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PWA control functions: from the projection of mpQP solution and back to the convexification by lifting

Songlin Yang¹, Sorin Olaru¹, and Pedro Rodriguez-Ayerbe¹

Abstract—This paper focuses on the geometric properties of the Piece-Wise Affine (PWA) feedback function as they appear from the optimal solution of the multi-parameter quadratic programming (mpQP) problem. Such optimization problems are popular formulations, for example, in the design of model-based predictive controllers (MPC) for discrete linear systems subject to input and state constraints. The paper considers such a PWA function as input data and provides a method for reconstructing a feasible convex set and a PWA curve within it, which retrieves the identical structure of the solution in the original parametric feasible set. The proposed method involves establishing and decomposing the topology structure of the polyhedral critical regions, which form the domain of the PWA function by means of a graph of interconnections. The regions are split into the boundary and interior collections using convex-concave lifting. The explicit solution is then merged based on the convex-concave liftings to reconstruct the feasible domain and the PWA curves.

I. INTRODUCTION

Model predictive control (MPC) has captured significant attention in the academic community due to its receding horizon optimal control formulation [1]. MPC uses a dynamic model of the process to predict the future evolution over a finite-time horizon and selects the optimal control actions with respect to a specified performance index. Furthermore, MPC can handle process constraints and multi-variable interactions in a unified formulation. Various researchers have recently studied MPC and their optimization-based control alternatives [2] from different angles [3]. The present paper focuses on the geometric characteristics of MPC [4] and aims to propose inverse optimal solutions [5] for the predictive controllers.

Explicit MPC (EMPC) emerged as a method to simplify online control computation by converting it into evaluating a piecewise affine function over polyhedral critical regions [6]. The geometrical properties of the optimal solution were exploited during the construction, but the EMPC implementation proved challenging due to the memory required for online evaluation. In [7], the authors reduce storage and evaluation time by assuming the initial state is contained in a given set, omitting irrelevant predictive trajectory regions. In [8], by enumerating the possible optimal active sets, the authors showed that EMPC solutions admit a closed-form solution which does not require the storage of critical regions. In [9], a parametrized polyhedra approach in the combined (input+parameter) space is used to analyse the

solution of EMPC. In [4], the authors construct the unconstrained critical region and then enumerate the others based on the combinations of active constraints.

Once the advantages of implicit and explicit MPC have been characterized, a series of studies [10] proved that explicit PWA formulations could be used to obtain inverse optimal QP formulations with lower computation footprint. Remarkably, inverse-optimal formulations have been constructed with only one supplementary dimension of the vector of arguments [5], thus offering a particularly compact QP for the PWA controller originated by the MPC formulation.

The inverse optimal solutions for PWA feedback functions obtained in the literature exploit constraint activation at the optimum for the entire range of the parameters. However, the original MPC formulation has a solution that is unconstrained around the equilibrium whenever this is located in the interior of the (state-input) feasible domain. In this context, and in contrast to prior research results, such as in [5] and [11], this paper aims to devise a new PWA function and a corresponding higher-dimensional polytope to uphold the geometric structure of the original control action, including the feasible domain and the constrained-unconstrained characteristics. Specifically, the PWA control is partitioned into two components: a linear function corresponding to the unconstrained critical region in the parameters' space and a constrained solution that is essentially a projection onto the boundary of the feasible set.

- A higher-dimensional polytope and a suitable PWA function are defined starting from an original continuous PWA control and its corresponding polyhedral critical region. Notably, the newly created polytope and the extended PWA function exhibit the same projection as the original PWA function.
- Drawing inspiration from the concept of convex lifting [12], we introduce a novel approach called convex-concave lifting to facilitate the construction of the higher-dimensional polytope.
- We examine the connection between the constructed polytope and the original mpQP problem and point out that the corresponding mpQP problem constructed using the proposed higher-dimensional polytope has a lower complexity.

Notation: In this paper, \mathbb{R}^n and \mathbb{N} denote the set of real numbers in n -dimensional space, the set of nonnegative integers, and $\mathcal{I}_N = [1, N] \cap \mathbb{N}$. The symbols $\mathbf{0}_{m \times n}$ and I_m represent a matrix of size $m \times n$ with all elements equal to zero and an m -dimensional identity matrix, respectively. If

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$\mathcal{P} \subset \mathbb{R}^n$ is a polyhedral set, then $\text{int}(\mathcal{P})$ denotes the set of interior points of \mathcal{P} and $\text{bd}(\mathcal{P})$ denotes the set of boundary points of \mathcal{P} , $\text{Proj}_{\mathbb{R}^m} \mathcal{P}$ represents the orthogonal projection of \mathcal{P} onto the subspace \mathbb{R}^m . The convex hull of a set $\{*\}$ is denoted by $\text{conv}\{*\}$. $\mathbb{B}^n = \{x \in \mathbb{R}^n : \|x\|_\infty \leq 1\}$.

II. BACKGROUND AND PROBLEM FORMULATION

A. From MPC to PWA control

Consider a discrete-time linear system:

$$x_{k+1} = Ax_k + Bu_k, \quad (1)$$

where the states $x_k \in \mathbb{R}^n$, and the inputs $u_k \in \mathbb{R}^m$ at time k are bounded by polyhedral sets:

$$\mathcal{X} = \{x \in \mathbb{R}^n | H_x x \leq b_x\}, \mathcal{U} = \{u \in \mathbb{R}^m | H_u u \leq b_u\},$$

where b_x, b_u, H_x , and H_u are known constant matrices, such that $0 \in \text{int}(\mathcal{X})$ and $0 \in \text{int}(\mathcal{U})$.

The model-based predictive controller with a N -steps receding horizon can be obtained by the iterative construction of the control sequence $\kappa_u^*(x) = [u_0^T, \dots, u_{N-1}^T]^T$ which solves the finite-time optimal control problem for the given initial state $x_0 = x$:

$$\kappa_u^*(x) = \arg \min_{\kappa_u} \sum_{k=0}^{N-1} u_k^T R u_k + \sum_{k=1}^N x_k^T Q x_k \quad (2a)$$

$$\text{s. t. } x_{k+1} = Ax_k + Bu_k, \quad (2b)$$

$$u_k \in \mathcal{U}, x_k \in \mathcal{X}, x_N \in \Omega \quad (2c)$$

with $R \succ 0, Q \succeq 0$ and $\Omega \subseteq \mathbb{R}^n$ a control invariant set with $0 \in \text{int}(\Omega)$. Without loss of generality, the problems to be solved at each sampling instant can be written:

$$\kappa_u^*(x) = \arg \min_{\kappa_u} \kappa_u^T H \kappa_u + x^T F \kappa_u, \quad (3a)$$

$$\text{s.t. } [x^T, \kappa_u^T] \in \mathcal{P}, \quad (3b)$$

where $x \in \underline{\mathcal{X}} = \text{Proj}_{\mathbb{R}^n} \mathcal{P} \subseteq \mathcal{X}$ is the current state playing the role of a parameter, $\kappa_u(x) \in \mathbb{R}^{mN}$ is the optimization vector and $H \succ 0$. The set \mathcal{P} is the parameterized feasible domain with the formulation

$$\mathcal{P} = \{[x^T, \kappa_u^T]^T | G \kappa_u \leq W + Ex, x \in \underline{\mathcal{X}}\}$$

where G, W, E are constructed from the mpQP problem (2). From the above notations it follows $\forall x \in \underline{\mathcal{X}}, \mathcal{P} \neq \emptyset$.

The solution to problem (3) takes the form of a piecewise affine (PWA) function of the system state x . In order to deal with those properties, the notion of polyhedral partition needs to be introduced.

Definition 1. A collection of polyhedral sets $\{\mathcal{X}_1, \dots, \mathcal{X}_N\}$ is called a polyhedral partition of $\underline{\mathcal{X}}$ if $\forall i, j \in \mathcal{I}_N, i \neq j$,

1. $\underline{\mathcal{X}} = \cup_{i=1}^N \mathcal{X}_i$;
2. $\text{int}(\mathcal{X}_i) \cap \text{int}(\mathcal{X}_j) = \emptyset$.

Such a polyhedral partition is next denoted as $\{\mathcal{X}_i\}_{\mathcal{I}_N}$.

Building upon the results presented in [6], the solution to the mpQP problem (2) can be expressed as a continuous PWA function of x , denoted as $\kappa_u^*(x)$.

$$\kappa_u^*(x) = f_i(x), x \in \mathcal{X}_i,$$

where $f_i : \mathbb{R}^n \rightarrow \mathbb{R}^{mN}$ is piecewise affine and \mathcal{X}_i is a polyhedron of polyhedral partition $\{\mathcal{X}_i\}_{\mathcal{I}_N}$.

B. Properties of the MPC controller

Similar to the literature on the geometric structure of MPC [4]- [9], the relationship between the unconstrained optimum and the feasible domain is privileged in the present work. The following propositions underline a fundamental property in this respect.

Proposition 1. If $0 \in \text{int}(\mathcal{X})$, $0 \in \text{int}(\mathcal{U})$ and $0 \in \text{int}(\Omega)$ in (2) then with respect to the solution of (3):

$$\mathcal{X}_1 = \{x \in \underline{\mathcal{X}} | \kappa_u^*(x) = -H^{-1}Fx\}$$

is a polyhedron set with $\text{int}(\mathcal{X}_1) \neq \emptyset$, and additionally,

$$\forall x \in \text{int}(\mathcal{X}_1), [x^T, \kappa_u^*(x)^T] \in \text{int}(\mathcal{P}).$$

Proof. At the origin, the optimal argument is $\kappa_u^*(0) = 0$, corresponding to a feasible interior point of the constraint set $0 \in \mathcal{P}$. Moreover, the unconstrained optimum is given by $\kappa_u^*(x) = -H^{-1}Fx$ and there exists a non-empty region $\mathcal{R} \subseteq \underline{\mathcal{X}}$ such that $(A - BH^{-1}F)x \in \underline{\mathcal{X}}, \forall x \in \mathcal{R}$. Once the non-emptiness of \mathcal{X}_1 is ensured, its boundaries are given by the KKT conditions represented by affine inequality constraints [6], which proves its polyhedral structure.

Additionally, suppose by contradiction that it exists $x \in \text{int}(\mathcal{X}_1)$ such that $[x^T, (-H^{-1}Fx)^T] \notin \text{int}(\mathcal{P})$. It means that $[x^T, (-H^{-1}Fx)^T] \in \text{bd}(\mathcal{P})$. and thus either $\kappa_u^*(x) \neq -H^{-1}Fx$ or $0 \in \text{bd}(\mathcal{P})$. The first case contradicts the definition of set \mathcal{X}_1 and the second with the assumption that both \mathcal{X}, Ω and \mathcal{U} contain the origin in their strict interior. \square

\mathcal{X}_1 is often denoted as the unconstrained critical region of the EMPC. In the geometry of the PWA control function, the importance of this set is underlined by the following result.

Proposition 2. If $0 \in \text{int}(\mathcal{X})$, $0 \in \text{int}(\mathcal{U})$ and $0 \in \text{int}(\Omega)$ in (2), then for $x \in \mathcal{X} \setminus \mathcal{X}_1$, the optimal solution for the mpQP problem (3) satisfies

$$[x^T, \kappa_u^*(x)^T] \in \text{bd}(\mathcal{P})$$

Proof. Suppose again by contradiction that it exists $x \in \mathcal{X} \setminus \mathcal{X}_1$ with $[x^T, \kappa_u^*(x)^T] \in \text{int}(\mathcal{P})$. From this second fact, it follows that the unconstrained optimum is feasible, i.e. $\kappa_u^*(x)^T = -H^{-1}Fx$, which leads to a contradiction as long as the optimality conditions should hold only for $x \in \mathcal{X}_1$. \square

Corollary 1. If $0 \in \text{int}(\underline{\mathcal{X}})$ and $0 \in \text{int}(\mathcal{U})$ then

$$\forall x \in \text{int}(\mathcal{X}_1), [x^T, \kappa_u^*(x)^T]^T \in \text{int}(\mathcal{P}_u),$$

where

$$\mathcal{P}_u := \text{conv} \{[x^T, \kappa_u^*(x)^T]^T : x \in \underline{\mathcal{X}}\}. \quad \square \quad (4)$$

Two important features of an MPC controller emerge from the aforementioned properties.

- The optimal solution $\kappa_u^*(x)$ consists of two parts: the unconstrained optimum for $x \in \mathcal{X}_1$, and the boundary solution for $x \in \mathcal{X}_i$ where $i \in \mathcal{I}_N \setminus \{1\}$.
- Only the first component of the sequence $\kappa_u^*(x)$ is used by MPC control action $\kappa_{pwa}(x)$ and will be denoted as

$$\kappa_{pwa}(x) = [I_m, \mathbf{0}_{m \times (mN-m)}] \kappa_u^*(x),$$

or explicitly as

$$\kappa_{pwa}(x) = F_i x + g_i, x \in \mathcal{X}_i, \quad (5)$$

where $F_i \in \mathbb{R}^{m \times n}$, $g_i \in \mathbb{R}^m$.

In the following sections, we combine these two properties to propose a problem of interest and its solution strategy.

C. Problems formulation

Starting from an mpQP problem (3) with a parameter vector $x \in \mathcal{X} \subset \mathbb{R}^n$ and the parameterized feasible domain $\mathcal{P} \subset \mathbb{R}^{n+mN}$, the corresponding solution $\kappa_u^*(x)$ for the problem (3) and control action $\kappa_{pwa}(x) : \mathbb{R}^n \rightarrow \mathbb{R}^m$ for the system (1) can be obtained with standard methods. With respect to the current developments, the control action $\kappa_{pwa}(x)$ defined over \mathcal{X} will be considered to be available in an explicit form.

The goal is to construct a feasible set $\mathcal{P}_z \subset \mathbb{R}^{n+m+n_z}$ starting from the continuous PWA control action $\kappa_{pwa}(x)$, such that

Prop 1: \mathcal{P}_z is a polytope, and it satisfies:

$$\text{Proj}_{\mathbb{R}^n} \mathcal{P}_z = \mathcal{X}.$$

Prop 2: There exists a continuous PWA vector function $\kappa_z(x) : \mathbb{R}^n \rightarrow \mathbb{R}^{m+n_z}$ subject to:

$$\kappa_{pwa}(x) = [I_m, \mathbf{0}_{m \times n_z}] \kappa_z(x). \quad (6)$$

Prop 3: $\forall x \in \text{int}(\mathcal{X}_1), [x^T, \kappa_z^T(x)]^T \in \text{int}(\mathcal{P}_z)$.

Prop 4: $\forall x \in \mathcal{X} \setminus \text{int}(\mathcal{X}_1), [x^T, \kappa_z^T(x)]^T \in \text{bd}(\mathcal{P}_z)$.

By imposing these four conditions, implicitly a reformulation of the MPC feasible domain and its optimal solution $(\mathcal{P}, \kappa_u^*)$ is achieved:

$$\begin{aligned} \mathcal{P} \subset \mathbb{R}^{n+mN} &\longrightarrow \mathcal{P}_z \subset \mathbb{R}^{n+m+n_z}, \\ \kappa_u^* : \mathbb{R}^n \rightarrow \mathbb{R}^{mN} &\longrightarrow \kappa_z : \mathbb{R}^n \rightarrow \mathbb{R}^{m+n_z}. \end{aligned}$$

If $m + n_z \ll mN$, the new pair $(\mathcal{P}_z, \kappa_z)$ preserves the topological structure of the pair $(\mathcal{P}, \kappa_u^*)$ while reducing its complexity. One can employ $(\mathcal{P}_z, \kappa_z)$ to formulate a new mpQP problem that is equivalent to (3) but has a lower complexity.

D. A scalar motivating example

Fig. 1 depicts a PWA function $\kappa_{pwa} : \mathbb{R} \rightarrow \mathbb{R}$ defined over a partition $\{\mathcal{X}_i\}_{\mathcal{I}_5}$ with 5 regions. The same figure depicts the solution pair $(\mathcal{P}_z, \kappa_z)$ satisfying:

$$\begin{aligned} \mathcal{P}_z \subset \mathbb{R}^3 : \text{Proj}_{\mathbb{R}} \mathcal{P} &= \cup_{i=1}^5 \mathcal{X}_i, \\ \kappa_z : \mathbb{R} \rightarrow \mathbb{R}^2 : \kappa_{pwa}(x) &= [1, 0] \kappa_z(x), \forall x \in \cup_{i=1}^5 \mathcal{X}_i. \end{aligned}$$

and fulfilling the conditions in the problem formulation.

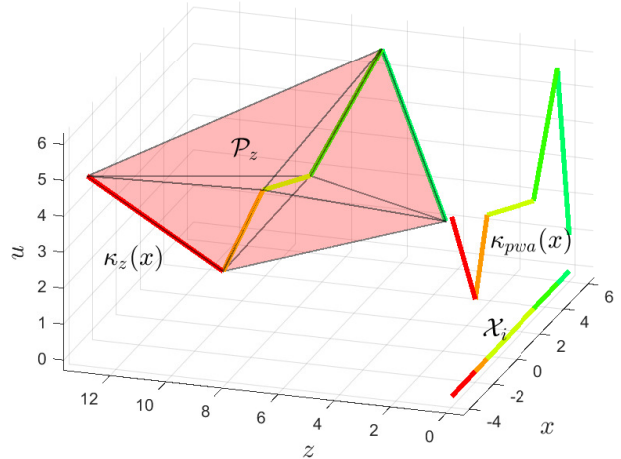


Fig. 1. Example of the solution sought for the pair $(\mathcal{P}_z, \kappa_z)$ based on an initial PWA function $\kappa_{pwa}(x)$.

III. PRELIMINARY RESULTS

Two instrumental notions are introduced in this section.

A. Convex lifting

Definition 2. [12] Given a polyhedral partition $\{\mathcal{X}_i\}_{\mathcal{I}_N}$ of a polyhedron $\mathcal{X} \subset \mathbb{R}^n$, $l(x) : \mathcal{X} \rightarrow \mathbb{R}$ is called a piecewise affine convex lifting if the following conditions hold:

- $l(x) = a_i^T x + b_i$ for $x \in \mathcal{X}_i$;
- $l(x)$ is continuous over \mathcal{X} ;
- $l(x) > a_j^T x + b_j$ for all $x \in \mathcal{X}_i \setminus \mathcal{X}_j$ with $i, j \in \mathcal{I}_N$ and $i \neq j$.

B. Interconnection graph corresponding to the topology of the PWA partition

The topological structure of a polyhedral partition $\{\mathcal{X}_i\}_{\mathcal{I}_N}$ characterizes the interconnectivity and distribution of the constituent regions \mathcal{X}_i according to the non-emptiness of their intersection. This is represented in terms of as a non-directional graph $(\mathcal{N}, \mathcal{E})_{\mathcal{I}_N}$, with \mathcal{N} denoting the regions and \mathcal{E} representing the links between these regions. If $\mathcal{X}_i \cap \mathcal{X}_j \neq \emptyset$, $\mathcal{E}(i, j) = \mathcal{E}(j, i) = 1$. Otherwise, $\mathcal{E}(i, j) = \mathcal{E}(j, i) = 0$.

Fig. 2 illustrates a topological structure graph for a 2-dimensional polyhedral partition.

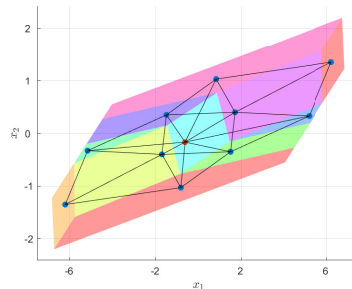


Fig. 2. A topological structure graph in the 2-dimensional partition.

IV. A BASIC CONSTRUCTIVE RESULT

To address the problem presented in subsection II.C, the topological structure graph of $\{\mathcal{X}_i\}_{\mathcal{I}_N}$ is classified and a novel convex-concave lifting technique to extend the variable to a higher-dimensional space for convexification purposes. The constructive process is completed with a third step where the PWA control law $\kappa_{pwa}(x)$ is combined with the lifting to establish a feasible set \mathcal{P}_z . The entire sequence of operations is illustrated through a uni-dimensional space example.

A. Decomposable interconnection graph within the partition

First, let us refer to the dimensional expansion lemma proposed in [5].

Lemma 1. [5] Let $\Gamma_s \subset \mathbb{R}^{d_s}$ be a full-dimensional polytope with the set of vertices $\mathcal{V}(\Gamma_s) = \{s^{(1)}, \dots, s^{(q)}\}$. For any finite set of points $\{s^{(1)}, \dots, s^{(q)}\} \subset \mathbb{R}^{d_t}$ defining a full-dimensional polytope, an extension of the family $\mathcal{V}(\Gamma_s)$ can be obtained in higher-dimensional space $\mathbb{R}^{d_s+d_t}$ for the concatenated vectors $[s^T, t^T]^T$ defining the set:

$$V_{s,t} := \left\{ \begin{bmatrix} s^{(1)} \\ t^{(1)} \end{bmatrix}, \dots, \begin{bmatrix} s^{(q)} \\ t^{(q)} \end{bmatrix} \right\}.$$

The polytope $\Gamma_{s,t} = \text{conv}(V_{s,t})$ satisfies: $V_{s,t} = \mathcal{V}(\Gamma_{s,t})$.

Remark 1. For two sets Γ_s and $\Gamma_{s,t}$, the following holds:

$$\text{Proj}_{\mathbb{R}^{d_s}} \Gamma_{s,t} = \Gamma_s.$$

According to Lemma 1, the set \mathcal{P}_u defined in Corollary 1 satisfies:

$$\text{Proj}_{\mathbb{R}^n} \mathcal{P}_u = \underline{\mathcal{X}}.$$

Typically, \mathcal{P}_u cannot satisfy the Prop 1–4 required to construct the feasible set \mathcal{P}_z . Consequently, an extra step is necessary to embed \mathcal{P}_u in a higher-dimensional polytope.

Definition 3. Let $(\mathcal{N}, \mathcal{E})_{\mathcal{I}_N}$ be a graph associated with $\{\mathcal{X}_i\}_{\mathcal{I}_N}$. A PWA function $f(x) : \bigcup_{i \in \mathcal{N}_{bd}} \mathcal{X}_i \rightarrow \mathbb{R}$ is called a boundary lifting function (BLF) if the following holds:

$$\mathcal{P}_f^i \subset \text{bd}(\mathcal{P}_f)$$

with $i \in \mathcal{N}_{bd} \subseteq \mathcal{I}_N$ and

$$\begin{aligned} \mathcal{P}_f^i &:= \text{conv} \{ [x^T, f(x)]^T : x \in \mathcal{X}_i \}, \\ \mathcal{P}_f &:= \text{conv} \left\{ [v^T, f(v)]^T : v \in \bigcup_{i \in \mathcal{N}_{bd}} \mathcal{V}(\mathcal{X}_i) \right\}. \end{aligned}$$

Remark 2. The existence of a BLF over $\bigcup_{i \in \mathcal{N}_{bd}} \mathcal{X}_i$ does not solve the stated problem but some connections between \mathcal{P}_f and \mathcal{P}_z can be established:

- $\text{Proj}_{\mathbb{R}^n} \mathcal{P}_z = \underline{\mathcal{X}}$. However, the set \mathcal{P}_f only guarantees that $\text{Proj}_{\mathbb{R}^n} \mathcal{P}_f \supseteq \bigcup_{i \in \mathcal{N}_{bd}} \mathcal{X}_i$.
- The set $\bigcup_{i \in \mathcal{N}_{bd}} \mathcal{X}_i$ is not required to be both compact and connected. Therefore, the PWA function $f(x)$ may be discontinuous.

Subsequently, we aim to identify multiple appropriate BLFs to build a polytope \mathcal{P}_z in a higher-dimensional space.

The domain of $f(x)$ allows us to partition the topology structure graph $(\mathcal{N}, \mathcal{E})_{\mathcal{I}_N}$ into two subgraphs: the boundary subgraph $(\mathcal{N}, \mathcal{E})_{\mathcal{N}_{bd}}$ and interior subgraph $(\mathcal{N}, \mathcal{E})_{\mathcal{N}_{in}}$. Here, $\mathcal{N}_{in} := \mathcal{I}_N \setminus \mathcal{N}_{bd}$. Fig. 3 illustrates an interior subgraph of $(\mathcal{N}, \mathcal{E})_{\mathcal{I}_N}$, as presented in Fig. 2.

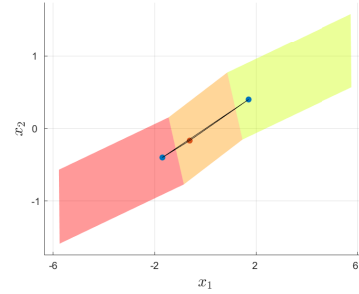


Fig. 3. An interior subgraph of the topology structure in Fig. 2.

B. Convex-concave construction

In order to create the feasible set \mathcal{P}_z , it is necessary to devise a BLF that elevates the regions in the interior subgraph $(\mathcal{N}, \mathcal{E})_{\mathcal{N}_{in}}$ to the boundary. A convex-concave lifting approach, derived from convex lifting, is suggested to facilitate this process.

Definition 4. Let $\{\mathcal{X}_i\}_{\mathcal{N}_{in}} \subseteq \{\mathcal{X}_i\}_{\mathcal{I}_N}$ be a collection of regions. A piecewise affine lifting function $g : \mathbb{R}^n \rightarrow \mathbb{R}$, defined as $g(x) = a_i^T x + b_i$ for $x \in \mathcal{X}_i$ where $i \in \mathcal{N}_{in}$, is categorized as a convex-concave lifting if it satisfies the following conditions:

- The set \mathcal{N}_{in} is partitioned into two groups:
 - 1) \mathcal{N}_{in}^{vex} , containing the convex items, and \mathcal{N}_{in}^{cave} , containing the concave items. These groups fulfill the conditions $\mathcal{N}_{in}^{vex} \cup \mathcal{N}_{in}^{cave} = \mathcal{N}_{in}$ and $\mathcal{N}_{in}^{vex} \cap \mathcal{N}_{in}^{cave} = \{1\}$.
- For all $i \in \mathcal{N}_{in}^{vex}$, the following conditions are satisfied:
 - 2) $g(x) > a_j^T x + b_j$ for all $x \in \mathcal{X}_i \setminus \mathcal{X}_j$ and all $j \neq i, j \in \mathcal{N}_{in}^{vex}$;
 - 3) $g(x) < a_j^T x + b_j, \forall x \in \mathcal{X}_i$ and $\forall j \in \mathcal{N}_{in}^{cave} \setminus \{1\}$.
- For all $i \in \mathcal{N}_{in}^{cave}$, the following conditions are satisfied:
 - 4) $g(x) < a_j^T x + b_j$ for all $x \in \mathcal{X}_i \setminus \mathcal{X}_j$ and all $j \neq i, j \in \mathcal{N}_{in}^{cave}$;
 - 5) $g(x) > a_j^T x + b_j, \forall x \in \mathcal{X}_i$ and $\forall j \in \mathcal{N}_{in}^{vex} \setminus \{1\}$.
- For all $i, j \in \mathcal{N}_{in}, i \neq j$, the following condition holds:
 - 6) $a_i^T x + b_i = a_j^T x + b_j, x \in \mathcal{X}_i \cap \mathcal{X}_j$.

Remark 3. A convex lifting is a convex-concave lifting with $\mathcal{N}_{in}^{cave} = \{1\}$.

The proposed convex-concave liftable conditions and their connection to a BLF for state space \mathcal{X} will be discussed next.

Definition 5. The collection $\{\mathcal{X}_i\}_{\mathcal{N}_{in}}$ is said to be valid for convex-concave liftable if

$$\bigcup_{i \in \mathcal{N}_{in}^{vex} \setminus \{1\}} \mathcal{X}_i \subset \mathcal{H}^+, \quad \bigcup_{i \in \mathcal{N}_{in}^{cave} \setminus \{1\}} \mathcal{X}_i \subset \mathcal{H}^-,$$

where \mathcal{H}^+ and \mathcal{H}^- are two halfspaces generated by a hyperplane $\mathcal{H} \subset \mathbb{R}^n$.

Remark 4. According to the above definition, a necessary condition for the application of the convex-concave lifting is the existence of a hyperplane to partition $\{\mathcal{X}_i\}_{\mathcal{N}_{in}}$ into two subsets. Fig. 2 and Fig. 3 illustrate this approach. For example, in the graph $(\mathcal{N}, \mathcal{E})_{\mathcal{I}_N}$, it may not be possible to divide the regions, except for the original region 1, into two groups using a hyperplane. However, a hyperplane can divide the interior subgraph $(\mathcal{N}, \mathcal{E})_{\mathcal{N}_{in}}$ into two groups once the region 1 is omitted. Thus, we refer to the set of regions $(\mathcal{N}, \mathcal{E})_{\mathcal{N}_{in}}$ generated from Fig. 3 as valid for convex-concave liftability.

In the rest of this subsection, the main objective is to propose an algorithm to effectively construct a convex-concave lifting for a valid collection $\{\mathcal{X}_i\}_{\mathcal{N}_{in}}$.

In practice, the indices of \mathcal{N}_{in} will be split into two sets such that:

$$\mathcal{N}_{in}^{vex} \cap \mathcal{N}_{in}^{cave} = \{1\}, \mathcal{N}_{in}^{vex} \cup \mathcal{N}_{in}^{cave} = \mathcal{N}_{in}. \quad (7)$$

If $\{\mathcal{X}_i\}_{\mathcal{N}_{in}}$ is valid for liftability, then \mathcal{N}_{in} can be directly partitioned. One of the two subsets is associated with the convex items, while the other part automatically will contain the concave items.

Let $(i, j) \in \mathcal{E}$ with $i, j \in (*)$, where $(*)$ denotes \mathcal{N}_{in} , \mathcal{N}_{in}^{vex} , or \mathcal{N}_{in}^{cave} . If $i, j \in \mathcal{N}_{in}^{vex}$, then conditions 2 in Definition 4 are convexity ones and can be described as:

$$\forall x \in \mathcal{X}_i \setminus (\mathcal{X}_i \cap \mathcal{X}_j), g_i(x) > g_j(x). \quad (8)$$

If $i, j \in \mathcal{N}_{in}^{cave}$, conditions 4 becomes:

$$\forall x \in \mathcal{X}_i \setminus (\mathcal{X}_i \cap \mathcal{X}_j), g_i(x) < g_j(x). \quad (9)$$

If $i, j \in \mathcal{N}_{in}$, condition 6 becomes:

$$\forall x \in \mathcal{X}_i \cap \mathcal{X}_j, g_i(x) = g_j(x). \quad (10)$$

Conditions 3 and 5 in Definition 4 are boundary conditions.

Algorithm 1 outlines the process for computing a_i and b_i for all $i \in \mathcal{N}_{in}$, in a convex-concave lifting.

C. Lifting a Polyhedral Partition

Suppose the collection $\{\mathcal{X}_i\}_{\mathcal{N}_{in}} \subseteq \{\mathcal{X}_i\}_{\mathcal{I}_N}$ is valid for convex-concave liftability. Let $g(x) : \bigcup_{\forall i \in \mathcal{N}_{in}} \mathcal{X}_i \rightarrow \mathbb{R}$ be a PWA function defined as $g(x) = a_i^T x + b_i, x_i \in \mathcal{X}_i$, which is a convex-concave lifting of $\{\mathcal{X}_i\}_{\mathcal{N}_{in}}$.

Proposition 3. Let $\{\mathcal{X}_i\}_{\mathcal{N}_{in}} \subseteq \{\mathcal{X}_i\}_{\mathcal{I}_N}$ be a convex-concave liftable collection, where \mathcal{N}_{in}^{vex} and \mathcal{N}_{in}^{cave} are the convex and concave items of \mathcal{N}_{in} , respectively. The following equation holds:

$$\mathcal{P}_g \subset \mathbb{R}^{n+1} : \text{Proj}_{\mathbb{R}^n} \mathcal{P}_g = \underline{\mathcal{X}}$$

with

$$\begin{aligned} \mathcal{P}_g = \{ & [x^T, z]^T \mid \text{s.t. } z \geq a_i^T x + b_i, z \leq a_j^T x + b_j, \\ & \forall x \in \underline{\mathcal{X}}, i \in \mathcal{N}_{in}^{vex} \setminus \{1\}, j \in \mathcal{N}_{in}^{cave} \setminus \{1\}, \end{aligned}$$

Algorithm 1 Construction a convex-concave lifting for a given collection $\{\mathcal{X}_i\}_{\mathcal{N}_{in}}$ of a polytope $\underline{\mathcal{X}} \subset \mathbb{R}^n$

Input: $\{\mathcal{X}_i\}_{\mathcal{N}_{in}}$ and a given constant $\epsilon > 0$.

Output: Parameter pairs $(a_i, b_i), \forall i \in \mathcal{N}_{in}$.

- 1: Divide the items set \mathcal{N}_{in} into two parts \mathcal{N}_{in}^{vex} and \mathcal{N}_{in}^{cave} .
- 2: Register all neighboring regions in $\{\mathcal{X}_i\}_{\mathcal{N}_{in}}, \{\mathcal{X}_i\}_{\mathcal{N}_{in}^{vex}}$, and $\{\mathcal{X}_i\}_{\mathcal{N}_{in}^{cave}}$.
- 3: For each $(i, j) \in \mathcal{E}$ and $i, j \in \mathcal{N}_{in}$

$$\forall v \in \mathcal{V}(\mathcal{X}_i \cap \mathcal{X}_j), a_i^T v + b_i = a_j^T v + b_j. \quad (11a)$$

- 4: For each $(i, j) \in \mathcal{E}$ with $i, j \in \mathcal{N}_{in}^{vex}$, add convexity conditions, $\forall v \in \mathcal{V}(\mathcal{X}_i), v \notin \mathcal{V}(\mathcal{X}_j)$:

$$a_i^T v + b_i \geq a_j^T v + b_j + \epsilon. \quad (11b)$$

- 5: For each $(i, j) \in \mathcal{E}$ $i, j \in \mathcal{N}_{in}^{cave}$, add concavity conditions, $\forall v \in \mathcal{V}(\mathcal{X}_i), v \notin \mathcal{V}(\mathcal{X}_j)$:

$$a_i^T v + b_i + \epsilon \leq a_j^T v + b_j. \quad (11c)$$

- 6: For $i \in \mathcal{N}_{in}^{vex} \setminus \{1\}, \forall j \in \mathcal{N}_{in}^{cave} \setminus \{1\}$, add boundary conditions, $\forall v \in \mathcal{V}(\mathcal{X}_i)$:

$$a_i^T v + b_i + \epsilon \leq a_j^T v + b_j. \quad (11d)$$

- 7: For $i \in \mathcal{N}_{in}^{cave} \setminus \{1\}, \forall j \in \mathcal{N}_{in}^{vex} \setminus \{1\}$, add boundary conditions, $\forall v \in \mathcal{V}(\mathcal{X}_i)$:

$$a_i^T v + b_i \geq a_j^T v + b_j + \epsilon. \quad (11e)$$

- 8: Solve a linear quadratic problem with variables $a_i, b_i, \forall i \in \mathcal{N}_{in}$:

$$\min_{a_i, b_i} \sum_{\forall i \in \mathcal{N}_{in}} (a_i^T a_i + b_i^T b_i) \text{ subject to (11)}. \quad (12)$$

if a_i, b_i are the solution to the optimization problem:

$$\min_{a_i, b_i} \sum_{\forall i \in \mathcal{N}_{in}} (a_i^T a_i + b_i^T b_i) \text{ subject to (11), (14)} \quad (13)$$

with (14) denoted as

$$\begin{aligned} a_i^T x + b_i & \leq a_j^T x + b_j, \\ \forall x \in \underline{\mathcal{X}}, i & \in \mathcal{N}_{in}^{vex} \setminus \{1\}, j \in \mathcal{N}_{in}^{cave} \setminus \{1\}. \end{aligned} \quad (14)$$

Proof. In the expression of \mathcal{P}_g , the condition $x \in \underline{\mathcal{X}}$ must be satisfied, which implies that:

$$\text{Proj}_{\mathbb{R}^n} \mathcal{P}_g \subseteq \underline{\mathcal{X}}.$$

Moreover, for any $x \in \underline{\mathcal{X}}$, there exists a real number $z_0 \in \mathbb{R}$ such that $[x^T, z_0]^T \in \mathcal{P}_g$. Therefore, we have:

$$\underline{\mathcal{X}} \subseteq \text{Proj}_{\mathbb{R}^n} \mathcal{P}_g.$$

As a result, the proposition is proven to be true. \square

Remark 5. To simplify the formulation, \mathcal{P}_g will be equivalently denoted as:

$$\mathcal{P}_g = \{ [x^T, z]^T \mid H_{z,x} x + H_{z,z} z \leq b_z \}. \quad (15)$$

Besides, to construct the convex-concave lifting $g(x)$ based on the optimization problem (13), a set of new constraints (14) must be considered. The aforementioned constraints correspond to boundary conditions and can be simplified to a finite set of constraints as follows:

- For $i \in \mathcal{N}_{in}^{veax} \setminus \{1\}$, $\forall j \in \mathcal{N}_{in}^{cave} \setminus \{1\}$, and $\forall v \in \mathcal{V}(\underline{\mathcal{X}})$:

$$a_i^T v + b_i \geq a_j^T v + b_j + \epsilon.$$

D. Composition of PWA functions

This subsection presents a methodology for building a set \mathcal{P}_z that leverages the decomposable interconnection graph approach detailed in Subsection A, in conjunction with convex-concave lifting techniques in Subsections B and C.

Given a polyhedral partition $\{\mathcal{X}_i\}_{\mathcal{I}_N}$ with topology $(\mathcal{N}, \mathcal{E})_{\mathcal{I}_N}$, we seek to construct a polytope \mathcal{P}_z that encapsulates the continuous PWA function $\kappa_{pwa} : \bigcup_{i=1}^N \mathcal{X}_i \rightarrow \mathbb{R}^m$. To achieve this objective, we begin by constructing a polytope $\mathcal{P}_u \subset \mathbb{R}^{n+m}$ defined in (16). The polytope can be represented equivalently as:

$$\mathcal{P}_u = \{[x^T, u^T]^T \in \mathbb{R}^{n+m} \mid H_{u,x}x + H_{u,u}u \leq b_u\}. \quad (16)$$

We begin by noting that $\text{Proj}_{\mathbb{R}^n} \mathcal{P}_u = \underline{\mathcal{X}}$. Additionally, we partition the topology $(\mathcal{N}, \mathcal{E})_{\mathcal{I}_N}$ into two subgraphs based on the following boundary condition:

$$(\mathcal{N}, \mathcal{E})_{\mathcal{N}_{bd}} : \forall i \in \mathcal{N}_{bd}, \mathcal{P}_u^i \subset bd(\mathcal{P}_u); \quad (17a)$$

$$(\mathcal{N}, \mathcal{E})_{\mathcal{N}_{in}} : \forall i \in \mathcal{N}_{in}, \mathcal{P}_u^i \cap \text{int}(\mathcal{P}_u) \neq \emptyset. \quad (17b)$$

with $\mathcal{P}_u^i = \text{conv}\{[v^T, \kappa_{pwa}^T(v)]^T : v \in \mathcal{V}(\mathcal{X}_i)\}$. Besides, sets \mathcal{N}_{bd} and \mathcal{N}_{in} satisfy:

$$\mathcal{N}_{bd} \cup \mathcal{N}_{in} = \mathcal{I}_N, \mathcal{N}_{bd} \cap \mathcal{N}_{in} = \emptyset, 1 \in \mathcal{N}_{in}.$$

Assumption 1. For the subgraph $(\mathcal{N}, \mathcal{E})_{\mathcal{N}_{in}}$, the relevant collection $\{\mathcal{X}_i\}_{\mathcal{N}_{in}}$ is valid for convex-concave liftability.

Based on Proposition 3 and Assumption 1, we can design a convex-concave lifting

$$g : \bigcup_{\forall i \in \mathcal{N}_{in}} \mathcal{X}_i \rightarrow \mathbb{R},$$

such that

$$\text{Proj}_{\mathbb{R}^n} \mathcal{P}_g = \underline{\mathcal{X}} \text{ and } \forall i \in \mathcal{N}_{in} \setminus \{1\}, \mathcal{P}_g^i \subset bd(\mathcal{P}_g)$$

with \mathcal{P}_g defined in (15) and \mathcal{P}_g^i denoted as

$$\mathcal{P}_g^i = \text{conv}\{[x^T, g^T(x)]^T : x \in \mathcal{X}_i\}. \quad (18)$$

Using PWA function $\kappa_{pwa}(x)$ and a convex-concave lifting $g(x)$, the following proposition constructs set \mathcal{P}_z .

Proposition 4. Given a continuous PWA function $\kappa_{pwa}(x)$ defined over a polyhedral partition $\{\mathcal{X}_i\}_{\mathcal{I}_N}$, we construct the set \mathcal{P}_u and a specialized convex-concave lifting $g(x)$ over a collection $\{\mathcal{X}_i\}_{\mathcal{N}_{in}}$. The polytope $\mathcal{P}_z \subset \mathbb{R}^{n+m+1}$ satisfies Prop 1–4 in II-C and can be expressed as:

$$\mathcal{P}_z = \{[x^T, u^T, z]^T \in \mathbb{R}^{n+m+1} \mid \text{subject to} \\ H_{u,x}x + H_{u,u}u \leq b_u, H_{z,x}x + H_{z,z}z \leq b_z\}.$$

Proof. The inclusion of $x \in \underline{\mathcal{X}}$ in \mathcal{P}_z implies that:

$$\text{Proj}_{\mathbb{R}^n} \mathcal{P}_z \subseteq \underline{\mathcal{X}}.$$

Additionally, for any $x \in \underline{\mathcal{X}}$, there exist two vectors $u_0 \in \mathbb{R}^m$ and $z_0 \in \mathbb{R}$ such that:

$$[x^T, u_0^T]^T \in \mathcal{P}_u, [x^T, z_0]^T \in \mathcal{P}_g$$

Therefore, we have:

$$[x^T, u_0^T, z_0]^T \in \mathcal{P}_z \Rightarrow \underline{\mathcal{X}} \subseteq \text{Proj}_{\mathbb{R}^n} \mathcal{P}_z.$$

Prop 1 imposed on \mathcal{P}_z according to II-C is proven to hold.

In order to satisfy Prop 2 in II-C, we define a piecewise affine function $\kappa_z : \bigcup_{i \in \mathcal{I}_N} \mathcal{X}_i \rightarrow \mathbb{R}^{m+1}$, given by:

$$\kappa_z(x) = \begin{bmatrix} \kappa_{pwa}(x) \\ z(x) \end{bmatrix}$$

with

$$z(x) = \begin{cases} g(x), & x \in \mathcal{X}_i, i \in \mathcal{N}_{in}, \\ g_c(x), & x \in \mathcal{X}_i, i \in \mathcal{N}_{bd}, \end{cases}$$

where function $g_c(x) : \mathbb{R}^n \rightarrow \mathbb{R}$ only needs to ensure the continuity of $z(x)$ over the entire state space $\underline{\mathcal{X}}$. Apparently, $\kappa_{pwa}(x) = [I_m, \mathbf{0}_{m \times 1}] \kappa_z(x)$.

Prop 3–4 in II-C are proven by analyzing various situations based on the value of x .

If $x \in \text{int}(\mathcal{X}_1)$, since $1 \in \mathcal{N}_{in}$, we have $[x^T, \kappa_{pwa}^T(x)]^T \in \text{int}(\mathcal{P}_u)$. This implies the existence of $\epsilon_1 > 0$ such that $[x^T, \kappa_{pwa}^T(x)]^T + \epsilon_1 \mathbb{B}^{n+m} \subset \mathcal{P}_u$. Based on the design process of convex-concave lifting, if $x \in \text{int}(\mathcal{X}_1)$, then $[x^T, g(x)]^T \in \text{int}(\mathcal{P}_g)$ holds true. Similarly, there is $\epsilon_2 > 0$ such that $[x^T, g(x)]^T + \epsilon_2 \mathbb{B}^{n+1} \subset \mathcal{P}_g$. Thus, we can choose $\epsilon_3 = \min\{\epsilon_1, \epsilon_2\}$ so that $[x^T, \kappa_{pwa}^T(x), g(x)]^T + \epsilon_3 \mathbb{B}^{n+m+1} \subset \mathcal{P}_z$.

If $x \in \mathcal{X}_i$ with $i \in \mathcal{N}_{in} \setminus \{1\}$, due to $i \in \mathcal{N}_{in}$, $[x^T, \kappa_{pwa}^T(x)]^T \in \text{int}(\mathcal{P}_u)$. It means $\exists \epsilon_1 > 0$, $[x^T, \kappa_{pwa}^T(x)]^T + \epsilon_1 \mathbb{B}^{n+m} \subset \mathcal{P}_u$. According to the design process of convex-concave lifting, $[x^T, g(x)]^T \in bd(\mathcal{P}_g)$, which means $\nexists \epsilon_2 > 0$, $[x^T, g(x)]^T + \epsilon_2 \mathbb{B}^{n+1} \subset \mathcal{P}_g$. It means $\nexists \epsilon > 0$, $[x^T, \kappa_{pwa}^T(x), g(x)]^T + \epsilon \mathbb{B}^{n+m+1} \subset \mathcal{P}_z$. Thus, $[x^T, \kappa_{pwa}^T(x), g(x)]^T \in bd(\mathcal{P}_z)$.

If $x \in \mathcal{X}_i$ with $i \in \mathcal{N}_{bd}$, we can not find $\epsilon > 0$, $[x^T, \kappa_{pwa}^T(x)]^T + \epsilon \mathbb{B}^{n+m} \subset \mathcal{P}_u$, which also means $\nexists \epsilon > 0$, $[x^T, \kappa_{pwa}^T(x), g_c(x)]^T + \epsilon \mathbb{B}^{n+m+1} \subset \mathcal{P}_z$ and $[x^T, \kappa_{pwa}^T(x), g_c(x)]^T \in bd(\mathcal{P}_z)$. \square

Algorithm 2 outlines the procedure for decomposing the construction task and building the constraints for the set \mathcal{P}_z .

Remark 6. The design of a convex-concave lifting for all regions except the original ones to the boundary in one step is impossible if Assumption 1 doesn't hold. In the following section, we will present the iterative process, which allows generalization by recursively leading to the validation of the Assumption 1.

Algorithm 2 Construction a feasible set \mathcal{P}_z for a given continuous PWA function κ_{pwa} defining over $\{\mathcal{X}_i\}_{\mathcal{I}_N}$.

Input: A continuous PWA function $\kappa_{pwa} : \bigcup_{i=1}^N \mathcal{X}_i \rightarrow \mathbb{R}^m$ and its domain $\mathcal{X} \subset \mathbb{R}^n$.

Output: A higher-dimensional polytope $\mathcal{P}_z \subset \mathbb{R}^{n+m+1}$ and a relevant continuous PWA function $\kappa_z(x) : \mathcal{X} \rightarrow \mathbb{R}^{m+1}$.

- 1: Establish the topology structure graph of $\{\mathcal{X}_i\}_{\mathcal{I}_N}$ denoted as $(\mathcal{N}, \mathcal{E})_{\mathcal{I}_N}$.
- 2: Construct the polytope $\mathcal{P}_u \subset \mathbb{R}^{n+m}$, denoted by (16), for the polyhedral partition $\{\mathcal{X}_i\}_{\mathcal{I}_N}$.
- 3: The graph $(\mathcal{N}, \mathcal{E})_{\mathcal{I}_N}$ can be partitioned into two parts based on \mathcal{P}_u , resulting in $(\mathcal{N}, \mathcal{E})_{\mathcal{N}_{bd}}$ and $(\mathcal{N}, \mathcal{E})_{\mathcal{N}_{in}}$, as shown in (17).
- 4: Construct a convex-concave lifting function $g(x)$ for subgraph $(\mathcal{N}, \mathcal{E})_{\mathcal{N}_{in}}$ such that the resulting set projection $\text{Proj}_{\mathbb{R}^n} \mathcal{P}_g = \mathcal{X}$, as defined in (15), is guaranteed.
- 5: Construct set \mathcal{P}_z and define the relevant continuous PWA function $\kappa_z : \mathcal{X} \rightarrow \mathbb{R}^{m+1}$ using Proposition 4 as a basis.

E. Exemplification for a scalar function

Consider a scalar PWA function:

$$\kappa_{pwa}(x) = \begin{cases} -1.5x - 4.75, & x \in \mathcal{X}_2 = \{x : -4 < x \leq -2\} \\ 2x + 2.25, & x \in \mathcal{X}_3 = \{x : -2 < x \leq -1\} \\ -0.25x, & x \in \mathcal{X}_1 = \{x : -1 < x \leq 3\} \\ 1.5x - 5.25, & x \in \mathcal{X}_4 = \{x : 3 < x \leq 5\} \\ -5x + 27.25, & x \in \mathcal{X}_5 = \{x : 5 < x \leq 6\} \end{cases}$$

By following the steps outlined in Algorithm 2, we obtained the required parameters in the following:

$$[H_{z,x}, H_{z,z}, b_z] = \begin{bmatrix} -0.0619 & -0.4952 & -0.8666 \\ 0.1592 & 0.0318 & 0.9867 \\ -0.0201 & 0.1806 & 0.9833 \\ -0.7276 & -0.4851 & 0.4851 \end{bmatrix},$$

$$[H_{u,x}, H_{u,u}, b_u] = \begin{bmatrix} 1.0000 & 0 & 6.0000 \\ -1.0000 & 0 & 4.0000 \\ -1.0000 & -1.0000 & 1.0000 \\ 0.7143 & 1.0000 & 2.1429 \end{bmatrix}$$

The function $z(x) : \mathbb{R} \rightarrow \mathbb{R}$ can be compactly denoted as $z(x) = a_i^T x + b_i$, where a_i and b_i are the i -th elements of vectors $[0, -1.6250, -1.000, -0.7143, -2.3214]$ and $[0, -2.2500, -1.0000, 2.1429, 10.1786]$, respectively.

For clarity, separate identical translation operations are applied to $\{\mathcal{X}_i\}_{\mathcal{I}_5}$, $\kappa_{pwa}(x)$ and \mathcal{P}_u , \mathcal{P}_g and $g(x)$, \mathcal{P}_z and $\kappa_z(x)$ in Fig. 4. $\kappa_z(x)$ is a curve within \mathcal{P}_z , and only when $x \in \text{int}(\mathcal{X}_1)$, the curve remains inside \mathcal{P}_z . Furthermore, the equations $\text{Proj}_{\mathbb{R}} \mathcal{P}_z = \text{Proj}_{\mathbb{R}} \mathcal{P}_u$ and $[1, 0]\kappa_z(x) = \kappa_{pwa}(x)$ hold, as depicted in Fig. 4.

V. GENERALIZATION

In this section, the set \mathcal{P}_z and a corresponding continuous PWA function $\kappa_z(x)$ are constructed in a general setting by relaxing assumption 1. An iterative graph decomposition approach is used to layer the convex-concave lifting technique to achieve this goal.

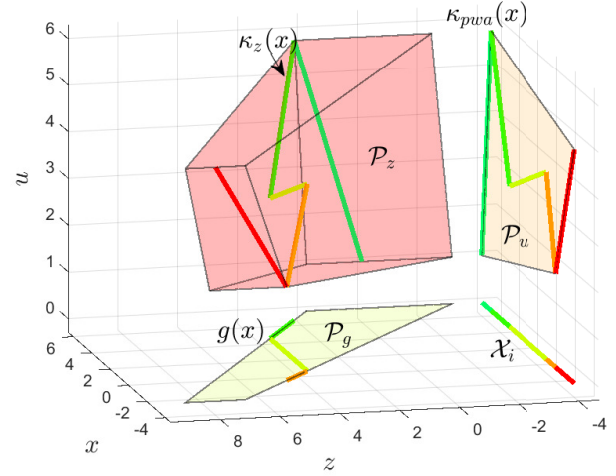


Fig. 4. Example of the solution sought for the pair $(\mathcal{P}_z, \kappa_z)$ for a scalar case.

A. Iterative decomposition of the interconnection graph

If Assumption 1 holds for subgraph $(\mathcal{N}, \mathcal{E})_{\mathcal{N}_{in}}$, i.e., the collection $\{\mathcal{X}_i\}_{\mathcal{N}_{in}}$ is valid for convex-concave liftability, a convex-concave lifting procedure can be constructed for $\{\mathcal{X}_i\}_{\mathcal{N}_{in} \setminus \{1\}}$ to the boundary of the lifting set while leaving region 1 still as the unconstrained region.

If Assumption 1 doesn't hold, the basic approach involves dividing subgraph $\{\mathcal{X}_i\}_{\mathcal{N}_{in}}$ into two new subgraphs, named $\{\mathcal{X}_i\}_{\mathcal{N}_{bd}^1}$ (i.e., liftable collection) and $\{\mathcal{X}_i\}_{\mathcal{N}_{in}^1}$ (i.e., unliftable collection), and then repeating the iteration on $\{\mathcal{X}_i\}_{\mathcal{N}_{in}^c}$ until $\{\mathcal{X}_i\}_{\mathcal{N}_{in}^c} = \{\mathcal{X}_i\}_{\{1\}}$. Here, $c \in \mathbb{N}$, $\{\mathcal{X}_i\}_{\mathcal{N}_{bd}^c}$ and $\{\mathcal{X}_i\}_{\mathcal{N}_{in}^c}$ are divided from $\{\mathcal{X}_c\}_{\mathcal{N}_{in}^{c-1}}$, and specifically, $\mathcal{N}_{in}^0 := \mathcal{N}_{in}$. The Algorithm 3 summarizes the constructing of the set \mathcal{P}_z up to the validation of the Assumption 1.

The main procedures involve finding a hyperplane \mathcal{H}^c in the 4-th step and achieving the objective by combining $\kappa_{pwa}(x)$ with several convex-concave lifting functions $g^j(x)$, where $j \in \{0, \dots, c\}$. While finding an optimal hyperplane is irrelevant to this study, it should be mentioned that the decomposition can also be achieved by excluding node 1 and decomposing the graph up to the partition in disjoint subgraphs. The following subsection presents a generic approach to deal with step 10.

B. General solution under layered convex-concave liftings

For the j -th convex-concave lifting of the liftable collection $\{\mathcal{X}_i\}_{\mathcal{N}_{in}^j}$ with $j \in \{0, \dots, c\}$, we denote the lifting as $g^j(x)$ and the corresponding \mathcal{P}_{g^j} similar to (15) and (18):

$$g^j : \bigcup_{\forall i \in \mathcal{N}_{in}^j} \mathcal{X}_i \rightarrow \mathbb{R},$$

$$\mathcal{P}_{g^j} = \{[x^T, z_j^T]^T \mid H_{z_j,x}x + H_{z_j,z_j}z_j \leq b_{z_j}\}.$$

Proposition 5. For a continuous PWA function $\kappa_{pwa}(x)$ defined over a polyhedral partition $\{\mathcal{X}_i\}_{\mathcal{I}_N}$, we have constructed set \mathcal{P}_u and a sequential convex-concave lifting $g^j(x)$ defined over a collection $\{\mathcal{X}_i\}_{\mathcal{N}_{in}^j}$ with $j \in \{0, \dots, c\}$.

Algorithm 3 Construction a feasible set \mathcal{P}_z for general case.

Input: A continuous PWA function $\kappa_{pwa} : \bigcup_{i=1}^N \mathcal{X}_i \rightarrow \mathbb{R}^m$ and its domain $\underline{\mathcal{X}} \subset \mathbb{R}^n$.

Output: A higher-dimensional polytope $\mathcal{P}_z \subset \mathbb{R}^{n+m+c}$ and a relevant continuous PWA function $\kappa_z(x) : \underline{\mathcal{X}} \rightarrow \mathbb{R}^{m+c}$ with $c \in \mathbb{N}$.

- 1: Initialize $c = 0$.
 - 2: Same steps 1-3 as in Algorithm 2.
 - 3: **while** $\{\mathcal{X}_i\}_{\mathcal{N}_{in}^c}$ is not convex-concave liftable **do**
 - 4: Split the graph $(\mathcal{N}, \mathcal{E})_{\mathcal{N}_{in}^c \setminus \{1\}}$ into two subgraphs by finding a hyperplane \mathcal{H}^c :
$$(\mathcal{N}, \mathcal{E})_{\mathcal{N}_{bd}^{c+1}} : \forall i \in \mathcal{N}_{bd}^{c+1}, \mathcal{H}^c \cap \mathcal{X}_i \neq \emptyset; \quad (19a)$$

$$(\mathcal{N}, \mathcal{E})_{\mathcal{N}_{in}^{c+1}} : \forall i \in \mathcal{N}_{in}^{c+1}, \mathcal{H}^c \cap \mathcal{X}_i = \emptyset. \quad (19b)$$
 - 5: Update parameters: $c = c + 1$.
 - 6: Update the item set: $\mathcal{N}_{in}^c = \mathcal{N}_{in}^c \cup \{1\}$.
 - 7: Construct a convex-concave lifting function $g^c(x)$ for subgraph $(\mathcal{N}, \mathcal{E})_{\mathcal{N}_{in}^c}$, ensuring the corresponding projection $\text{Proj}_{\mathbb{R}^n} \mathcal{P}_{g^c} = \underline{\mathcal{X}}$ as defined in equation (15).
 - 8: **end while**
 - 9: Repeat Step 7.
 - 10: By combining the functions $\kappa_{pwa}(x)$ and $g^j(x)$ with $j \in \{0, \dots, c\}$, we generate set \mathcal{P}_z and define the appropriate function $\kappa_z : \underline{\mathcal{X}} \rightarrow \mathbb{R}^{m+c}$.
-

Polytope $\mathcal{P}_z \subset \mathbb{R}^{n+m+c}$ satisfies Prop 1–4 in II-C and can be expressed as:

$$\mathcal{P}_z = \{ [x^T, u^T, z^T]^T \in \mathbb{R}^{n+m+c} \mid \text{subject to: } j \in \{0, \dots, c\} \\ H_{u,x}x + H_{u,u}u \leq b_u, H_{z_j,x}x + H_{z_j,z_j}z_j \leq b_{z_j} \}$$

with z_j the j -th element of vector $z \in \mathbb{R}^c$.

Proof. The proof process is similar to that in Proposition 4. \square

C. A link with low complexity mpQP alternatives to MPC

This subsection explains constructing an mpQP problem using \mathcal{P}_z and the function $\kappa_z(x)$ proposed in the previous subsection. We also analyze the relationship between the new mpQP problem and the original problem in (3).

Consider an mpQP problem:

$$\kappa_z(x) = \arg \min_{u,z} f(x, u, z) \quad (20a)$$

$$\text{subject to: } [x^T, u^T, z^T]^T \in \mathcal{P}_z, \quad (20b)$$

where $\kappa_z(x)$ and \mathcal{P}_z are provided by proposition 5.

Upon comparing problems (3) and (20), the follows hold:

- They share the same domain of parameter x :

$$\text{Proj}_{\mathbb{R}^n} \mathcal{P} = \text{Proj}_{\mathbb{R}^n} \mathcal{P}_z = \underline{\mathcal{X}}.$$

- The first m elements of the optimal solution coincide:

$$\kappa_{pwa}(x) = [I_m, \mathbf{0}_{m \times (m-1)N}] \kappa_u^*(x) = [I_m, \mathbf{0}_{m \times c}] \kappa_z(x).$$

- Their optimal solutions share the same geometric structures: the solution for the unconstrained region \mathcal{X}_1

remains within the feasible domain. In contrast, the solution for other regions lies on the boundary.

- The variables in problem (3) have a dimension of mN , while those in problem (20) have a dimension of $m+c$. Typically, we have

$$mN \gg m+c,$$

indicating that problem (20) is less complex and can be solved faster online than problem (3).

Remark 7. Designing a suitable cost function $f(x, u, z)$ to ensure the feasibility of problem (20) is a challenging task and will be the subject of further studies.

VI. CONCLUSIONS

Starting from the optimal control law obtained by means of an mpQP problem, an analysis of the polyhedral partition's topology has been made to decompose the regions into the boundary (saturated one) and interior parts. Different strategies exploiting the convex-concave liftability of the interior regions' help replace the interior regions on the boundary in a higher dimension. The polytope obtained by successive liftings is one of the most important steps for building inverse-optimal solutions, as it provides the feasible domain in terms of linear inequalities. Future work will build upon this methodology and detail the selection of the optimal index.

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