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Robust PWA control Based on State Space Partitions

Songlin Yang¹, Sorin Olaru¹, Pedro Rodriguez-Ayerbe¹

Abstract—Under the convex lifting for the state partitions, this paper studies the robust piecewise affine (PWA) control laws for linear discrete-time systems with state and input constraints. In particular, unlike the study of perturbations of state space partitions in vertex or half-plane representations, each state partition in this study is the smallest perturbation element. Practically, a maximum admissible region collection (MARC) is defined to inscribe the maximum perturbation region of each state partition. Based on MARC, which guarantees the positive invariance of the original controller, a fragility of convex lifting (FCL) is proposed to realize a robust PWA control with an implicit controller form. A numerical example illustrates the effectiveness of the new robust PWA controller.

I. INTRODUCTION

Controller fragility is a major problem in the performance of feedback control systems due to inaccurate controller execution [1]. Inaccuracies in control actions are due to various reasons: analogue-to-digital conversion, truncation, quantization, the resolution of sensing devices and the associated measurement accuracy [2]. These negative factors affect the performance of the controller at the time of implementation. It is assumed that the fragility analysis of a classical linear controller is relatively straightforward after passing through the parameter sensitivity tool. In contrast, robustness analysis of PWA controllers defined in a state space partition is challenging.

The assessment of the robustness of controllers concerning their internal representation requires an in-depth analysis of the design methodologies and the structural implementation details. In recent years, many researchers have investigated the robustness of PWA control laws from different perspectives. For a linear system with PWA control laws, an explicit robustness margin assuming a polytopic uncertainty constraint on the nominal system was proposed in [3]. In [4], a delay margin based on positive invariance of the nominal closed-loop dynamics in the PWA formulation was proposed for PWA systems in the presence of variable time delay. Different from the robustness analysis of the system model and the corresponding PWA controller, in [5], [6], several novel ways of reducing the memory footprint of PWA controller laws were proposed to realize the high accuracy, low precision, and memory efficient embedded model predictive control. In [7], researchers investigated the impact of unavoidable errors on the system performance during the practical application of PWA control laws. In [8], the authors showed that the non-overlapping and the

invariance properties of a PWA controller are preserved when the perturbation occurs on the state regions' vertices.

Besides, convex lifting is also an effective tool for the robust PWA control problem. In [9], authors shows that a suitable convex lifting can guarantee the convergence of the closed-loop dynamics of constrained linear systems affected by bounded additive disturbances and polytopic uncertainties. In [10], the drawback of [9], which the closed-loop stability is not in the sense of Lyapunov is worked out by means of a suitable construction of control Lyapunov function based on the convex lifting. In this paper, the authors propose the fragility of convex lifting to explore the robust PWA control based on the closed relationship between the convex lifting and the state partitions of PWA control laws.

Unlike the vertex-based or hyperplane-based fragility analysis in [8], [11], and [12], this paper focuses on the fragility of state partitions directly. In this work, the minimum perturbation elements of the state partitions are state regions, which is realized by the convex lifting. In terms of the fragility of the PWA control laws, the novel points of this work are:

- The robust PWA control law is realized, which has a special perturbation mode in the state space and maintains the invariance of the state regions after perturbation. Similarly study the robust PWA control based on state space partitions, in [8], [11], set the state partitions in the vertices representation and only researched the robustness under the perturbation of one or two vertices. In [12], authors explored the robust PWA control under the state partitions in the halfplane representation. In this paper, we thought the minimum perturbation element of the state partitions is one region, which does not care about the representations of the state partitions;
- A fragility of convex lifting is proposed to construct an implicit PWA control law. Normally, a convex lifting is established based on a fixed state partition. In this paper, to authors' knowledge, we firstly consider the fragility problem of the convex lifting. Then, a novel concept fragility of convex lifting and relevant algorithm are established to answer this problem. It is worth mentioning that a state partition can establish a convex lifting, vice versa.

This paper is organized as follows. Section II is dedicated to the preliminaries and the necessary knowledge related to PWA functions. Section III proposes a maximum admissible region and its constructional conditions. Section IV concentrates on the fragility of convex lifting: its definition, propositions, and associated algorithm. A numerical example

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in Section V exhibits the effectivity of the proposed robust PWA controller.

Generalities and basic notions

In this paper, \mathbb{R}^n , \mathbb{R}_+ , $\mathbb{R}_{>}$, \mathbb{N} , \mathcal{I}_N and \mathbb{B}^n denote the set of real numbers in n -dimensional space, the set of nonnegative real numbers, the set of positive real numbers, the set of nonnegative integers, $\{1, 2, \dots, N\}$, and $\{x \in \mathbb{R}^n \mid \|x\|_\infty \leq 1\}$, respectively. Also, $\mathcal{V}(S)$, $\text{int}(S)$, and ∂S denote the set of vertices, the interior, and the boundary of polytope S , respectively. We use $\text{conv}(S)$ to denote the convex hull of S . Given two sets $S_1, S_2 \in \mathbb{R}^d$, we denote a new set: $S_1 \setminus S_2 = \{x \in \mathbb{R}^d : x \in S_1, x \notin S_2\}$. Meanwhile, the Minkowski sum of two sets, denoted by $S_1 \oplus S_2$, is defined as $S_1 \oplus S_2 = \{x_1 + x_2 : x_1 \in S_1, x_2 \in S_2\}$. For two column vectors: $x, y \in \mathbb{R}^d$, $x = [x_1, x_2, \dots, x_d]^T$, $y = [y_1, y_2, \dots, y_d]^T$, $x \leq y$ means $x_i \leq y_i, \forall i \in \mathcal{I}_d$. $\mathbf{1}_{m \times n}$ ($\mathbf{0}_{m \times n}$), $\mathbf{1}_m$ and \mathbf{I}_n denote a $m \times n$ matrix with its elements equal to one (zero), a column vector with its elements equal to one, and a n -dimensional identity matrix, respectively.

II. PRELIMINARIES

A. PWA control

Consider a discrete-time linear system:

$$x_{k+1} = Ax_k + Bu_k, \quad (1)$$

where $x_k \in \mathbb{R}^{n_x}$, $u_k \in \mathbb{R}^{n_u}$ are the state vector and control input vector respectively at time k . The states and inputs are bounded by polytopes as

$$\mathcal{X} = \{x \in \mathbb{R}^{n_x} \mid H_x x \leq b_x\}, \mathcal{U} = \{u \in \mathbb{R}^{n_u} \mid H_u u \leq b_u\},$$

where b_x, b_u, H_x , and H_u are constant matrices with suitable dimension. Below lists some relevant notions.

Definition 1. A collection of polyhedral sets $\{\mathcal{X}_1, \dots, \mathcal{X}_N\}$ is called a polyhedral partition of \mathcal{X} if $\forall i, j \in \mathcal{I}_N, i \neq j$,

1. $\mathcal{X} = \bigcup_{i=1}^N \mathcal{X}_i$;
2. $\text{int}(\mathcal{X}_i) \cap \text{int}(\mathcal{X}_j) = \emptyset$.

Such a polyhedral partition is next denoted as $\{\mathcal{X}_i\}_{\mathcal{I}_N}$.

Remark 1. In definition 1, condition 1 ensures any point in \mathcal{X} belongs to at least one region \mathcal{X}_i , while condition 2 guarantees non-overlapping among different regions.

For system (1), we consider the PWA control law as

$$u_k = \kappa_{pwa}(x_k), \forall k \in \mathbb{N}, \quad (2)$$

where $\kappa_{pwa}(x)$ is a PWA function over $\{\mathcal{X}_i\}_{\mathcal{I}_N}$:

$$\kappa_{pwa} : \left\{ \bigcup_{i \in \mathcal{I}_N} \mathcal{X}_i \right\} \rightarrow \mathcal{U}, \quad (3)$$

$$\kappa_{pwa}(x) = F_i x + g_i, \text{ if } x \in \mathcal{X}_i, i \in \mathcal{I}_N,$$

Definition 2. The polyhedral set \mathcal{X} is positively invariant with respect to system (w.r.t) (1) if $\forall x \in \mathcal{X}$, $(Ax + B\kappa_{pwa}(x)) \in \mathcal{X}$.

Assumption 1. In the current study, we assume that PWA control law (2) has been designed to guarantee the positive invariance of \mathcal{X} with respect to (1).

B. Problem formulation

We are interested with such a problem: as shown in (4) when polyhedral partition $\{\mathcal{X}_i\}_{\mathcal{I}_N}$ is perturbed to new form, could we still keep \mathcal{X} is positively invariant w.r.t system (1)?

$$\left. \begin{array}{l} \mathcal{X}_i \rightarrow \tilde{\mathcal{X}}_i, \\ \kappa_{pwa}(x) \rightarrow \tilde{\kappa}_{pwa}(x), \end{array} \right\} \Rightarrow (Ax + B\tilde{\kappa}_{pwa}(x)) \in \mathcal{X}, \quad (4)$$

where $\tilde{\kappa}_{pwa}(x) = F_i x + g_i$, if $x \in \tilde{\mathcal{X}}_i, i \in \mathcal{I}_N$.

In the literature [8], [11], and [12], researchers connected the perturbation problem to the specific set representations. In detail, the converted result are:

$$\begin{array}{l} \mathcal{X}_i \rightarrow \tilde{\mathcal{X}}_i \mapsto v_i \rightarrow \tilde{v}_i, \text{ vertex-representation} \\ \mathcal{X}_i \rightarrow \tilde{\mathcal{X}}_i \mapsto H_i \rightarrow \tilde{H}_i, \text{ hyperplane-representation,} \end{array}$$

where v_i and H_i (or \tilde{v}_i and \tilde{H}_i) are the vertex and hyperplane of \mathcal{X}_i (or $\tilde{\mathcal{X}}_i$).

The drawback of realizing perturbation of $\{\mathcal{X}_i\}_{\mathcal{I}_N}$ based on specific set representations is that it is difficult to achieve the whole region's perturbation in these approaches. You have to perturb the vertex or hyperplane one-by-on-one, which is coupled and high conservativeness.

In this paper, inspired by the set relationship in [13], we considered the state region \mathcal{X}_i as a minimum element, and used the convex lifting to study the perturbation problem.

III. A FEASIBLE STRATEGY

For perturbation problem (4), the key point is to define and calculate the feasible perturbation region of \mathcal{X}_i . In this section, we propose a notion MARC and a proposition to complete these tasks.

A. MAXIMUM ADMISSIBLE REGION COLLECTION

Definition 3. Consider system (1) with PWA control law (2), and relevant polyhedral partition $\{\mathcal{X}_i\}_{\mathcal{I}_N}$, a collection of polyhedral sets $\{\bar{\mathcal{X}}_1, \dots, \bar{\mathcal{X}}_N\}$ is called the maximum admissible regions collection (MARC) of $\{\mathcal{X}_i\}_{\mathcal{I}_N}$ if for all $i \in \mathcal{I}_N$, there are

- $\forall x_k \in \bar{\mathcal{X}}_i$, there are $x_{k+1} \in \mathcal{X}$ and $\tilde{\kappa}_{pwa}(x_k) \in \mathcal{U}$;
- $\forall \epsilon > 0, \forall x_k \in \bar{\mathcal{X}}_i \oplus \epsilon \mathbb{B}^{n_x}$, there are $x_{k+1} \in \mathcal{X}$ and $\tilde{\kappa}_{pwa}(x_k) \in \mathcal{U}$

$\bar{\mathcal{X}}_i$ is called the maximum admissible set (MAS) of \mathcal{X}_i . Such a MARC is next denoted as $\{\bar{\mathcal{X}}_i\}_{\mathcal{I}_N}$.

Remark 2. Obviously, for each $i \in \mathcal{I}_N$, $\mathcal{X}_i \subset \bar{\mathcal{X}}_i$. And $\{\bar{\mathcal{X}}_i\}_{\mathcal{I}_N}$ is not a polyhedral partition.

In perturbation problem (4), if $\tilde{\mathcal{X}}_i \subset \bar{\mathcal{X}}_i, \forall i \in \mathcal{I}_N$, the problem is feasible. Thus, we define a feasible solution of problem (4) based on MARC.

Definition 4. Consider system (1) with PWA control law (2), relevant polyhedral partition $\{\mathcal{X}_i\}_{\mathcal{I}_N}$ and its MARC $\{\bar{\mathcal{X}}_i\}_{\mathcal{I}_N}$, a polyhedral partition $\{\tilde{\mathcal{X}}_i\}_{\mathcal{I}_N}$ is called an admissible perturbation polyhedral partition (APPP) of $\{\mathcal{X}_i\}_{\mathcal{I}_N}$ if:

- $\mathcal{X} = \bigcup_{i=1}^N \tilde{\mathcal{X}}_i$;
- $\tilde{\mathcal{X}}_i \subset \bar{\mathcal{X}}_i$.

Remark 3. An APPP $\{\tilde{\mathcal{X}}_i\}_{\mathcal{I}_N}$ provides a special perturbation case of the original polyhedral partition $\{\mathcal{X}_i\}_{\mathcal{I}_N}$. Each partition \mathcal{X}_i perturbs to its admissible perturbation set $\tilde{\mathcal{X}}_i$ in the same time, and new collection of perturbation region $\tilde{\mathcal{X}}_i$ is still a polyhedral partition, which means the annoying phenomenon overlapping and regions hole disappears in the forme defined APPP.

B. THE PROCESS OF CONSTRUCTION

Proposition 1. Consider system (1) with PWA control law (2), and relevant polyhedral partition $\{\mathcal{X}_i\}_{\mathcal{I}_N}$. The MAS $\bar{\mathcal{X}}_i$ of \mathcal{X}_i has explicit representation:

$$\bar{\mathcal{X}}_i = \{x | H_x(A + BF_i)x \leq b_x - H_x Bg_i\} \cap \mathcal{X}_i^u, \quad (5a)$$

$$\mathcal{X}_i^u = \{x | H_u F_i x \leq b_u - H_u g_i\}. \quad (5b)$$

Proof. Firstly, for a random $x_k \in \bar{\mathcal{X}}_i, i \in \mathcal{I}_N$, if $\tilde{\kappa}_{pwa}(x_k) = F_i x_k + g_i$, then the following inequality holds:

$$H_x(A + B\tilde{\kappa}_{pwa}(x_k)) \leq b_x \text{ and } H_u \tilde{\kappa}_{pwa}(x_k) \leq b_u.$$

So if the input signal $u_k = \tilde{\kappa}_{pwa}(x_k)$, there are $x_{k+1} \in \mathcal{X}$ and $\tilde{\kappa}_{pwa}(x_k) \in \mathcal{U}$.

Next, assuming $\exists \epsilon_0 > 0, \forall x_k \in \bar{\mathcal{X}}_i \oplus \epsilon_0 \mathbb{B}^{n_x}, i \in \mathcal{I}_N$, there are $x_{k+1} \in \mathcal{X}$ and $\tilde{\kappa}_{pwa}(x_k) \in \mathcal{U}$. Given one special point $x_k^0 \in \partial \bar{\mathcal{X}}_i$, e.g., $e_l^T H_x(A + BF_i)x_k^0 = e_l^T(b_x - H_x Bg_i)$, we select

$$x_k^1 = x_k^0 + z^0$$

with $z^0 = \text{sgn}(b_x - Bg_i)\epsilon_0 x_k^0 / \|x_k^0\|$. Because $x_k^0 \in \bar{\mathcal{X}}_i$ and $\|z^0\|_\infty = \epsilon_0, x_k^1 \in \bar{\mathcal{X}}_i \oplus \epsilon_0 \mathbb{B}^{n_x}$. Thus, if we set $u_k = \tilde{\kappa}_{pwa}(x_k) = F_i x_k + g_i$, we can get $x_{k+1}^1 = (A + BF_i)x_k^0 + Bg_i + (A + BF_i)z^0$. There are $e_l^T H_x x_{k+1}^1 = e_l^T b_x + e_l^T \text{sgn}(b_x - Bg_i)\epsilon_0 \|x_k^0\| (b_x - Bg_i)$. Because $e_l^T \text{sgn}(b_x - Bg_i)\epsilon_0 \|x_k^0\| (b_x - Bg_i) > 0, x_{k+1}^1 \notin \bar{\mathcal{X}}_i$. The assumption not holds. \square

Remark 4. Proposition 1 provides the explicit formulation of a MAS of a state region. Based on this result, we can next design an APPP for the original polyhedral partition $\{\mathcal{X}_i\}_{\mathcal{I}_N}$.

IV. EFFECTIVE CONSTRUCTURE APPROACH BASED ON CONVEX LIFTING

In this section, the convex lifting is utilized to solve perturbation problem (4). In further step, a notion FCL is proposed to establish a robust PWA control law for system (1).

A. CONVEX LIFTING AND CONSTRUCTION CONDITION

Definition 5. Given a polyhedral partition $\{\mathcal{X}_i\}_{\mathcal{I}_N}$ of a polyhedron $\mathcal{X} \subset \mathbb{R}^{n_x}$, $l(x)$ is called a convex lifting if the following conditions hold true:

- $l(x) = a_i^T x + b_i$ for $x \in \mathcal{X}_i$;
- $l(x)$ is continuous over \mathcal{X} ;
- $l(x) > a_j^T x + b_j$ for all $x \in \mathcal{X}_i \setminus \mathcal{X}_j$ with $i, j \in \mathcal{I}_N$ and $i \neq j$.

Lemma 1. For two convex sets S_1 and $S_2, S_1 \subset S_2$ if $\text{int}S_1 \cap \partial S_2 = \emptyset$ and $S_1 \cap \text{int}S_2 \neq \emptyset$.

Proof. $\text{int}S_1 \cap \partial S_2 = \emptyset$ induces that $S_1 \subset S_2$ or $\text{int}S_1 \cap S_2 = \emptyset$. If $\text{int}S_1 \cap S_2 = \emptyset$ holds, there is also $S_1 \cap \text{int}S_2 = \emptyset$, which violates the second condition. Thus, $S_1 \subset S_2$. \square

Proposition 2. For a PWA control law (2) with a polyhedral partition $\{\mathcal{X}_i\}_{\mathcal{I}_N}$ and its MARC $\{\bar{\mathcal{X}}_i\}_{\mathcal{I}_N}$. A partition $\{\tilde{\mathcal{X}}_1, \dots, \tilde{\mathcal{X}}_N\}$ is an APPP of $\{\mathcal{X}_i\}_{\mathcal{I}_N}$ if $\forall x \in \tilde{\mathcal{X}}_i$, there is

$$i = \arg \max_{r \in \mathcal{I}_N} a_r^T x + b_r, \text{ s. t. } x \in \mathcal{X}, \quad (6)$$

where a_i and b_i satisfy

$$a_i^T v_i^j + b_i \leq a_j^T v_i^j + b_j, \forall v_i^j \in \mathcal{V}(\bar{\mathcal{X}}_i^j), \quad (7a)$$

$$a_i^T v_i^{che} + b_i \geq a_j^T v_i^{che} + b_j + \epsilon, \forall i \neq j, i, j \in \mathcal{I}_N, \quad (7b)$$

where $\bar{\mathcal{X}}_i^j := \partial(\bar{\mathcal{X}}_i \cap \mathcal{X}) \cap \mathcal{X}_j, v_i^{che}$ is the chebyshev center of \mathcal{X}_i , and ϵ is a known small enough positive constant.

Proof. Firstly, we show that $\forall i \in \mathcal{I}_N, \tilde{\mathcal{X}}_i \subset \bar{\mathcal{X}}_i$. Combining (6) and (7a), one knows $v_i^j \notin \text{int}\tilde{\mathcal{X}}_i$. Further, $\text{int}\tilde{\mathcal{X}}_i \cap \bar{\mathcal{X}}_i^j = \emptyset$. Because of $\bigcup_{\forall j \in \mathcal{I}_N, j \neq i} \bar{\mathcal{X}}_i^j = \bar{\mathcal{X}}_i \cap \mathcal{X}, \text{int}\tilde{\mathcal{X}}_i \cap \partial \bar{\mathcal{X}}_i = \emptyset$. According to (15b), $v_i^{che} \in \text{int}\mathcal{X}_i \subset \bar{\mathcal{X}}_i$ and $v_i^{che} \in \tilde{\mathcal{X}}_i$. Based two sets relationship in Lemma 1, $\tilde{\mathcal{X}}_i \subset \bar{\mathcal{X}}_i$ holds.

Secondly, we certify the partition $\{\tilde{\mathcal{X}}_1, \dots, \tilde{\mathcal{X}}_N\}$ is a polyhedral partition defined over \mathcal{X} . a_i and b_i define a series of linear function over state space \mathcal{X} , thus $\bigcup_{\forall i \in \mathcal{I}_N} \tilde{\mathcal{X}}_i = \mathcal{X}$ holds. Assuming $\exists i, j \in \mathcal{I}_N, i \neq j$ makes $\text{int}\tilde{\mathcal{X}}_i \cap \text{int}\tilde{\mathcal{X}}_j \neq \emptyset$. Based on optimization problem (6), if $x_0 \in \text{int}\tilde{\mathcal{X}}_i \cap \text{int}\tilde{\mathcal{X}}_j$, for all $x \in x_0 \oplus \epsilon_0 \mathbb{B}^d$ with ϵ_0 a small enough positive number, i and j both are the solution of optimization problem (6), which means $i = j$. This conclusion contradicts the assumption. Thus $\text{int}\tilde{\mathcal{X}}_i \cap \text{int}\tilde{\mathcal{X}}_j = \emptyset, \forall i, j \in \mathcal{I}_N, i \neq j$ hold, the partition $\{\tilde{\mathcal{X}}_1, \dots, \tilde{\mathcal{X}}_N\}$ is a polyhedral partition.

Combining with Definition 4, the partition $\{\tilde{\mathcal{X}}_1, \dots, \tilde{\mathcal{X}}_N\}$ is an APPP of $\{\mathcal{X}_i\}_{\mathcal{I}_N}$. \square

Remark 5. Obviously, with the parameters provided in Proposition 2, a PWA function

$$l(x) = a_i^T x + b_i, \forall x \in \tilde{\mathcal{X}}_i, i \in \mathcal{I}_N$$

is a convex lifting of $\{\tilde{\mathcal{X}}_i\}_{\mathcal{I}_N}$. Meanwhile, the new polyhedral partition $\{\mathcal{X}_i\}_{\mathcal{I}_N}$ inscribed by a_i and b_i is a feasible perturbation case for problem (4).

If we define a series new sets $\Phi_i, i \in \mathcal{I}_N$ as:

$$\Phi_i = \{(a_i, b_i) | \text{s.t. (7)}\},$$

each pairs $(a_i, b_i) \in \Phi_i$ define a new PWA control law $\tilde{\kappa}_{pwa}(x)$ for system (1). It means that Φ_i is a feasible set of the variables a_i and b_i establishing a suitable PWA control law for problem (4). For constraints (7) and the relevant feasible set Φ_i , here lists some properties of them:

- 1) There exists some redundant constraints in (7) to cause a high conservativeness result in the designing process for $\{\tilde{\mathcal{X}}_1, \dots, \tilde{\mathcal{X}}_N\}$;

2) Feasible set Φ_i may be not full dimensional, which means for $(a_i, b_i) \in \Phi_i$, $\exists \epsilon_0 > 0$, makes $(a_i, b_i) \oplus \epsilon_0 \mathbb{B}^{n_x+1} \subset \Phi_i$.

Figure 1 illustrates Proposition 2 and its features. In subfigure (a), $v := v_{2,3}^2 = v_{2,3}^3$ with $v_{2,3}^2 \in \mathcal{V}(\bar{\mathcal{X}}_2^2)$ and $v_{2,3}^3 \in \mathcal{V}(\bar{\mathcal{X}}_3^3)$. Based on the Proposition 2, two constraints can be established and be deduced:

$$\left. \begin{aligned} a_2^T v + b_2 &\leq a_3^T v + b_3, \\ a_3^T v + b_3 &\leq a_2^T v + b_2, \end{aligned} \right\} \Rightarrow a_2^T v + b_2 = a_3^T v + b_3.$$

The result means $v \in \tilde{\mathcal{X}}_2 \cap \tilde{\mathcal{X}}_3$. It is intuitive that there exists $\tilde{\mathcal{X}}_{2,0}$ and $\tilde{\mathcal{X}}_{3,0}$ satisfy $\tilde{\mathcal{X}}_{2,0} \subset \bar{\mathcal{X}}_2$, $\tilde{\mathcal{X}}_{3,0} \subset \bar{\mathcal{X}}_3$, and $v \in \tilde{\mathcal{X}}_{2,0} \cap \tilde{\mathcal{X}}_{3,0}$. The constraints based on point v increase the conservativeness of the perturbation. Actually, they can be removed from the constraints without affecting the effectiveness of the Proposition 2.

In subfigure (b), one part of region \mathcal{X}_2 is the boundary of its MAS $\bar{\mathcal{X}}_2$, which led to the freedom of (a_2, b_2) is limited.

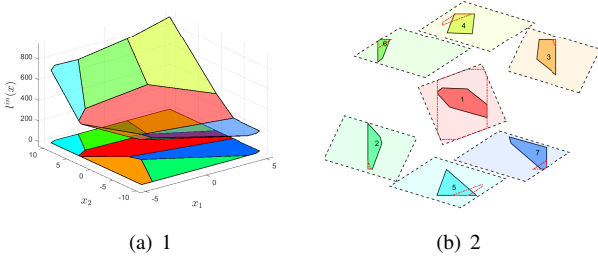


Fig. 1. A perturbation hyperplane in its fragility margin.

B. ROBUST PWA CONTROL BASED ON CONVEX LIFTING

In the following part, a robust PWA control law is established.

Corollary 1. Consider system (1) with a polyhedral partition $\{\mathcal{X}_i\}_{\mathcal{I}_N}$. A convex lifting of $\{\mathcal{X}_i\}_{\mathcal{I}_N}$ is denoted as

$$l(x) = a_i^T x + b_i, \forall x \in \mathcal{X}_i$$

with $a_i \in \mathbb{R}^{n_x}$, $b_i \in \mathbb{R}$. The following statements hold true:

- $\mathcal{X}_i = \{x \in \mathcal{X} | a_i^T x + b_i \geq a_j^T x + b_j, \forall j \in \mathcal{I}_N, j \neq i\}$;
- If a_i and b_i is perturbed to new values: $a_i \rightarrow \tilde{a}_i$ and $b_i \rightarrow \tilde{b}_i$, a relevant PWA function $\tilde{l}(x) = \tilde{a}_i^T x + \tilde{b}_i, \forall x \in \mathcal{X}_i$ is a convex lifting of a new polyhedral partition $\{\tilde{\mathcal{X}}_i\}_{\mathcal{I}_N}$ with

$$\tilde{\mathcal{X}}_i = \{x \in \mathcal{X} | \tilde{a}_i^T x + \tilde{b}_i \geq \tilde{a}_j^T x + \tilde{b}_j, \forall j \in \mathcal{I}_N, j \neq i\}.$$

Definition 6. Consider system (1) with a PWA control law (2) over a polyhedral partition $\{\mathcal{X}_i\}_{\mathcal{I}_N}$. A PWA control law is represented as

$$\tilde{\kappa}_{pwa}(x) = F_i x + g_i, i = \arg \max_{r \in \mathcal{I}_N} \tilde{l}(x) \quad (8)$$

with $\tilde{l}(x)$ a convex lifting of $\{\tilde{\mathcal{X}}_i\}_{\mathcal{I}_N}$:

$$\begin{aligned} \tilde{l}(x) &= \tilde{a}_i^T x + \tilde{b}_i, \forall x \in \tilde{\mathcal{X}}_i, \\ \tilde{\mathcal{X}}_i &= \{x \in \mathcal{X} | \tilde{a}_i^T x + \tilde{b}_i \geq \tilde{a}_j^T x + \tilde{b}_j, \forall j \in \mathcal{I}_N, j \neq i\}. \end{aligned}$$

$\tilde{\kappa}_{pwa}(x)$ is robust if $\forall \tilde{a}_i \in \Phi_i^a \subset \mathbb{R}^{n_x}, \tilde{b}_i \in \Phi_i^b \subset \mathbb{R}, i \in \mathcal{I}_N$, and for all $x \in \mathcal{X}$, there is

$$A + B\tilde{\kappa}_{pwa}(x) \in \mathcal{X}.$$

Meanwhile, the pair of set (Φ^a, Φ^b) with

$$\Phi^a = \Phi_1^a \times \Phi_2^a \times \dots \times \Phi_i^a \dots \times \Phi_N^a, \quad (9a)$$

$$\Phi^b = \Phi_1^b \times \Phi_2^b \times \dots \times \Phi_i^b \dots \times \Phi_N^b \quad (9b)$$

is called the fragility of convex lifting.

Remark 6. If $\Phi_i^a = \{a_i\}$ and $\Phi_i^b = \{b_i\}$ for all $i \in \mathcal{I}_N$ with a_i and b_i known constants, robust PWA control law (8) degrades to a normal PWA control law as (3). A FCL, as a criterion to reflect the robustness of a control, is expected to be intuitive and large.

Explicit FCL and Pre-processing

An explicit formulation for FCL Φ_a and Φ_b is expired:

$$\Phi_i^a = a_i \oplus \delta_i^a \mathbb{B}^{n_x}, \Phi_i^b = b_i \oplus \delta_i^b \mathbb{B} \quad (10)$$

with variables $a_i \in \mathbb{R}^{n_x}$, δ_i^a, b_i , and $\delta_i^b \in \mathbb{R}$. This form is intuitive, and we only need to construct a feasible problem to maximum the size of these sets.

Based on Proposition 2 and the relevant analysis, we know that constraints (7) ensure the feasible set for FCL (Φ^a, Φ^b) . The special form of Φ_i^a and Φ_i^b should be the subset of the feasible set:

$$\Phi_i^a \times \Phi_i^b \subset \Phi_i.$$

A pre-processing should be adapted in order to get an FCL in form (10) based on the features of feasible set Φ_i .

First step: for system (1), a PWA control law (2), a polyhedral partition $\{\mathcal{X}_i\}_{\mathcal{I}_N}$ and its MARC $\{\bar{\mathcal{X}}_i\}_{\mathcal{I}_N}$, we initialize an auxiliary APPP $\{\mathcal{X}_i^o\}_{\mathcal{I}_N}$ of $\{\mathcal{X}_i\}_{\mathcal{I}_N}$ satisfying:

$$\exists \epsilon > 0, (\mathcal{X}_i^o \oplus \epsilon \mathbb{B}^{n_x}) \cap \mathcal{X} \subset \bar{\mathcal{X}}_i \cap \mathcal{X}. \quad (11)$$

In this way, an FCL in formulation (12) is possible.

Second step: in constraints (7), we replace set \mathcal{X}_i with new auxiliary set \mathcal{X}_i^o , and remove the redundant constraints. In this way, an FCL with low conservativeness is available.

Assumption 2. After two steps processing, an FCL in formulation (12) with low conservativeness is available.

One Proposition is proposed to construct an FCL with formulation (12) under Assumption 2.

Proposition for constructing FCL

Proposition 3. Consider system (1) with PWA control law (2), a polyhedral partition $\{\mathcal{X}_i\}_{\mathcal{I}_N}$, MARC $\{\tilde{\mathcal{X}}_i\}_{\mathcal{I}_N}$, and an auxiliary APPP $\{\mathcal{X}_i^o\}_{\mathcal{I}_N}$. The pair of sets (Φ^a, Φ^b) with the formulation (9) and (10) are an FCL if variables $a_i, \delta_i^a, b_i,$ and δ_i^b satisfy:

$$\begin{aligned} \bar{\phi}(v_i^j, a_i, \delta_i^a) + b_i + \bar{b}_i &\leq \underline{\phi}(v_i^j, a_j, \delta_j^a) + b_j + \underline{b}_j, & (12a) \\ \underline{\phi}(v_i^{che}, a_i, \delta_i^a) + b_i + \underline{b}_i &\geq \bar{\phi}(v_i^{che}, a_j, \delta_j^a) + b_j + \bar{b}_j + \epsilon, & (12b) \end{aligned}$$

where $\forall v_i^j \in \mathcal{V}(\bar{\mathcal{X}}_i^{o,j}) \setminus \mathcal{V}_i^{o,j}$ with $\bar{\mathcal{X}}_i^{o,j} := \partial(\bar{\mathcal{X}}_i \cap \mathcal{X}) \cap \mathcal{X}_j^o$ and $\mathcal{V}_i^{o,j}$ being the redundant points set of $\bar{\mathcal{X}}_i^{o,j}$, v_i^{che} is the chebyshev center of \mathcal{X}_i^o , and ϵ is a known small enough positive constant. Besides, $\bar{\phi}$ and $\underline{\phi}$ represent two function mapping:

$$\bar{\phi}, \underline{\phi}: \mathbb{R}^d \times \mathbb{R}^d \times \mathbb{R}_+ \rightarrow \mathbb{R}, \quad (13a)$$

$$\bar{\phi}(x, y, z) = x^T y + \sum_{i=1}^d \max\{-zx(i), zx(i)\}, \quad (13b)$$

$$\underline{\phi}(x, y, z) = x^T y + \sum_{i=1}^d \min\{-zx(i), zx(i)\}. \quad (13c)$$

Proof. On the basis of Definitions 4 and 6, if we can evidence that for a random series parameters $\tilde{a}_i \in \Phi_i^a, \tilde{b}_i \in \Phi_i^b, \forall i \in \mathcal{I}_N$, there is a relevant APPP $\{\tilde{\mathcal{X}}_i\}_{\mathcal{I}_N}$ of $\{\mathcal{X}_i\}_{\mathcal{I}_N}$ guaranteeing that $\forall x \in \tilde{\mathcal{X}}_i, i$ is the solution of optimization problem (6), and then (Φ^a, Φ^b) is an FCL.

Apparently, v_i^j and v_i^{che} are known vectors. For random variables $\tilde{a}_i \in \Phi_i^a, \tilde{b}_i \in \Phi_i^b, \forall i \in \mathcal{I}_N$, there are:

$$\tilde{a}_i^T v + \tilde{b}_i \leq \bar{\phi}(v, a_i, \delta_i^a) + b_i + \bar{b}_i, \quad (14a)$$

$$\tilde{a}_i^T v + \tilde{b}_i \geq \underline{\phi}(v, a_i, \delta_i^a) + b_i + \underline{b}_i \quad (14b)$$

with $v \in \{\bar{v}_i^j, v_i^{che}\}$. Combining (12) and (14), we induce that

$$\tilde{a}_i^T v_i^j + \tilde{b}_i \leq \tilde{a}_j^T v_i^j + \tilde{b}_j, \quad (15a)$$

$$\tilde{a}_i^T v_i^{che} + \tilde{b}_i \geq \tilde{a}_j^T v_i^{che} + \tilde{b}_j + \epsilon. \quad (15b)$$

We defined a new polyhedron as:

$$\tilde{\mathcal{X}}_i := \{x \in \mathcal{X} | \tilde{a}_i^T x + \tilde{b}_i \geq \tilde{a}_j^T x + \tilde{b}_j, \forall j \in \mathcal{I}_N \text{ and } j \neq i\}.$$

In this way, i is the solution of an optimization problem:

$$\max_{r \in \mathcal{I}_N} \tilde{a}_r^T x + \tilde{b}_r, \text{ s. t. } x \in \mathcal{X}$$

with $(\tilde{a}_i, \tilde{b}_i)$ satisfying constraints (15). So, combining Proposition 2, the sets collection $\{\tilde{\mathcal{X}}_1, \dots, \tilde{\mathcal{X}}_N\}$ is an APPP of $\{\mathcal{X}_i\}_{\mathcal{I}_N}$. Thus, (Φ^a, Φ^b) is an FCL. \square

Remark 7. Proposition 3 provides a feasible domain for an FCL with formulation (9) and (10).

Optimization problem and algorithm

In general, on the one hand, we expect maximum the volume of the FCL. On the other hand, we hope minimum the difference between the original polyhedral partition $\{\mathcal{X}_i\}_{\mathcal{I}_N}$ and its APPP $\{\tilde{\mathcal{X}}_i\}_{\mathcal{I}_N}$. An optimization problem is constructed to satisfy these objectives.

$$\begin{aligned} \min_{a_i, \delta_i^a, b_i, \delta_i^b} & \sum_{i=1}^N \|a_i - a_{i,0}\| + \|b_i - b_{i,0}\| - \delta_i^a - \delta_i^b, & (16) \\ \text{s.t.} & \delta_i^a \geq \epsilon, \delta_i^b \geq \epsilon, & (12), \end{aligned}$$

where $a_{i,0}$ and $b_{i,0}, \forall i \in \mathcal{I}_N$ are the parameters of a convex lifting for polyhedral partition $\{\mathcal{X}_i\}_{\mathcal{I}_N}$. Using items $\|a_i - a_{i,0}\|$ and $\|b_i - b_{i,0}\|$, we can minimize the difference between $\{\mathcal{X}_i\}_{\mathcal{I}_N}$ and its APPP. Meanwhile, the value of δ_i^a and δ_i^b represent the volume of Φ_i^a : the larger former the larger later.

An Algorithm 1 demonstrates the whole constructing process of a robust PWA control law for a constrained linear discrete-time system as (1).

Algorithm 1 Robust PWA control law and relevant FCL

Require: System (1), a PWA control law (2), relevant polyhedral partitions $\{\mathcal{X}_i\}_{\mathcal{I}_N}$ over the state space \mathcal{X} ;

Ensure: (Φ_a, Φ_b) and $\tilde{\kappa}_{pwa}(x)$;

- 1: Construction a convex lifting $l(x) = a_{i,0}^T + b_{i,0}$ and the MARC $\{\tilde{\mathcal{X}}_i\}_{\mathcal{I}_N}$ for $\{\mathcal{X}_i\}_{\mathcal{I}_N}$ based on Proposition 1;
 - 2: Pre-processing $\{\mathcal{X}_i\}_{\mathcal{I}_N}$ and $\{\tilde{\mathcal{X}}_i\}_{\mathcal{I}_N}$ to get an APPP $\{\mathcal{X}_i^o\}_{\mathcal{I}_N}$ satisfying (11) and identify the redundant points set $\mathcal{V}_i^{o,j}$ mentioned in Proposition 3;
 - 3: Establishing an optimization problem (16), and getting the optimization solution $a_i, \delta_i^a, b_i, \delta_i^b$ for all $i \in \mathcal{I}_N$;
 - 4: The FCL (Φ_a, Φ_b) and a robust control law $\tilde{\kappa}_{pwa}(x)$ based on Proposition 2.
-

V. NUMERICAL EXAMPLE

Consider a two order linear discrete-time system:

$$x_{k+1} = Ax_k + Bu_k \quad (17)$$

with the states and inputs constraints

$$-5 \leq [1 \ 0]x_k \leq 5, \|u_k\|_\infty \leq 5, \forall k \geq \mathbb{N}.$$

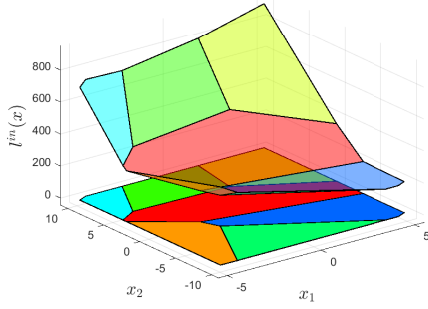
The system parameter matrices are:

$$A = \begin{bmatrix} 0.3 & 0.7 \\ -1.2 & 0.4 \end{bmatrix}, B = \begin{bmatrix} 0.85 \\ -0.75 \end{bmatrix}.$$

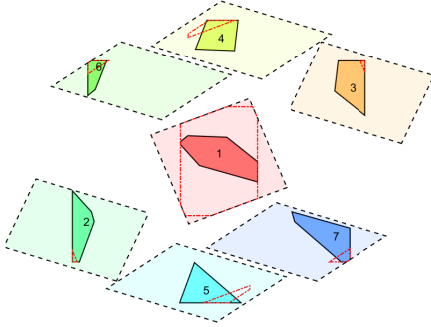
A PWA controller $u_k = \kappa_{pwa}(x_k)$ with a formulation (3) has been obtained in an explicit form (MPT 3.0 toolbox [14]). The necessary parameters are denoted as $Q = \mathbb{I}_2, R = 1$, a prediction step 2, an LQR terminal set and a terminal state cost

$$P = \begin{bmatrix} 2.3164 & -0.6129 \\ -0.6129 & 1.9116 \end{bmatrix}.$$

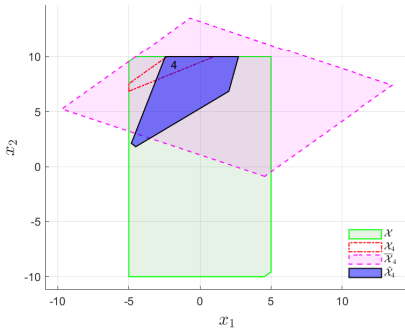
For system (17), the parameters $F_i \in \mathbb{R}^2$ and $g_i \in \mathbb{R}$ for all $i \in \mathcal{I}_7$ of $\kappa_{pwa}(x_k)$ are listed in TABLE I.



(a) 1



(b) 2



(c) 2

Fig. 2. A perturbation hyperplane in its fragility margin.

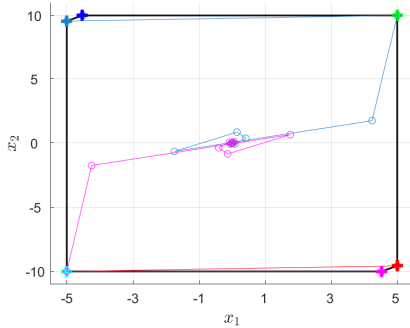


Fig. 3. A perturbation hyperplane in its fragility margin.

In order to illustrate the effectiveness of the proposed

fragility of convex lifting, two simulation scenarios were shown:

- **Scenario 1:** the fragility of convex lifting (Φ^a, Φ^b) was calculated, and the controversy of original convex lifting with a random case in the fragility of convex lifting $\tilde{a} \in \Phi^a, \tilde{b} \in \Phi^b$ were shown;
- **Scenario 2:** with initial state $x_0 \in \mathcal{V}(\mathcal{X})$, the state trajectories were exhibited.

Scenario 1: Under the process of Algorithm 1, the parameters of the fragility of convex lifting (Φ^a, Φ^b) for system (17) is provided as

$$\begin{aligned}\Phi^a &= \Phi_1^a \times \Phi_2^a \times \cdots \times \Phi_7^a, \\ \Phi^b &= \Phi_1^b \times \Phi_2^b \times \cdots \times \Phi_7^b\end{aligned}$$

with

$$\Phi_i^a = a_i \oplus \Xi \mathbb{B}^2 \subset \mathbb{R}^2, \Phi_i^b = b_i \oplus \Xi \mathbb{B} \subset \mathbb{R},$$

where $\Xi = 0.0235$, a_i , and b_i are noted in TABLE I. We set two different convex lifting for system (17) to show the effectiveness of (Φ^a, Φ^b). These convex lifting are denoted as:

$$l^{in}(x) = a_i^{in} x + b_i^{in}, l^{out}(x) = a_i^{out} x + b_i^{out}, i \in \mathcal{I}_7$$

with $a_i^{in} = a_i + [1, 1]^T \Xi \in \Phi_i^a$, $b_i^{in} = b_i - \Xi \in \Phi_i^b$, $a_i^{out} \notin \Phi_i^a$, and $b_i^{out} \notin \Phi_i^b$.

Figure 1a demonstrates the convex lifting $l^{in}(x)$ and its relevant polyhedral partition $\{\mathcal{X}_i^{in}\}_{\mathcal{I}_7}$ over the state space \mathcal{X} . Figure 1b shows the relationship among the original polyhedral partition $\{\mathcal{X}_i\}_{\mathcal{I}_7}$, the MARC $\{\bar{\mathcal{X}}_i\}_{\mathcal{I}_7}$, and $\{\mathcal{X}_i^{in}\}_{\mathcal{I}_7}$. Apparently, $\mathcal{X}_i^{in} \subset \bar{\mathcal{X}}_i$. Figure 1c exhibits that if $a_i^{out} \notin \Phi_i^a$ and $b_i^{out} \notin \Phi_i^b$, $\exists i \in \mathcal{I}_7$ causes $\mathcal{X}_i^{out} \subset \bar{\mathcal{X}}_i$, where \mathcal{X}_i^{out} is one partition of the polyhedral partition $\{\mathcal{X}_i^{out}\}_{\mathcal{I}_7}$ of $l^{out}(x)$. It means the invariance property is unsatisfied.

Scenario 2: Under the given convex lifting $l^{in}(x)$, the current PWA controller for system (17) was constructed as:

$$\kappa_{pwa}(x) = F_i x + g_i, i = \arg \max_{r \in \mathcal{I}_N} (a_r^{in})^T x + b_r^{in}.$$

Assuming the initial state x_0 traverses the points set $\mathcal{V}(\mathcal{X})$, and Figure 2 illustrates the relevant trajectories from time instant 0 to 1000. Intuitively, all the state stay inside of \mathcal{X} , which means the invariance of \mathcal{X} is satisfied under $\kappa_{pwa}(x)$ with a perturbation convex lifting $l^{in}(x)$.

VI. CONCLUSIONS

A convex lifting based approach is verified effective to deal with the perturbation of a state space partition with a polyhedron form of a PWA controller for a discrete time linear system. Furthermore, we propose a fragility of convex lifting to guarantee the positive invariance of the robustness of a PWA controller with an implicit judgement condition, which is in the convex lifting form. In next steps, a more effective strategy to get a fragility of convex lifting with less conservativeness should be focused.

TABLE I
PARAMETERS FOR THE FRAGILITY OF CONVEX LIFTING

	1	2	3	4	5	6	7
F_i^T	-0.6786	0	0	-0.3529	-0.3529	-0.3529	-0.3529
	-0.2027	0	0	-0.8235	-0.8235	-0.8235	-0.8235
g_i	0	5.0	-5.0	5.8824	-5.8824	5.8824	-5.8824
a_i	16.7319	-69.0562	31.2889	17.3957	-28.1551	-2.2461	0
	43.9251	-12.0397	53.0654	56.0568	-22.6566	61.9989	0
b_i	75.9569	-191.5256	14.1226	21.8529	-130.1280	-95.4561	0

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