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On the inverse optimality of a class of PWA functions through liftings

Songlin Yang¹, Sorin Olaru¹, and Pedro Rodriguez-Ayerbe¹

Abstract—This paper focuses on the (re-)construction of the optimal solution for the multi-parameter quadratic programming (mpQP) problems. Optimization problems of this nature are widely employed in the formulation of model-based predictive controllers (MPC) for discrete linear systems, wherein input and state constraints are imposed. This study examines the geometric characteristics of the explicit solution of an mpQP problem and introduces a novel convex-concave lifting technique to synthesize an equivalent mpQP problem. Whenever the solution corresponds to a PWA function, the present approach maintains the structure and control laws for the original systems. A new (less complex) cost function and a corresponding feasible domain are constructed through lifting for the equivalent optimization problem. The effectiveness of this strategy is demonstrated through an illustrative example.

I. INTRODUCTION

Parametric convex programming (PCP) is a mathematical framework that has gained considerable attention in the fields of computational geometry, operational research, and control theory [1] through the parametrization of the optimal choice with respect to context parameters or updated measurements. In control theory, researchers employ PCP for controller design, such as model-based predictive control (MPC), and to analyze system dynamics across different scenarios [2]. MPC can handle process constraints and multi-variable interactions in a unified formulation. Various researchers have recently studied MPC and their alternatives [3] from different optimization-related angles [4]. The present paper focuses on the geometric characteristics of MPC [5] and aims to propose inverse optimal solutions for a class of predictive controllers.

Explicit MPC (EMPC) emerged as a method to simplify online control computation by converting it into evaluating a piecewise affine (PWA) function over polyhedral critical regions [6]. The geometrical properties of the optimal solution were exploited during the construction, but the EMPC implementation was restricted by the memory required for online evaluation. In [7], the authors reduce storage and evaluation time by assuming the initial state is contained in a given set, omitting irrelevant predictive trajectory regions. In [8], the authors proposed a novel approach for solving an mpQP problem. Their method employed an implicit representation of the Karush-Kuhn-Tucker conditions utilizing ramp functions. The effectiveness of the proposed strategy was verified through a straightforward computer code implementation with modest memory demands. In [9], a parametrized polyhedra approach in the combined

(input+parameter) space is used to analyse the solution of EMPC. In [5], the authors construct the unconstrained critical region and then enumerate the others based on the combinations of active constraints.

Once the pros/cons of implicit and explicit MPC have been documented, the research moved its focus [10] to prove that explicit PWA control functions could be seen as input data for inverse optimal QP formulations with lower computation footprint. Remarkably, inverse-optimal formulations have been constructed with only one supplementary dimension of the vector of arguments [1], thus offering a particularly compact QP for the PWA controller originated by the MPC formulation.

As a contribution to the existing literature (e.g., [1] and [11]), the present study proposes a new approach for inverse optimal solutions for a particular class of PWA feedback functions. The main contribution of this context is listed in the following:

- We propose constructing an mpQP problem for each PWA function of a scalar parameter. Furthermore, whenever the PWA function is a solution of an mpQP, the novel mpQP problem is shown to maintain the same geometric structure as the optimal solution of the original mpQP problem.
- Drawing inspiration from the concept of convex lifting [12], we introduce a novel approach called convex-concave lifting to guarantee the geometric structure of the PWA feedback control.

The paper is structured as follows. Section II introduces the preliminaries and presents the problem formulation for reconstructing an mpQP problem. Section III defines and analyzes convex-concave lifting, followed by a two-step approach for reconstructing an mpQP problem: constructing a feasible domain and proposing an appropriate cost function. Section IV provides a numerical example, and Section V concludes the paper.

Notation: In this paper, \mathbb{R}^n and \mathbb{N} denote the set of real numbers in n -dimensional space, the set of nonnegative integers, and $\mathcal{I}_N = [1, N] \cap \mathbb{N}$. The symbols $\mathbf{0}_{m \times n}$ and I_m represent a matrix of size $m \times n$ with all elements equal to zero and an m -dimensional identity matrix, respectively. If $\mathcal{P} \subset \mathbb{R}^n$ is a polyhedral set, then $\text{int}(\mathcal{P})$ denotes the set of interior points of \mathcal{P} and $\text{bd}(\mathcal{P})$ denotes the set of boundary points of \mathcal{P} , $\text{Proj}_{\mathbb{R}^m} \mathcal{P}$ represents the orthogonal projection of \mathcal{P} onto the subspace \mathbb{R}^m . The convex hull of a set $\{*\}$ is denoted by $\text{conv}\{*\}$. $\mathbb{B}^n = \{x \in \mathbb{R}^n : \|x\|_\infty \leq 1\}$.

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II. PRELIMINARIES

A. A PWA control based on MPC

Consider a discrete-time linear system:

$$x_{k+1} = Ax_k + Bu_k, \quad (1)$$

where the states $x_k \in \mathbb{R}^n$, and the inputs $u_k \in \mathbb{R}^m$ at time k are bounded by polyhedral sets:

$$\mathcal{X} = \{x \in \mathbb{R}^n | H_x x \leq b_x\}, \mathcal{U} = \{u \in \mathbb{R}^m | H_u u \leq b_u\},$$

where b_x, b_u, H_x , and H_u are known constant matrices, such that $0 \in \text{int}(\mathcal{X})$ and $0 \in \text{int}(\mathcal{U})$.

An N-steps receding horizon finite-time optimal control problem for the system (1) with initial state $x_0 = x$:

$$\kappa_u^*(x) = \arg \min_{\kappa_u} \sum_{k=0}^{N-1} u_k^T R u_k + \sum_{k=1}^N x_k^T Q x_k \quad (2a)$$

$$\text{s. t. } x_{k+1} = Ax_k + Bu_k, \quad (2b)$$

$$u_k \in \mathcal{U}, x_k \in \mathcal{X}, x_N \in \Omega \quad (2c)$$

where $\kappa_u^*(x) = [u_0^T, \dots, u_{N-1}^T]^T$ denotes the control sequence, $R \succ 0, Q \succeq 0$, and $\Omega \subseteq \mathbb{R}^n$ is a control invariant set with $0 \in \text{int}(\Omega)$. Without loss of generality, the problems to be solved at each sampling instant can be written:

$$\kappa_u^*(x) = \arg \min_{\kappa_u} \kappa_u^T H \kappa_u + x^T F \kappa_u, \quad (3a)$$

$$\text{s. t. } [x^T, \kappa_u^T] \in \mathcal{P}, \quad (3b)$$

where $x \in \mathcal{X} = \text{Proj}_{\mathbb{R}^n} \mathcal{P} \subseteq \mathcal{X}$ is the current state playing the role of a parameter, $\kappa_u(x) \in \mathbb{R}^{mN}$ is the optimization vector and $H \succ 0$. The set \mathcal{P} is the parameterized feasible domain with the formulation

$$\mathcal{P} = \{[x^T, \kappa_u^T]^T | G \kappa_u \leq W + Ex, x \in \mathcal{X}\},$$

where G, W, E are constructed from the mpQP problem (2). From the above notations it follows $\forall x \in \mathcal{X}, \mathcal{P} \neq \emptyset$.

In [6], the solution of problem (2) was expressed as a continuous PWA function of x , denoted as $\kappa_u^*(x)$:

$$\kappa_u^*(x) = f_i(x), x \in \mathcal{X}_i,$$

where $f_i : \mathbb{R}^n \rightarrow \mathbb{R}^{mN}$ is piecewise affine and \mathcal{X}_i is a polyhedron of polyhedral partition.

Definition 1. A collection of polyhedral sets $\{\mathcal{X}_1, \dots, \mathcal{X}_N\}$ is called a polyhedral partition of \mathcal{X} if $\forall i, j \in \mathcal{I}_N, i \neq j$,

1. $\mathcal{X} = \bigcup_{i=1}^N \mathcal{X}_i$;
2. $\text{int}(\mathcal{X}_i) \cap \text{int}(\mathcal{X}_j) = \emptyset$.

Such a polyhedral partition is next denoted as $\{\mathcal{X}_i\}_{\mathcal{I}_N}$.

B. Geometric structure

Similar to the literature on the geometric structure of MPC [5]- [9], the relationship between the unconstrained optimum and the feasible domain is privileged in the present work.

Proposition 1. If $0 \in \text{int}(\mathcal{X})$, $0 \in \text{int}(\mathcal{U})$ and $0 \in \text{int}(\Omega)$ in (2) then with respect to the solution of (3):

$$\mathcal{X}_1 = \{x \in \mathcal{X} | \kappa_u^*(x) = -H^{-1}Fx\}$$

is a non-empty polyhedron set, denoted as the unconstrained critical region. And additionally,

$$\begin{aligned} \forall x \in \text{int}(\mathcal{X}_1), [x^T, \kappa_u^*(x)^T] &\in \text{int}(\mathcal{P}), \\ \forall x \in \mathcal{X} \setminus \mathcal{X}_1, [x^T, \kappa_u^*(x)^T] &\in \text{bd}(\mathcal{P}). \end{aligned}$$

Proof. At the origin, the optimal argument is $\kappa_u^*(0) = 0$, corresponding to a feasible interior point of the constraint set $0 \in \mathcal{P}$. Moreover, the unconstrained optimum is given by $\kappa_u^*(x) = -H^{-1}Fx$ and there exists a non-empty region $\mathcal{R} \subseteq \mathcal{X}$ such that $(A - BH^{-1}F)x \in \mathcal{X}, \forall x \in \mathcal{R}$. Once the non-emptiness of \mathcal{X}_1 is ensured, its boundaries are given by the KKT conditions represented by affine inequality constraints [6], which proves its polyhedral structure.

Additionally, suppose by contradiction that it exists $x \in \text{int}(\mathcal{X}_1)$ such that $[x^T, (-H^{-1}Fx)^T] \notin \text{int}(\mathcal{P})$. It means that $[x^T, (-H^{-1}Fx)^T] \in \text{bd}(\mathcal{P})$. and thus either $\kappa_u^*(x) \neq -H^{-1}Fx$ or $0 \in \text{bd}(\mathcal{P})$. The first case contradicts the definition of set \mathcal{X}_1 and the second with the assumption that both \mathcal{X}, Ω and \mathcal{U} contain the origin in their strict interior.

Suppose again by contradiction that it exists $x \in \mathcal{X} \setminus \mathcal{X}_1$ with $[x^T, \kappa_u^*(x)^T] \in \text{int}(\mathcal{P})$. From this second fact, it follows that the unconstrained optimum is feasible, i.e. $\kappa_u^*(x)^T = -H^{-1}Fx$, which leads to a contradiction as long as the optimality conditions should hold only for $x \in \mathcal{X}_1$. \square

Remark 1. The optimal solution $\kappa_u^*(x)$ consists of two parts: the unconstrained optimum for $x \in \mathcal{X}_1$, and the boundary solution for $x \in \mathcal{X}_i$ where $i \in \mathcal{I}_N \setminus \{1\}$.

As a solution of an N-steps receding horizon optimal problem, only the first component of the sequence $\kappa_u^*(x)$ is used by MPC control action $\kappa_{pwa}(x)$, denoted as

$$\kappa_{pwa}(x) = [I_m, \mathbf{0}_{m \times (mN-m)}] \kappa_u^*(x),$$

or explicitly as

$$\kappa_{pwa}(x) = F_i x + g_i, x \in \mathcal{X}_i,$$

where $F_i \in \mathbb{R}^{m \times n}$, $g_i \in \mathbb{R}^m$. Because \mathcal{X}_i is the unconstrained region, the following equations hold:

$$F_1 = -[I_m, \mathbf{0}_{m \times (mN-m)}] H^{-1}F, g_1 = 0. \quad (4)$$

Corollary 1. If $0 \in \text{int}(\mathcal{X})$ and $0 \in \text{int}(\mathcal{U})$ then

$$\forall x \in \text{int}(\mathcal{X}_1), [x^T, \kappa_{pwa}(x)^T]^T \in \text{int}(\mathcal{P}_u),$$

where

$$\mathcal{P}_u := \text{conv} \{[x^T, \kappa_{pwa}(x)^T]^T : x \in \mathcal{X}\}. \quad \square \quad (5)$$

C. General problem formulation

To simplify the mpQP problem (3), one is interested in formulating an equivalent optimization problem using the geometric structure of problem (3) as pointed out in Proposition 1.

The objective here is to construct an equivalent mpQP problem with the new formulation:

$$\kappa_y^*(x) = \arg \min_y J(x, y), \quad (6a)$$

$$\text{s. t. } [x^T, y^T] \in \mathcal{P}_y, \quad (6b)$$

where the relevant parameters satisfy:

Prop 1: \mathcal{P}_y is a polytope, and it satisfies:

$$\text{Proj}_{\mathbb{R}^n} \mathcal{P}_y = \underline{\mathcal{X}}.$$

Prop 2: $\kappa_y^*(x) : \mathbb{R}^n \rightarrow \mathbb{R}^{m+n_y}$ subject to:

$$\kappa_{pwa}(x) = [I_m, \mathbf{0}_{m \times n_y}] \kappa_y^*(x). \quad (7)$$

Prop 3: $\forall x \in \text{int}(\mathcal{X}_1), [x^T, (\kappa_y^*(x))^T]^T \in \text{int}(\mathcal{P}_y)$.

Prop 4: $\forall x \in \mathcal{X} \setminus \text{int}(\mathcal{X}_1), [x^T, (\kappa_y^*(x))^T]^T \in \text{bd}(\mathcal{P}_y)$.

D. A particular class of feedback laws

In the present paper, we will concentrate on the specific class of feedback laws:

$$\kappa_{pwa} : \mathbb{R}^n \rightarrow \mathbb{R}^m \quad (8)$$

$$\kappa_{pwa}(x) = \begin{bmatrix} \kappa_{pwa}^1(x_{i_1}) \\ \vdots \\ \kappa_{pwa}^m(x_{i_m}) \end{bmatrix} \quad (9)$$

where $i_1, \dots, i_m \in \mathbb{N}_{[1,m]}$ are indices pointing to the dependence of each function $\kappa_{pwa}^i(x)$ on one and only one of the components of the state vector x . As such the general problem can be reduced to the construction of a m scalar PWA functions $\kappa_{pwa}^i : \mathbb{R} \rightarrow \mathbb{R}$. For the sake of simplicity, in the remainder of the paper, we will use the notation κ_{pwa} for functions from \mathbb{R} to \mathbb{R} .

Fig. 1 depicts such a PWA function $\kappa_{pwa} : \mathbb{R} \rightarrow \mathbb{R}$ defined over a partition $\{\mathcal{X}_i\}_{\mathcal{I}_5}$ with 5 regions. The same figure depicts a pair $(\mathcal{P}_y, \kappa_y)$, with the following properties

$$\mathcal{P}_y \subset \mathbb{R}^3 : \text{Proj}_{\mathbb{R}} \mathcal{P}_y = \cup_{i=1}^5 \mathcal{X}_i,$$

$$\kappa_y^* : \mathbb{R} \rightarrow \mathbb{R}^2 : \kappa_{pwa}(x) = [1, 0] \kappa_y^*(x), \forall x \in \cup_{i=1}^5 \mathcal{X}_i.$$

which represent the relevant elements for the construction of an mpQP problem, which is able to retrieve as the optimal solution the original PWA function.

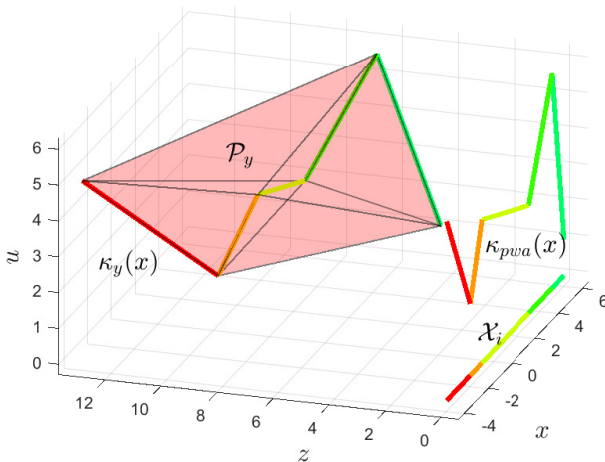


Fig. 1. Example of the solution sought for the pair $(\mathcal{P}_y, \kappa_y)$ based on an initial PWA function $\kappa_{pwa}(x)$.

III. MAIN RESULT

The construction of an equivalent mpQP (6) with *Prop 1 – 4* can be divided into two steps. Firstly, a pair $(\mathcal{P}_y, \kappa_y)$ satisfying *Prop 1, 3, 4* is constructed. Secondly, leveraging $(\mathcal{P}_y, \kappa_y)$, a cost function (6a) and a set of constraints (6b) are proposed to ensure *Prop 2*.

A. Convex-concave lifting

1) *Basic notions and definitions:* Prior to the effective construction procedure, a novel convex-concave lifting technique is introduced to ensure the fulfilment of *Prop 3* and *4* in the mpQP (6).

Lemma 1. [12] Given a polyhedral partition $\{\mathcal{X}_i\}_{\mathcal{I}_N}$ of a polyhedron $\underline{\mathcal{X}} \subset \mathbb{R}^n$, $l(x) : \underline{\mathcal{X}} \rightarrow \mathbb{R}$ is called a piecewise affine convex lifting if the following conditions hold:

- $l(x) = a_i^T x + b_i$ for $x \in \mathcal{X}_i$;
- $l(x)$ is continuous over $\underline{\mathcal{X}}$;
- $l(x) > a_j^T x + b_j$ for all $x \in \mathcal{X}_i \setminus \mathcal{X}_j$ with $i, j \in \mathcal{I}_N$ and $i \neq j$.

Definition 2. A PWA function $g : \mathbb{R}^n \rightarrow \mathbb{R}$, defined as $g(x) = a_i^T x + b_i$ for $x \in \mathcal{X}_i$ with $i \in \mathcal{I}_N$, is categorized as a convex-concave lifting if it satisfies the following conditions:

- The index set \mathcal{I}_N is partitioned into two groups:
 - 1) \mathcal{I}_N^{veex} and \mathcal{I}_N^{cave} , containing the convex items and the concave items, respectively. They fulfill that

$$\mathcal{I}_N^{veex} \cup \mathcal{I}_N^{cave} = \mathcal{I}_N \text{ and } \mathcal{I}_N^{veex} \cap \mathcal{I}_N^{cave} = \{1\}.$$

- For all $i \in \mathcal{I}_N^{veex}$, the following conditions are satisfied:
 - 2) $g(x) > a_j^T x + b_j$ for all $x \in \mathcal{X}_i \setminus \mathcal{X}_j$ and all $j \neq i, j \in \mathcal{I}_N^{veex}$;
 - 3) $g(x) < a_j^T x + b_j, \forall x \in \mathcal{X}_i$ and $\forall j \in \mathcal{I}_N^{cave} \setminus \{1\}$.
- For all $i \in \mathcal{I}_N^{cave}$, the following conditions are satisfied:
 - 4) $g(x) < a_j^T x + b_j$ for all $x \in \mathcal{X}_i \setminus \mathcal{X}_j$ and all $j \neq i, j \in \mathcal{I}_N^{cave}$;
 - 5) $g(x) > a_j^T x + b_j, \forall x \in \mathcal{X}_i$ and $\forall j \in \mathcal{I}_N^{veex} \setminus \{1\}$.
- For all $i, j \in \mathcal{I}_N, i \neq j$, the following condition holds:
 - 6) $a_i^T x + b_i = a_j^T x + b_j, x \in \mathcal{X}_i \cap \mathcal{X}_j$.

Remark 2. The proposed convex-concave lifting has the following properties:

- A convex-concave lifting is a convex lifting if

$$\mathcal{I}_N^{cave} = \{1\}.$$

- The collection $\{\mathcal{X}_i\}_{\mathcal{I}_N}$ is convex-concave liftable is

$$\forall i \in \mathcal{I}_N, \mathcal{X}_i \subset \mathbb{R}.$$

Assumption 1. The polyhedral partition $\{\mathcal{X}_i\}_{\mathcal{I}_N}$, it is convex-concave liftable.

Remark 3. Assumption 1 guarantees the existence of a PWA function $g(x)$ defined over $\{\mathcal{X}_i\}_{\mathcal{I}_N}$. Remark 2 highlights that if $\{\mathcal{X}_i\}_{\mathcal{I}_N}$ is defined in a one-dimensional space, Assumption 1 is satisfied. In this study, our focus is on a polyhedral partition $\{\mathcal{X}_i\}_{\mathcal{I}_N}$ that satisfies Assumption 1.

Based on the analysis of Definition 2, we present Algorithm 1 for constructing a convex-concave lifting applicable to a liftable polyhedral partition $\{\mathcal{X}_i\}_{\mathcal{I}_N}$.

Algorithm 1 Construction a convex-concave lifting for a given collection $\{\mathcal{X}_i\}_{\mathcal{I}_N}$ of a polytope $\mathcal{X} \subset \mathbb{R}^n$

Input: $\{\mathcal{X}_i\}_{\mathcal{I}_N}$ and a given constant $\epsilon > 0$.

Output: Parameter pairs $(a_i, b_i), \forall i \in \mathcal{I}_N$.

- 1: Partition the set of indices \mathcal{I}_N into \mathcal{I}_N^{vex} and \mathcal{I}_N^{cave} .
- 2: Register all neighboring regions in $\{\mathcal{X}_i\}_{\mathcal{I}_N}$, $\{\mathcal{X}_i\}_{\mathcal{I}_N^{vex}}$, and $\{\mathcal{X}_i\}_{\mathcal{I}_N^{cave}}$, and store them in a set of pairs \mathcal{E} .
- 3: For each $(i, j) \in \mathcal{E}$ and $i, j \in \mathcal{I}_N$

$$\forall v \in \mathcal{V}(\mathcal{X}_i \cap \mathcal{X}_j), a_i^T v + b_i = a_j^T v + b_j. \quad (10a)$$

- 4: For each $(i, j) \in \mathcal{E}$ with $i, j \in \mathcal{I}_N^{vex}$, add convexity conditions, $\forall v \in \mathcal{V}(\mathcal{X}_i), v \notin \mathcal{V}(\mathcal{X}_j)$:

$$a_i^T v + b_i \geq a_j^T v + b_j + \epsilon. \quad (10b)$$

- 5: For each $(i, j) \in \mathcal{E}, i, j \in \mathcal{I}_N^{cave}$, add concavity conditions, $\forall v \in \mathcal{V}(\mathcal{X}_i), v \notin \mathcal{V}(\mathcal{X}_j)$:

$$a_i^T v + b_i + \epsilon \leq a_j^T v + b_j. \quad (10c)$$

- 6: For $i \in \mathcal{I}_N^{vex} \setminus \{1\}, \forall j \in \mathcal{I}_N^{cave} \setminus \{1\}$, add boundary conditions, $\forall v \in \mathcal{V}(\mathcal{X}_i)$:

$$a_i^T v + b_i + \epsilon \leq a_j^T v + b_j. \quad (10d)$$

- 7: For $i \in \mathcal{I}_N^{cave} \setminus \{1\}, \forall j \in \mathcal{I}_N^{vex} \setminus \{1\}$, add boundary conditions, $\forall v \in \mathcal{V}(\mathcal{X}_i)$:

$$a_i^T v + b_i \geq a_j^T v + b_j + \epsilon. \quad (10e)$$

- 8: Solve a linear quadratic problem with variables $a_i, b_i, \forall i \in \mathcal{I}_N$:

$$\min_{a_i, b_i} \sum_{\forall i \in \mathcal{I}_N} (a_i^T a_i + b_i^T b_i) \text{ subject to (10)}. \quad (11)$$

2) *Application of a convex-concave lifting:* Let us consider a given polyhedral partition $\{\mathcal{X}_i\}_{\mathcal{I}_N}$ consisting of an unconstrained region indexed by 1, a convex index set denoted as \mathcal{I}_N^{vex} , and a concave index set denoted as \mathcal{I}_N^{cave} . We define a PWA function as follows:

$$g(x) = a_i^T x + b_i, \forall x \in \mathcal{X}_i, i \in \mathcal{I}_N$$

where $a_1 = 0$ and $b_1 = 0$. This function serves as a convex-concave lifting defined over the polyhedral partition $\{\mathcal{X}_i\}_{\mathcal{I}_N}$. Subsequently, we can define a series of polytopes:

$$\Gamma_{[x^T z]^T} := \text{conv} \left\{ [v^T, g(v)]^T : v \in \cup_{i=1}^N \mathcal{V}(\mathcal{X}_i) \right\}, \quad (12a)$$

$$\mathcal{F}_{[x^T z]^T}^i := \text{conv} \left\{ [v^T, g(v)]^T : v \in \mathcal{V}(\mathcal{X}_i) \right\}, i \in \mathcal{I}_N. \quad (12b)$$

In accordance with Definition 2, sets $\mathcal{X}_i, \mathcal{X}, \Gamma_{[x^T z]^T}$ and $\mathcal{F}_{[x^T z]^T}^i$ exhibit the following relationships:

- $\text{Proj}_{\mathbb{R}^m} \Gamma_{[x^T z]^T} = \mathcal{X}, \text{Proj}_{\mathbb{R}^m} \mathcal{F}_{[x^T z]^T}^i = \mathcal{X}_i$.
- $\forall i \in \mathcal{I}_N \setminus \{1\}, \mathcal{F}_{[x^T z]^T}^i$ is a facet of $\Gamma_{[x^T z]^T}$.

$$\bullet \text{int}(\mathcal{F}_{[x^T z]^T}^1) \subset \text{int}(\Gamma_{[x^T z]^T}).$$

Using the aforementioned properties as conditions, we retrieve the convex-concave lifting $g(x)$ from the set $\Gamma_{[x^T z]^T}$ by means of the following proposition.

Proposition 2. The given PWA function $g(x)$ is the optimal solution to the optimization problem below:

$$\min_z z^T z \text{ s.t. } [x^T, z]^T \in \Gamma_{[x^T z]^T}. \quad (13)$$

Proof. The unconstrained solution of mpQP (13) is

$$z^{un} = 0.$$

Combing with the group of polytopes $\mathcal{F}_{[x^T z]^T}^i$, we prove the proposition by dividing $\Gamma_{[x^T z]^T}$ into three subsets:

$$\Gamma_{[x^T z]^T} = \Gamma_{[x^T z]^T}^{vex} \cup \Gamma_{[x^T z]^T}^{un} \cup \Gamma_{[x^T z]^T}^{cave}$$

with

$$\Gamma_{[x^T z]^T}^{vex} = \mathcal{S}(\mathcal{I}_N^{vex} \setminus \{1\}), \Gamma_{[x^T z]^T}^{un} = \mathcal{S}(\{1\}),$$

$$\Gamma_{[x^T z]^T}^{cave} = \mathcal{S}(\mathcal{I}_N^{cave} \setminus \{1\}),$$

$$\mathcal{S} : \mathbb{N} \rightarrow \mathbb{R}^{m+1},$$

$$\mathcal{S}(S) = \text{conv} \left\{ [v^T, z]^T : v \in \cup_{i \in S} \mathcal{V}(\mathcal{X}_i), z \in \mathbb{R} \right\},$$

where Figure 2 illustrates these sets. In Figure 2, the sets $\mathcal{I}_N^{vex} \setminus \{1\}, \{1\}$ and $\mathcal{I}_N^{cave} \setminus \{1\}$ are denoted as S_1, S_2 and S_3 , respectively.

Case 1: $[x^T, z]^T \in \Gamma_{[x^T z]^T}^{un}$ For $\Gamma_{[x^T z]^T}^{un}$, the following relation holds:

$$\text{int}(\mathcal{F}_{[x^T z]^T}^1) \subset \text{int}(\Gamma_{[x^T z]^T}^{un}),$$

and $g(x) = 0$ in the mean time. Thus, the subsequent relation is valid:

$$[x^T z]^T \in \Gamma_{[x^T z]^T}^{un} \Rightarrow g(x) = z^{un} = 0.$$

Case 2: $[x^T, z]^T \in \Gamma_{[x^T z]^T}^{vex}$

In this case, $i \in \mathcal{I}_N^{vex} \setminus \{1\}$, $g(x)$ degenerates into a convex lifting of $\{\mathcal{X}_i\}_{\mathcal{I}_N^{vex} \setminus \{1\}}$, and mpQP (13) can convert to:

$$\min_z z \text{ s.t. } [x^T, z]^T \in \Gamma_{[x^T z]^T}^{vex}.$$

According to the definition of convex lifting, we get:

$$[x^T z]^T \in \Gamma_{[x^T z]^T}^{vex} \Rightarrow g(x) = z^* = a_i^T x + b_i, x \in \mathcal{X}_i$$

with $i \in \mathcal{I}_N^{vex} \setminus \{1\}$.

Case 3: $[x^T, z]^T \in \Gamma_{[x^T z]^T}^{cave}$

The analysis procedure follows a similar approach as shown in case 2. Notably, the mpQP (13) is converted as:

$$\max_z z \text{ s.t. } [x^T, z]^T \in \Gamma_{[x^T z]^T}^{cave}.$$

Thus, the proposition is proved. \square

Remark 4. Proposition 2 outlines a procedure to retrieve a convex-concave lifting $g(x)$ using a polytope $\Gamma_{[x^T : z]^T}$. This insight will guide us in constructing an equivalent problem (6) following a similar approach.

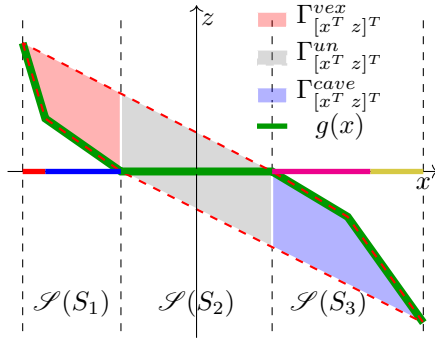


Fig. 2. A convex-concave lifting $g(x)$ and its relevant sets

B. Step 1: Constructing of parameter feasible set \mathcal{P}_y and feasible solution $\kappa_y(x)$

First, let us refer to the dimensional expansion lemma proposed in [1].

Lemma 2. [1] Let $\Gamma_s \subset \mathbb{R}^{d_s}$ be a full-dimensional polytope with the set of vertices $\mathcal{V}(\Gamma_s) = \{s^{(1)}, \dots, s^{(q)}\}$. For any finite set of points $\{s^{(1)}, \dots, s^{(q)}\} \subset \mathbb{R}^{d_t}$ defining a full-dimensional polytope, an extension of the family $\mathcal{V}(\Gamma_s)$ can be obtained in higher-dimensional space $\mathbb{R}^{d_s+d_t}$ for the concatenated vectors $[s^T, t^T]^T$ defining the set:

$$V_{s,t} := \left\{ \begin{bmatrix} s^{(1)} \\ t^{(1)} \end{bmatrix}, \dots, \begin{bmatrix} s^{(q)} \\ t^{(q)} \end{bmatrix} \right\}.$$

The polytope $\Gamma_{s,t} = \text{conv}(V_{s,t})$ satisfies: $V_{s,t} = \mathcal{V}(\Gamma_{s,t})$.

Remark 5. For two sets Γ_s and $\Gamma_{s,t}$, the following holds:

$$\text{Proj}_{\mathbb{R}^{d_s}} \Gamma_{s,t} = \Gamma_s.$$

Corollary 2. If $\mathcal{V}(\Gamma_s)$ and $V_{s,t}$ are rearranged such that

$$\left\{ \begin{bmatrix} s_1^{(1)} \\ s_2^{(1)} \end{bmatrix}, \dots, \begin{bmatrix} s_1^{(q)} \\ s_2^{(q)} \end{bmatrix} \right\}, \left\{ \begin{bmatrix} s_1^{(1)} \\ t^{(1)} \end{bmatrix}, \dots, \begin{bmatrix} s_1^{(q)} \\ t^{(q)} \end{bmatrix} \right\},$$

then the result in Lemma 2 holds:

$$\Gamma_{s,t} = \text{conv}(V_{s,t}) \Rightarrow V_{s,t} = \mathcal{V}(\Gamma_{s,t}). \quad \square$$

Based on the Assumption 1, let the convex-concave liftable collection of polyhedra $\{\mathcal{X}_i\}_{\mathcal{I}_N}$ and $g(x) : \bigcup_{i \in \mathcal{I}_N} \mathcal{X}_i \rightarrow \mathbb{R}$ be a PWA function defined as $g(x) = a_i^T x + b_i, x_i \in \mathcal{X}_i$, representing such a convex-concave lifting of $\{\mathcal{X}_i\}_{\mathcal{I}_N}$.

Proposition 3. Let \mathcal{I}_N^{conv} and $\mathcal{I}_N^{concave}$ be the subsets indices of \mathcal{I}_N corresponding to the convex and concave liftings. There exist a polytope $\mathcal{P}_y \subset \mathbb{R}^{n+m+1}$ and a PWA function $\kappa_y : \mathbb{R}^n \rightarrow \mathbb{R}^{m+1}$ satisfying Prop 1–4 in II-C. Moreover, they can explicitly be constructed as:

$$\mathcal{P}_y := \text{conv} \left\{ [v^T, y^T]^T : v \in \bigcup_{i=1}^N \mathcal{V}(\mathcal{X}_i), y = \kappa_y(v) \right\},$$

$$\kappa_y(x) := [\kappa_{pwa}(x)^T, g(x)^T]^T, x \in \underline{\mathcal{X}},$$

where $g(x)$ is a convex-concave lifting over the given polyhedral partition $\{\mathcal{X}_i\}_{\mathcal{I}_N}$.

Proof. To prove Prop 1 in II-C, one has:

$$\underline{\mathcal{X}} = \text{conv} \bigcup_{i=1}^N \mathcal{V}(\mathcal{X}_i) \Rightarrow \text{Proj}_{\mathbb{R}^n} \mathcal{P}_y = \underline{\mathcal{X}}.$$

Also, with respect to Prop 2 in II-C, it is clear that

$$\kappa_{pwa}(x) = [I_m, \mathbf{0}_{m \times 1}] \kappa_y(x).$$

To validate Prop 3-4 in II-C, one can use \mathcal{P}_u and $\Gamma_{[x^T, z]^T}$, as defined in (5) and (12a) respectively. From the definitions, we derive that:

$$\text{Proj}_{\mathbb{R}^{n+m}} \mathcal{P}_y = \mathcal{P}_u, \text{Proj}_{\mathbb{R}^n} \mathcal{P}_u = \underline{\mathcal{X}},$$

$$\text{Proj}_{\mathbb{R}^{[1:n, n+m+1]}} \mathcal{P}_y = \Gamma_{[x^T, z]^T}, \text{Proj}_{\mathbb{R}^n} \Gamma_{[x^T, z]^T} = \underline{\mathcal{X}}.$$

If $x \in \text{int}(\mathcal{X}_1)$, since set \mathcal{X}_1 is the unconstrained region, we have $[x^T, \kappa_{pwa}(x)^T]^T \in \text{int}(\mathcal{P}_u)$. It follows the existence of $\epsilon_1 > 0$ such that $[x^T, \kappa_{pwa}(x)^T]^T + \epsilon_1 \mathbb{B}^{n+m} \subset \mathcal{P}_u$. Based on the design procedure of convex-concave lifting, if $x \in \text{int}(\mathcal{X}_1)$, then $[x^T, g(x)^T]^T \in \text{int}(\Gamma_{[x^T, z]^T})$ holds true. Similarly, there is $\epsilon_2 > 0$ such that $[x^T, g(x)^T]^T + \epsilon_2 \mathbb{B}^{n+1} \subset \Gamma_{[x^T, z]^T}$. Thus, we can choose $\epsilon_3 = \min\{\epsilon_1, \epsilon_2\}$ so that $[x^T, \kappa_{pwa}(x)^T, g(x)^T]^T + \epsilon_3 \mathbb{B}^{n+m+1} \subset \mathcal{P}_y$.

If $x \in \mathcal{X}_i$ with $i \in \mathcal{I}_N \setminus \{1\}$, one can not find $\epsilon > 0$, $[x^T, g(x)^T]^T + \epsilon \mathbb{B}^{n+1} \subset \Gamma_{[x^T, z]^T}$, which also means $\nexists \epsilon > 0$, $[x^T, \kappa_{pwa}(x)^T, g(x)^T]^T + \epsilon \mathbb{B}^{n+m+1} \subset \mathcal{P}_y$ and $[x^T, \kappa_{pwa}(x)^T, g_c(x)^T]^T \in \text{bd}(\mathcal{P}_y)$.

Thus, Prop 3 and 4 in II-C are verified. \square

C. Step 2: Constructing an equivalent mpQP with a cost function $J(x, [u^T, z]^T)$

Proposition 4. Consider the mpQP problem (6) with the parameter feasible set \mathcal{P}_y and a PWA function $\kappa_y(x)$ defined according to the Proposition 3. The optimal solution to the mpQP problem (6) satisfies the following equation:

$$\kappa_y^*(x) \equiv \kappa_y(x), \forall x \in \underline{\mathcal{X}},$$

if the cost function is synthesised as

$$J(x, [u^T, z]^T) = \alpha z^T z + 0.5 u^T u - u^T F_1 x$$

with $\alpha \in \mathbb{R}$ a large enough positive constant.

Proof. By utilizing the cost function, we obtain the unconstrained optimal solution $\kappa_y^{un}(x)$, expressed as:

$$\kappa_y^{un}(x) = \begin{bmatrix} F_1 \\ 0 \end{bmatrix} x.$$

Additionally, as x serves as a parameter in the optimization problem, the cost function $J(x, [u^T, z]^T)$ can be equivalently transformed into:

$$\Rightarrow \alpha z^T z + 0.5(u - F_1 x)^T (u - F_1 x)$$

$$\Rightarrow \left(\begin{bmatrix} u \\ z \end{bmatrix} - \kappa_y^{un}(x) \right)^T \begin{bmatrix} I_m & 0 \\ 0 & 2\alpha \end{bmatrix} \left(\begin{bmatrix} u \\ z \end{bmatrix} - \kappa_y^{un}(x) \right). \quad (15)$$

In the subsequent analysis, we will establish the proposition by considering the cases where x belongs to different regions.

Case 1: $x \in \mathcal{X}_1$

As shown in Proposition 3, when $x \in \mathcal{X}_1$, there are:

$$\kappa_y(x) = [F_1^T, 0]^T x \Rightarrow \kappa_y^*(x) = \kappa_y^{un}(x) = \kappa_y(x).$$

Case 2: $x \notin \mathcal{X}_1$

When $x \notin \mathcal{X}_1$, there are:

$$z \neq 0 \Rightarrow [x^T, (\kappa_y^{un}(x))^T]^T \notin \text{int}(P_y).$$

In this situation, the optimization problem can be equivalently converted as

$$\begin{aligned} \min_{[u^T, z]^T} J(x, [u^T, z]^T) \text{ in type (15),} \\ \text{s.t. } [x^T, u^T, z]^T \in \text{bd}(P_y). \end{aligned}$$

For sufficiently large positive constant α , in conjunction with formulation (15), the factor influencing the cost function will converge to the term $z^T z$. According to Proposition 3, the optimal trajectory on the boundary, minimizing $z^T z$, satisfies:

$$z^* = g(x).$$

In the same time, with the help of the construction process of \mathcal{P}_y ,

$$u^* = \kappa_{pwa}(x)$$

when z sets to the boundary value. Thus, we have

$$\kappa_y^*(x) = \begin{bmatrix} u^* \\ z^* \end{bmatrix} = \begin{bmatrix} \kappa_{pwa}(x) \\ g(x) \end{bmatrix} = \kappa_y(x).$$

Thus, $\forall x \in \mathcal{X}, \kappa_y^*(x) \equiv \kappa_y(x)$ is proved. \square

IV. ILLUSTRATION

Consider a scalar PWA function:

$$\kappa_{pwa}(x) = \begin{cases} -1.5x - 4.75, & x \in \mathcal{X}_2 = \{x : -4 < x \leq -2\} \\ 2x + 2.25, & x \in \mathcal{X}_3 = \{x : -2 < x \leq -1\} \\ -0.25x, & x \in \mathcal{X}_1 = \{x : -1 < x \leq 3\} \\ 1.5x - 5.25, & x \in \mathcal{X}_4 = \{x : 3 < x \leq 5\} \\ -5x + 27.25, & x \in \mathcal{X}_5 = \{x : 5 < x \leq 6\} \end{cases}$$

By following the steps outlined in Algorithm 1, we have obtained a convex-concave lifting function $g : \mathbb{R} \rightarrow \mathbb{R}$ defined as $g(x) = a_i x + b_i$. To simplify without loss of generality, we apply linear mapping, represented as:

$$a_i = 10000a_i^*, b_i = 10000b_i^*,$$

where a_i^* and b_i^* represent the optimal solutions of the constrained optimization problem (11), subject to the constraints $a_1 = 0$ and $b_1 = 0$. The resulting lifting parameters, a_i and b_i , are obtained by assigning the i -th elements of the vectors as follows:

$$\begin{bmatrix} -2.0000 & -1.0000 & 0 & -0.8571 & -1.8571 \\ -3.0000 & -1.0000 & 0 & 2.5714 & 7.5714 \end{bmatrix},$$

Finally, we formulate an mpQP problem (6) by incorporating the cost function and parameter feasible set as follows:

$$\begin{aligned} J(x, [u, z]^T) &= 0.5u^2 + 0.25xu + 100z^2, \\ \mathcal{P}_y &= \{[x, u, z]^T \in \mathbb{R}^T : H_t[x, u, z]^T \leq b_t\} \end{aligned}$$

with

$$H_t = \begin{bmatrix} -0.3353 & 0.7115 & -0.3435 \\ -0.3195 & 0.2452 & -0.8323 \\ -0.6286 & 0.1143 & -0.4000 \\ -0.4583 & -0.1549 & -0.7681 \\ 0.2966 & 0.0322 & 0.4023 \\ 0.2947 & -0.0417 & 0.2709 \\ 0.1626 & -0.4182 & 0.3950 \\ 0.1610 & -0.4295 & 0.3758 \end{bmatrix}, b_t = \begin{bmatrix} 0.5132 \\ 0.3808 \\ 0.6571 \\ 0.4195 \\ 0.8655 \\ 0.9154 \\ 0.8016 \\ 0.8052 \end{bmatrix}.$$

For clarity, separate identical translation operations are applied to $\{\mathcal{X}_i\}_{\mathcal{X}_5}$, $\text{Proj}_{[x,u]^T} \kappa_y^*(x)$, $\Gamma_{[x,z]^T}$, \mathcal{P}_y and $\kappa_y^*(x)$ in Fig. 3. $\kappa_y^*(x)$ is the optimal solution of the mpQP problem, and only when $x \in \text{int}(\mathcal{X}_1)$, the curve remains inside \mathcal{P}_y . Furthermore, the equations $\text{Proj}_{[x,u]^T} \kappa_y^*(x) = \kappa_{pwa}(x)$, as depicted in Fig. 3.

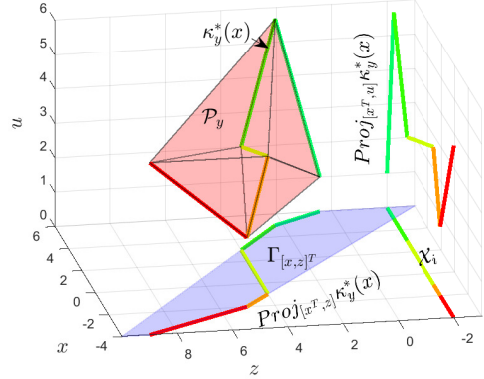


Fig. 3. Example of the solution sought for the pair $(\mathcal{P}_y, \kappa_y^*)$.

V. CONCLUSIONS

After delineating the geometric structure of the optimal solution for an mpQP problem, a polyhedral partition was implemented to partition the regions into the boundary (saturated) and interior components. To preserve the boundary-interior structure in a lifting space, a convex-concave lifting technique was introduced. By leveraging a high-dimensional parametric feasible set and a relevant cost function, an equivalent mpQP problem was constructed, considering the characteristics of convex-concave lifting. It should be noted that the current convex-concave liftability conditions can be regarded as conservative. However, it has been pointed out that it covers at least the class of systems with PWA control laws that can be decoupled to dependency on scalar components of the state space. Additional endeavours will be dedicated to addressing the conservativeness of the proposed strategy and extending its applicability to complex systems.

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