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Subjective Expected Utility on Arbitrary State Spaces*

Thai Ha-Huy [†]

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Abstract

This article considers Savage's theorem in a configuration relaxing the *technical* axioms P6 and P7 that ensure a *continuum* nature on the set of states. With the only enrichment on fundamentals being the connectivity of the outcomes set, we show that a weakened version of the *Independence* property is sufficient to establish a utility function, a subjective probability, and an expected utility behavior. The proof does not require the existence of a pair event, an idea initiated by Ramsey (1931) and applied by Gul (1992).

KEYWORDS: Savage theorem, *sure-thing principle*, Independence axiom, expected utility representation.

JEL CLASSIFICATION: D11, D90.

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1 Introduction

1.1 Motivation and result

After seven decades since the publication of “The foundation of Statistics,” (Savage (1954)) and scholars continue to admire this tour de force, calling it a “crowning glory of decision theory”.¹ Based on a list of seven axioms, this legendary contribution reconciled the ideas of de Finetti (1937) about subjective probability and those of von Neumann and Morgenstein (1947) about expected utility behavior. The proof of Savage does not rely on the mathematical structures of de Finetti or von Neumann - Morgenstein, which are crucial for using the separate theorem in convex analysis.

Besides the famous and perhaps most discussed Savage axiom, the *sure-thing principle*, the structure of Savage’s world also relies on ”technical axioms” *P6* and *P7* which ensure a *continuum* nature on the set of states. This *continuum* property allows Savage to divide the set of states by two equal parts and, then by four, eight, sixteen and so on. We can therefore measure every event by passing through the limit a sequence of events with probabilities in the form $\frac{k}{2^n}$, with $0 \leq k \leq 2^n$.²

Naturally, efforts have been done to extend the result of Savage to configurations that encompass the possibility of *atoms*, events that can not be divided into smaller non-null ones. It is not surprising that we must make a trade-off between the richness of the set of states and the set of outcomes. The existence of atoms is problematic.

The seminal contribution of Gul (1992) proposes an idea initiated by Ramsey (1931), assuming the existence of a *pair event*, an event that the decision maker believes has the same chance to occur as its complementary one. This choice is clearly

¹See Kreps (1988).

²The proof of Savage is long and complicated. For a more simple proof and better explanations, see Gilboa (2009), chapter 10, and Abdellaoui and Wakker (2020).

intuitive: if a subjective probability exists, the probability of this event should be $\frac{1}{2}$. Gul (1992) followed the approach by Ramsey (1931) rather than Savage and defines the probability of an event through a utility function. Instead of the *sure-thing principle*, Gul (1992) assumed an *Independence* property, which states that combining two acts with a third one using a non-null event does not alter the comparison between them.

This article is a study of Savage's theorem in a configuration where the *continuum* property of the set of states may not be satisfied. We proposed a weaker condition used in the literature with three considerations. First, there is no restriction on the set of states. It may be finite or infinite, contain atoms or not, countable or continuum. Second, we present a weakened version of *Independence* axiom, requiring its verification with *one* special non-null event. Third, we do not assume the existence of a pair event.

The main idea of our approach is as follows. Given every two outcomes x and y , using the event in *Independence* axiom, we construct an outcome that can be considered their *midpoint*, or equivalently, an outcome that represents the an equi-probability of $\frac{1}{2}$ to x and y . Following the construction midpoints between x , y and their midpoint, and so on, we can find the equivalent outcome of every distribution that attributes a probability of the form $\frac{k}{2^n}$ to x and $1 - \frac{k}{2^n}$ to y . For a given event, a lottery that wins x if that event occurs and y if not, is equivalent to the limit of a sequence of midpoints. Hence, the probability of that event can be determined. We can then prove the existence of an expected utility representation of the comparison between acts.

1.2 Related literature

For a finite number of states, [Kraft et al. \(1959\)](#) and [Scott \(1964\)](#) considered *cancellation* as a necessary and sufficient condition for the existence of a probability measure. [Kraft et al. \(1959\)](#) also gave a counterexample to prove that the additivity is not strong enough for positive answer when the number of states is bigger than or equal to 5.

For an infinite number of states, [Chateauneuf and Jaffray \(1984\)](#) and [Chateauneuf \(1985\)](#) considered the problem under the *Archimedean* property and prove that this condition is sufficient for the establishment of a probability measure. The curious readers can refer to the excellent reviews of [Fishburn \(1986, 1989\)](#) and [Mackenzie \(2019\)](#).

The article by [Gilboa and Schmeidler \(1989\)](#), a “second Big Bang after [Savage \(1954\)](#),”³ opens a huge body of literature for the configurations where Savage’s famous *sure-thing principle* is not satisfied, and paves the way for numerous contemporaneous developments of the *ambiguity* literature. For a survey, see [Etner et al. \(2012\)](#).

To the best of our knowledge, the idea of establishing a midpoint without passing through a pair event appeared first in [Ghirardato et al. \(2003\)](#) and is in application after by [Ghirardato and Pennesi \(2020\)](#). In these two contributions, the authors used an *essential event* to determine a midpoint of two given outcomes, which they called a *preference average*. Applying this idea to an order that represents *biseparable references*, the midpoints serve to construct probability mixtures of acts. They can therefore enjoy a rich structure similarly to [Anscombe and Aumann \(1962\)](#) and [Gilboa and Schmeidler \(1989\)](#).

³See [Karni et al. \(2022\)](#).

Without relying on the *continuum* nature of the set of states neither of the set of outcomes, [Mackenzie \(2019, 2020\)](#) studied different *atoms swarming* properties to establish probabilistic beliefs. His results did not require the usual conditions such as no atoms, cancellation, or solvability. In our opinion, *atoms swarming* conditions may be considered a beautiful bridge linking the configurations with discrete and continuum states.

Adding a continuity property may allow us to change some conditions' statements from "for every" to "exists," as shown in the weakened *Independence* property of this article. An interesting discussion about this situation may be found in the article by [Segal \(2023\)](#).

1.3 Organization

The article is organized as follows. Section 2 presents the fundamental notions. Section 3 describes the establishment of the subjective probability and expected utility behavior. All proofs are given in the Appendix.

2 Fundamentals

Let S be the set of states and \mathcal{A} an algebra of events on S . The set S can be a continuum, discrete, atomless, or hybrid type which contains continuum subsets and atoms. For each event $A \in \mathcal{A}$, let A^c be its complementary event: $A^c = S \setminus A$.

The set of outcome X is endowed with a topology τ . Denote by \mathcal{F}_0 the set of measurable *finite-value* acts from S with algebra \mathcal{A} to X .

$$\mathcal{F}_0 = \{f : S \rightarrow X \text{ such that } f \text{ is measurable and } f(S) \text{ is finite}\}.$$

For a measurable partition A_1, A_2, \dots, A_n of S , outcomes x_1, x_2, \dots, x_n in X , denote by $x_{1,A_1}x_{2,A_2} \dots x_{n,A_n}$ the act that obtains outcome x_k for $s \in A_k$. For example, $x_A y_{A^c}$ denotes the act that takes value x if $s \in A$ and value y otherwise. In the same spirit, for every acts f and g , $f_A g_{A^c}$ denotes the act that is equal to $f(s)$ on A and $g(s)$ on A^c .

Let \mathcal{P}_0 be the set of finite support probability distributions on X . For p_1, p_2, \dots, p_n belonging to the interval $[0, 1]$ such that $\sum_{k=1}^n p_k = 1$ and $x_1, x_2, \dots, x_n \in X$, let $(p_1 : x_1, p_2 : x_2, \dots, p_n : x_n)$ be the random distribution on X which takes value x_k with probability p_k .

The comparison between acts is defined by an order \succeq . The strict and equivalent comparisons are denoted by \succ and \sim , respectively. Before establishing some fundamental properties on order \succeq , let us define some notations. An event A is called a null-event if for every outcomes x and y , act h , we have $x_A h_{A^c} \sim y_A h_{A^c}$. The outcomes set X can be considered the set of constant acts and a subset of \mathcal{F}_0 .

For a given sequence of outcomes $\{x_n\}_{n=0}^\infty$, we use the notion convergence in \succeq to an outcome x , in the sense that for every $y \succ x \succ z$, there exists N such that for $n \geq N$, $y \succ x_n \succ z$. We use this notion to distinguish the convergence with respect to the topology τ of X . Throughout this work, when we say that some sequence converges to some outcome, we always refer to the convergence in \succeq .

Axiom F1. Fundamental

- i) Completeness, transitivity, monotonicity and non-triviality *For every $f, g \in \mathcal{F}_0$, either $f \succeq g$ or $g \succeq f$. If $f \succeq g$ and $g \succeq h$, then $f \succeq h$. For every non-null even A*

$$x \succeq y \text{ if and only if } x_A g_{A^c} \succeq y_A g_{A^c}.$$

There exist $\bar{z}, \underline{z} \in X$ such that $\bar{z} \succ \underline{z}$.

ii) Weak comparative probability For any $A, B \in \mathcal{A}$ and $x \succ y, x' \succ y'$,

$$x_A y_{A^c} \succeq x_B y_{B^c} \text{ if and only if } x'_A y'_{A^c} \succeq x'_B y'_{B^c}.$$

iii) Continuity For any $x \in X$, the sets $\{y \in X \text{ such that } y \succeq x\}$ and $\{y \in X \text{ such that } x \succeq y\}$ are closed with respect to the τ -topology. Moreover, the space (X, τ) is connected and separable.⁴

Conditions (i) and (ii) are standard in literature and are presented in the contribution by Savage (1954). Condition (iii) assumes a *continuum* property on the set of outcomes.

Thanks to *Weak comparative probability* property, we can define an order on the set of events as follows, $A \succeq_\ell B$ if there exists $x \succ y$ such that

$$x_A y_{A^c} \succeq x_B y_{B^c}.$$

The comparison using \succeq_ℓ does not depend on the choice of outcomes x and y .⁵

3 Expected utility representation

3.1 Independence axiom and the determination of the midpoint

The first purpose of this section is to provide a *mixture* of acts that replaces the lack of linear structure being in use in Anscombe and Aumann (1962), and Gilboa and Schmeidler (1989).

⁴We can not split X into two disjoint closed subsets, and there exists a countable and dense subset of X .

⁵See Gilboa (2009), Chapter 10.

Definition 1. For every acts $f, g \in \mathcal{F}_0$, a non-null event H , define the mixture of f and g through H an act \tilde{f} satisfying: for any $s \in S$,

$$\tilde{f}(s) \sim f(s)_H h(s)_{H^c}.$$

We denote by $Hf + H^c g$ a mixing act of f and g through H . One must avoid confusing $Hf + H^c g$ with $f_H g_{H^c}$. The former can be considered as a *convex combination* act of f and g with weighted parameters defined using the event set H , while the latter is an act equal to f on H and to g on H^c . From now on, we always assume axioms *Fundamental* and *Independence* on the order \succeq .

Axiom A1. Independence *There exists an event H such that both H and H^c are non-null and for all acts $f, g, h \in \mathcal{F}_0$,*

$$f \succeq g \text{ if and only if } Hf + H^c h \succeq Hg + H^c h.$$

Mixing f and g with h using event H does not change the comparison between f and g . The intuition is clear once a probability measure μ on the set of events is established. The mixing act between f and g through H is the act $\mu(H)f + \mu(H^c)g$ that corresponds each state s to a lottery that gives $f(s)$ with probability $\mu(H)$ and $g(s)$ with the probability $1 - \mu(H)$. The *independence* axiom condition therefore requires that $f \succeq g$ if and only if $\mu(H)f + \mu(H^c)h \succeq \mu(H)h + \mu(H^c)h$. A more detailed intuition is presented in Gul (1992). It is worth noting that here we require *Independence* for only one event H , instead of every non-null event.

The relation between the *independence* axiom and the *sure-thing principle* is an important question. In Gul (1992), where the states space S is finite, *independence* implies the *sure-thing principle*. Proposition 1 states the same conclusion for a general set of states S .

Proposition 1. *The sure-thing principle is satisfied. For every non-null event A , acts f, g, h, \tilde{h} ,*

$$f_A h_{A^c} \succeq g_A h_{A^c} \text{ if and only if } f_A \tilde{h}_{A^c} \succeq g_A \tilde{h}_{A^c}.$$

Under *Independence*, using event H , we can determine the midpoint of two outcomes x and y , which can be considered an equivalent outcome of uniform distribution. This allows us to overcome the lack of a pair event, as proposed in [Ramsey \(1931\)](#), and used by [Gul \(1992\)](#) in a configuration where the set S is finite.

Lemma 1. *For events x and y , define sequences $\{x_n\}_{n=0}^\infty$ and $\{y_n\}_{n=0}^\infty$ as follows. Let $x_0 = x$ and $y_0 = y$. For every $n \geq 0$, let*

$$x_{n+1} \sim x_{n,H} y_{n,H^c} \text{ and } y_{n+1} \sim y_{n,H} x_{n,H^c}.$$

Then both $\{x_n\}_{n=0}^\infty$ and $\{y_n\}_{n=0}^\infty$ converge to an outcome z that we denote as

$$z \sim \frac{1}{2}x \oplus \frac{1}{2}y.$$

Let us present some intuition for the definition of the midpoint and the result of Lemma 1. Assume for an instance that a subjective probability has been established. Let p be the probability of event H . For every n , outcomes x_n and y_n are equivalent correspondingly to distributions $(p_n : x, (1 - p_n) : y)$ and $((1 - q_n) : x, q_n : y)$, with $p_0 = 1, q_0 = 0$ and for $n \geq 0$,

$$p_{n+1} = p \times p_n + (1 - p) \times q_n,$$

$$q_{n+1} = (1 - p) \times p_n + p \times q_n.$$

Assume that $x \succeq y$ and $H \succeq_\ell H^c$. By induction, $x_n \succeq y_n, p_n \geq q_n$ and $p_n + q_n = 1$ for every n . Since $\{p_n\}_{n=0}^\infty$ is decreasing and $\{q_n\}_{n=0}^\infty$ is increasing, they converge respectively to p^* and q^* and $p^* \geq q^*$. This implies $p^* = q^* = \frac{1}{2}$. Hence, the

sequences $\{x_n\}_{n=0}^{\infty}$ and $\{y_n\}_{n=0}^{\infty}$ have the same limit, that is an equivalent outcome of distribution $(\frac{1}{2} : x, \frac{1}{2} : y)$ and this outcome can be considered a midpoint of x and y . The cases $H^c \succeq_{\ell} H$ or $y \succeq x$ are similar.

We can therefore work with distributions of the form $(\frac{1}{2} : x, \frac{1}{2} : y)$ without relying on a pair event set. Following [Debreu \(2015\)](#),⁶ Theorem 1, chapter 9, as [Gul \(1992\)](#), we obtain a utility function u that represents the comparison between these special distributions.

Lemma 2. *There exists a utility function u such that for every $x, y, x', y' \in X$,*

$$\frac{1}{2}x \oplus \frac{1}{2}y \succeq \frac{1}{2}x' \oplus \frac{1}{2}y' \text{ if and only if } \frac{1}{2}(u(x) + u(y)) \geq \frac{1}{2}(u(x') + u(y')).$$

The function u is unique up to a strictly increasing transformation.

3.2 Expected utility representation

We use midpoint z as an equivalent outcome of distribution $(\frac{1}{2} : x, \frac{1}{2} : y)$. The construction of equivalence of distributions on the set $\{x, y\}$ is intuitive. Using $\frac{1}{2}x \oplus \frac{1}{2}z$, we obtain an equivalent outcome of the distribution $(\frac{3}{4} : x, \frac{1}{4} : y)$. Similarly, $\frac{1}{2}z \oplus \frac{1}{2}y$ represents $(\frac{1}{4} : x, \frac{3}{4} : y)$, and so on. Continuing with this line of reasoning, we can create equivalent representations of every distribution of the form $(\frac{k}{2^n} : x, \frac{2^n-k}{2^n} : y)$, for $0 \leq k \leq 2^n$. Taking the limits when n converges to infinity, we construct the representation of every distribution which takes at most two values: $(p : x, (1-p) : y)$, with $x, y \in X$ and $0 \leq p \leq 1$. Moreover, if z is equivalent to $(p : x, (1-p) : y)$, then $u(z) = pu(x) + (1-p)u(y)$.

⁶This is a posthumous publication of the article “*Topological methods in cardinal utility theory*” by Gerard Debreu, *Mathematical Methods in the Social Sciences*, 1959.

Proposition 2. *There exists a unique probability measure μ and a utility function u such that for every events A, B and outcomes x, y, x', y' , $x_A y_{A^c} \succeq x'_B y'_{B^c}$ is equivalent to*

$$\mu(A)u(x) + (1 - \mu(A))u(y) \geq \mu(B)u(x') + (1 - \mu(B))u(y').$$

Once the subjective probability has been established, an act $f = x_{1,A_1} x_{2,A_2} \dots x_{n,A_n}$, with a partition A_1, A_2, \dots, A_n of S , can be considered an equivalence of the distribution $(\mu(A_1) : x_1, \mu(A_2) : x_2, \dots, \mu(A_n) : x_n)$. Therefore, the comparison of distributions can be realized through equivalent outcomes. We obtain Savage's theorem in a new context without relying on the *continuum* nature of the set of states.

Theorem 1. *There exists unique finitely additive probability measure μ and unique utility function u (up to a strictly increasing affine transformation) such that for every acts f and g belonging to \mathcal{F}_0 :*

$$f \succeq g \text{ if and only if } \int_S u(f(s)) \mu(ds) \geq \int_S u(g(s)) \mu(ds).$$

4 Appendix

4.1 Proof of Proposition 1

Consider a non-null event A . Assume that for acts f, g, h , we have $f_A h_{A^c} \succeq g_A h_{A^c}$.

We want to prove that for every $\tilde{h} \in \mathcal{F}_0$, $f_A \tilde{h}_{A^c} \succeq g_A \tilde{h}_{A^c}$.

First, we prove the following claim: if there is some $\hat{h} \in \mathcal{F}_0$ such that for every $s \in S$,

$$\tilde{h}(s) \sim h(s)_H \hat{h}(s)_{H^c},$$

then $f_A h_{A^c} \succeq g_A h_{A^c}$ if and only if $f_A \tilde{h}_{A^c} \succeq g_A \tilde{h}_{A^c}$.

This claim is a direct consequence of the following two:

i) $f_A h_{A^c} \succeq g_A h_{A^c}$ if and only if

$$f_A \tilde{h}_{A^c} \succeq (Hg + H^c f)_A \tilde{h}_{A^c}.$$

ii) $f_A \tilde{h}_{A^c} \succeq g_A \tilde{h}_{A^c}$ if and only if

$$f_A \tilde{h}_{A^c} \succeq (Hg + H^c f)_A \tilde{h}_{A^c}.$$

First, we prove (i). Indeed, from *Independence* axiom, $f_A h_{A^c} \succeq g_A h_{A^c}$ if and only if

$$H(f_A h_{A^c}) + H^c(f_A \hat{h}_{A^c}) \succeq H(g_A h_{A^c}) + H^c(f_A \hat{h}_{A^c}),$$

which is equivalent to

$$f_A \tilde{h}_{A^c} \succeq (Hg + H^c f)_A \tilde{h}_{A^c}.$$

Now, consider (ii). Using once again the *Independence* axiom, $f_A \tilde{h}_{A^c} \succeq g_A \tilde{h}_{A^c}$ if and only if

$$H(f_A \tilde{h}_{A^c}) + H^c(f_A \tilde{h}_{A^c}) \succeq H(g_A \tilde{h}_{A^c}) + H^c(f_A \tilde{h}_{A^c}),$$

which is equivalent to

$$f_A \tilde{h}_{A^c} \succeq (Hg + H^c f)_A \tilde{h}_{A^c}.$$

Hence $f_A h_{A^c} \succeq g_A h_{A^c}$ if and only if $f_A \tilde{h}_{A^c} \succeq g_A \tilde{h}_{A^c}$. The claim is proven.

Fix outcome x such that $\bar{z} \succ x \succ \underline{z}$. We will prove that

$$f_A h_{A^c} \succeq g_A h_{A^c} \text{ if and only if } f_A x_{A^c} \succeq g_A x_{A^c}.$$

Indeed, consider the sequence of acts $h^0, h^1, \dots, h^n, \dots \in \mathcal{F}_0$ defined as

$$h^0 = h,$$

$$h^{n+1} = Hh^n + H^c x \text{ for } n \geq 0.$$

Using the same arguments as the case $\tilde{h} = Hh + H^c\hat{h}$, we have $f_A h_{A^c} \succeq g_A h_{A^c}$ is equivalent to $f_A h_{A^c}^1 \succeq g_A h_{A^c}^1$, which is equivalent to $f_A h_{A^c}^2 \succeq g_A h_{A^c}^2$ and so on. By induction, for every n , $f_A h_{A^c} \succeq g_A h_{A^c}$ is equivalent to $f_A h_{A^c}^n \succeq g_A h_{A^c}^n$. The sequence of acts $\{h^n\}_{n=0}^\infty$ converges to x , in the sense that for every $y \succ x \succ z$, there exists N such that for $n \geq N$, $s \in S$,

$$y \succ h^n(s) \succ z.$$

Let n be sufficiently big,

$$h^n(s)_{H\bar{z}_{H^c}} \succ x \succ h^n(s)_{\underline{z}_{H^c}},$$

for every $s \in S$. This implies the existence of $h^* \in \mathcal{F}_0$ such that for every $s \in S$,

$$h^n(s)_H h^*(s)_{H^c} \sim x.$$

This is equivalent to $Hh^n + H^c h^* \sim x$. Hence $f_A h_{A^c}^n \succeq g_A h_{A^c}^n$ is equivalent to $f_A x_{A^c} \succeq g_A x_{A^c}$. Applying the same arguments for \tilde{h} , we get $f_A \tilde{h}_{A^c} \succeq g_A \tilde{h}_{A^c}$ if and only if $f_A x_{A^c} \succeq g_A x_{A^c}$.

The *sure-thing principle* is proven.

4.2 Proof of Lemma 1

Consider the case $H \succeq_\ell H^c$ and $x \succeq y$. Then we have

$$x_0 \succeq x_1 \succeq \dots \succeq x_n \succeq \dots \succeq y_n \succeq y_{n-1} \succeq \dots \succeq y_1 \succeq y_0.$$

Let Z be the set of outcomes z such that $x_n \succeq z$ for every $n \geq 0$. This set is closed with respect to the topology τ of X . Let \bar{Z}^c be the close envelope of the open set Z^c . The connectivity of X implies that $Z \cap \bar{Z}^c \neq \emptyset$. Fix $z \in Z \cap \bar{Z}^c$. This outcome is found on the boundaries of Z and Z^c . We will prove that z can be considered

as an infimum of the sequence $\{x_n\}_{n=0}^\infty$, in the sense that for every outcome $y \succ z$, there exists n such that $y \succ x_n \succeq z$. Indeed, the contrary implies that z belongs to the interior of the set Z : a contradiction. Using the same argument, we prove the existence of an outcome z' that can be considered a supremum of the sequence $\{y_n\}_{n=0}^\infty$.

To finish the proof, we must prove $z \sim z'$. Assume the contrary, $z \succ z'$. Then $z \succ z_H z'_{H^c} \succ z'$. The continuous nature of the set of outcomes X ensures the existence of $y \succ z$ and $z' \succ y'$ such that $z \succ y_H y'_{H^c} \succ z'$. For n sufficiently big, $y \succ x_n \succeq z \succ z' \succeq y_n \succ y'$, with a direct consequence that $z \succ x_{n,H} y_{n,H^c} \sim x_{n+1}$: a contradiction. Hence, both $\{x_n\}_{n=0}^\infty$ and $\{y_n\}_{n=0}^\infty$ converge to z .

In the case $H^c \succeq_\ell H$ and $x \succeq y$, we use the same arguments, with the only observation that the sequences $\{x_{2n}\}_{n=0}^\infty$ and $\{y_{2n+1}\}_{n=0}^\infty$ are decreasing, the sequences $\{x_{2n+1}\}_{n=0}^\infty$ and $\{y_{2n}\}_{n=0}^\infty$ are increasing. They converge all to a midpoint of x and y .

The case $y \succeq x$ now becomes trivial. It is obvious that $\frac{1}{2}x \oplus \frac{1}{2}y \sim \frac{1}{2}y \oplus \frac{1}{2}x$ for every x and y .

4.3 Proof of Lemma 2

The proof of Lemma 2 is long and will be presented in two steps.

(1) Step 1: We prove that for every outcomes x, y, x', y' , we have

$$\frac{1}{2} \left(\frac{1}{2}x \oplus \frac{1}{2}y \right) \oplus \frac{1}{2} \left(\frac{1}{2}x' \oplus \frac{1}{2}y' \right) \sim \frac{1}{2} \left(\frac{1}{2}x \oplus \frac{1}{2}x' \right) \oplus \frac{1}{2} \left(\frac{1}{2}y \oplus \frac{1}{2}y' \right). \quad (1)$$

(2) Step 2: We prove that for every outcomes $x_1, x_2, x_3, y_1, y_2, y_3$, if

$$\frac{1}{2}x_2 \oplus \frac{1}{2}y_1 \succeq \frac{1}{2}x_1 \oplus \frac{1}{2}y_2 \text{ and } \frac{1}{2}x_3 \oplus \frac{1}{2}y_2 \succeq \frac{1}{2}x_2 \oplus \frac{1}{2}y_3, \quad (2)$$

then

$$\frac{1}{2}x_3 \oplus \frac{1}{2}y_1 \succeq \frac{1}{2}x_1 \oplus \frac{1}{2}y_3.$$

The proof of Step 1 requires two preparation statements. First, consider x, y, x', y', z, z' such that

$$z \sim x_H y_{H^c} \text{ and } z' \sim x'_H y'_{H^c}.$$

Fix $w, v, t \in X$ satisfying:

$$w \sim x_H x'_{H^c},$$

$$v \sim y_H y'_{H^c},$$

$$t \sim w_H v_{H^c}.$$

Then we have $t \sim z_H z'_{H^c}$.

Indeed, by *Independence* axiom, the mixture of acts $x_H y_{H^c}$ and $x'_H y'_{H^c}$ using H is equivalent to the mixture between z and z' using H . We obtain:

$$\begin{aligned} t &\sim w_H v_{H^c} \\ &\sim H(x_H y_{H^c}) + H^c(x'_H y'_{H^c}) \\ &\sim Hz + H^c z' \\ &\sim z_H z'_{H^c}. \end{aligned}$$

Second statement: consider x, y, x', y', z, z' such that

$$z \sim \frac{1}{2}x \oplus \frac{1}{2}y \text{ and } z' \sim \frac{1}{2}x' \oplus \frac{1}{2}y'.$$

Fix $w, v, t \in X$ satisfying:

$$w \sim x_H x'_{H^c},$$

$$v \sim y_H y'_{H^c},$$

$$t \sim \frac{1}{2}w \oplus \frac{1}{2}v.$$

Then we have $t \sim z_H z'_{H^c}$.

Let $x_0 = x$, $x'_0 = x'$, $y_0 = y$, $y'_0 = y'$, $w_0 = w$, and $v_0 = v$. Consider sequences $\{x_n\}_{n=0}^\infty$, $\{x'_n\}_{n=0}^\infty$, $\{y_n\}_{n=0}^\infty$, $\{y'_n\}_{n=0}^\infty$, $\{w_n\}_{n=0}^\infty$, $\{v_n\}_{n=0}^\infty$ being defined as:

$$\begin{aligned} x_{n+1} &\sim x_{n,H} y_{n,H^c} \text{ and } y_{n+1} \sim y_{n,H} x_{n,H^c}, \\ x'_{n+1} &\sim x'_{n,H} y'_{n,H^c} \text{ and } y'_{n+1} \sim y'_{n,H} x'_{n,H^c}, \\ w_{n+1} &\sim w_{n,H} v_{n,H^c} \text{ and } v_{n+1} \sim v_{n,H} w_{n,H^c}. \end{aligned}$$

Let sequence $\{t_n\}_{n=0}^\infty$ be defined as

$$t_n = w_{n,H} v_{n,H^c},$$

for every n .

Using the first statement, by induction, we have, for every n ,

$$\begin{aligned} w_n &\sim x_{n,H} x'_{n,H^c}, \\ v_n &\sim y_{n,H} y'_{n,H^c}. \end{aligned}$$

Let n goes to infinity, we have x_n and y_n converge to the midpoint z of x and y , x'_n and y'_n to the midpoint z' of x' and y' , and w_n and v_n to the midpoint t of w and v . Hence, $t \sim z_H z'_{H^c}$.

We finish Step 1 in the proof. Consider x, y, x', y', z, z' such that

$$z \sim \frac{1}{2}x \oplus \frac{1}{2}y \text{ and } z' \sim \frac{1}{2}x' \oplus \frac{1}{2}y'.$$

Fix $w, v, t \in X$ satisfying:

$$\begin{aligned} w &\sim \frac{1}{2}x \oplus \frac{1}{2}x', \\ v &\sim \frac{1}{2}y \oplus \frac{1}{2}y', \\ t &\sim \frac{1}{2}w \oplus \frac{1}{2}v. \end{aligned}$$

We will prove that $t \sim \frac{1}{2}z \oplus \frac{1}{2}z'$.

Let $w_0 = x$, $w'_0 = x'$, $v_0 = y$, $v'_0 = y'$, $z_0 = z$, $z'_0 = z'$. Consider sequences $\{w_n\}_{n=0}^\infty$, $\{v_n\}_{n=0}^\infty$, $\{z_n\}_{n=0}^\infty$, $\{z'_n\}_{n=0}^\infty$ being defined as:

$$\begin{aligned} w_{n+1} &\sim w_{n,H}w'_{n,H^c} \text{ and } w'_{n+1} \sim w'_{n,H}w_{n,H^c}, \\ v_{n+1} &\sim v_{n,H}v'_{n,H^c} \text{ and } v'_{n+1} \sim v'_{n,H}v_{n,H^c}, \\ z_{n+1} &\sim z_{n,H}z'_{n,H^c} \text{ and } z'_{n+1} \sim z'_{n,H}z_{n,H^c}. \end{aligned}$$

Let

$$t_n \sim \frac{1}{2}w_n \oplus \frac{1}{2}v_n.$$

Since w_n and v_n converge respectively to w and v , t_n converges to $t \sim \frac{1}{2}w \oplus \frac{1}{2}v$. By the second statement and induction, we have

$$t_n \sim z_{n,H}z'_{n,H^c}.$$

Since z_n and z'_n converge to a midpoint of z and z' , t_n converges to $t \sim \frac{1}{2}z \oplus \frac{1}{2}z'$. The claim in Step 1 is proven.

Now, we begin Step 2.

Assume (2). Consider the case where we can "decrease" y_1 and "increase" y_3 such that the comparisons in (2) become "equivalent". Precisely, assume the existence of outcomes y'_1 and y'_3 such that $y_1 \succeq y'_1$ and $y'_3 \succeq y_3$, satisfying

$$\frac{1}{2}x_2 \oplus \frac{1}{2}y'_1 \sim \frac{1}{2}x_1 \oplus \frac{1}{2}y_2 \text{ and } \frac{1}{2}x_3 \oplus \frac{1}{2}y_2 \sim \frac{1}{2}x_2 \oplus \frac{1}{2}y'_3.$$

Then

$$\frac{1}{2} \left(\frac{1}{2}x_3 \oplus \frac{1}{2}y_2 \right) \oplus \frac{1}{2} \left(\frac{1}{2}x_2 \oplus \frac{1}{2}y'_1 \right) \sim \frac{1}{2} \left(\frac{1}{2}x_1 \oplus \frac{1}{2}y_2 \right) \oplus \frac{1}{2} \left(\frac{1}{2}x_2 \oplus \frac{1}{2}y'_3 \right).$$

By (1), we obtain

$$\frac{1}{2} \left(\frac{1}{2}x_3 \oplus \frac{1}{2}y'_1 \right) \oplus \frac{1}{2} \left(\frac{1}{2}x_2 \oplus \frac{1}{2}y_2 \right) \sim \frac{1}{2} \left(\frac{1}{2}x_1 \oplus \frac{1}{2}y'_3 \right) \oplus \frac{1}{2} \left(\frac{1}{2}x_2 \oplus \frac{1}{2}y_2 \right).$$

Since two sides of this equivalence relation share the common term $\frac{1}{2}x_2 \oplus \frac{1}{2}y_2$,

$$\frac{1}{2}x_3 \oplus \frac{1}{2}y'_1 \sim \frac{1}{2}x_1 \oplus \frac{1}{2}y'_3.$$

Recall that $y_1 \succeq y'_1$ and $y'_3 \succeq y_3$. We obtain

$$\frac{1}{2}x_3 \oplus \frac{1}{2}y_1 \succeq \frac{1}{2}x_1 \oplus \frac{1}{2}y_3. \quad (3)$$

Consider the general case with (2). Fix x such that $\bar{z} \succ x \succ \underline{z}$. For every outcome $w \in \{x_1, x_2, x_3, y_1, y_2, y_3\}$, let the sequence $\{w(n)\}_{n=0}^\infty$ be defined as: $w(0) = w$, and for every $n \geq 0$,

$$w(n+1) = \frac{1}{2}w(n) \oplus \frac{1}{2}x.$$

By induction and using the result in Step 1, we can prove that (2) is equivalent to

$$\frac{1}{2}x_2(n) \oplus \frac{1}{2}y_1(n) \succeq \frac{1}{2}x_1(n) \oplus \frac{1}{2}y_2(n) \text{ and } \frac{1}{2}x_3(n) \oplus \frac{1}{2}y_2(n) \succeq \frac{1}{2}x_2(n) \oplus \frac{1}{2}y_3(n),$$

for every $n \geq 0$.

Since when n tends to infinity, six sequences all converge to x , there exists n sufficiently big such that when we replace $y_1(n)$ by \underline{z} and $y_3(n)$ by \bar{z} , the comparisons are reversed:

$$\frac{1}{2}x_1(n) \oplus \frac{1}{2}y_2(n) \succ \frac{1}{2}x_2(n) \oplus \frac{1}{2}\underline{z} \text{ and } \frac{1}{2}x_2(n) \oplus \frac{1}{2}\bar{z} \succ \frac{1}{2}x_3(n) \oplus \frac{1}{2}y_2(n).$$

This implies the existence of $y'_1(n)$ and $y'_3(n)$ such that $y_1(n) \succeq y'_1(n)$, $y'_3(n) \succeq y_3(n)$, and

$$\frac{1}{2}x_2(n) \oplus \frac{1}{2}y'_1(n) \sim \frac{1}{2}x_1(n) \oplus \frac{1}{2}y_2(n) \text{ and } \frac{1}{2}x_3(n) \oplus \frac{1}{2}y_2(n) \sim \frac{1}{2}x_2(n) \oplus \frac{1}{2}y'_3(n).$$

Then we have

$$\frac{1}{2}x_3(n) \oplus \frac{1}{2}y_1(n) \succeq \frac{1}{2}x_1(n) \oplus \frac{1}{2}y_3(n),$$

which is equivalent to (3). The claim in Step 2 is proven. By Theorem 1, chapter 9 in [Debreu \(2015\)](#), there exists a utility function u such that $\frac{1}{2}x \oplus \frac{1}{2}y \succeq \frac{1}{2}x' \oplus \frac{1}{2}y'$ if and only if $\frac{1}{2}u(x) + \frac{1}{2}u(y) \geq \frac{1}{2}u(x') + \frac{1}{2}u(y')$.

4.4 Proof of Proposition 2

Consider a construction of the following sequence $\{z^{k,2^n}\}$, with $n \geq 0$ and $0 \leq k \leq 2^n$.

For $n = 1$, fix outcomes $z^{0,2}$, $z^{1,2}$ and $z^{2,2}$ as:

$$\begin{aligned} z^{0,2} &= y, \\ z^{1,2} &\sim \frac{1}{2}x \oplus \frac{1}{2}y, \\ z^{2,2} &= x. \end{aligned}$$

For $n \geq 1$, $0 \leq k \leq 2^{n+1}$, fix the elements $z^{k,2^{n+1}} \in X$ as:

$$\begin{aligned} z^{k,2^{n+1}} &= z^{k',2^n} \text{ if } k = 2k', \text{ with } 0 \leq k' \leq 2^n, \\ z^{k,2^{n+1}} &\sim \frac{1}{2}z^{2k',2^n} \oplus \frac{1}{2}z^{2k'+1,2^n} \text{ if } k = 2k' + 1, \text{ with } 0 \leq k' \leq 2^n - 1. \end{aligned}$$

The following assertions are intuitive and can be proven by induction. Assume that $x \succ y$.

a) For every $0 \leq k \leq n$,

$$x \succ z^{2^n-1,2^n} \succ \dots \succ z^{k+1,2^n} \succ z^{k,2^n} \succ \dots \succ z^{1,2^n} \succ y.$$

b) For every $0 \leq k \leq n$,

$$u(z^{k,2^n}) = \frac{k}{2^n}u(x) + \left(1 - \frac{k}{2^n}\right)u(y).$$

c) For any event $A \in \mathcal{A}$ and n , there exists unique $k_n(A)$ such that:

$$z^{k_n(A)+1,2^n} \succ x_A y_{A^c} \succeq z^{k_n(A),2^n}.$$

Fix an event $A \in \mathcal{A}$ and outcomes $x \succ y$, consider the sequence $\{(k_n(A), 2^n)\}_{n=0}^\infty$ such that for every n ,

$$z^{k_n(A)+1,2^n} \succ x_A y_{A^c} \succeq z^{k_n(A),2^n}.$$

Observe that the sequence $\left\{\frac{k_n(A)}{2^n}\right\}_{n \geq 0}$ is increasing in respect to n . We may define the probability measure of A using outcomes x and y as

$$\mu^{x,y}(A) = \lim_{n \rightarrow \infty} \frac{k_n(A)}{2^n}.$$

However, the sequence $\{(k_n(A), 2^n)\}_{n=0}^{\infty}$ and the limit may depend on the choice of x and y . Under the *Independence* axiom, we can discard this possibility and prove that the value of $\mu^{x,y}(A)$ is independent with respect to the choice of x and y .

Now, we begin the main part of the proof of this Proposition. First, we prove the independence of $\mu^{x,y}(A)$ with respect to the choice of x, y .

Fix any $x^*, y^* \in X$ such that $x^* \succ y^*$. Consider an event $A \in \mathcal{A}$. Let $p = \mu^{x^*, y^*}(A)$, with the same construction of μ^{x^*, y^*} , using the sequences of $\{z^{k, 2^n}\}$.

First, we prove that under any other choice of $x \sim z^{k, 2^n}$ and $y \sim z^{k', 2^n}$ with $k > k'$:

$$\mu^{x,y}(A) = \mu^{x^*, y^*}(A).$$

The proof will be given by induction. Consider first the case $n = 1$. Consider $z^{1,2}$ and $z^{0,2}$. Let $x' = z^{1,2}$, and $y' = z^{0,2} = y^*$. Let $p = \mu^{x^*, y^*}(A)$ and $p' = \mu^{x', y'}(A)$.

Recall that,

$$\begin{aligned} u(x_A y_{A^c}) &= pu(x^*) + (1-p)u(y^*), \\ u(x'_A y'_{A^c}) &= p'u(x') + (1-p')u(y') \\ &= \frac{p'}{2}(u(x) + u(y)) + (1-p')u(y') \\ &= \frac{p'}{2}u(x^*) + \left(1 - \frac{p'}{2}\right)u(y^*). \end{aligned}$$

Since $y' = y^*$, using the first statement in the proof of Lemma 2, we get

$$x'_A y'_{A^c} \sim \frac{1}{2}(x_A y_{A^c}) \oplus \frac{1}{2}y^*.$$

This implies

$$\begin{aligned}
u(x'_A y'_{A^c}) &= \frac{1}{2}(u(x_A y_{A^c}) + u(y^*)) \\
&= \frac{1}{2}(pu(x^*) + (1-p)u(y^*)) + \frac{1}{2}u(y^*) \\
&= \frac{p}{2}u(x^*) + \left(1 - \frac{p}{2}\right)u(y^*).
\end{aligned}$$

Hence $p = p'$, or $\mu^{x',y'}(A) = \mu^{x,y}(A)$.

For the case of the choice x and $z^{1,2}$, we use the same arguments. The conclusion is immediate for $x = x^*, y = y^*$.

Now assume that the assertion is true for any number n . We will prove that it is also true for $n + 1$. Consider any $0 \leq k' \leq k \leq 2^{n+1}$. By the construction of the sequence $\{z^{k,2^n}\}_{n=0}^\infty$, there exist $x, x', y, y' \in \{z^{k,2^n}\}_{k=0}^{2^n}$ such that $x \succeq y, x' \succeq y'$ and

$$\begin{aligned}
z^{k,2^{n+1}} &\sim \frac{1}{2}x \oplus \frac{1}{2}x' \\
z^{k',2^{n+1}} &\sim \frac{1}{2}y \oplus \frac{1}{2}y'.
\end{aligned}$$

Let $t \sim z_A^{k,2^{n+1}} z_{A^c}^{k',2^{n+1}}, v \sim x_A y_{A^c}, w \sim x'_A y'_{A^c}$. Applying the first step in the proof of Lemma 2, the equivalence $t \sim \frac{1}{2}w \oplus \frac{1}{2}v$ is satisfied. Hence

$$\begin{aligned}
u(t) &= \frac{1}{2}(u(w) + u(v)) \\
&= \frac{1}{2}(pu(x) + (1-p)u(y) + pu(x') + (1-p)u(y')) \\
&= p \left(\frac{1}{2}(u(x) + u(x')) \right) + (1-p) \left(\frac{1}{2}(u(y) + u(y')) \right) \\
&= pu \left(z^{k,2^{n+1}} \right) + (1-p)u \left(z^{k',2^{n+1}} \right).
\end{aligned}$$

This implies

$$\mu^{z^{k,2^{n+1}}, z^{k',2^{n+1}}}(A) = \mu^{x,y}(A) = \mu^{x^*,y^*}(A).$$

Consider now any x, y such that $x^* \succeq x \succeq y \succeq y^*$. Let $\{z^{k_n, 2^n}\}_{n=0}^\infty$ and $\{z^{k'_n, 2^n}\}_{n=0}^\infty$ be sequences that converge correspondingly to x and y . Then

$$\begin{aligned} u(x_A y_{A^c}) &= \lim_{n \rightarrow \infty} u\left(z_A^{k_n, 2^n} z_{A^c}^{k'_n, 2^n}\right) \\ &= pu(x) + (1-p)u(y), \end{aligned}$$

which is equivalent to

$$\mu^{x,y}(A) = \mu^{x^*,y^*}(A).$$

For any $x \succeq y$ and $x' \succeq y'$, fix x^* and y^* such that $x^* \succeq x, x'$ and $y, y' \succeq y^*$, we have

$$\mu^{x,y}(A) = \mu^{x^*,y^*}(A) = \mu^{x',y'}(A).$$

Hence the choice of value $\mu(A)$ does not depend on the choice of x, y .

Now, we prove that μ is a finitely countable probability. We will prove that for every events A and B such that $A \cap B = \emptyset$,

$$\mu(A \cup B) = \mu(A) + \mu(B).$$

Define $C = (A \cup B)^c$. Let $x \succ y$. Since $x_A x_{B \cup C} \succeq x_{A \cup B} y_C \succeq x_A y_{B \cup C}$, there exists outcome w such that

$$x_A x_{B \cup C} = x_{A \cup B} y_C \sim x_A w_{B \cup C}.$$

Applying the *sure-thing principle* by replacing x by y on the event A , we get

$$x_B y_{A \cup C} = y_A x_{B \cup C} \sim y_A w_{B \cup C}.$$

From $x_{A \cup B} y_C \sim x_A w_{B \cup C}$ and $x_B y_{A \cup C} \sim y_A w_{B \cup C}$ we get

$$\mu(A \cup B)u(x) + (1 - \mu(A \cup B))u(y) = \mu(A)u(x) + (1 - \mu(A))u(w), \quad (4)$$

$$\mu(B)u(x) + (1 - \mu(B))u(y) = \mu(A)u(y) + (1 - \mu(A))u(w). \quad (5)$$

Taking (4) minus (5), we obtain

$$(u(x) - u(y))\mu(A \cup B) = (u(x) - u(y))(\mu(A) + \mu(B)).$$

Hence,

$$\mu(A \cup B) = \mu(A) + \mu(B).$$

4.5 Proof of Theorem 1

By Proposition 2, there exists unique probability measure μ and a utility function such that for every events $A, B \in \mathcal{A}$, outcomes $x, y, x', y' \in X$, the comparison $x_A y_{A^c} \succeq x'_B y'_{B^c}$ is equivalent to:

$$\mu(A)u(x) + \mu(A^c)u(y) \geq \mu(B)u(x') + \mu(B^c)u(y').$$

Suppose that the assertion of the theorem is true for the acts which take almost $n - 1$ different values, with some $n \geq 2$. We will prove that it is verified for n different values.

For a partition $\{A_k\}_{k=1}^n$ of S , let $f = x_{1,A_1}x_{2,A_2} \dots x_{n,A_n}$, with $\{A_k\}_{k=1}^n$ a partition of S . Fix a constant outcome $v \in X$. For $1 \leq k \leq n$, let $p_k = \mu(A_k)$.

We will prove that

$$f \succeq v \text{ if and only if } \sum_{k=1}^n p_k u(x_k) \geq u(v).$$

Fix $w \in X$ such that

$$x_{1,A_1}x_{2,A_2}x_{3,A_3} \dots x_{n,A_n} \sim x_{1,A_1}w_{\cup_{k=2}^n A_k}.$$

By the *sure-thing principle* property, replacing x_1 by x_2 , this implies

$$x_{2,A_1}x_{2,A_2} \dots x_{n,A_n} \sim x_{2,A_1}w_{\cup_{k=2}^n A_k},$$

which is equivalent to

$$p_1u(x_2) + p_2u(x_2) + p_3u(x_3) + \dots + p_nu(x_n) = p_1u(x_2) + (p_2 + p_3 + \dots + p_n)u(w).$$

Hence

$$u(w) = \frac{1}{\sum_{k=2}^n p_k} \sum_{k=2}^n p_k u(x_k).$$

This allows us to deduce the value of f :

$$\begin{aligned} u(x_{1,A_1}w_{\cup_{k=2}^n A_k}) &= p_1u(x_1) + \left(\sum_{k=2}^n p_k \right) u(w) \\ &= \sum_{k=1}^n p_k u(x_k). \end{aligned}$$

We have $f \succeq v$ if and only if $x_{1,A_1}w_{\cup_{k=2}^n A_k} \succeq v$, which is equivalent to

$$\sum_{k=1}^n p_k u(x_k) \geq u(v).$$

For every acts $f = x_{1,A_1}x_{2,A_2} \dots x_{n,A_n}$ and $g = y_{1,B_1}y_{2,B_2} \dots y_{m,B_m}$, by considering v such that $v \sim g$, one has

$$f \succeq g \text{ if and only if } \sum_{k=1}^n \mu(A_k)u(x_k) \geq \sum_{k=1}^m \mu(B_k)u(y_k).$$

The proof is completed.

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