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# Instabilities in generic second order traffic models with relaxation

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## Abstract

We prove the existence of weak solutions for a class of second order traffic models with relaxation, without requiring the sub-characteristic stability condition to hold. Therefore, large oscillations may arise from small perturbations of equilibria, capturing the formation of stop-and-go waves observed in reality. An analysis of the corresponding travelling waves completes the study.

*2020 Mathematics Subject Classification:* 35L65; 90B20

**Key words:** Macroscopic traffic flow models; hyperbolic systems of conservation laws; relaxation; sub-characteristic condition.

## 1 Introduction

Macroscopic traffic flow models have been used since several decades [20, 24] to describe the spatio-temporal evolution of aggregated quantities, like vehicle density and average speed. In particular, second order models [2, 22, 28, 29], which consist in the mass conservation equation coupled with a momentum balance equation accounting for the speed dynamics, have been developed to describe specific traffic characteristics that cannot be captured by scalar models, such as scattered fundamental diagrams and the appearance and persistence of stop-and-go waves. Indeed, traffic equilibria are generally unstable [4, 27], and small perturbances in drivers' behaviour can generate so called "phantom" jams with no apparent cause. The same behaviour can be observed at a macroscopic scale considering relaxation source terms that violate the usual stability conditions [14, 26].

In this paper, we study a Generic Second Order Model (GSOM) [18] with relaxation, which extends the well-known Aw-Rascle-Zhang system [2, 29], allowing for more general speed formulation. In the literature, hyperbolic systems of balance laws with relaxation source terms are usually studied under a stability condition ensuring well-posedness and convergence results as the relaxation parameter tends to zero [7, 8]. We recall that the stability condition is related to the dissipativity of the evolution equation obtained by a first

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order expansion around equilibria, as observed in [7]. Under this hypothesis, one can recover uniform compactness estimates, which ensure convergence to the equilibrium equation, see e.g. [17, 19, 16, 25] for traffic flow applications.

Aiming at capturing the traffic flow instabilities observed in reality, here we drop the stability assumption, and we study the model properties for a fixed positive relaxation parameter. In particular, we provide a global existence proof for weak solutions based on wave-front tracking approximations and an analysis of travelling waves for the system and its Chapman-Enskog expansion. Our results are general enough to hold for a class of traffic flow models, without too specific assumptions, and they show that these models are able to capture the formation and persistence of stop-and-go waves. This paves the way to the design of control strategies to dampen traffic oscillations in a macroscopic framework, in the spirit of [15].

The presentation is organized as follows. We introduce the GSOM system with relaxation in Section 2 and we describe the corresponding Riemann problem in Section 3. Section 4 provides the compactness estimates on the sequence of approximate solutions, which are used to pass to the limit providing the existence of weak solutions in Section 5. Travelling waves are studied in Sections 6 and 7.

## 2 Model description

We consider the system

$$\begin{cases} \partial_t \rho + \partial_x(\rho v) = 0, \\ \partial_t(\rho w) + \partial_x(\rho w v) = \rho \frac{V(\rho) - v}{\tau}, \end{cases} \quad x \in \mathbb{R}, \quad t > 0, \quad (2.1)$$

defined on an invariant domain of the form

$$\Omega := \left\{ U = (\rho, w) \in \mathbb{R}^2 : \rho \in [0, R(w_{max})], w \in [w_{min}, w_{max}] \right\}, \quad (2.2)$$

for some  $0 < w_{min} \leq w_{max} < +\infty$ . Above, the average speed of vehicles is a function of the density  $\rho = \rho(t, x)$  and of a Lagrangian vehicle property  $w = w(t, x)$ , namely  $v = \mathcal{V}(\rho, w)$  for some speed function  $\mathcal{V} : \Omega \rightarrow \mathbb{R}_{\geq 0}$  satisfying [13]:

$$\mathcal{V}(\rho, w) \geq 0, \quad \mathcal{V}(0, w) = w, \quad (2.3a)$$

$$2\mathcal{V}_\rho(\rho, w) + \rho\mathcal{V}_{\rho\rho}(\rho, w) < 0 \text{ for } w > 0, \quad (2.3b)$$

$$\mathcal{V}_w(\rho, w) > 0, \quad (2.3c)$$

$$\forall w > 0 \quad \exists R(w) > 0 : \quad \mathcal{V}(R(w), w) = 0. \quad (2.3d)$$

As in [13], we observe that (2.3b) implies that  $Q(\rho, w) := \rho\mathcal{V}(\rho, w)$  is strictly concave and  $\mathcal{V}_\rho(\rho, w) < 0$  for  $w > 0$ , if  $\mathcal{V}$  is a  $\mathbf{C}^2$  function in  $\rho$ . We also remark that in (2.3d) we can have  $R(w) = \bar{R}$  for all  $w > 0$ .

Moreover, we assume the equilibrium speed  $V : \mathbb{R}^+ \rightarrow \mathbb{R}^+$  is a non-increasing function ( $V'(\rho) \leq 0$ ) such that  $V(\rho) = 0$  for  $\rho \geq \rho_{max} > 0$ .

Notice that, setting  $\mathcal{V}(\rho, w) = w - p(\rho)$  for a suitable ‘‘pressure’’ function  $p$ , system (2.1) corresponds to the Aw-Rascle-Zhang (ARZ) model [2, 29]. We also remark that, taking  $w = \bar{w}$  constant, we recover the classical Lighthill-Whitham-Richards (LWR) model [20, 24].

Under the above hypotheses, system (2.1) is strictly hyperbolic for  $\rho > 0$ , with eigenvalues

$$\lambda_1(\rho, w) = \mathcal{V}(\rho, w) + \rho \mathcal{V}_\rho(\rho, w), \quad \lambda_2(\rho, w) = \mathcal{V}(\rho, w), \quad (2.4)$$

and corresponding eigenvectors

$$r_1(\rho, w) = \begin{pmatrix} -1 \\ 0 \end{pmatrix}, \quad r_2(\rho, w) = \begin{pmatrix} \mathcal{V}_w(\rho, w) \\ -\mathcal{V}_\rho(\rho, w) \end{pmatrix}, \quad (2.5)$$

with the first characteristic field being genuinely non-linear and the second linearly degenerate. The associated Riemann invariants [11, Chapter 7.3] are

$$z_1(\rho, w) = \mathcal{V}(\rho, w), \quad z_2(\rho, w) = w.$$

Since  $\mathcal{V}_\rho(\rho, w) < 0$  and  $\mathcal{V}(0, w) = w$ , the range of  $v = \mathcal{V}(\rho, w)$  is given by  $v \in [0, w]$  for any  $w \in [w_{min}, w_{max}]$ . Therefore, the inverse function  $\rho = \mathcal{R}(v, w)$  is uniquely defined in the invariant domain

$$\mathcal{W} := \left\{ W = (v, w) \in \mathbb{R}^2 : 0 \leq v \leq w, w \in [w_{min}, w_{max}] \right\}. \quad (2.6)$$

We remark that, in the  $(v, w)$  coordinates, the curves  $\rho = \mathcal{R}(v, w) = const$  are parallel to the first bisector [23, Section 5]. Indeed, we get:

$$\begin{aligned} \mathcal{R}(v, w) = const &\implies \mathcal{R}_v + \mathcal{R}_w = 0 \\ \mathcal{V}(\mathcal{R}(v, w), w) = v &\implies \mathcal{V}_\rho \mathcal{R}_v = 1 \text{ and } \mathcal{V}_\rho \mathcal{R}_w + \mathcal{V}_w = 0 \end{aligned}$$

Replacing  $\mathcal{R}_v = 1/\mathcal{V}_\rho$  and  $\mathcal{R}_w = -\mathcal{V}_w/\mathcal{V}_\rho$  in the first relation, we get  $\mathcal{V}_w = 1$  on  $\mathcal{R}(v, w)$  level curves.

We recall that weak solutions of (2.1) corresponding to an initial datum  $U(0, x) = U_0(x) = (\rho_0(x), w_0(x)) \in \mathbf{L}^\infty(\mathbb{R}, E)$  are defined as

**Definition 1.** Let  $u_0 = (\rho_0, \rho_0 w_0)^T \in \mathbf{L}^1(\mathbb{R}; \mathbb{R}_+^2)$  and  $T > 0$  be given. Then  $u = (\rho, \rho w)^T \in \mathbf{C}^0([0, T], \mathbf{L}^1(\mathbb{R}; \mathbb{R}_+^2))$  is a weak solution to the Cauchy problem for (2.1) if, for all  $\varphi \in \mathbf{C}_c^1(\mathbb{R} \times \mathbb{R}; \mathbb{R})$ , it holds

$$\int_{\mathbb{R}} u_0(x) \varphi(0, x) dx + \int_0^T \int_{\mathbb{R}} [u \partial_t \varphi + F(u) \partial_x \varphi](t, x) dx dt + \int_0^T \int_{\mathbb{R}} G(u) \varphi(t, x) dx dt = 0, \quad (2.7)$$

where

$$F(u) = \begin{bmatrix} \rho v \\ \rho w v \end{bmatrix}, \quad G(u) = \begin{bmatrix} 0 \\ \rho \frac{V(\rho) - v}{\tau} \end{bmatrix}.$$

For smooth solutions, system (2.1) is equivalent to:

$$\begin{cases} \partial_t \rho + \partial_x(\rho v) = 0, \\ \partial_t v + \lambda_1(\rho, w) \partial_x v = \frac{V(\rho) - v}{\tau}, \end{cases} \quad x \in \mathbb{R}, \quad t > 0, \quad (2.8)$$

and

$$\begin{cases} \partial_t \rho + \partial_x(\rho v) = 0, \\ \partial_t w + \lambda_2(\rho, w) \partial_x w = \frac{V(\rho) - v}{\tau}, \end{cases} \quad x \in \mathbb{R}, t > 0. \quad (2.9)$$

We also recall that the *sub-characteristic condition* [7, 8] writes

$$\mathcal{V}_\rho(\rho, w) \leq V'(\rho) \leq 0 \quad \text{for } w \text{ s.t. } \mathcal{V}(\rho, w) = V(\rho), \quad (2.10)$$

see e.g. Figure 1.

If (2.10) is satisfied, it is known [16] that (weak entropic) solutions to (2.1) converge to the (weak entropic) solutions of

$$\partial_t \rho + \partial_x(\rho V(\rho)) = 0, \quad x \in \mathbb{R}, t > 0, \quad (2.11)$$

for  $\tau \searrow 0$  or  $t \rightarrow \infty$  (using the change of variables  $\tilde{t} = t/\tau$  and  $\tilde{x} = x/\tau$ ).

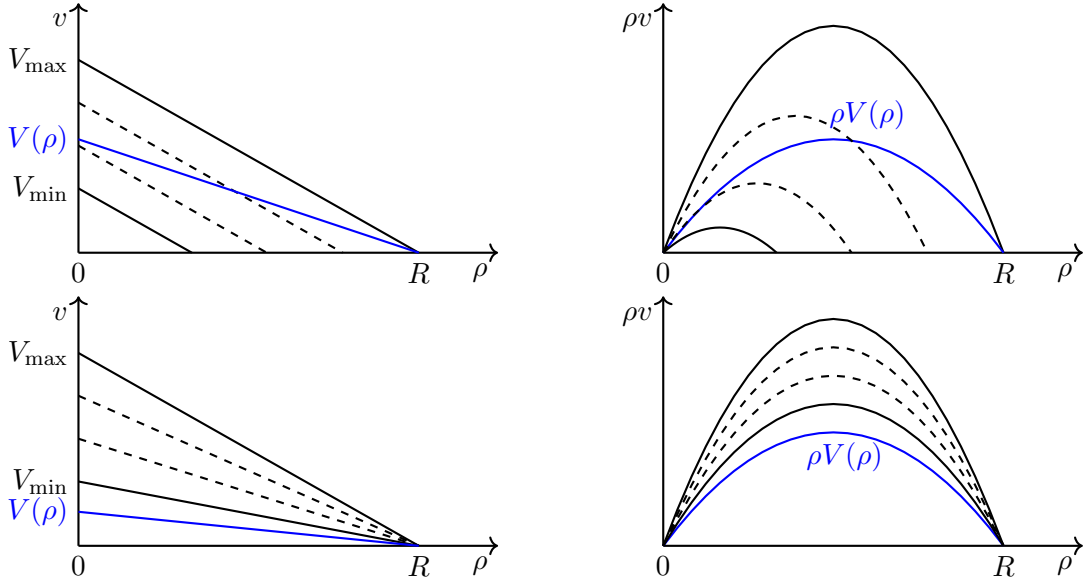


Figure 1: Speed-density (left) and flow-density (right) fundamental diagrams for the GSOM model (2.1) corresponding to the choices (satisfying the sub-characteristic condition (2.10)): Top:  $\mathcal{V}(\rho, w) = w - \alpha\rho$ , with  $w \in [V_{\min}, V_{\max}]$  and  $V(\rho) = \beta(R - \rho)$  with  $\alpha \geq \beta > 0$ ; Bottom:  $\mathcal{V}(\rho, w) = w(R - \alpha\rho)$ , with  $w \in [V_{\min}, V_{\max}]$  and  $V(\rho) = \beta(R - \rho)$  with  $\alpha \geq \beta > 0$ .

The equilibrium curve  $V(\rho) = V(\mathcal{R}(v, w))$  is implicitly defined by

$$V(\mathcal{R}(v, \phi(v))) = v.$$

Differentiating w.r.to  $v$  we obtain

$$V'(\rho) (\mathcal{R}_v + \mathcal{R}_w \phi'(v)) = 1,$$

which gives

$$\phi'(v) = \frac{1}{\mathcal{R}_w} \left( \frac{1}{V'(\rho)} - \mathcal{R}_v \right) = \frac{1}{\mathcal{V}_w} \left( 1 - \frac{\mathcal{V}_\rho}{V'(\rho)} \right).$$

Since  $\mathcal{V}_w > 0$ , we get  $\phi'(v) \leq 0$  if and only if  $\mathcal{V}_\rho \leq V'(\rho)$ , see (2.10).

We are interested in investigating the behaviour of (2.1) when (2.10) is violated. Therefore, we assume that there exists a point  $v_{cr}$ , resp.  $w_{cr} = \phi(v_{cr})$ ,  $\rho_{cr} = \mathcal{R}(v_{cr}, w_{cr})$ , such that  $\phi'(v_{cr}) = 0$ , i.e.

$$\mathcal{V}_\rho(\rho_{cr}, w_{cr}) = \mathcal{V}_\rho(\rho_{cr}, \phi(v_{cr})) = V'(\rho_{cr}). \quad (2.12)$$

We will distinguish the two cases:

$$\phi'(v) < 0 \quad \text{if } v < v_{cr}, \quad \phi'(v) > 0 \quad \text{if } v > v_{cr}, \quad (2.13a)$$

and

$$\phi'(v) > 0 \quad \text{if } v < v_{cr}, \quad \phi'(v) < 0 \quad \text{if } v > v_{cr}. \quad (2.13b)$$

Moreover, we observe that, for  $v > v_{cr}$  for case (2.13a), resp.  $v < v_{cr}$  for case (2.13b),  $\phi'(v) < 1$  if  $V'(\mathcal{R}(v, \phi(v))) > \mathcal{V}_\rho(\mathcal{R}(v, \phi(v)), \phi(v)) + V'(\mathcal{R}(v, \phi(v)))\mathcal{V}_w(\mathcal{R}(v, \phi(v)), \phi(v))$ .

In case (2.13a), the equilibrium curve intersects the vacuum line  $v = w$  at  $\hat{v} = \hat{w}$  implicitly defined by  $\mathcal{V}(0, \hat{w}) = V(0)$ , see Figure 2, left. We remark that in this case all invariant domains must include vacuum states.

In case (2.13b), the equilibrium curve intersects the congestion line  $v = 0$  at  $\hat{w}$  implicitly defined by  $\mathcal{V}(\mathcal{R}(\hat{w}), \hat{w}) = 0 = V(\rho_{max})$ , see Figure 2, right. We remark that in this case all invariant domains must include congestion states.

We also define  $\check{\rho}, \check{v}$  such that

$$\check{v} = \mathcal{V}(\check{\rho}, \hat{w}) = V(\check{\rho}). \quad (2.14)$$

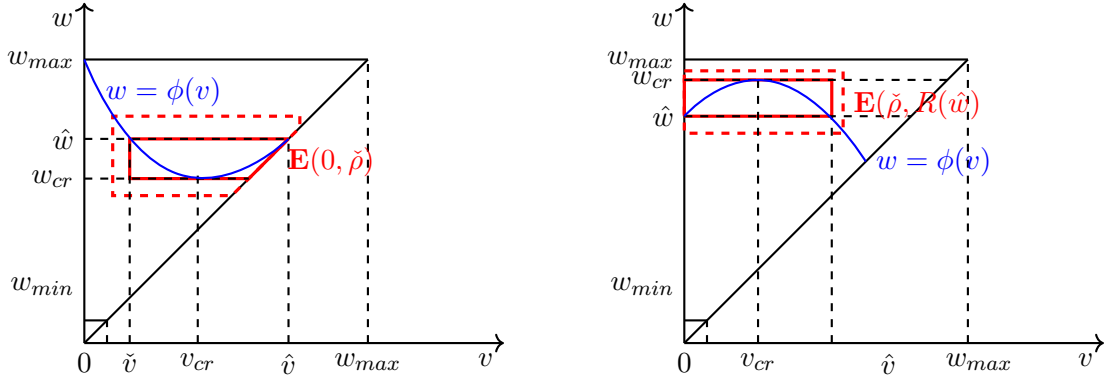


Figure 2: Examples of non-monotone equilibrium curves  $w = \phi(v)$  and minimal invariant domains  $\mathbf{E}$  (continuous red line) and general invariant domains (dashed red line): left, convex case (2.13a); right, concave case (2.13b).

### 3 The Riemann problem

Let us consider (2.1) with initial data

$$(\rho, w)(0, x) = \begin{cases} U_L = (\rho_L, w_L) & \text{if } x < 0, \\ U_R = (\rho_R, w_R) & \text{if } x > 0, \end{cases} \quad (3.1)$$

and set  $v_L = \mathcal{V}(\rho_L, w_L)$ ,  $v_R = \mathcal{V}(\rho_R, w_R)$ .

We observe that, if the subcharacteristic condition (2.10) is satisfied, weak solutions of (2.1), (3.1) tend to weak solutions of the Riemann problem

$$\begin{cases} \partial_t \rho + \partial_x(\rho V(\rho)) = 0, & x \in \mathbb{R}, t > 0, \\ \rho(0, x) = \begin{cases} \bar{\rho}_L & \text{if } x < 0, \\ \bar{\rho}_R & \text{if } x > 0, \end{cases} \end{cases}$$

as  $\tau \searrow 0$  (or  $t \rightarrow \infty$ ), where  $\bar{\rho}_{L,R} = \mathcal{R}(\bar{v}_{L,R}, \bar{w}_{L,R})$  are the projection of  $U_L, U_R$  on the equilibrium curve implicitly defined by

$$\begin{aligned} \bar{v}_{L,R} &= V(\mathcal{R}(\bar{v}_{L,R}, \bar{v}_{L,R} + w_{L,R} - v_{L,R})), \\ \bar{w}_{L,R} &= \bar{v}_{L,R} + w_{L,R} - v_{L,R}, \end{aligned}$$

see Figure 3.

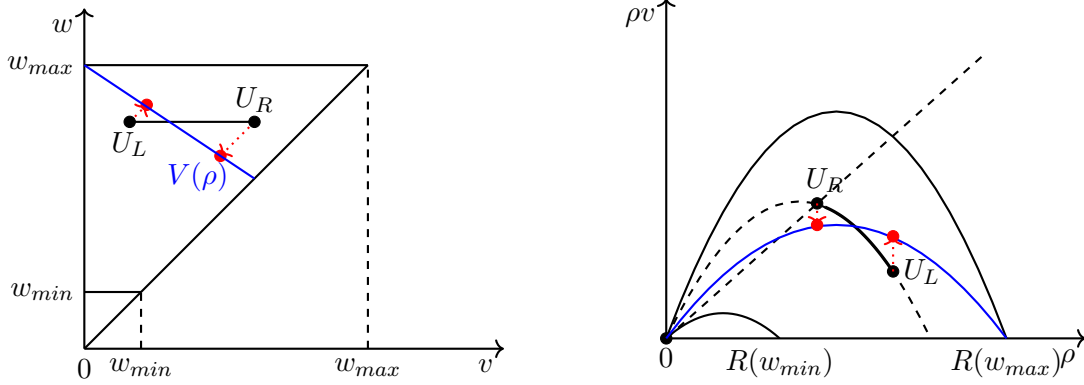


Figure 3: Illustration of the limit behaviour of the solutions to the Riemann problem (2.1), (3.1) under the subcharacteristic condition (2.10).

### 3.1 The homogeneous Riemann problem

We recall here the detailed construction of weak (entropy) solutions of problem

$$\begin{cases} \partial_t \rho + \partial_x(\rho v) = 0, \\ \partial_t(\rho w) + \partial_x(\rho w v) = 0, \end{cases} \quad x \in \mathbb{R}, t > 0, \quad (3.2)$$

with initial conditions (3.1).

In general, the left state  $U_L$  is connected to  $U_M = (\rho_M, w_M)$  by a first family wave (rarefaction or shock), i.e.  $z_2(\rho_L, w_L) = w_L = w_M = z_2(\rho_M, w_M)$ , and  $U_M$  is connected to  $U_R$  by a contact-discontinuity with  $z_1(\rho_M, w_M) = v_M = \mathcal{V}(\rho_M, w_M) = v_R = z_1(\rho_R, w_R)$ . Thus, the intermediate state  $U_M$  is identified by the system of equations

$$\begin{cases} w_M = w_L, \\ v_M = v_R, \\ \rho_M = \mathcal{R}(v_R, w_L). \end{cases}$$

If  $w_L \leq v_R$ , we set  $\rho_M = 0$ , meaning that  $U_M$  corresponds to the vacuum.

The propagation speed  $\sigma$  of a shock wave between two states  $U_-$  and  $U_+$  is given by the Rankine-Hugoniot condition

$$\sigma(U_-, U_+) = \frac{\rho_+ v_+ - \rho_- v_-}{\rho_+ - \rho_-}. \quad (3.3)$$

In this work, we will rely on the following definition of solutions of (2.1), (3.1), see also [12]. Since the presence of vacuum states prevents uniqueness even if an entropy condition is enforced, we refer to [1] for an alternative construction.

**Definition 2.** For any  $U_L, U_R \in \Omega$ , the Riemann solver

$$\mathcal{RS} : \Omega \times \Omega \rightarrow \mathbf{C}^0 \left( ]0, +\infty[; \mathbf{L}_{loc}^1(\mathbb{R}; \Omega) \right), \quad (U_L, U_R) \mapsto \mathcal{RS}(U_L, U_R)$$

is defined as follows:

1. If  $(v_R, w_R) \in \mathcal{W}$  and  $v_L \geq v_R$ , then

$$\mathcal{RS}(U_L, U_R)(t, x) = \begin{cases} U_L & \text{if } x < \sigma(U_L, U_M)t, \\ U_M & \text{if } \sigma(U_L, U_M)t < x < v_R t, \\ U_R & \text{if } x > v_R t, \end{cases}$$

with  $\sigma$  defined in (3.3).

2. If  $(v_L, w_L) \in \mathcal{W} \setminus \{v = w\}$ , and  $v_L \leq v_R < w_L$ , then

$$\mathcal{RS}(U_L, U_R)(t, x) = \begin{cases} U_L & \text{if } x < \lambda_1(\rho_L, w_L)t, \\ \hat{U} & \text{if } \lambda_1(\rho_L, w_L)t < x < \lambda_1(\rho_M, w_M)t, \\ U_M & \text{if } \lambda_1(\rho_M, w_M)t < x < v_R t, \\ U_R & \text{if } x > v_R t, \end{cases}$$

with  $\hat{U} = (\rho, w_L)$  solving  $\lambda_1(\rho, w_L) = \frac{x}{t}$ .

3. If  $(v_L, w_L) \in \mathcal{W} \setminus \{v = w\}$  and  $w_L \leq v_R$ , then

$$\mathcal{RS}(U_L, U_R)(t, x) = \begin{cases} U_L & \text{if } x < \lambda_1(\rho_L, w_L)t, \\ \hat{U} & \text{if } \lambda_1(\rho_L, w_L)t < x < \lambda_1(0, w_L)t, \\ U_M & \text{if } \lambda_1(0, w_L)t < x < v_R t, \\ U_R & \text{if } x > v_R t, \end{cases}$$

with  $\hat{U} = (\rho, w_L)$  solving  $\lambda_1(\rho, w_L) = \frac{x}{t}$  and  $U_M = (0, w_L)$ .

4. If  $v_L = w_L$  and  $v_R = w_R$ , then

$$\mathcal{RS}(U_L, U_R)(t, x) = \begin{cases} U_L & \text{if } x < 0, \\ U_R & \text{if } x > 0. \end{cases} \quad (3.4)$$



### 3.2 The relaxation Riemann problem

We now turn to the solution of the relaxation step, consisting of the system of ODEs

$$\begin{cases} \partial_t \rho = 0, \\ \partial_t v = \frac{V(\rho) - v}{\tau} = \partial_t w, \end{cases} \quad x \in \mathbb{R}, \quad t > 0, \quad (3.5)$$

with Riemann-like initial datum (3.1), whose solutions are given by

$$\begin{cases} \rho(t, x) = \rho(0, x), \\ v(t, x) = V(\rho(0, x)) + (v(0, x) - V(\rho(0, x))) e^{-t/\tau}, \end{cases} \quad x \in \mathbb{R}, \quad t > 0. \quad (3.6)$$

Invariant domains for this step must include segments parallel to the line  $v = w$  joining any point of the domain with the equilibrium curve.

From the above analysis, we easily see that the invariant domains for the relaxed Riemann problem (2.1), (3.1) must have edges parallel to the Riemann invariant level curves and contain properly a portion of the equilibrium curve, see Figure 2 and definitions (4.1), (4.2) below.

## 4 Wave-front tracking approximations

The construction of wave-front tracking (WFT) approximate solutions is based on a two step process, that successively solves the homogeneous system (3.2) for a piece-wise constant initial datum, and then integrates the source term contained in the ODE (3.5), see [16] and references therein.

To define the invariant domains, we need to distinguish the two cases illustrated in Figure 2, namely the case in which the equilibrium curve  $w = \phi(v)$  is a convex function and the case in which it is concave.

Under the hypothesis (2.13a), invariant domains take the form

$$\mathbf{E} = \mathbf{E}[0, \bar{\rho}] = \{U = (\rho, w) : \rho \in [0, \bar{\rho}], \mathcal{V}(\rho, w) \in [\bar{v}, v_M], w \in [w_m, \bar{w}]\}, \quad (4.1)$$

with  $\bar{\rho} \geq \check{\rho}$ ,  $v_M \geq \hat{v}$ ,  $w_m \leq w_{cr}$  and  $\bar{v} = \mathcal{V}(\bar{\rho}, \bar{w}) = V(\bar{\rho})$ , where  $\check{\rho}, \hat{v}$  and  $w_{cr}$  are chosen as in (2.12), (2.14).

Under the hypothesis (2.13b), invariant domains take the form

$$\mathbf{E} = \mathbf{E}[\bar{\rho}, R(\hat{w})] = \{U = (\rho, w) : \rho \in [\bar{\rho}, R(w_M)], \mathcal{V}(\rho, w) \in [0, \bar{v}], w \in [\bar{w}, w_M]\}, \quad (4.2)$$

with  $\bar{\rho} \leq \check{\rho}$ ,  $w_M \geq w_{cr}$  and  $\bar{v} = \mathcal{V}(\bar{\rho}, \bar{w}) = V(\bar{\rho})$ , where  $\check{\rho}$  and  $w_{cr}$  are chosen as in (2.12), (2.14).

Let us consider initial data  $U_0 = (\rho_0, w_0) : \mathbb{R} \rightarrow \mathbf{E}$  such that  $\text{TV}(w_0) + \text{TV}(\mathcal{V}(\rho_0, w_0)) < +\infty$ . For any  $T > 0$ , we consider a sequence of time-steps  $\Delta t^\nu > 0$ ,  $\nu \in \mathbb{N}$  such that  $\Delta t^\nu \rightarrow 0$  and we partition the interval  $[0, T[$  in intervals of the form  $[n\Delta t^\nu, (n+1)\Delta t^\nu[$ ,  $n \in \mathbb{N}$ . We denote with  $U^\nu(t, x) = (\rho^\nu, w^\nu)(t, x)$ ,  $t \in [0, T]$ ,  $x \in \mathbb{R}$ , the sequence of WFT approximate solutions of (2.1) constructed as detailed below:

1. Define a sequence of piece-wise constant functions  $U_0^\nu = (\rho_0^\nu, w_0^\nu) \in \mathbf{E}$  satisfying

$$\begin{aligned} \mathrm{TV}(\mathcal{V}(\rho_0^\nu, w_0^\nu)) &\leq \mathrm{TV}(\mathcal{V}(\rho_0, w_0)), & \|\mathcal{V}(\rho_0^\nu, w_0^\nu) - \mathcal{V}(\rho_0, w_0)\|_{\mathbf{L}^\infty} &\leq \frac{1}{\nu}, & \|\rho_0^\nu - \rho_0\|_{\mathbf{L}^1} &\leq \frac{1}{\nu}, \\ \mathrm{TV}(w_0^\nu) &\leq \mathrm{TV}(w_0), & \|w_0^\nu - w_0\|_{\mathbf{L}^\infty} &\leq \frac{1}{\nu}, & \|w_0^\nu - w_0\|_{\mathbf{L}^1} &\leq \frac{1}{\nu}, \end{aligned}$$

and, for each  $\nu \in \mathbb{N}$ , the piece-wise constant function  $U_0^\nu$  has a finite number of discontinuities.

2. Solve the homogeneous system (3.2) corresponding to the Riemann problems arising at discontinuities for  $t \in [0, \Delta t^\nu[$  using the WFT method and name  $U^\nu(t, \cdot)$ ,  $t \in [0, \Delta t^\nu[$ , the corresponding piece-wise constant function [3].
3. At  $t = \Delta t^\nu$ , we define

$$\begin{aligned} \rho^\nu(\Delta t^\nu, \cdot) &= \rho^\nu(\Delta t^\nu -, \cdot), \\ w^\nu(\Delta t^\nu, \cdot) &= w^\nu(\Delta t^\nu -, \cdot) + \frac{\Delta t^\nu}{\tau} [V(\rho^\nu(\Delta t^\nu, \cdot)) - \mathcal{V}(U^\nu(\Delta t^\nu -, \cdot))]. \end{aligned}$$

Note that  $\rho$  is conserved during this second step, while  $w$  (and  $v = \mathcal{V}(\rho, w)$ ) is updated according to (3.5).

Observe that we also have

$$\mathcal{V}(U^\nu(\Delta t^\nu, \cdot)) = \mathcal{V}(U^\nu(\Delta t^\nu -, \cdot)) + \frac{\Delta t^\nu}{\tau} [V(\rho^\nu(\Delta t^\nu, \cdot)) - \mathcal{V}(U^\nu(\Delta t^\nu -, \cdot))], \quad (4.3)$$

see (3.5).

4. Treat  $U^\nu(\Delta t^\nu, \cdot)$  as a new piece-wise constant initial condition and repeat the previous steps 2–3 to define the solution  $U^\nu(t, \cdot)$  for each  $t \in [0, T]$ , for any  $T > 0$  fixed.

#### 4.1 $\mathbf{L}^\infty$ estimates

**Proposition 1.** *For  $\Delta t^\nu \leq \tau / \max\{1, \|\mathcal{V}_w\|_\infty\}$ , the set  $\mathbf{E}$  is an invariant domain for the proposed WFT scheme.*

*Proof.* Let us assume  $U_0(x) \in \mathbf{E}$  for all  $x \in \mathbb{R}$ . System (3.2) being of Temple class, Step 2 above clearly preserves the inequalities on the Riemann invariants  $w, \mathcal{V}(\rho, w)$ .

Concerning the relaxation Step 3., if  $v_- \geq V(\rho)$  we have that

$$V(\rho) = \left(1 - \frac{\Delta t}{\tau}\right) V(\rho) + \frac{\Delta t}{\tau} V(\rho) \leq \left(1 - \frac{\Delta t}{\tau}\right) v_- + \frac{\Delta t}{\tau} V(\rho) = v_+ \leq v_-,$$

where we used (4.3) and the hypothesis  $\Delta t \leq \tau$ . Hence,  $v_+ \in [\bar{v}, v_M]$  (resp.  $v_+ \in [0, \bar{v}]$ ).

On the other hand

$$w_+ = w_- + \frac{\Delta t}{\tau} (V(\rho) - v_-) \leq w_-$$

and, developing

$$\mathcal{V}(\rho, w_-) = \mathcal{V}(\rho, \phi(V(\rho))) + \mathcal{V}_w(\rho, \tilde{w}) (w_- - \phi(V(\rho))) = V(\rho) + \mathcal{V}_w(\rho, \tilde{w}) (w_- - \phi(V(\rho)))$$

for some  $\tilde{w}$ ,

$$\begin{aligned} w_+ &= w_- - \frac{\Delta t}{\tau} \mathcal{V}(\rho, w_-) + \frac{\Delta t}{\tau} V(\rho) \\ &= w_- \left( 1 - \frac{\Delta t}{\tau} \mathcal{V}_w(\rho, \tilde{w}) \right) + \frac{\Delta t}{\tau} \mathcal{V}_w(\rho, \tilde{w}) \phi(V(\rho)) \geq \phi(V(\rho)), \end{aligned}$$

where we used  $\Delta t \leq \tau / \|\mathcal{V}_w\|_\infty$ . Therefore,  $w_+ \in [w_m, \bar{w}]$  (resp.  $w_+ \in [\bar{w}, w_M]$ ).

The case  $v_- < V(\rho)$  is treated analogously.  $\square$

## 4.2 BV estimates

**Proposition 2.** *Let us assume there exists  $c > 0$  such that  $\mathcal{V}_\rho(\rho, w) \leq -c$  for all  $(\rho, w) \in \mathbf{E}$ . For any  $t > 0$ , the total variation of the Riemann invariants*

$$\text{TV}(W(U^\nu(t, \cdot))) := \text{TV}(w^\nu(t, \cdot)) + \text{TV}(\mathcal{V}(U^\nu(t, \cdot)))$$

of the WFT approximate solution satisfies the uniform bound

$$\text{TV}(W(U^\nu(t, \cdot))) \leq \text{TV}(W(U_0)) e^{Kt/\tau}, \quad (4.4)$$

where  $K := \frac{2}{c} \|V'\|_\infty \max\{1, \|\mathcal{V}_w\|_\infty\}$ . In particular, when  $\tau \searrow 0$ , the estimate (4.4) blows up.

*Proof.* Notice that, when we solve the homogeneous system (3.2), the total variation of both Riemann invariants is non-increasing in time since we are dealing with a Temple-class system. We thus focus on the evolution of the total variation at step 3, corresponding to (3.5).

We recall that, at  $t_n = n\Delta t^\nu$  and dropping the index  $\nu$  for simplicity,

$$\begin{aligned} \rho(t_n+, x) &= \rho(t_n-, x), \\ v(t_n+, x) &= v(t_n-, x) + \frac{\Delta t}{\tau} (V(\rho) - v)(t_n-, x), \\ w(t_n+, x) &= w(t_n-, x) + \frac{\Delta t}{\tau} (V(\rho) - v)(t_n-, x). \end{aligned}$$

Therefore, at each jump in the approximate solution, we have

$$\begin{aligned} |v_r^+ - v_l^+| &= \left| v_r^- - v_l^- + \frac{\Delta t}{\tau} (V(\rho_r) - V(\rho_l)) - \frac{\Delta t}{\tau} (v_r^- - v_l^-) \right| \\ &= \left| \left( 1 - \frac{\Delta t}{\tau} \right) (v_r^- - v_l^-) + \frac{\Delta t}{\tau} (V(\rho_r) - V(\rho_l)) \right| \\ &\leq \left( 1 - \frac{\Delta t}{\tau} \right) |v_r^- - v_l^-| + \frac{\Delta t}{\tau} \|V'\|_\infty |\rho_r - \rho_l|. \end{aligned}$$

From the relation  $\rho = \mathcal{R}(v, w)$ , we get

$$\begin{aligned} |\rho_r - \rho_l| &\leq \|\mathcal{R}_v\|_\infty |v_r^- - v_l^-| + \|\mathcal{R}_w\|_\infty |w_r^- - w_l^-| \\ &\leq \frac{1}{c} |v_r^- - v_l^-| + \frac{\|\mathcal{V}_w\|_\infty}{c} |w_r^- - w_l^-| \end{aligned}$$

by the relations  $\mathcal{R}_v = 1/\mathcal{V}_\rho$  and  $\mathcal{R}_w = -\mathcal{V}_w/\mathcal{V}_\rho$ . Hence

$$|v_r^+ - v_l^+| \leq \left(1 + \frac{\Delta t}{\tau} \left(\frac{\|V'\|_\infty}{c} - 1\right)\right) |v_r^- - v_l^-| + \frac{\Delta t}{\tau} \frac{\|V'\|_\infty \|\mathcal{V}_w\|_\infty}{c} |w_r^- - w_l^-|. \quad (4.5)$$

Similar estimates lead to

$$|w_r^+ - w_l^+| \leq \frac{\Delta t}{\tau} \left(\frac{\|V'\|_\infty}{c} + 1\right) |v_r^- - v_l^-| + \left(1 + \frac{\Delta t}{\tau} \frac{\|V'\|_\infty \|\mathcal{V}_w\|_\infty}{c}\right) |w_r^- - w_l^-|. \quad (4.6)$$

Summing (4.5) and (4.6), we obtain

$$\text{TV}(W(U(t_n+, \cdot))) \leq \left(1 + K \frac{\Delta t}{\tau}\right) \text{TV}(W(U(t_n-, \cdot)))$$

with

$$K := \frac{2}{c} \max \left\{ \|V'\|_\infty, \|V'\|_\infty \|\mathcal{V}_w\|_\infty \right\},$$

which gives (4.4).  $\square$

### 4.3 $L^1$ Lipschitz continuity in time

To get the BV estimates in space and time, following [6], see also [16, Lemma 2.3], we need the next result.

**Proposition 3.** *Under the same assumption as in Proposition 2, let us assume  $\text{TV}(W(U_0)) = M < +\infty$ . Then there exist a constant  $C_M > 0$  independent of  $\tau$  and a constant  $L_\tau > 0$ , such that,  $\forall a < b$  and  $\forall 0 \leq s < t$ , the WFT approximate solutions satisfy*

$$\int_a^b |\rho^\nu(t, x) - \rho^\nu(s, x)| \leq C_M e^{Kt/\tau} (t - s), \quad (4.7)$$

$$\int_a^b |w^\nu(t, x) - w^\nu(s, x)| \leq \left(C_M e^{Kt/\tau} + L_\tau\right) (t - s + \Delta t). \quad (4.8)$$

In particular, both estimates above blow up as  $\tau \searrow 0$ .

*Proof.* Let  $s$  and  $t \in \mathbb{R}$  such that  $0 \leq s < t$ . If there are no time-steps between  $s$  and  $t$ , (4.7) and (4.8) are true for any  $L_\tau \geq 0$ , as a direct application of Temple-class system properties, see [16]. We suppose now that there are  $N + 1$  time-steps between  $s$  and  $t$ :

$$s \leq k\Delta t \leq (k+1)\Delta t \leq \dots \leq (N+k)\Delta t \leq t,$$

so that  $N\Delta t \leq t - s$ .

Let  $a < b$  given and  $x \in ]a, b[$ . We can then estimate

$$\begin{aligned} & |\rho^\nu(t, x) - \rho^\nu(s, x)| \\ &= \left| \rho^\nu(t, x) - \rho^\nu((N+k)\Delta t, x) + \sum_{i=k}^{k+N-1} [\rho^\nu((i+1)\Delta t, x) - \rho^\nu(i\Delta t, x)] + \rho^\nu(k\Delta t, x) - \rho^\nu(s, x) \right| \end{aligned}$$

$$\leq |\rho^\nu(t, x) - \rho^\nu((N+k)\Delta t, x)| + \sum_{i=k}^{k+N-1} |\rho^\nu((i+1)\Delta t, x) - \rho^\nu(i\Delta t, x)| + |\rho^\nu(k\Delta t, x) - \rho^\nu(s, x)|.$$

Since  $\rho^\nu$  does not change through the splitting process, we can apply the previous property between two consecutive time-steps to obtain

$$\begin{aligned} & \int_a^b |\rho^\nu(t, x) - \rho^\nu(s, x)| dx \\ & \leq C_M e^{Kt/\tau} \left[ t - (N+k)\Delta t + \sum_{i=k}^{N+k-1} ((i+1)\Delta t - i\Delta t) + k\Delta t - s \right] \\ & = C_M e^{Kt/\tau} [t - (N+k)\Delta t + (N+k)\Delta t - k\Delta t + k\Delta t - s] = C_M e^{Kt/\tau} (t - s). \end{aligned}$$

Concerning  $w^\nu$ , which is modified at each splitting step, we have to consider an additional term:

$$\begin{aligned} & \sum_{i=k}^{N+k} \int_a^b |w^\nu(i\Delta t+, x) - w^\nu(i\Delta t-, x)| dx = \frac{\Delta t}{\tau} \sum_{i=k}^{N+k} \int_a^b |(V(\rho^\nu) - v^\nu)(i\Delta t-, x)| dx \\ & \leq \frac{\Delta t}{\tau} (N+1)(b-a) \sup_{U^\nu \in \mathbf{E}} |V(\rho^\nu) - v^\nu| \\ & \leq L_\tau (t - s + \Delta t), \end{aligned}$$

with  $L_\tau = \frac{b-a}{\tau} \sup_{U^\nu \in \mathbf{E}} |V(\rho^\nu) - v^\nu|$ . Summing to the other terms, we get (4.8).  $\square$

## 5 Existence of weak solutions

The uniform bounds derived in Section 4 allow to apply Helly's Theorem to state the existence of a subsequence of WFT approximate solutions, still denoted by  $\{W^\nu\}_\nu$ , converging in  $\mathbf{L}_{\text{loc}}^1$  to a function  $W$ . It now remains to prove that  $u = u(W)$  is a weak solution of (2.1).

**Theorem 1.** *Let  $U_0 : \mathbb{R} \rightarrow \mathbf{E}$  with  $\text{TV}(W(U_0)) < +\infty$  and let  $W$  be the limit function of the sequence  $\{W^\nu\}_\nu$  of WFT approximate solutions as  $\nu \rightarrow \infty$ . Then  $u = u(W)$  is a weak solution of (2.1) with initial data  $u_0 = u(W(U_0))$  in the sense of Definition 1.*

*Proof.* Let  $T > 0$  be a given finite time horizon and consider  $\varphi \in \mathbf{C}_c^1([-\infty, T] \times \mathbb{R}; \mathbb{R})$ . We define  $N^\nu$  so that  $T = N^\nu \Delta t^\nu + \beta^\nu$ ,  $\beta^\nu \in [0, \Delta t^\nu[$ . Following [10], we observe that

$$\begin{aligned} & \int_{k\Delta t^\nu}^{(k+1)\Delta t^\nu} \int_{\mathbb{R}} [u(W^\nu) \partial_t \varphi + F(u(W^\nu)) \partial_x \varphi] (t, x) dx dt \\ & = \int_{\mathbb{R}} \varphi((k+1)\Delta t^\nu, x) u(W^\nu((k+1)\Delta t^\nu-, x)) dx - \int_{\mathbb{R}} \varphi(k\Delta t^\nu, x) u(W^\nu(k\Delta t^\nu+, x)) dx, \end{aligned}$$

since  $u^\nu := u(W^\nu)$  are weak solutions of the homogeneous system (3.2) in each interval  $]k\Delta t, (k+1)\Delta t[ \times \mathbb{R}$  by construction. Therefore, remembering that  $\rho^\nu(k\Delta t+) = \rho^\nu(k\Delta t-)$ , we get

$$\int_0^T \int_{\mathbb{R}} [u(W^\nu) \partial_t \varphi + F(u(W^\nu)) \partial_x \varphi] (t, x) dx dt$$

$$\begin{aligned}
&= \left( \sum_{k=0}^{N^\nu-1} \int_{k\Delta t^\nu}^{(k+1)\Delta t^\nu} + \int_{N^\nu\Delta t^\nu}^T \right) \int_{\mathbb{R}} [u(W^\nu)\partial_t\varphi + F(u(W^\nu))\partial_x\varphi] (t, x) dxdt \\
&= \sum_{k=0}^{N^\nu-1} \int_{\mathbb{R}} \varphi((k+1)\Delta t^\nu, x) u(W^\nu((k+1)\Delta t^\nu-, x)) dx \\
&\quad - \sum_{k=0}^{N^\nu} \int_{\mathbb{R}} \varphi(k\Delta t^\nu, x) [u(W^\nu(k\Delta t^\nu-, x)) + \Delta t G(u(W^\nu(k\Delta t^\nu-, x)))] dx \\
&= - \sum_{k=0}^{N^\nu} \Delta t \int_{\mathbb{R}} \varphi(k\Delta t^\nu, x) G(u(W^\nu(k\Delta t^\nu-, x))) dx - \int_{\mathbb{R}} \varphi(0, x) u(W_0^\nu(x)) dx
\end{aligned}$$

Passing to the limit as  $\nu \rightarrow +\infty$  in the above equality, by Lebesgue dominated convergence theorem we obtain

$$\begin{aligned}
&\int_0^T \int_{\mathbb{R}} [u(W)\partial_t\varphi + F(u(W))\partial_x\varphi] (t, x) dxdt \\
&= - \int_0^T \int_{\mathbb{R}} \varphi(t, x) G(u(W(t, x))) dx - \int_{\mathbb{R}} \varphi(0, x) u(W_0(x)) dx,
\end{aligned}$$

which concludes the proof.  $\square$

**Remark 1.** *In the present setting, the violation of the subcharacteristic condition (2.10) contradicts the existence of an entropy, entropy-flux pairs for system (2.1), i.e. functions  $\eta, q : \mathbb{R}_+^2 \rightarrow \mathbb{R}$  such that*

$$\begin{aligned}
\nabla\eta^T(z)DF(z) &= \nabla^T q(z), \\
\nabla\eta^T(z)G(z) &\leq 0,
\end{aligned}$$

for  $z \in \mathbb{R}_+^2$ . We refer to [7, Theorem 2.1] for details.

## 6 Traveling wave analysis

In this section, we aim at investigating the behaviour of travelling waves for system (2.1) violating the stability condition (2.10). We refer to [14, 26, 21] for a similar analysis conducted on other models.

We look for traveling wave solutions of (2.8) of the form  $(\rho(\xi), v(\xi))$ , where  $\xi = (x - \sigma t)/\tau$ :

$$-\sigma\rho' + (\rho v)' = 0, \tag{6.1a}$$

$$-\sigma v' + \lambda_1(\rho, w)v' = V(\rho) - v. \tag{6.1b}$$

Integrating (6.1a), we get

$$\rho v = \sigma\rho + m \quad \implies \quad \rho = \frac{m}{v - \sigma} = \frac{m}{\mathcal{V}(\rho, w) - \sigma} \quad \iff \quad \mathcal{V}(\rho, w) = \sigma + \frac{m}{\rho}.$$

for some  $m \in \mathbb{R}$ . From (6.1b), we then get

$$v' = \frac{V(\rho) - v}{\lambda_1(\rho, w) - \sigma} = \frac{(v - \sigma)(V - v)}{(v - \sigma)^2 + m\mathcal{V}_\rho} \quad \text{or} \quad \rho' = \frac{\rho}{\sigma - v} \frac{V(\rho) - v}{\lambda_1(\rho, w) - \sigma}. \tag{6.2}$$

Note that, if (2.10) holds, then the denominator  $\lambda_1(\rho, w) - \sigma \neq 0$ , otherwise there exists a sonic point  $v_\sigma$  such that

$$v_\sigma - \sigma = \pm \sqrt{-m\mathcal{V}_\rho} \quad \Longrightarrow \quad v_\sigma = \sigma \pm \sqrt{-m\mathcal{V}_\rho}.$$

Since  $\mathcal{V}_\rho \leq 0$ , we need to take  $m > 0$ . We regularize (6.2) requiring  $v_\sigma = V(m/(v_\sigma - \sigma))$ .

In a periodic setting, two states  $(\rho_\pm, v_\pm)$  will be connected alternatively by a jump discontinuity satisfying the Rankine-Hugoniot condition

$$\sigma = \frac{\rho_+ v_+ - \rho_- v_-}{\rho_+ - \rho_-}, \quad (6.3)$$

and a solution of (6.2). We recall that entropy admissible jump discontinuities must satisfy  $v_- \geq v_+$ . In particular, if  $v_- > v_+$  (shock) we must have  $w_- = w_+$ . Therefore, for the solution of (6.2) going from  $v_+$  to  $v_-$ , we must have  $v' \geq 0$ . Besides, we have  $m = \rho_\pm(v_\pm - \sigma) \geq 0$  since  $v_\pm \geq \sigma$ . This confirms the existence of a sonic point  $v_\sigma$ . (Note that in the case of a contact discontinuity, it holds  $v_- = v_+ = \sigma$  and therefore  $v' = 0$ ).

Let us analyze (6.2) better. We can rewrite the denominator as

$$(v - \sigma)^2 - (\sqrt{-m\mathcal{V}_\rho})^2 = \left( (v - \sigma) - \sqrt{-m\mathcal{V}_\rho} \right) \left( (v - \sigma) + \sqrt{-m\mathcal{V}_\rho} \right).$$

Since  $v > \sigma$ , the denominator has a unique zero of multiplicity one at

$$v_\sigma = \sigma + \sqrt{-m\mathcal{V}_\rho}.$$

Solution of (6.2) being increasing along  $\xi$ , we have

$$\begin{aligned} v_+ < \sigma + \sqrt{-m\mathcal{V}_\rho} &\quad \Longrightarrow \quad (v_+ - \sigma)^2 - (\sqrt{-m\mathcal{V}_\rho})^2 < 0, \\ v_- > \sigma + \sqrt{-m\mathcal{V}_\rho} &\quad \Longrightarrow \quad (v_- - \sigma)^2 - (\sqrt{-m\mathcal{V}_\rho})^2 > 0. \end{aligned}$$

Therefore, to have  $v' \geq 0$  in (6.2), the numerator must satisfy

$$\begin{aligned} V(\rho) &\leq v = \mathcal{V}(\rho, w_\pm) \quad \text{for } \rho_\sigma \leq \rho \leq \rho_+, \\ V(\rho) &\geq v = \mathcal{V}(\rho, w_\pm) \quad \text{for } \rho_- \leq \rho \leq \rho_\sigma. \end{aligned}$$

In particular,  $V(\rho_\sigma) = v_\sigma = \sigma + \frac{m}{\rho_\sigma}$ . See Figure 4.

To summarize, the recipe to construct a periodic wave oscillating between states  $(\rho_\pm, v_\pm)$  (referred to as ‘‘jamiton’’ in [14, 26]) is the following:

1. For a prescribed downstream state  $(\rho_+, v_+)$  such that  $v_+ = \mathcal{V}(\rho_+, w_+) > V(\rho_+)$  compute  $\rho_\sigma$  implicitly defined by  $V(\rho_\sigma) = \mathcal{V}(\rho_\sigma, w_+) = v_\sigma$ . Then, from the identities

$$\begin{cases} v_+ = \sigma + \frac{m}{\rho_+}, \\ v_\sigma = \sigma + \frac{m}{\rho_\sigma}, \end{cases}$$

we recover

$$\begin{cases} m = \frac{\rho_\sigma \rho_+}{\rho_+ - \rho_\sigma} (v_\sigma - v_+), \\ \sigma = v_+ \frac{\rho_+}{\rho_+ - \rho_\sigma} + v_\sigma \frac{\rho_\sigma}{\rho_\sigma - \rho_+}. \end{cases}$$

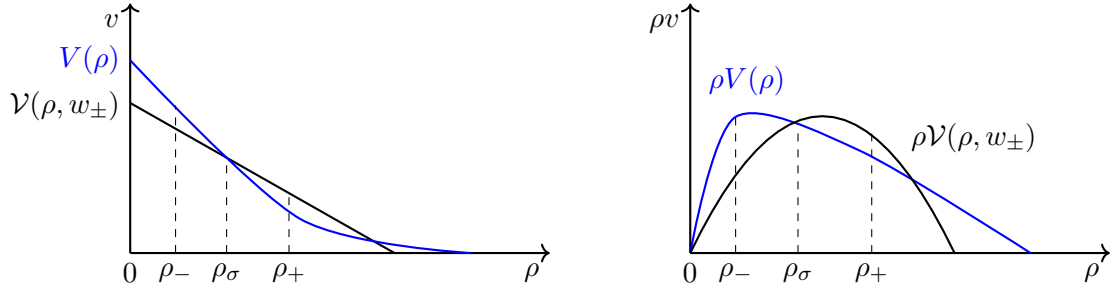


Figure 4: Speed-density (left) and flow-density (right) representations of the curves involved in the construction of the traveling wave profiles, where we set  $\mathcal{V}(\rho, w) = w - \rho$ ,  $V(\rho) = V_{max} (1 - \exp(C(1 - R/\rho)))$ .

2. Determine the upstream state  $(\rho_-, v_-)$ , with  $v_- = \mathcal{V}(\rho_-, w_+)$  using the Rankine-Hugoniot condition (6.3).
3. Integrate (6.2) from  $v(0) = v_+$  to  $v(\pi) = v_-$ . The density is then given by

$$\rho(\xi) = \frac{m}{v(\xi) - \sigma}.$$

The period of the travelling wave is given by  $\pi$ .

4. The total number of vehicles involved is then given by

$$N = \int_0^\pi \rho(\xi) d\xi.$$

See Figure 5.

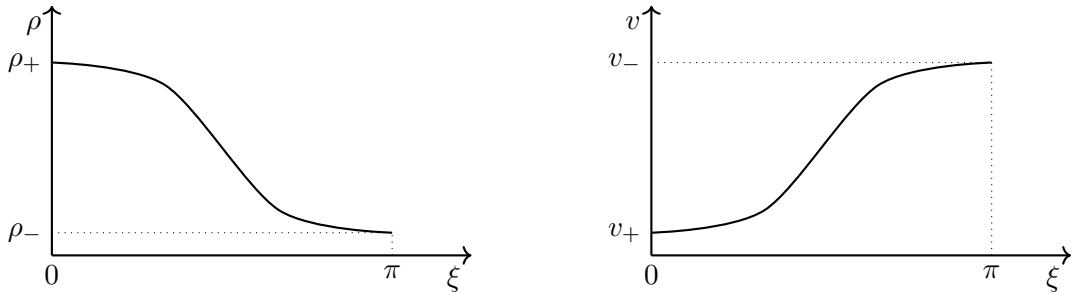


Figure 5: Density and speed profiles of the jamiton solution of (6.1).

## 7 Chapman-Enskog expansion

Another classical way of investigating the stability of equilibria of relaxation systems is to perform a formal expansion with respect to the relaxation parameter, see e.g. [5, 7, 9].

We consider the equations in  $(\rho, v)$  coordinates (2.8), which we recall here

$$\partial_t \rho + \partial_x(\rho v) = 0, \tag{7.1a}$$



$$\partial_t v + \lambda_1(\rho, w) \partial_x v = \frac{V(\rho) - v}{\tau}. \quad (7.1b)$$

Assuming smooth solutions, from (7.1b) we recover:

$$v = V(\rho) - \tau (\partial_t v + \lambda_1(\rho, w) \partial_x v) = V(\rho) - \tau (V'(\rho) \partial_t \rho + \lambda_1(\rho, w) \partial_x v).$$

Substituting (7.1a), i.e.  $\partial_t \rho = -\partial_x(\rho v) = -\partial_x(\rho V(\rho)) + \mathcal{O}(\tau)$  and  $\partial_x v = \partial_x V(\rho) + \mathcal{O}(\tau)$ , we get

$$v = V(\rho) - \tau (\lambda_1(\rho, w) \partial_x V(\rho) - V'(\rho) \partial_x(\rho V(\rho))) + \mathcal{O}(\tau^2).$$

Replacing this last expression in (7.1a) and truncating to the first order we get

$$\partial_t \rho + \partial_x(\rho V(\rho)) = \tau \partial_x \left( \rho V'(\rho) (\lambda_1(\rho, w) - (\rho V(\rho))') \partial_x \rho \right). \quad (7.2)$$

Since  $V'(\rho) \leq 0$ , the diffusive equation (7.2) is stable if and only if  $\lambda_1(\rho, w) \leq (\rho V(\rho))'$ , i.e. (2.10) holds.

Alternatively, the formal Chapman-Enskog expansion in  $v = \sum_{k=0}^{\infty} \tau^k v_k$  leads to:

$$\begin{aligned} \partial_t \rho + \sum_{k=0}^{\infty} \tau^k \partial_x(\rho v_k) &= 0, \\ \sum_{k=0}^{\infty} \tau^k \partial_t v_k + \lambda_1(\rho, w) \sum_{k=0}^{\infty} \tau^k \partial_x v_k &= \frac{V(\rho)}{\tau} - \sum_{k=0}^{\infty} \tau^{k-1} v_k. \end{aligned}$$

The second identity yields

$$\begin{aligned} v_0 &= V(\rho), \\ \partial_t v_0 + \lambda_1(\rho, w) \partial_x v_0 &= -v_1, \\ \partial_t v_k + \lambda_1(\rho, w) \partial_x v_k &= -v_{k+1}. \end{aligned}$$

Replacing in the first identity we get

$$\begin{aligned} \partial_t \rho + \partial_x(\rho V(\rho)) &= \partial_t \rho + \partial_x(\rho v_0) \\ &= - \sum_{k=1}^{\infty} \tau^k \partial_x(\rho v_k) \\ &= \sum_{k=0}^{\infty} \tau^{k+1} \partial_x \left( \rho (\partial_t v_k + \lambda_1(\rho, w) \partial_x v_k) \right), \end{aligned}$$

which, to the first order  $k = 0$ , is

$$\begin{aligned} \partial_t \rho + \partial_x(\rho V(\rho)) &= \tau \partial_x \left( \rho (\partial_t v_0 + \lambda_1(\rho, w) \partial_x v_0) \right) \\ &= \tau \partial_x \left( \rho V'(\rho) (\partial_t \rho + \lambda_1(\rho, w) \partial_x \rho) \right) \\ &= \tau \partial_x \left( \rho V'(\rho) (\lambda_1(\rho, w) - (\rho V(\rho))') \partial_x \rho \right) \end{aligned} \quad (7.3)$$

as in (7.2).

Considering travelling wave solutions of (7.2) of the form  $\rho(t, x) = \rho(\xi)$ , where  $\xi = (x - \sigma t)/\tau$ , we get

$$-\sigma\rho' + (\rho V(\rho))' = \left( \rho V'(\rho) (\lambda_1(\rho, w) - (\rho V(\rho))') \rho' \right)'. \quad (7.4)$$

By integration, we obtain

$$\rho' = \frac{\rho(V(\rho) - \sigma) - m}{\rho V'(\rho) (\lambda_1(\rho, w) - (\rho V(\rho))')}, \quad (7.5)$$

which is coherent with (6.2) setting  $m = \rho(\sigma - v)$  and assuming  $\sigma \sim (\rho V(\rho))'$ , which gives  $\rho V'(\rho) = \sigma - V(\rho) \sim \sigma - v$ .

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