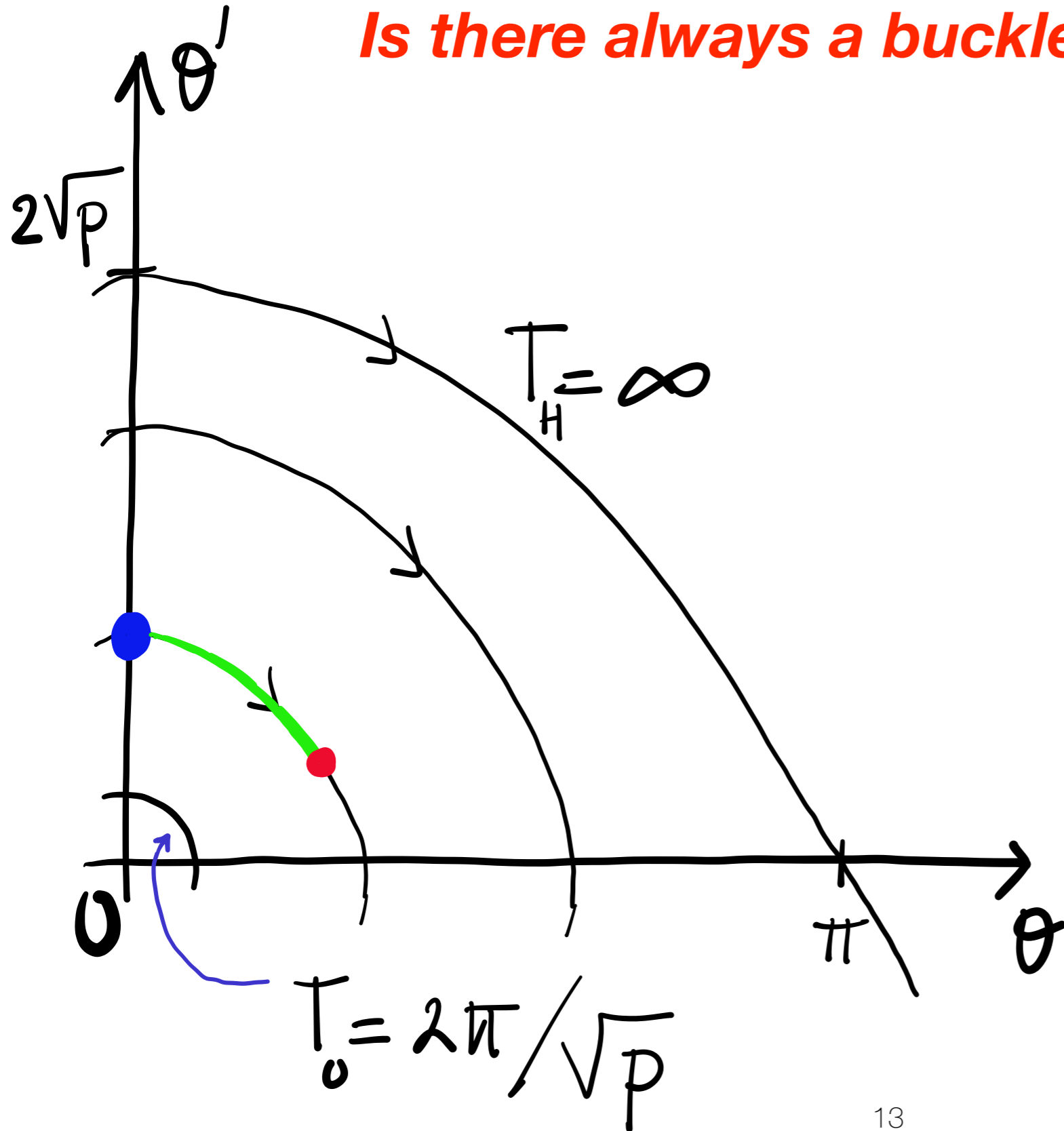
# Buckling of a beam

*Is there always a buckled solution?*



# Buckling of a beam

*Is there always a buckled solution?*



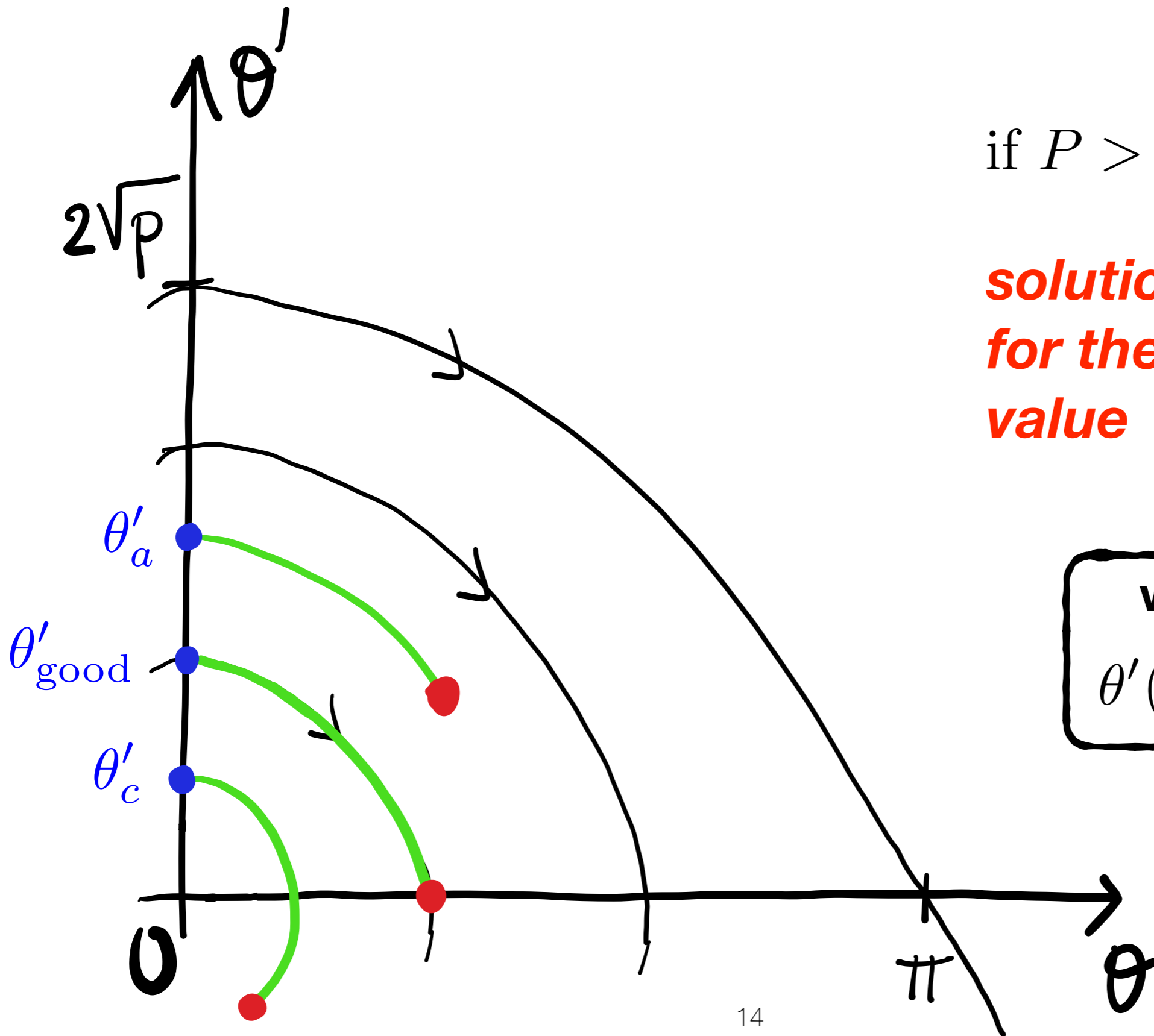
$$\text{if } \frac{T_0}{4} > L$$

$$\Rightarrow \frac{\pi}{2\sqrt{P}} > L$$

$$\Rightarrow \frac{\pi^2}{4L^2} > P$$

***no solution!***

# Buckling of a beam



if  $P > \frac{\pi^2}{4L^2}$

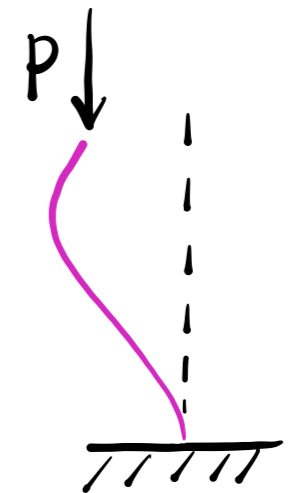
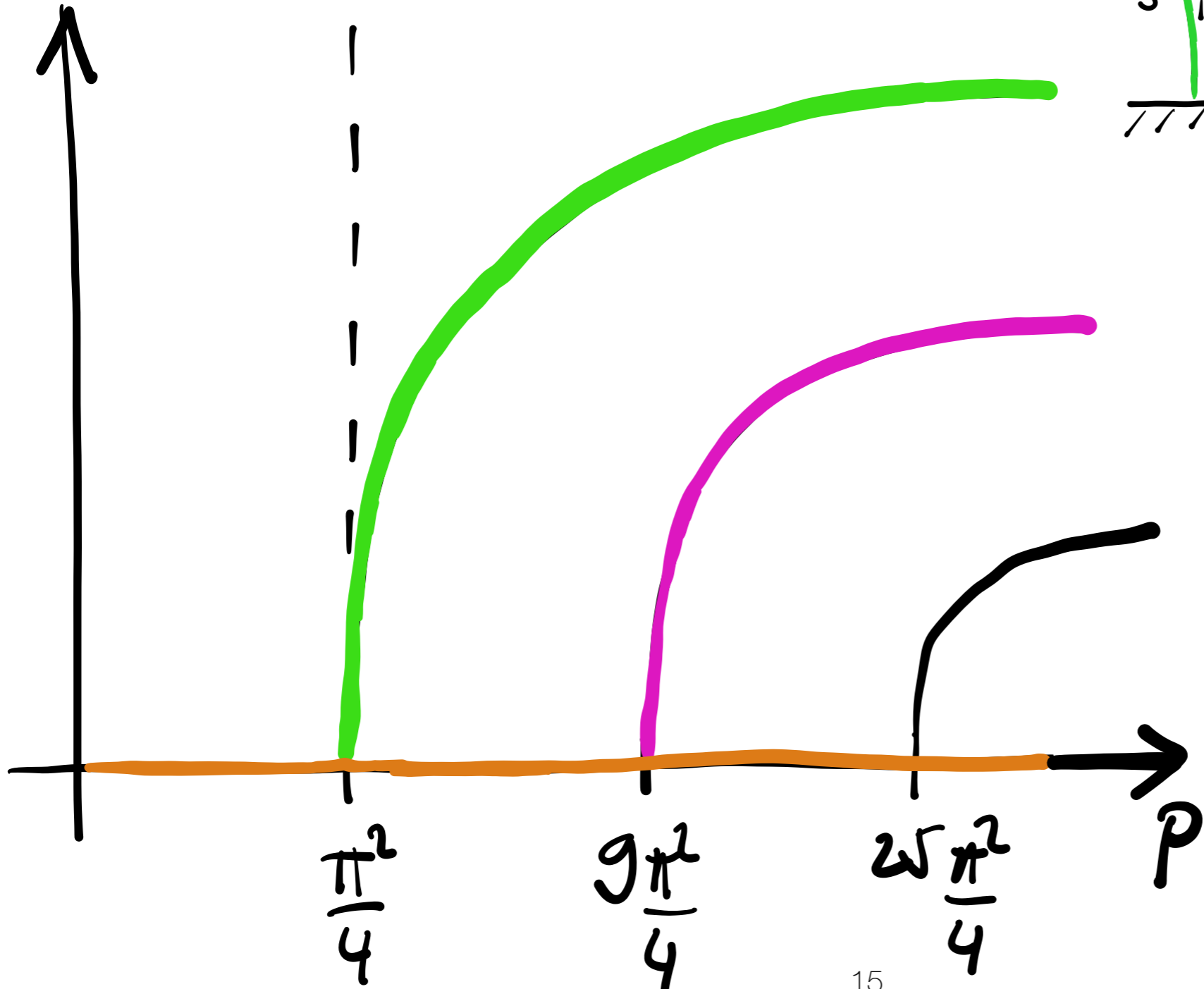
***solution only  
for the right  $\theta'(0)$   
value***

**we must take**  
 $\theta'(0) = \theta'_{\text{good}}(P)$

# All solutions: bifurcation diagram

(with no loss of generality, we set  $L = 1$ )

$\theta'_{\text{good}}$



(L = 1)

# All solutions: bifurcation diagram

**Boundary Value Problem**

$$EI \theta'' + P \sin \theta(s) = 0$$

$$\theta(0) = 0, \quad \theta'(1) = 0$$

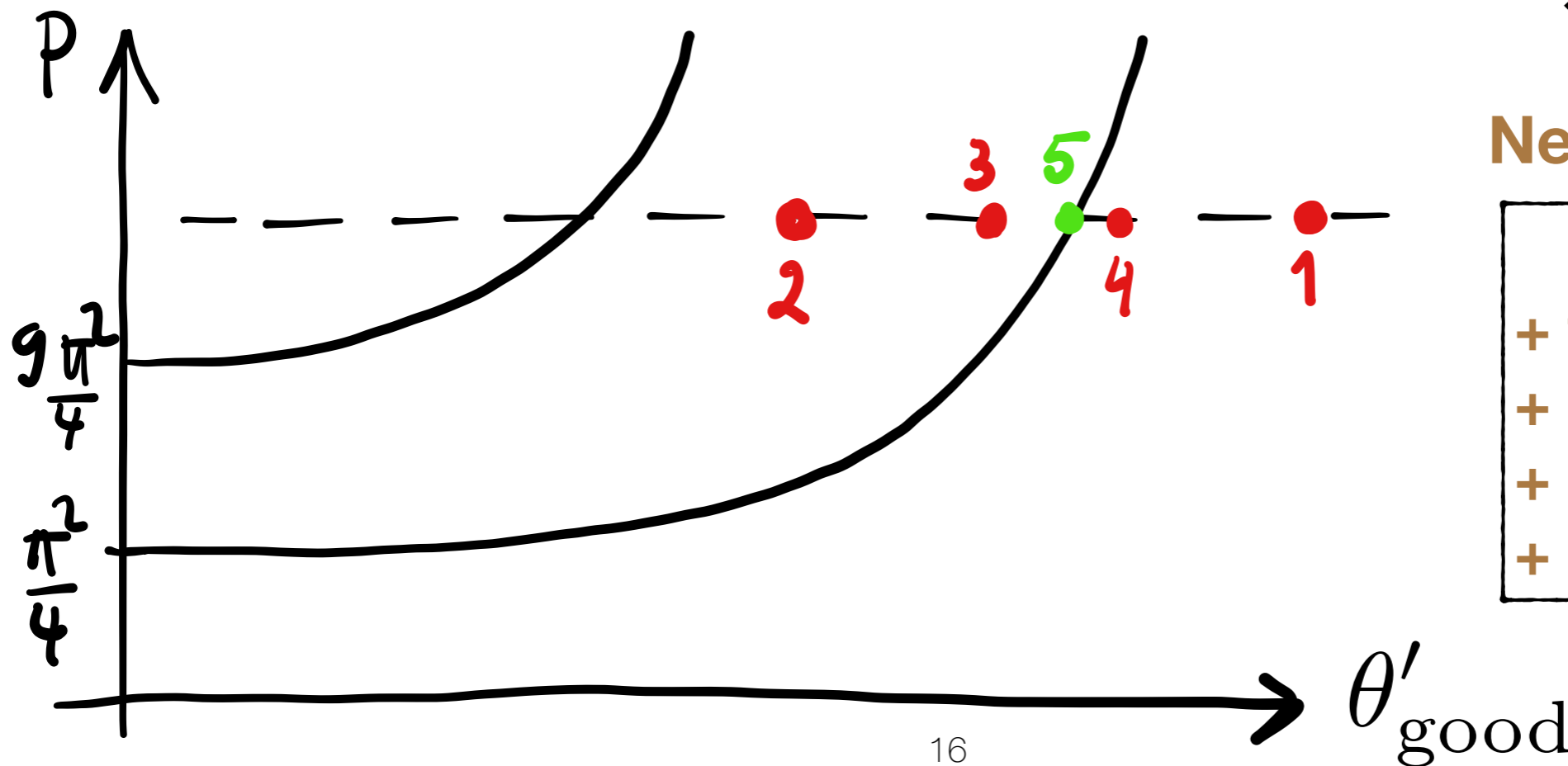
**Initial Value Problem**

$$EI \theta'' + P \sin \theta(s) = 0$$

$$\theta(0) = 0, \quad \theta'(0) = \theta'_{\text{good}}$$

the solution:  $\theta(s) = \phi(s, P, \theta'_{\text{good}})$

boundary condition:  $\phi_{,s}(1, P, \theta'_{\text{good}}) = \varphi(P, \theta'_{\text{good}}) = 0$

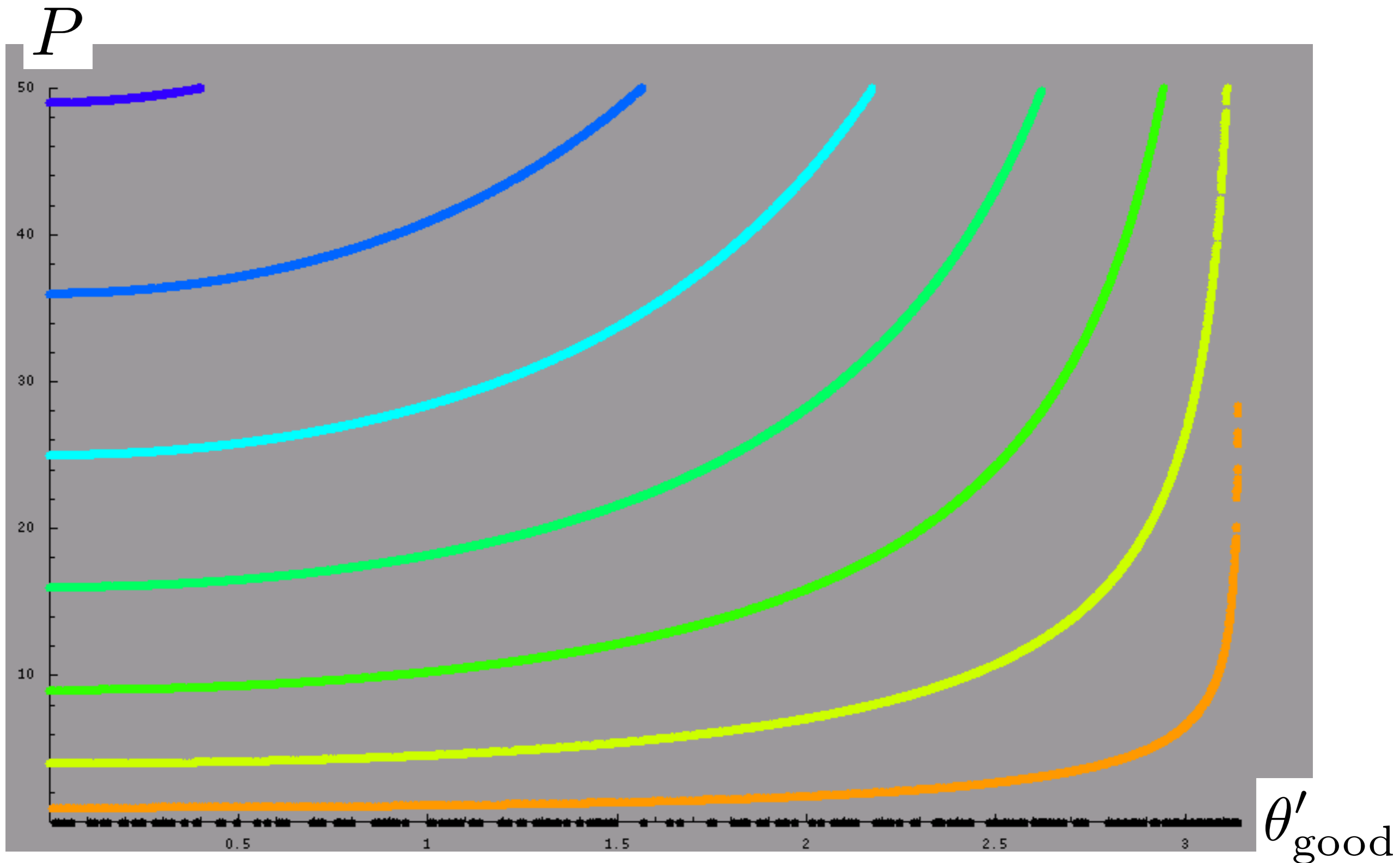


## Newton-Raphson

- Recipe**
- + fix P
  - + seed  $\theta'_{\text{good}}$
  - + newton steps
  - + solution found!



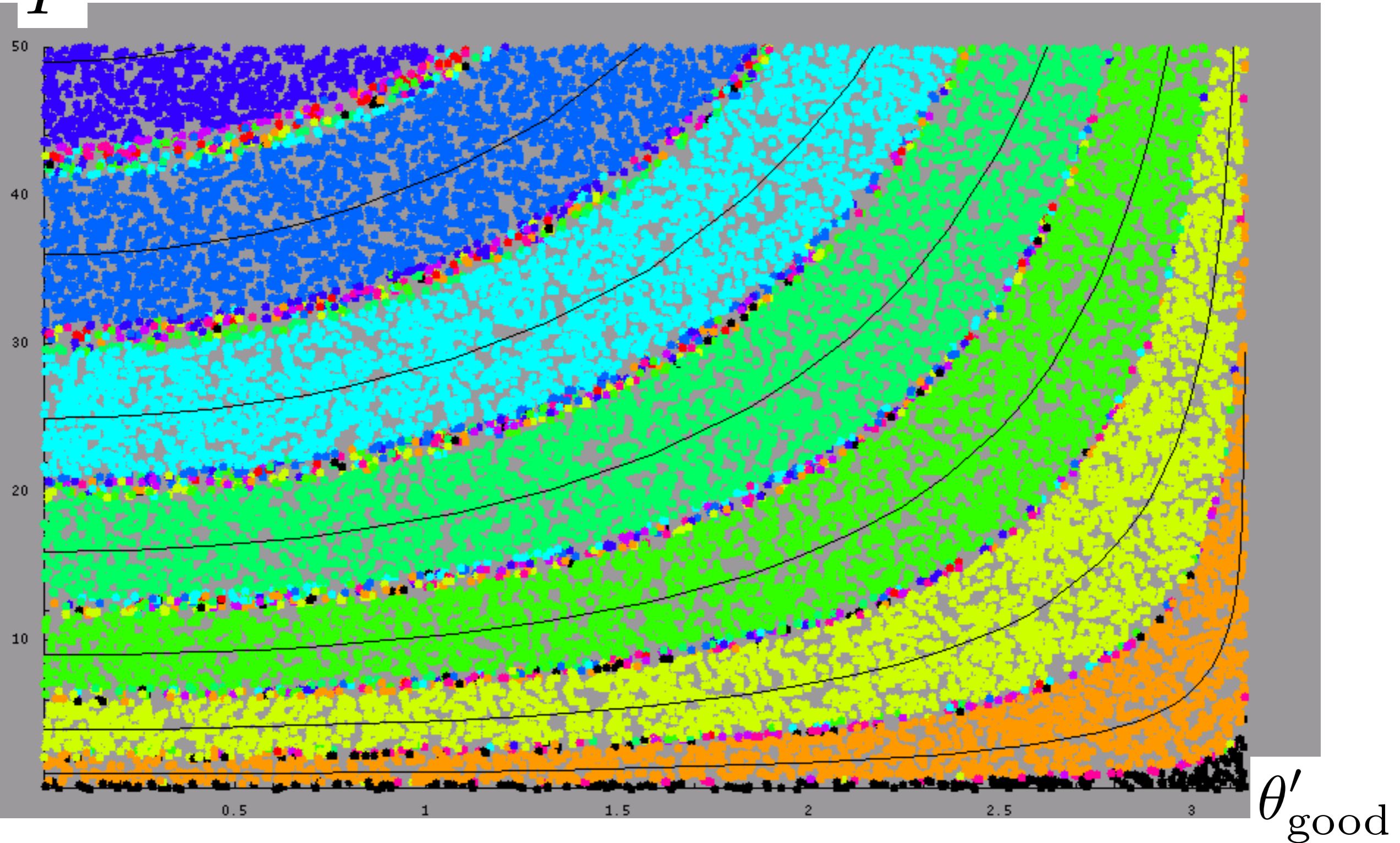
# Bifurcation diagram: cloud search



# Bifurcation diagram: cloud search

20 000 points (few seconds)

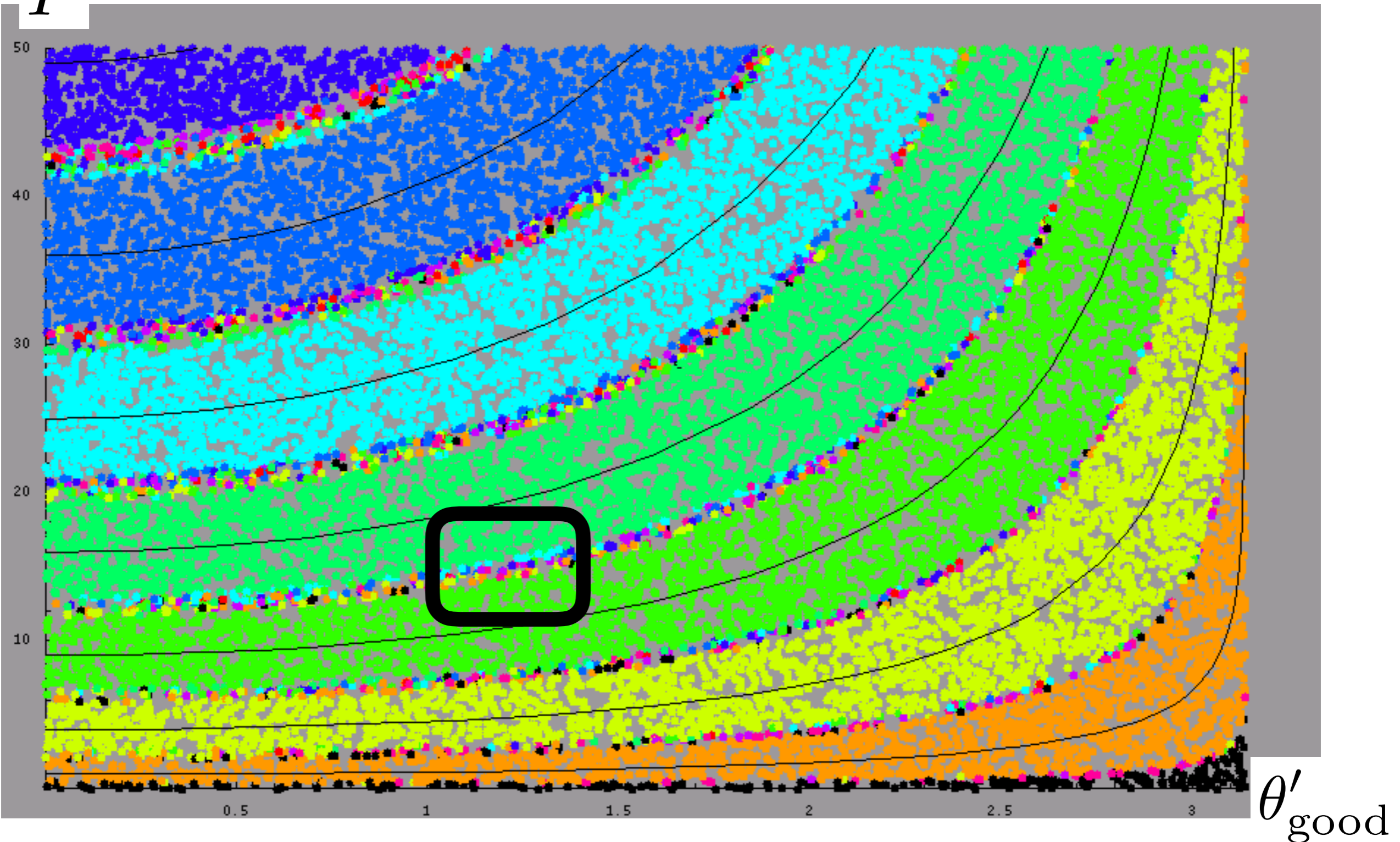
$P$



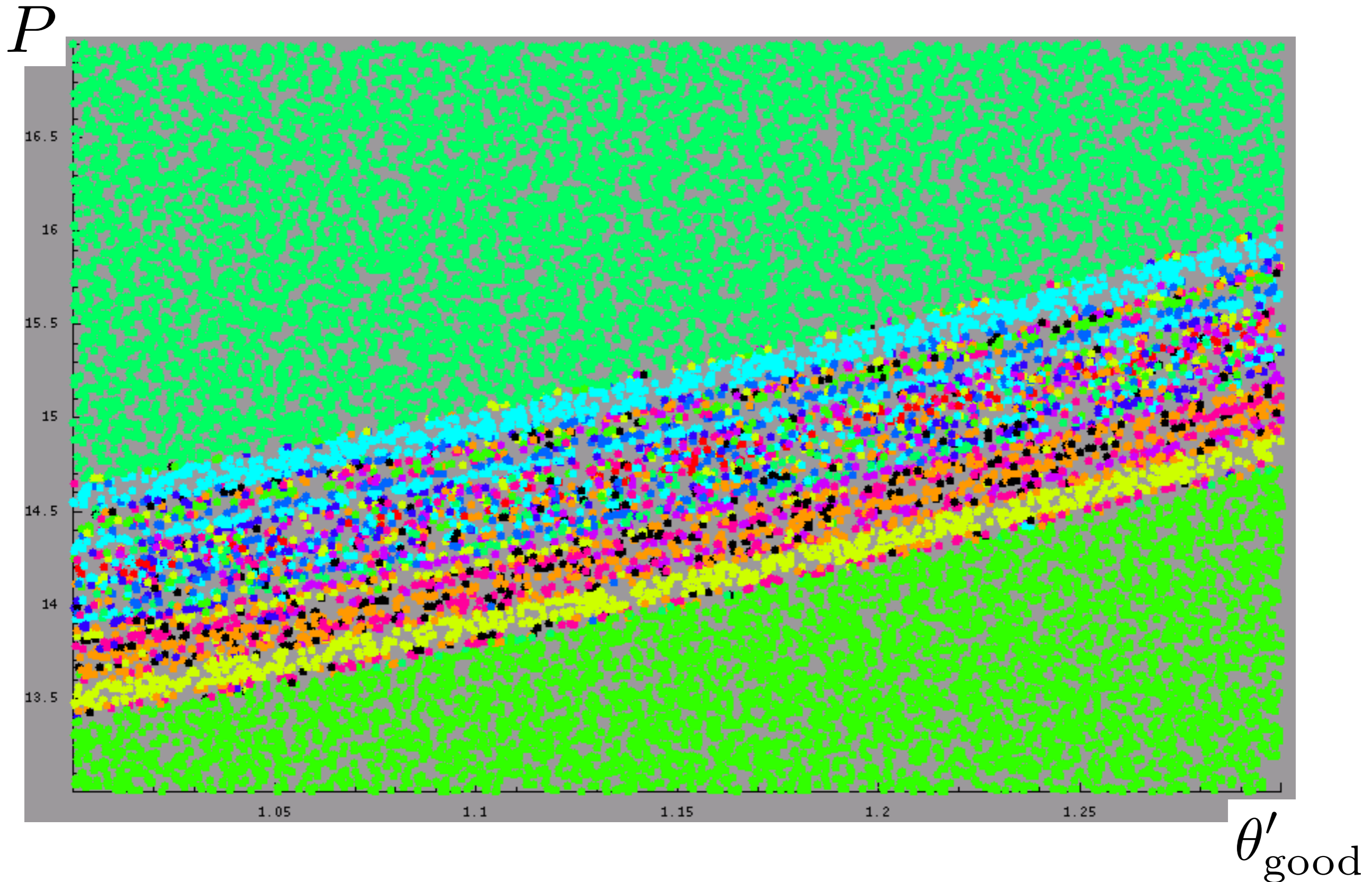
# Bifurcation diagram: cloud search

20 000 points (few seconds)

$P$



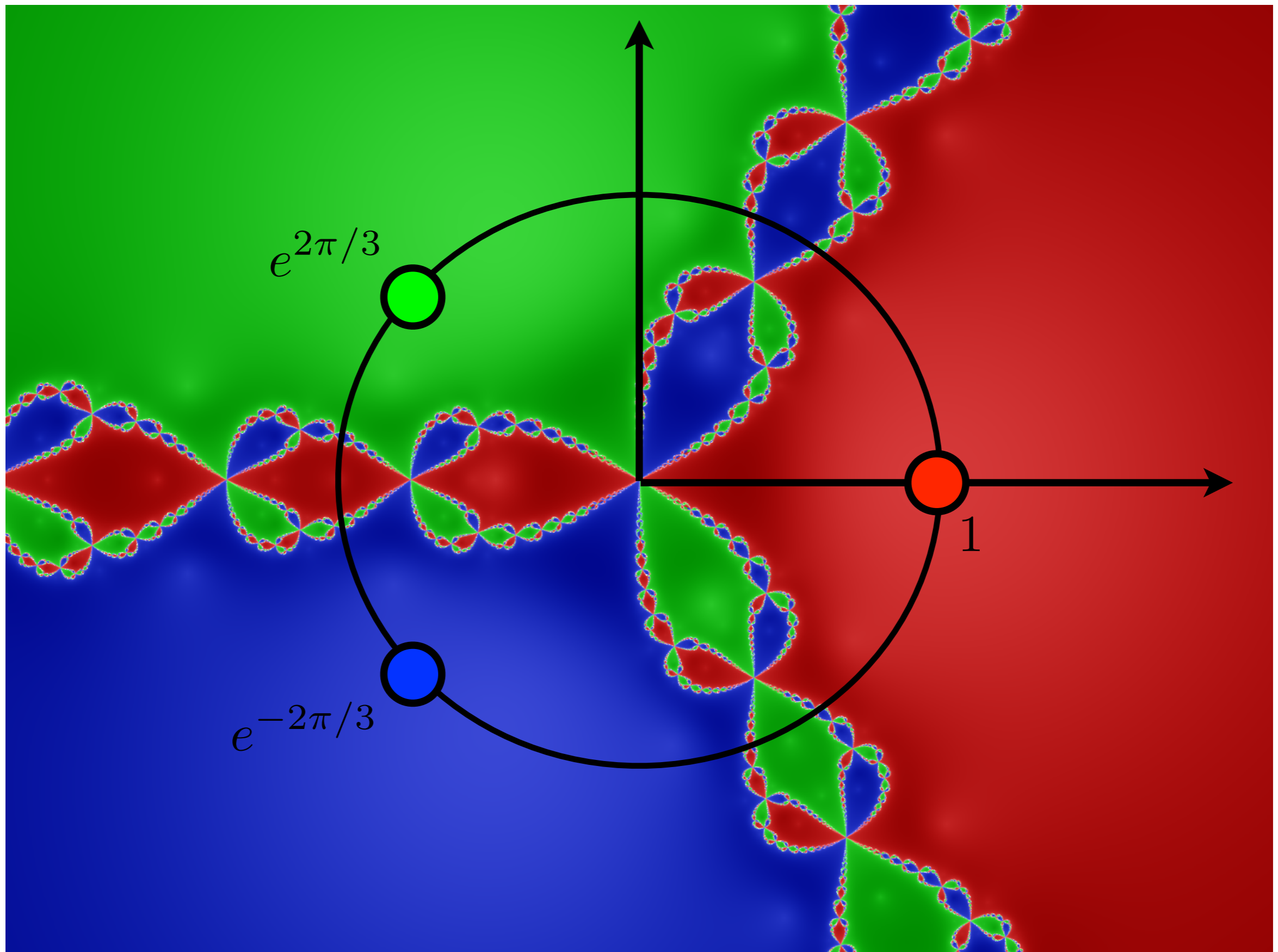
# Bifurcation diagram: fractal basin boundaries



# Cubic roots of unity

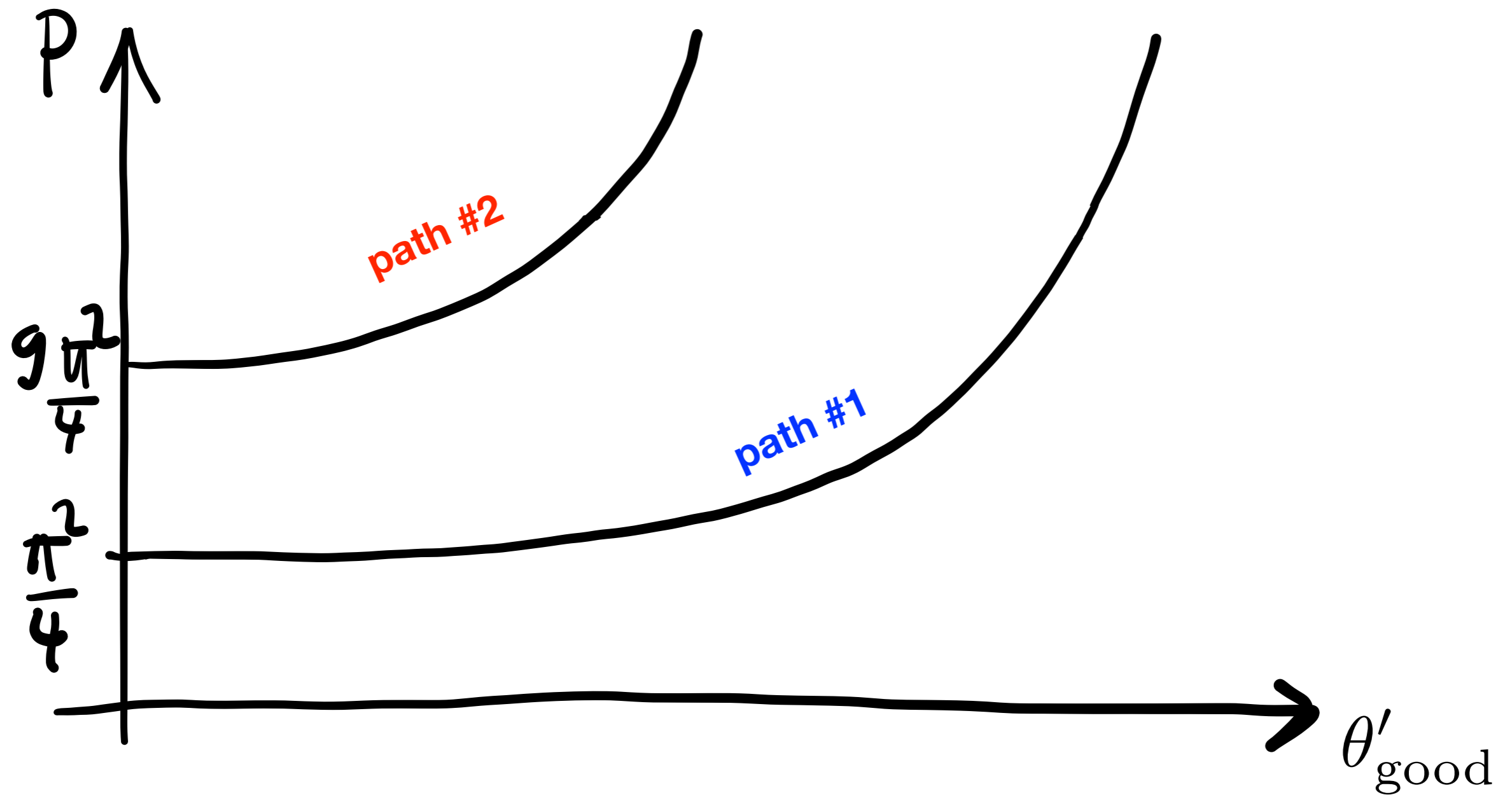
$$z^3 = 1$$

wikipedia.org



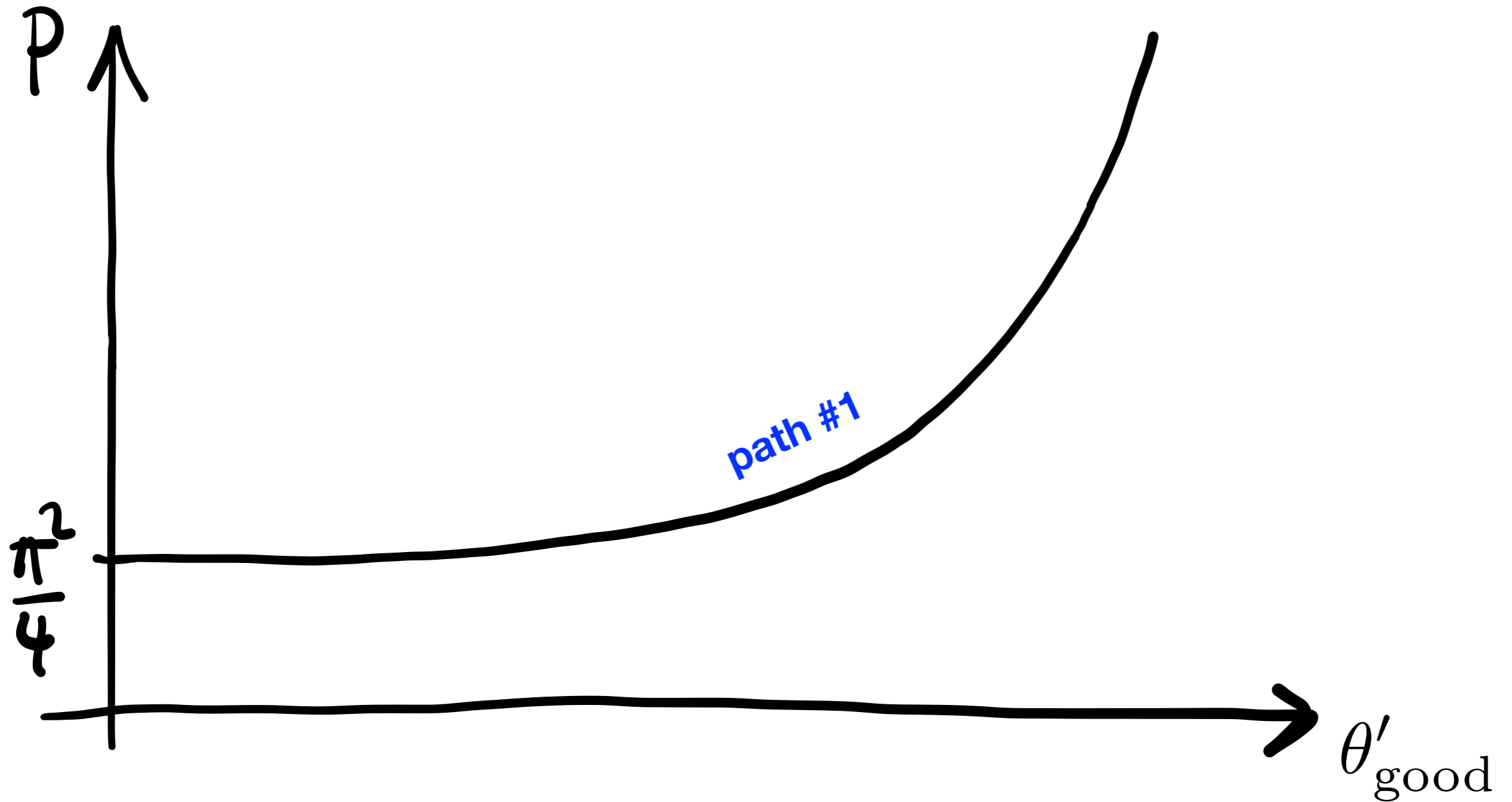
# Numerical path following (continuation)

when number of unknowns is large ( $>10$ ), cloud search is too long



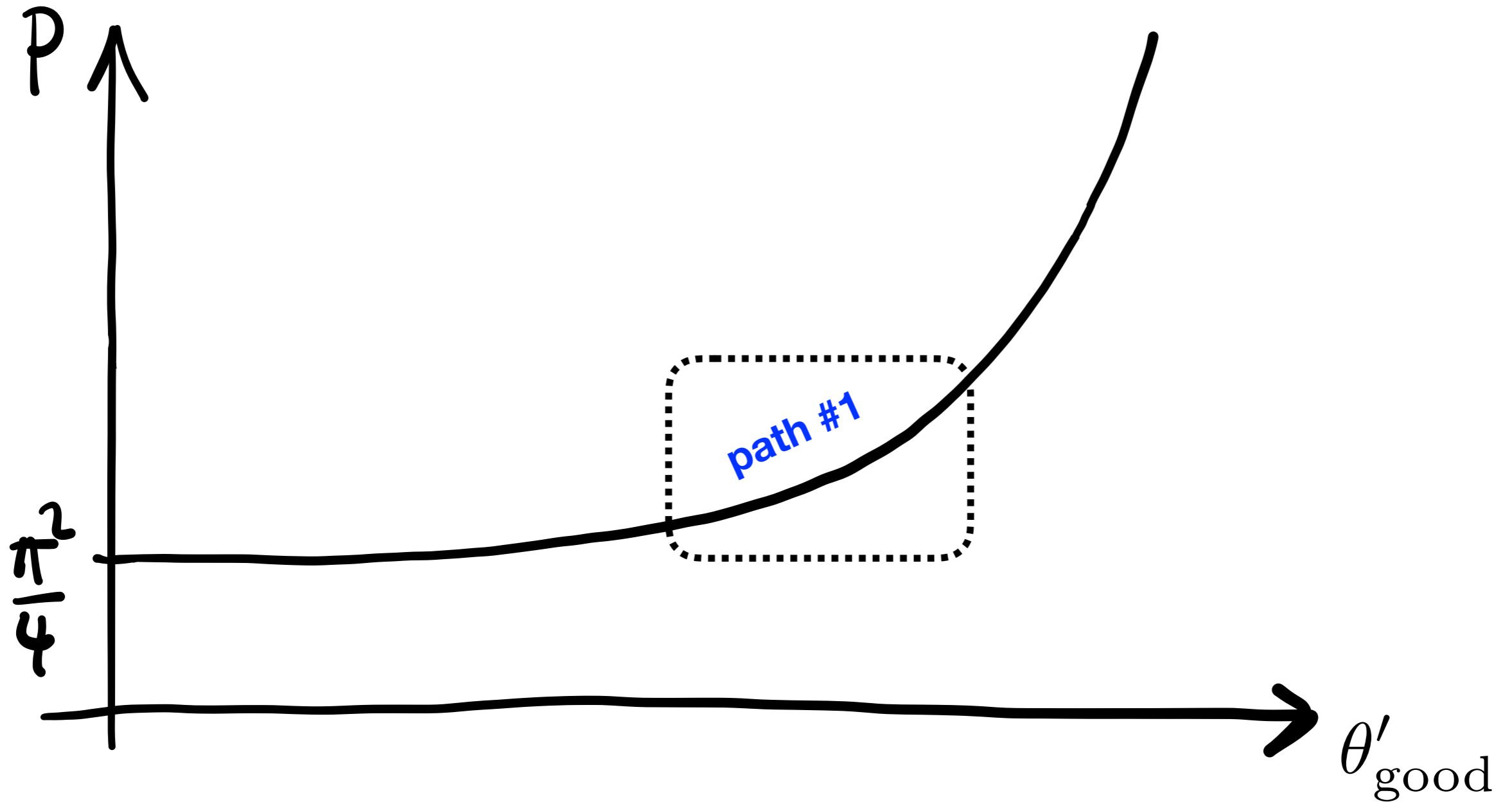
# Numerical path following (continuation)

compute path#1, then path#2, etc



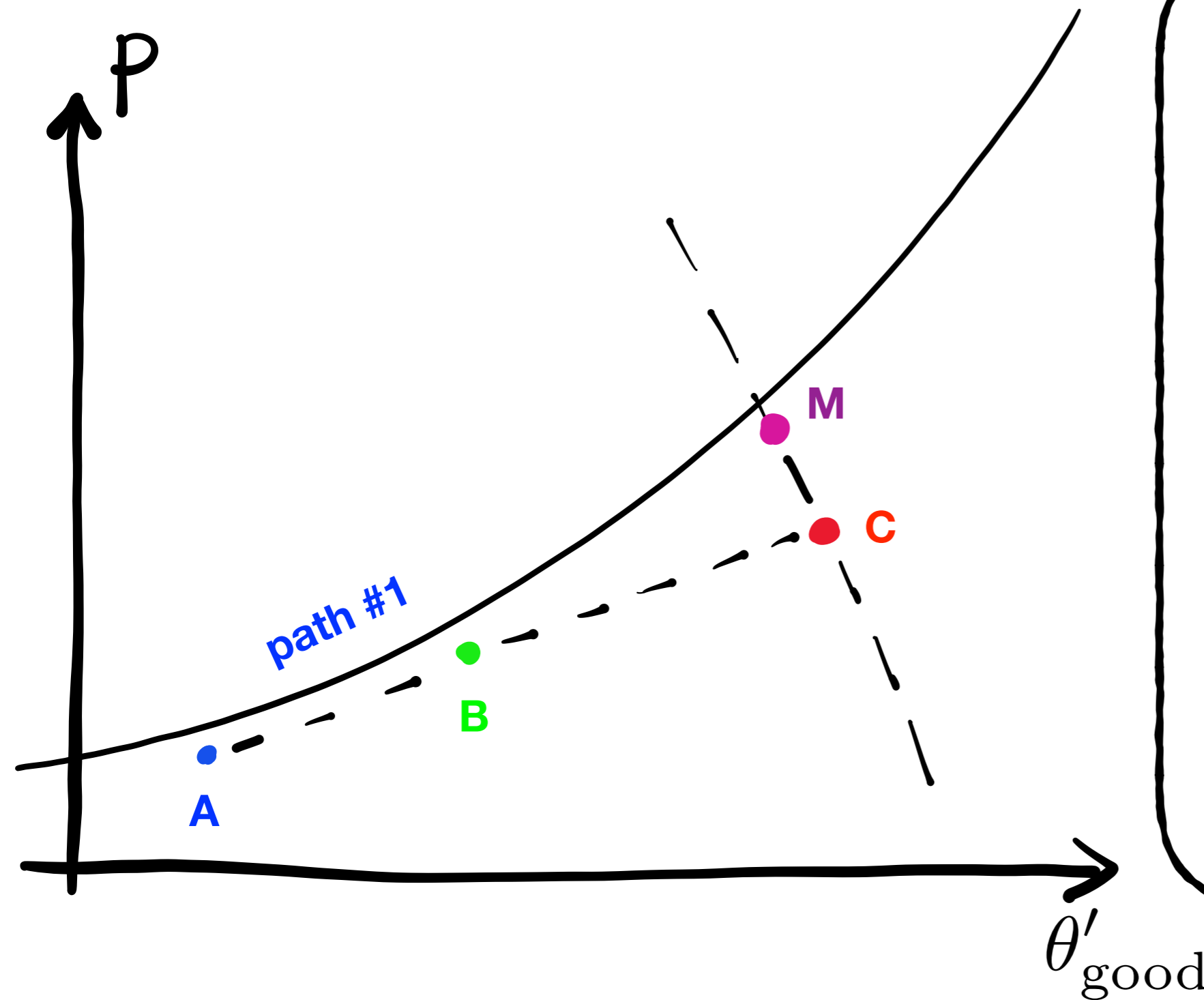
# Numerical path following (continuation)

compute path#1, then path#2, etc





# Numerical path following (continuation)



## Recipe

A and B are known points  
C is taken such  
that  $AC = \text{stepsize } AB$

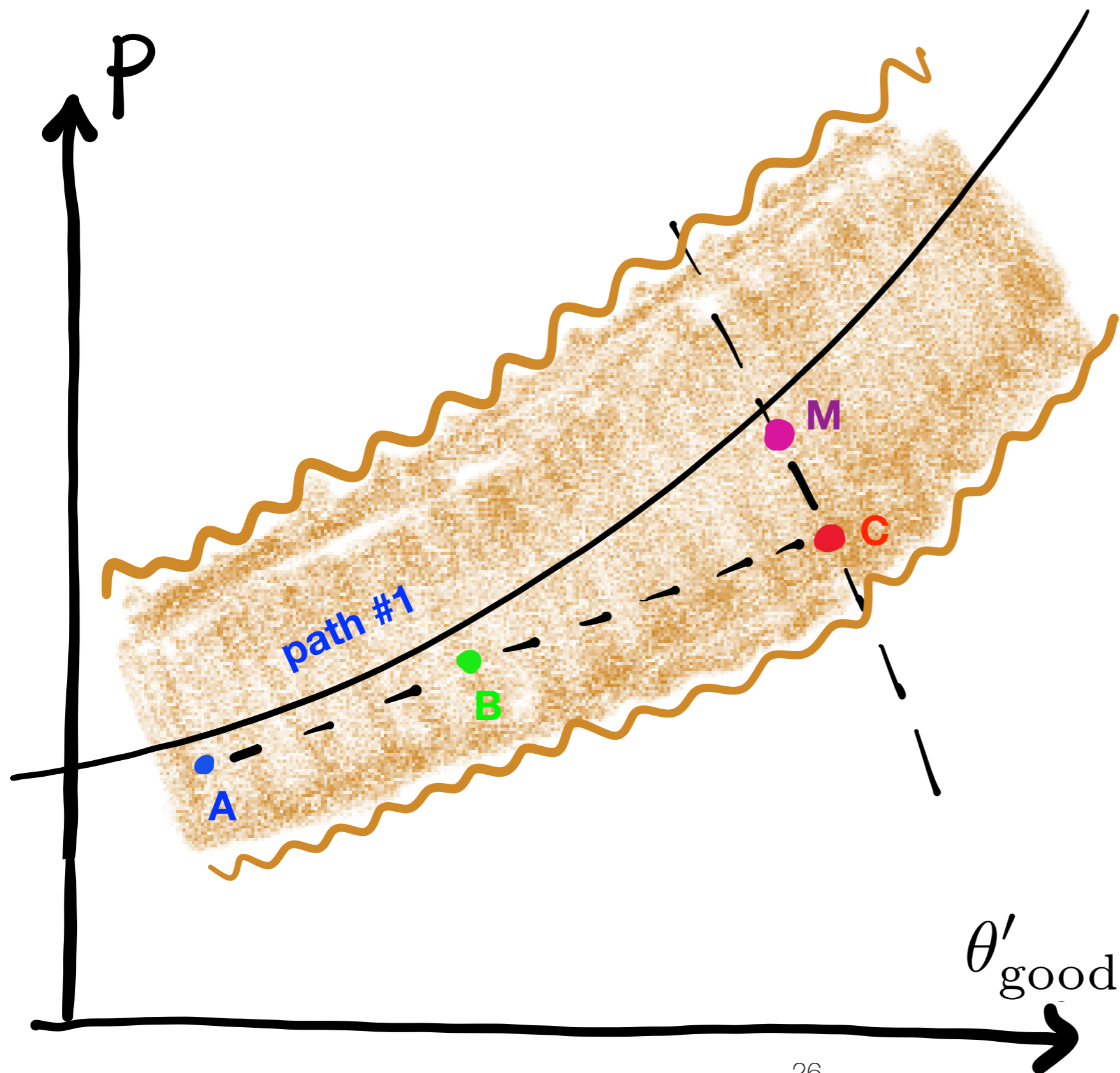
$$\begin{cases} \varphi(P, \theta'_{\text{good}}) = 0 \\ \mathbf{CM} \cdot \mathbf{AB} = 0 \end{cases}$$

2 equations  
2 unknowns

Newton is started  
with C as seed

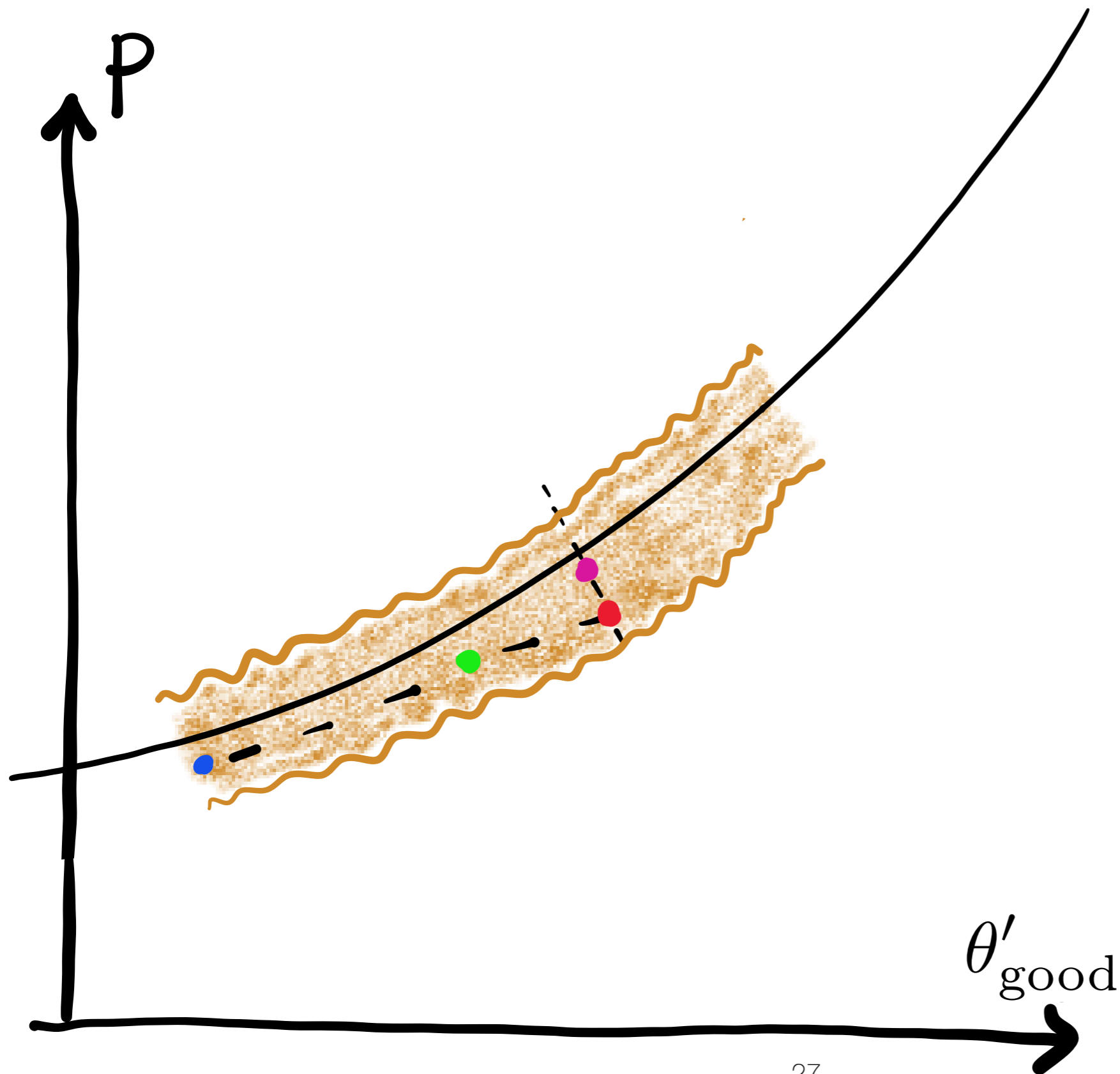
see e.g. books by Keller 1976, Allgower 1990, Ascher 1995, Dankowicz 2013

# Numerical path following (continuation)



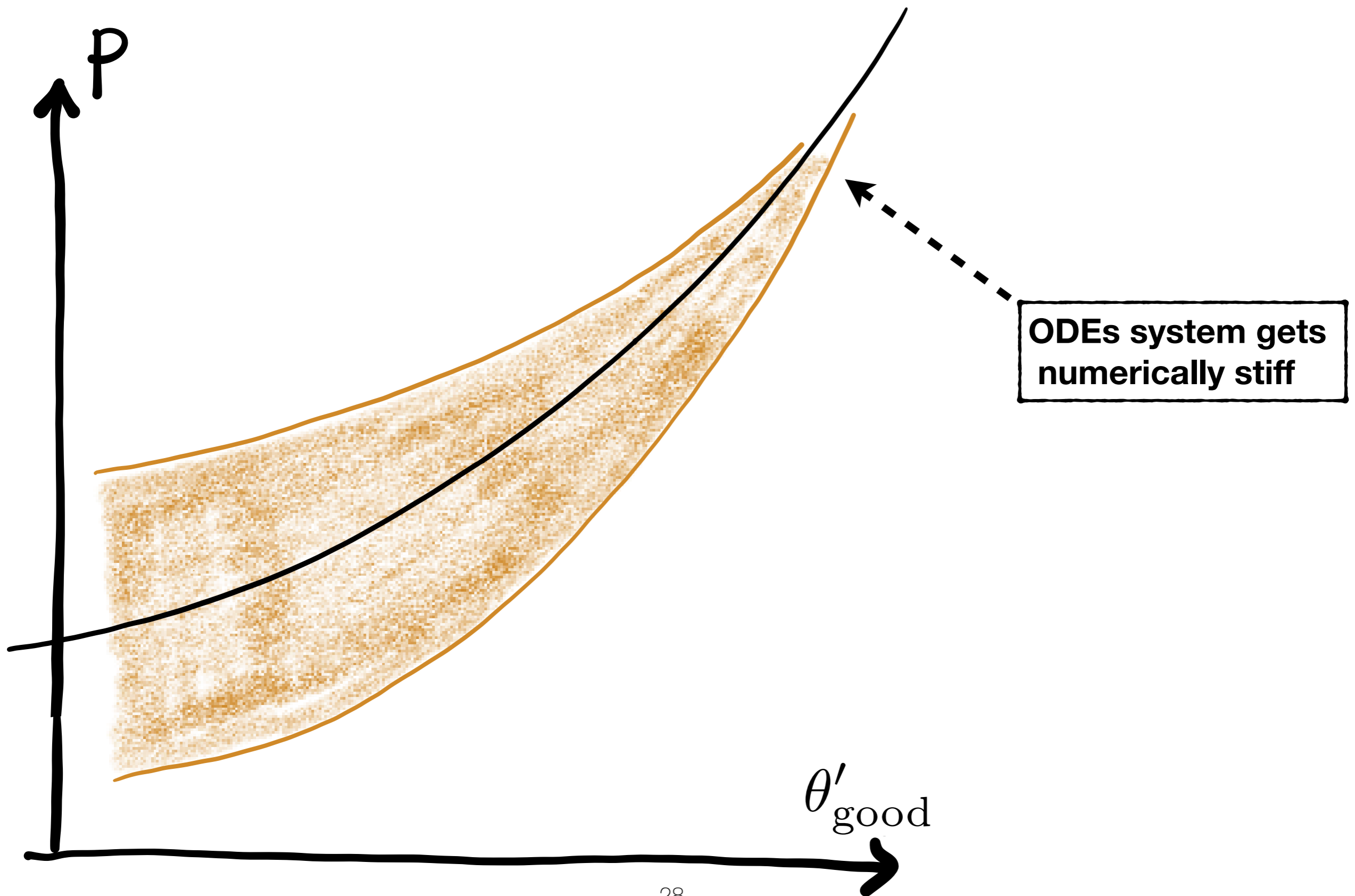
this works fine if  
the seed (point C)  
is inside the **basin  
of attraction**  
of Newton method

# Numerical path following (continuation)

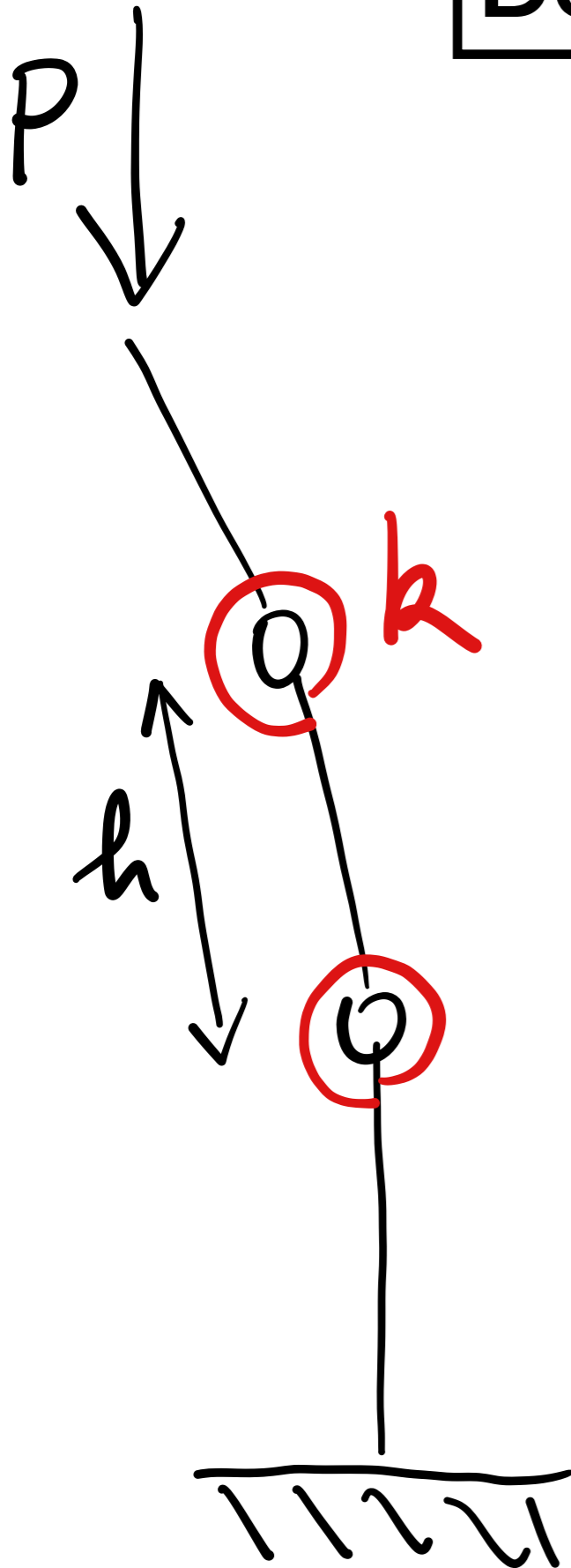


if **basin of attraction**  
is narrower  
=> **small step sizes**

# Numerical path following (continuation)



# Domokos' ghost solutions



$$EI \theta'' + P \sin \theta(s) = 0$$

**Discretization of differential equations**

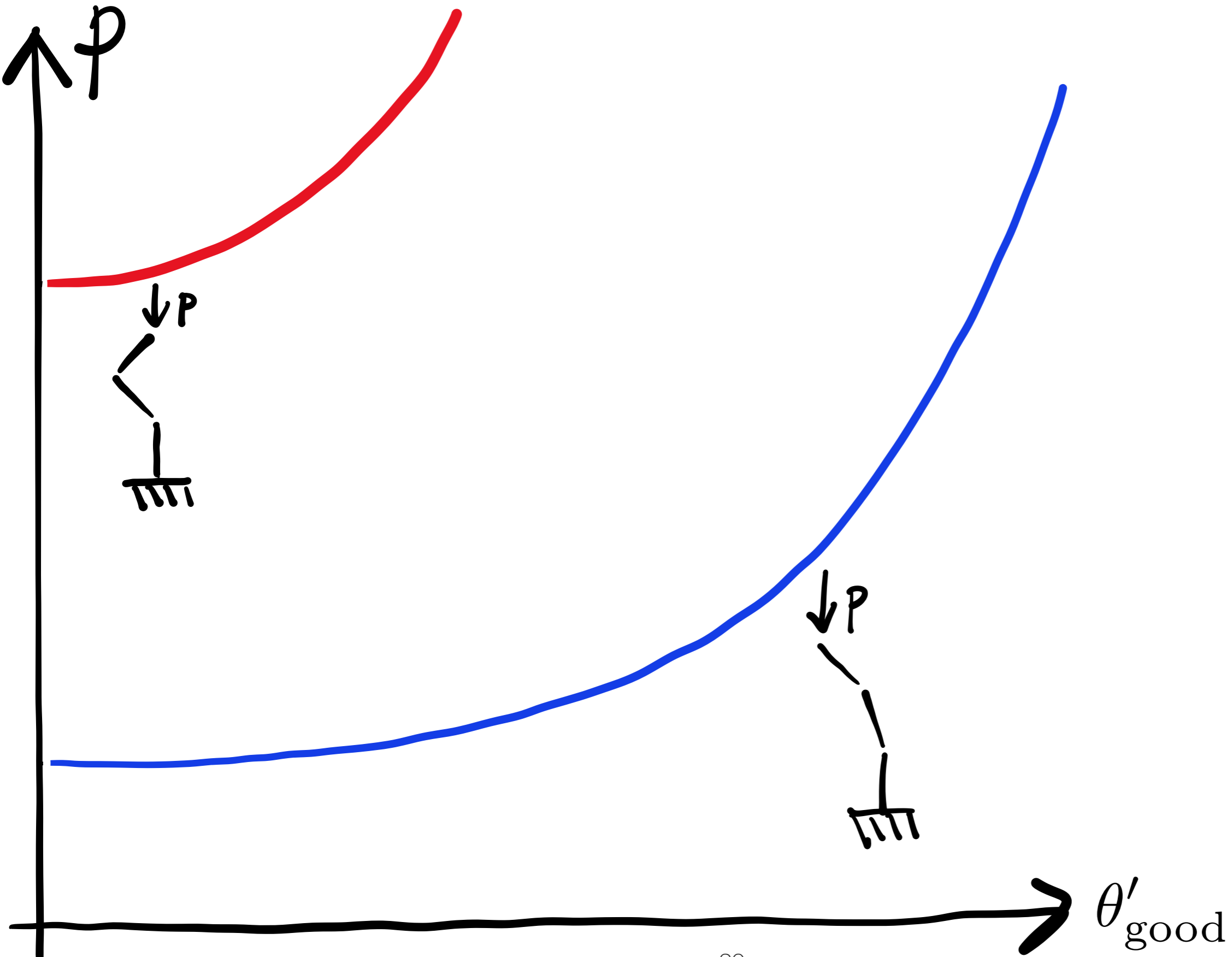
**e.g. finite differences, collocation, spectral, finite elements, etc**

$$k \frac{\theta_{i+1} - 2\theta_i + \theta_{i-1}}{h^2} + P \sin \theta_i = 0$$

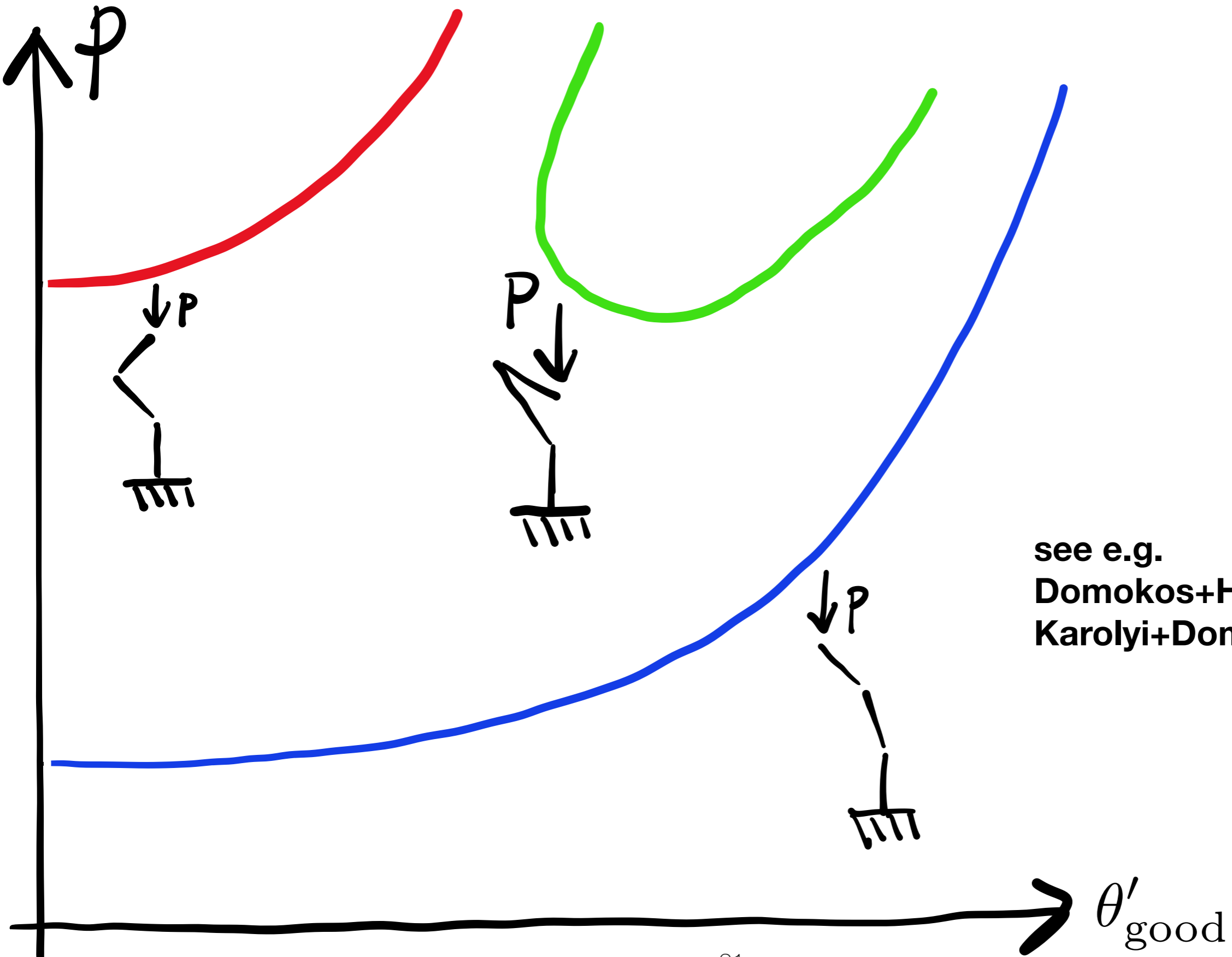
(with  $EI = kL$ )

**Rigid bars linked with torsional springs**

# Domokos' ghost solutions

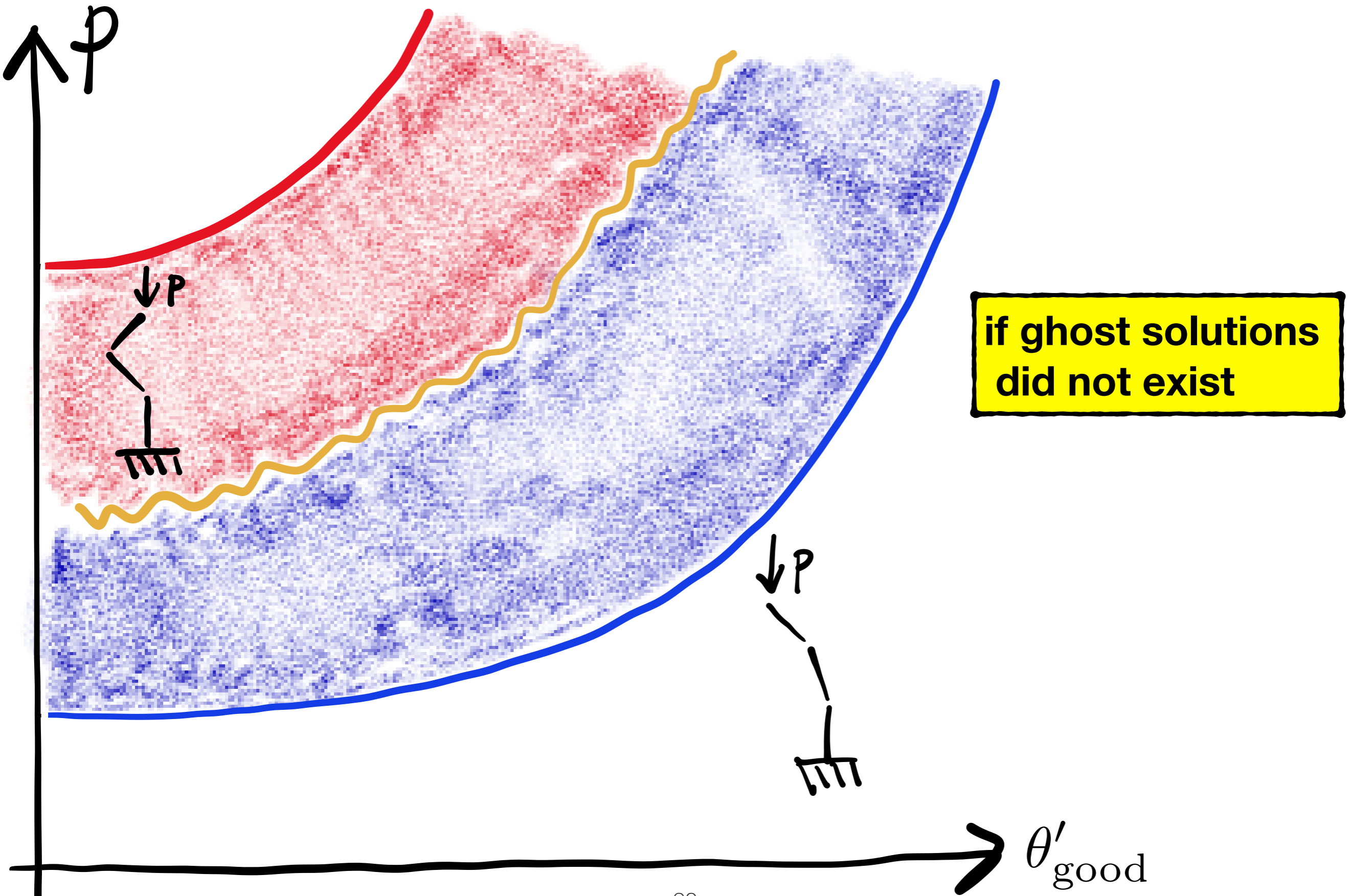


# Domokos' **ghost** solutions



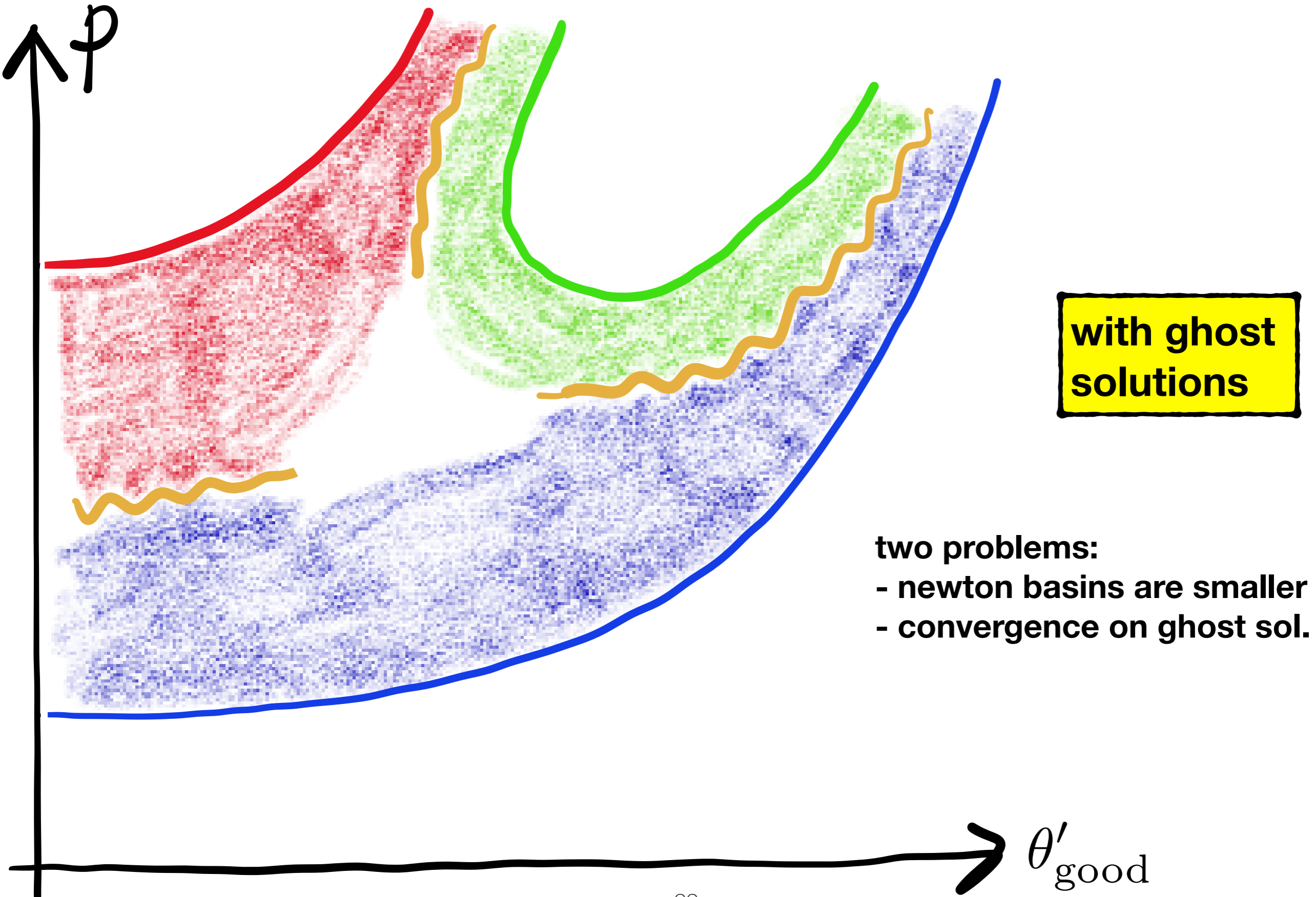
see e.g.  
Domokos+Holmes(1993)  
Karolyi+Domokos(1999)

# Domokos' ghost solutions





# Domokos' ghost solutions



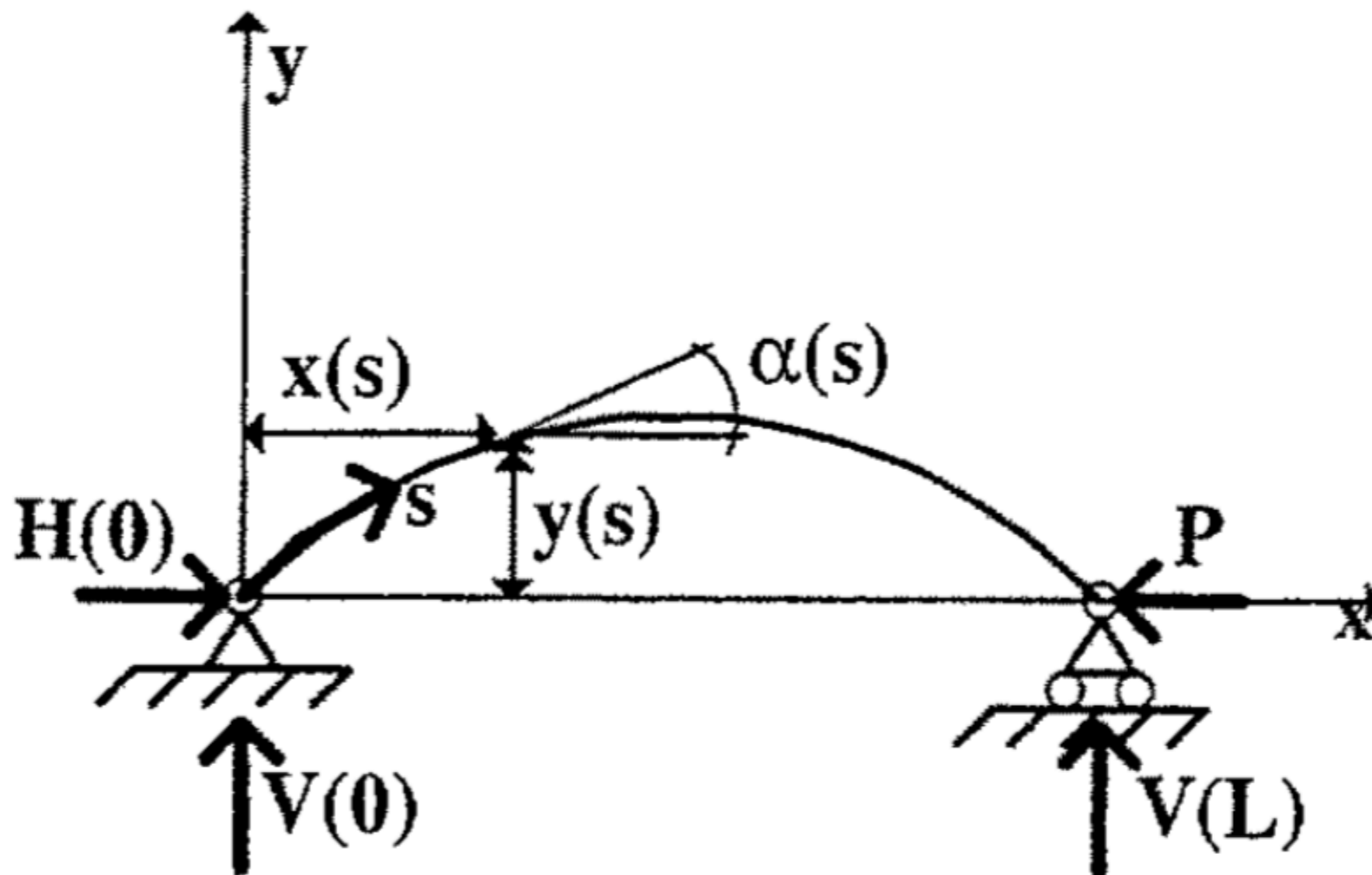
# Domokos' ghost solutions

## Euler's Problem, Euler's Method, and the Standard Map; or, the Discrete Charm of Buckling

G. Domokos<sup>1</sup> and P. Holmes<sup>2</sup>

J. Nonlinear Sci. Vol. 3: pp. 109–151 (1993)

Journal of  
Nonlinear  
Science  
© 1993 Springer-Verlag New York Inc.

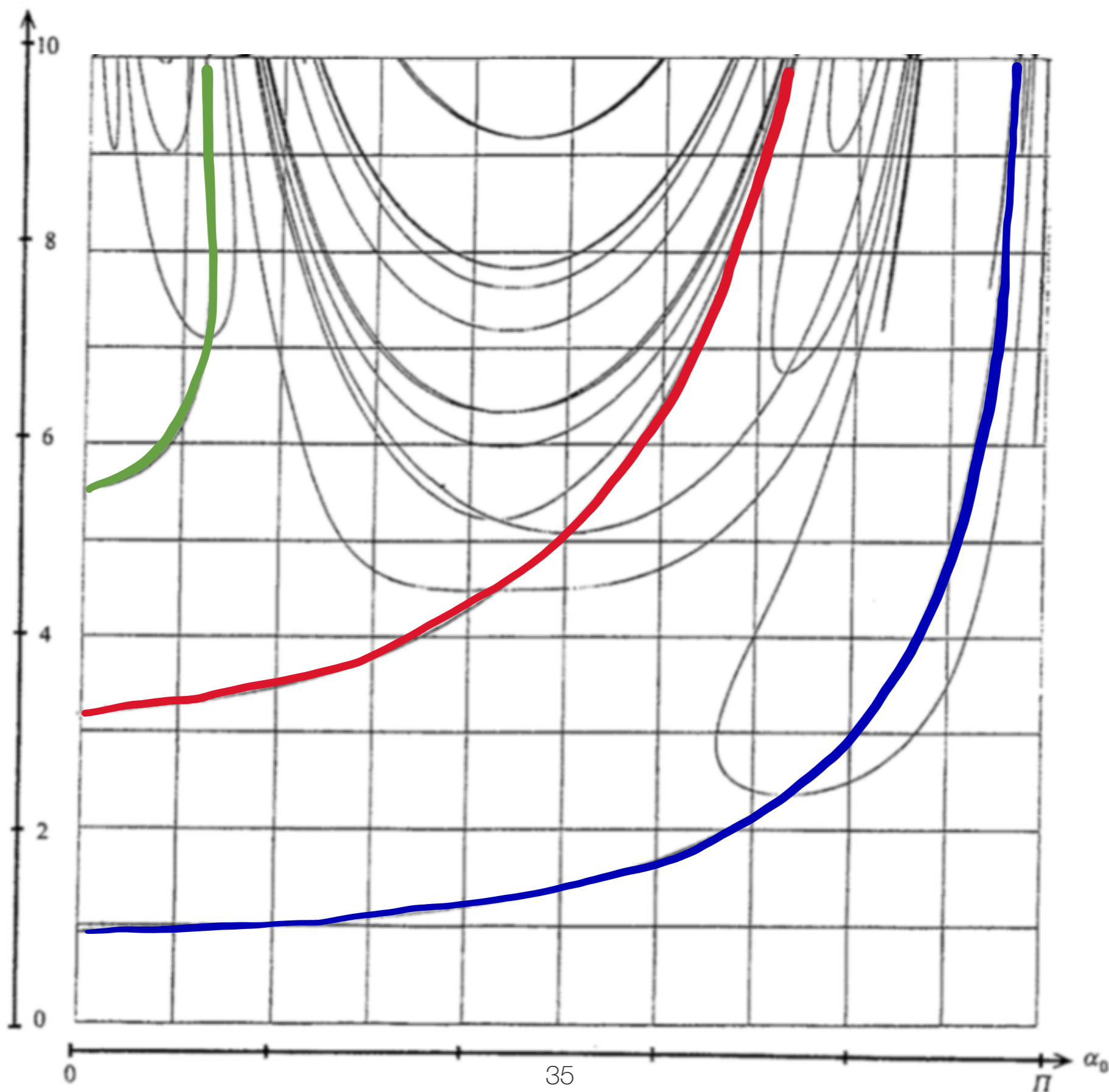


# Domokos' ghost solutions

Domokos+Holmes(1993)

$$\frac{P}{\pi^2}$$

**N = 4  
bars**



# Domokos' ghost solutions

Domokos+Holmes(1993)

$$\frac{P}{\pi^2}$$

10

**N = 4  
bars**

