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► To cite this version:

Long Hu, Guillaume Olive. Minimal null control time of some 1D hyperbolic balance laws with constant coefficients and properties of related kernel equations. 2023. hal-04318642

HAL Id: hal-04318642

<https://hal.science/hal-04318642v1>

Preprint submitted on 1 Dec 2023

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1 Minimal null control time of some 1D hyperbolic balance
2 laws with constant coefficients and properties of related
3 kernel equations

4 Long Hu* Guillaume Olive†

5 December 1, 2023

6 **Abstract**

7 In this work, we study the null controllability by one-sided boundary controls of one-
8 dimensional hyperbolic balance laws with constant coefficients. Our first result shows that,
9 when the system has only one negative or positive speed, the minimal null control time
10 of such systems depends on some orthogonality conditions for a particular sequence. This
11 sequence is explicit in function of the coefficients of the system but it is defined by a nonlinear
12 recurrence relation. Our second result then completes the previous one by giving explicit
13 bounds on the number of orthogonality conditions that have to be checked in two nontrivial
14 situations. The proofs rely on a careful analysis of the so-called kernel equations associated
15 with the system, including a new well-posedness result. Our results are also valid for the
16 finite-time stabilization property.

17 **Keywords.** Hyperbolic systems; Minimal control time; Backstepping method; Nonlinear
18 recurrence relation

19 **2010 Mathematics Subject Classification.** 35L40, 93B05, 93D15, 11B

20 **1 Introduction and main results**

21 **1.1 Problem description**

22 In this paper, we are interested in the null controllability properties of a class of one-dimensional
23 (1D) hyperbolic system with constant coefficients (see e.g. [BC16, Chap. 1] for applications).
24 The equations describing such phenomenons are

25
$$\frac{\partial y}{\partial t}(t, x) + \Lambda \frac{\partial y}{\partial x}(t, x) = My(t, x). \quad (1a)$$

26 Above, $t \in (0, T)$ is the time variable, $T > 0$, $x \in (0, 1)$ is the space variable and the state
27 is $y : (0, T) \times (0, 1) \rightarrow \mathbb{R}^n$ ($n \geq 2$). The matrix $\Lambda \in \mathbb{R}^{n \times n}$ will always be assumed diagonal
28 $\Lambda = \text{diag}(\lambda_1, \dots, \lambda_n)$, with $m \geq 1$ negative speeds and $p \geq 1$ positive speeds ($m + p = n$):

29
$$\lambda_1 < \dots < \lambda_m < 0 < \lambda_{m+1} < \dots < \lambda_{m+p}. \quad (1b)$$

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30 The matrix $M \in \mathbb{R}^{n \times n}$ couples the equations of the system inside the domain and will be called
 31 the internal coupling matrix. We will consider an initial condition at time $t = 0$:

$$32 \quad y(0, x) = y^0(x). \quad (1c)$$

33 Let us now discuss the boundary conditions. The structure of Λ induces a natural splitting
 34 of the state into components corresponding to negative and positive speeds, denoted respectively
 35 by y_- and y_+ . For the above system to be well-posed in $(0, T) \times (0, 1)$ with an initial condition
 36 at time $t = 0$, we then need to add boundary conditions at $x = 1$ for y_- and at $x = 0$ for y_+ . We
 37 will consider the following type of boundary conditions, motivated by its numerous applications
 38 (see again [BC16]):

$$39 \quad y_-(t, 1) = u(t), \quad y_+(t, 0) = Qy_-(t, 0). \quad (1d)$$

40 The function u is the so-called control, it will be at our disposal. It only acts on one part of
 41 the boundary and, on the other part of the boundary, the equations are coupled by the matrix
 42 $Q \in \mathbb{R}^{p \times m}$. This matrix will be called the boundary coupling matrix. In what follows, (1a), (1c)
 43 and (1d) together will be referred to as system (1).

We recall that system (1) is well-posed in the following functional setting: for every $T > 0$,
 $y^0 \in L^2(0, 1)^n$ and $u \in L^2(0, T)^m$, there exists a unique solution y to system (1) with

$$y \in C^0([0, T]; L^2(0, 1)^n) \cap C^0([0, 1]; L^2(0, T)^n).$$

44 By solution we mean “solution along the characteristics”. We refer for instance to [Cor+21] for a
 45 proof of this well-posedness result in such a setting (see also [BC16, Appendix A] when $u = 0$).

46 The regularity $C^0([0, T]; L^2(0, 1)^n)$ of the solution allows us to consider control problems in
 47 the space $L^2(0, 1)^n$:

Definition 1.1. Let $T > 0$ be fixed. We say that system (1) is null controllable in time T if,
 for every $y^0 \in L^2(0, 1)^n$, there exists $u \in L^2(0, T)^m$ such that the corresponding solution y to
 system (1) satisfies

$$y(T, \cdot) = 0.$$

48 Since controllability in time T_1 implies controllability in any time $T_2 \geq T_1$, it is natural to
 49 try to find the smallest possible control time, the so-called “minimal control time”.

50 **Definition 1.2.** For any Λ, M and Q as above, we denote by $T_{\text{inf}}(\Lambda, M, Q) \in [0, +\infty]$ the
 51 minimal null control time of system (1), that is

$$52 \quad T_{\text{inf}}(\Lambda, M, Q) = \inf \{T > 0 \mid \text{System (1) is null controllable in time } T\}. \quad (2)$$

53 The time $T_{\text{inf}}(\Lambda, M, Q)$ is named “minimal” null control time according to the current litera-
 54 ture, despite it is not always a minimal element of the set. We keep this naming here, but we use
 55 the notation with the “inf” to avoid eventual confusions. The goal of this article is to characterize
 56 $T_{\text{inf}}(\Lambda, M, Q)$ in some new situations.

In order to state our results and those of the literature, we need to introduce the following
 times:

$$T_i = \frac{1}{-\lambda_i} \quad \text{if } i \leq m, \quad T_i = \frac{1}{\lambda_i} \quad \text{if } i \geq m + 1.$$

57 The time T_i is the time needed for the controllability of a single equation (the transport equation)
 58 with speed λ_i . Note that the assumption (1b) implies in particular the following order relation:

$$59 \quad \begin{cases} T_1 \leq \dots \leq T_m, \\ T_n \leq \dots \leq T_{m+1}. \end{cases} \quad (3)$$

60 1.2 Literature

61 Here, we briefly describe the results of the literature about the null controllability of system (1).
 62 All the results below are also valid for space-dependent versions of this system.

- 63 • It was first proved in the celebrated survey [Rus78] that system (1) is null controllable in
 64 any time $T \geq T_{m+1} + T_m$. A strength of this result is that it is valid for any M and Q .
 65 However, it was also observed in that paper that the minimal control time can be smaller
 66 than $T_{m+1} + T_m$. Finding the minimal control time even in the simpler case $M = 0$ was
 67 then left as an open problem.
- 68 • For $M = 0$, the minimal null control time was eventually found in [Wec82]. The author
 69 gave an explicit expression of this time in terms of some indices related to Q .
- 70 • Finding the minimal null control time for arbitrary M and Q is still an open challenging
 71 problem. There has been a recent resurgence on the characterization of this time. A first
 72 result in this direction was obtained in [CN19] and then completed in [CN21a]. Therein,
 73 the authors introduced a class of boundary coupling matrices Q for which they showed that
 74 the minimal null control time is smaller than $T_{m+1} + T_m$, whatever the internal coupling
 75 matrix M is.
- 76 • For full row rank boundary coupling matrices ($\text{rank } Q = p$), the minimal null control time
 77 was found in [HO21a]. In this case, it has been shown that this time is the same as for the
 78 system without internal coupling ($M = 0$).
- 79 • For systems of $n = 2$ equations, the minimal null control time was found in [Cor+13] and
 80 [HO21b]. In particular, it has been shown in the second reference that this time depends
 81 on the internal coupling matrix M when the boundary coupling matrix is zero. This is a
 82 feature that was not observed nor highlighted in all the other works and that the results
 83 of the present paper will also share.
- 84 • Finally, the smallest and largest values that the minimal null control time can take with
 85 respect to the internal coupling matrix M were found in [HO22].

86 Other related works include for instance [Cor+21; CN21b; MA22] about time-dependent
 87 versions of system (1) and [Li10; LR10; Hu15; CN20; CN22] for quasilinear versions of this
 88 system (in a C^1 framework).

89 1.3 Notations and important definitions

90 To state the main results of this article we first need to introduce some notations.

91 All along this article, we denote by A^T the transpose of a matrix A . For any integer $N \geq 1$,
 92 $\mathbb{R}_0^{N \times N}$ denotes the set of matrices of size $N \times N$ with diagonal entries all equal to zero. The matrix
 93 Id_N denotes the $N \times N$ identity matrix. A matrix (or matrix-valued function) of size $N_1 \times N_2$
 94 will simply be denoted using the corresponding lowercase letter when $N_2 = 1$ (e.g. $Q \in \mathbb{R}^{p \times m}$
 95 will be denoted by $q \in \mathbb{R}^p$ when $m = 1$). The inner product of two vectors $v_1, v_2 \in \mathbb{R}^{n-1}$ will be
 96 denoted by $\langle v_1, v_2 \rangle$.

97 Let us now introduce a sequence that will play a key role throughout this paper. For any
 98 $i \in \{1, \dots, n\}$, we first define the following quantities.

- For every $r, j \in \{1, \dots, n\}$, we denote by

$$\alpha_{rj} = \frac{m_{rj}}{\lambda_i - \lambda_j} \quad \text{if } j \neq i, \quad \alpha_{ri} = \frac{m_{ri}}{\lambda_i}.$$

- Let then $A, D \in \mathbb{R}^{(n-1) \times (n-1)}$ be the matrices defined by

$$A = (\alpha_{rj})_{r,j \neq i}, \quad D = \text{diag} \left(\frac{\lambda_i + \lambda_j}{\lambda_i - \lambda_j} \right)_{j \neq i},$$

and let $\psi^0, w \in \mathbb{R}^{n-1}$ be the vectors defined by

$$\psi^0 = (\alpha_{ij})_{j \neq i}, \quad v = \left(-\frac{1}{2} \alpha_{ji} \right)_{j \neq i}.$$

- Let $(\psi^r)_{r \in \mathbb{N}} \subset \mathbb{R}^{n-1}$ be the sequence defined by ψ^0 and

$$\psi^1 = A^\top \psi^0, \quad \psi^r = A^\top \psi^{r-1} + D \sum_{\ell=0}^{r-2} \langle v, \psi^{r-2-\ell} \rangle \psi^\ell, \quad \forall r \geq 2. \quad (4)$$

- Finally, for $q \in \mathbb{R}^{n-1}$, let $b \in \mathbb{R}^{n-1}$ be the nonzero vector defined by

$$b = (\beta_j)_{j \neq i}, \quad \beta = -\Lambda \begin{pmatrix} 1 \\ q \end{pmatrix}. \quad (5)$$

All the above quantities depend on the index i but we omit it for clarity.

With the previous notations, we introduce the following sets. For any $k \in \{2, \dots, n+1\}$, we denote by \mathcal{C}_k the set of $(M, q) \in \mathbb{R}_0^{n \times n} \times \mathbb{R}^{n-1}$ such that, for every $2 \leq i < k$, we have $q_{i-1} = 0$ and

$$\langle b, \psi^r \rangle = 0, \quad \forall r \in \mathbb{N}. \quad (6)$$

Here, we use the convention that $\mathcal{C}_2 = \mathbb{R}_0^{n \times n} \times \mathbb{R}^{n-1}$. Additionally, we will denote by $\mathcal{C}_{n+2} = \emptyset$. Note that we then have $\mathcal{C}_2 \supset \mathcal{C}_3 \supset \dots \supset \mathcal{C}_{n+1} \supset \mathcal{C}_{n+2}$.

1.4 Main results and comments

The first result of this article is the following characterization of the minimal null control time in the case of one negative speed.

Theorem 1.3. *Assume that $m = 1$. Let us denote by*

$$\tau_i = \max \{T_1 + T_i, T_2\} \quad \text{if } 2 \leq i \leq n, \quad \tau_{n+1} = \max \{T_1, T_2\}, \quad (7)$$

(we have $\tau_2 \geq \tau_3 \geq \dots \geq \tau_{n+1}$ from (3)). Then, for any $M \in \mathbb{R}_0^{n \times n}$ and $q \in \mathbb{R}^{n-1}$, we have:

1. $T_{\inf}(\Lambda, M, q) \in \{\tau_2, \dots, \tau_{n+1}\}$. Moreover, the infimum is always reached (in (2)).
2. For any $k \in \{2, \dots, n+1\}$, we have

$$T_{\inf}(\Lambda, M, q) = \tau_k \iff (M, q) \in \mathcal{C}_k \setminus \mathcal{C}_{k+1}.$$

We recall that the set \mathcal{C}_k is defined at the end of Section 1.3.

Remark 1.4. Theorem 1.3 solves the open problem raised at the end of [HO21b, Section 5] for systems with constant coefficients.

120 **Remark 1.5.** Theorem 1.3 remains valid if we replace everywhere in this article the null con-
 121 trollability property by the finite-time stabilization property by L^2 bounded feedbacks (that
 122 is when the control u is looked under the more particular form $u(t) = \int_0^1 r(\xi)^\top y(t, \xi) d\xi$ with
 123 $r \in L^2(0, 1)^n$). This easily follows from the proof below.

Even though the set \mathcal{C}_k is explicit in function of the parameters of the system, the orthogo-
 nality conditions (6) that define this set can be difficult to study in general because the sequence
 $(\psi^r)_{r \in \mathbb{N}}$ is defined by a nonlinear recurrence relation. Note however that there always exists an
 integer $N \geq 1$ such that

$$\psi^r \in \text{span} \{ \psi^s \mid s \leq N - 1 \}, \quad \forall r \geq N,$$

124 so that

$$125 \quad \langle b, \psi^r \rangle = 0, \quad \forall r \in \mathbb{N} \iff \langle b, \psi^r \rangle = 0, \quad \forall r \in \{0, \dots, N - 1\}. \quad (8)$$

126 This means that (6) only needs to be checked for the first N values of r . However, such a N
 127 depends on the sequence and it is a priori unknown, so that, in practice, we do not know when
 128 we have to stop checking the orthogonality conditions. Our second result provides information
 129 on this issue in two particular cases:

Theorem 1.6. *Let $i \in \{1, \dots, n\}$ be fixed. Define*

$$N_\psi = \min \{ N \geq 1 \mid (8) \text{ holds} \}.$$

130 *We have $N_\psi \leq 3$ for $n = 3$ and $N_\psi \leq 6$ for $n = 4$.*

131 **Remark 1.7.** It would be interesting to find a bound of N_ψ with respect to n for arbitrary n .

132 Our main results can for instance be combined to deduce a very explicit characterization of
 133 the minimal null control time in the following particular case:

134 **Corollary 1.8.** *Assume that $m = 1$ and $p = 2$. Then, for any $M \in \mathbb{R}_0^{3 \times 3}$ and $q \in \mathbb{R}^2$, we have:*

1. $T_{\text{inf}}(\Lambda, M, q) = \max \{ T_1, T_2 \}$ if, and only if, (M, q) satisfies

$$q = 0, \quad m_{21} = m_{31} = 0.$$

2. $T_{\text{inf}}(\Lambda, M, q) = \max \{ T_1 + T_3, T_2 \}$ if, and only if, (M, q) satisfies

$$q = 0, \quad m_{21} = m_{23} = 0, \quad m_{31} \neq 0,$$

or

$$q_1 = 0, \quad q_2 \neq 0 \quad \text{and} \quad \left(m_{21} = m_{23} = 0 \quad \text{or} \quad \begin{cases} m_{21} = rsm_{23}, \\ m_{31} = r^2sm_{13}, \\ m_{32} = -rm_{12}, \end{cases} \right),$$

135 where $r = -\frac{\lambda_3 q_2}{\lambda_1}$ and $s = \frac{\lambda_2 - \lambda_1}{\lambda_2 - \lambda_3}$.

136 3. In all the other situations, $T_{\text{inf}}(\Lambda, M, q) = T_1 + T_2$.

137 For $p = 3$, there is no simple presentation as for $p = 2$, even though the orthogonality
 138 conditions are explicit (see also Remark 1.10 below) and we know that we only have to check
 139 the first six conditions. Therefore, we only give a nontrivial example:

Example 1.9. Let $\sigma \in \mathbb{R} \setminus \{0\}$ be arbitrary and consider system (1) with

$$\Lambda = \text{diag}(-1, 1, 2, 3), \quad M = \begin{pmatrix} 0 & -3 & 1/(2\sigma) & 0 \\ 2 & 0 & 0 & -2 \\ 3\sigma & 0 & 0 & -\sigma \\ 0 & 2 & 1/(2\sigma) & 0 \end{pmatrix}, \quad q = \begin{pmatrix} 0 \\ 0 \\ 1/3 \end{pmatrix}.$$

- For $i = 2$, we have $b = (1 \ 0 \ -1)^\top$ and

$$\psi^0 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \quad \psi^1 = -\frac{1}{\sigma} \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \quad \psi^2 = -\frac{3}{2}\psi^0, \quad \psi^3 = -3\psi^1, \quad \psi^4 = \frac{9}{2}\psi^0, \quad \psi^5 = \frac{45}{4}\psi^1.$$

- For $i = 3$, we have the same b and

$$\psi^0 = \sigma \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \quad \psi^1 = -\sigma \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \quad \psi^2 = -\frac{3}{4}\psi^0, \quad \psi^3 = -\frac{3}{2}\psi^1, \quad \psi^4 = \frac{9}{8}\psi^0, \quad \psi^5 = \frac{45}{16}\psi^1.$$

140 From Theorems 1.3 and 1.6, we deduce that $T_{\text{inf}}(\Lambda, M, q) = \tau_4 = 4/3$.

141 **Remark 1.10.** For arbitrary n , we will see that the orthogonality conditions (6) are satisfied if
142 one of the following three conditions holds:

143 (C1) $\text{Kal}(A, v)^\top \psi^0 = \text{Kal}(A, b)^\top \psi^0 = 0$, where $\text{Kal}(A, h) = (h|Ah|\dots|A^{n-2}h) \in \mathbb{R}^{(n-1) \times (n-1)}$
144 denotes the Kalman matrix of (A, h) , for any $h \in \mathbb{R}^{n-1}$.

145 (C2) There exists $\emptyset \neq J \subsetneq \{1, \dots, n-1\}$ such that $\psi_j^0 = a_{rj} = b_r = 0$ for every $j \notin J$ and $r \in J$.

(C3) $\langle b, \psi^0 \rangle = 0$ and there exists $j_0 \in \{1, \dots, n-1\}$ such that $b_{j_0} = 0$ and $\text{rank } \Delta_{j_0} = 1$, where

$$\Delta_{j_0} = \begin{pmatrix} (D - d_{j_0})\psi^0 & A^\top e_{j_0} \\ A^\top \psi^0 & v_{j_0} \psi^0 - \langle v, \psi^0 \rangle e_{j_0} \end{pmatrix} \in \mathbb{R}^{2(n-1) \times 2},$$

146 where d_{j_0} is the j_0 -th diagonal entry of D and e_{j_0} is the j_0 -th canonical vector of \mathbb{R}^{n-1} .

147 We will also see that, for $n = 3$ (resp. $n = 4$), it is necessary that one of the conditions (C1),
148 (C2) (resp. (C1), (C2), (C3)) holds (it is however preferable to use Theorem 1.6 in these cases).

149 The rest of this article is organized as follows. In Section 2, we use the equivalence between
150 the controllability of system (1) and that of a simpler system to obtain a characterization of
151 this property in terms of some orthogonality conditions for the derivatives at the origin of any
152 solution to the so-called kernel equations. In Section 3, we compute these derivatives for a
153 particular solution and we obtain a general formula for this solution. In Section 4, we study the
154 orthogonality conditions associated with the previous solution and we deduce our main results. In
155 Section 5, we supplement our results by studying the structure of the solution associated with the
156 orthogonality conditions. Finally, in Appendix A, we give a simple proof of the characterization
157 of the controllability properties for the equivalent system and, in Appendix B, we prove the
158 existence of a solution to the kernel equations by a new approach.

159 2 An equivalent system and the kernel equations

160 The first step in the proof of our results is to consider a system which is equivalent to our initial
161 system from a control point of view.

162 **Lemma 2.1.** *For any $T > 0$, system (1) is null controllable in time T if, and only if, so is the*
163 *system*

$$164 \begin{cases} \frac{\partial \tilde{y}}{\partial t}(t, x) + \Lambda \frac{\partial \tilde{y}}{\partial x}(t, x) = F(x)\tilde{y}_-(t, 0), \\ \tilde{y}_-(t, 1) = \tilde{u}(t), \quad \tilde{y}_+(t, 0) = Q\tilde{y}_-(t, 0), \\ \tilde{y}(0, x) = \tilde{y}^0(x), \end{cases} \quad (9)$$

165 where $F \in C^0([0, 1])^{n \times m}$ is defined by

$$166 F(x) = -K(x, 0)\Lambda \begin{pmatrix} \text{Id}_m \\ Q \end{pmatrix}, \quad (10)$$

167 and $K \in C^0(\overline{\mathcal{T}})^{n \times n}$ is any solution to

$$168 \begin{cases} \Lambda \frac{\partial K}{\partial x}(x, \xi) + \frac{\partial K}{\partial \xi}(x, \xi)\Lambda + K(x, \xi)M = 0, \\ \Lambda K(x, x) - K(x, x)\Lambda = M, \end{cases} \quad (11)$$

169 in the closure of the triangle $\mathcal{T} = \{(x, \xi) \in \mathbb{R}^2 \mid 0 < \xi < x < 1\}$.

By solution to (11) we mean solution along the characteristics. This result is by now well-known: it consists in using the invertible spatial transformation

$$\tilde{y}(t, x) = y(t, x) - \int_0^x K(x, \xi)y(t, \xi) d\xi,$$

170 in order to transform a solution of system (1) into a solution of system (9) (see e.g. [Hu+19,
171 Section 2.2]). This idea is the starting point of the so-called backstepping method for partial
172 differential equations and introduced more specifically for hyperbolic systems of two equations in
173 [Cor+13]. Equations (11) are thus called the kernel equations. The difficult point is not so much
174 the result of the previous lemma but rather to prove that (11) actually has at least a solution.
175 It follows from the results of [Hu+16] that there are many solutions to the kernel equations (11)
176 in $\overline{\mathcal{T}}$.

177 **Remark 2.2.** The choice of solution to the kernel equations (11) does not affect the controllability
178 properties of system (1) because all the corresponding systems (9)-(10) are equivalent from
179 a control point of view.

180 Now, two problems naturally arise:

- 181 1. Can we characterize the null controllability of the equivalent system (9) in function of Λ, Q
182 and F ?
- 183 2. If so, can this characterization be presented explicitly in terms of Λ, Q and M ?

184 These problems are still open in general. One particular case where the first problem has been
185 completely solved is the case $m = 1$ (one negative speed). This was done in [HO21b, Section 5].

Theorem 2.3. *Assume that $m = 1$. Then, system (9) is null controllable in time T if, and only if,*

$$T \geq \max_{2 \leq i \leq n} \{T_1 + T_i^*, T_2\},$$

where

$$T_i^* = \begin{cases} T_i & \text{if } q_{i-1} \neq 0, \\ T_i(1 - \ell(f_i)) & \text{if } q_{i-1} = 0, \end{cases}$$

186 where $\ell(f_i) = \sup I(f_i)$ with $I(f_i) = \{\ell \in (0, 1) \mid f_i = 0 \text{ in } (0, \ell)\}$, if $I(f_i) \neq \emptyset$, and $\ell(f_i) = 0$
187 otherwise.

188 The second problem could not be solved though because, even if the conditions for f are
189 explicit, the “map” $M \mapsto f$ (“defined” by (11)-(10), with $m = 1$) is quite complicated. It was
190 left as an open problem in the same paper. This is precisely where our main results step in.

191 From the above result of [HO21b] we see that the values at $x = 0$ of f and its derivatives
192 (assuming it is smooth) can affect the minimal null control time $T_{\text{inf}}(\Lambda, -, q, f)$ of the system
193 ($T_i^* = T_i$ if $f_i^{(N)}(0) \neq 0$ for some $N \geq 0$). Our idea is to show that these values in fact completely
194 characterize $T_{\text{inf}}(\Lambda, -, q, f)$ because M is constant and that we can explicitly relate them to M
195 thanks to the kernel equations.

196 It is clear that $T_{\text{inf}}(\Lambda, -, q, f)$ is solely characterized by $f(0), f'(0), f''(0)$, etc. if we have

$$197 \quad f \text{ is analytic in a neighborhood of } [0, 1]. \quad (12)$$

198 Under such an assumption, Theorem 2.3 takes a simpler form:

199 **Corollary 2.4.** *Assume that $m = 1$, let $q \in \mathbb{R}^{n-1}$ be given and assume (12). Then, we have:*

- 200 1. $T_{\text{inf}}(\Lambda, -, q, f) \in \{\tau_2, \dots, \tau_{n+1}\}$ (recall (7)). Moreover, the infimum is always reached.
201 2. For any $k \in \{2, \dots, n+1\}$, we have

$$T_{\text{inf}}(\Lambda, -, q, f) = \tau_k \iff (q, f) \in \mathcal{S}_k \setminus \mathcal{S}_{k+1},$$

202 where, for every $k \in \{2, \dots, n+1\}$, \mathcal{S}_k is the set of $(q, f) \in \mathbb{R}^{n-1} \times C^0([0, 1])^n$ such that
203 $q_{i-1} = f_i = 0$ for every $2 \leq i < k$ (we use the convention that $\mathcal{S}_2 = \mathbb{R}^{n-1} \times C^0([0, 1])^n$,
and $\mathcal{S}_{n+2} = \emptyset$).

204 This result is immediate from the previous theorem but we give a simple and direct proof in
205 Appendix A. Note that it is the complete analogue of Theorem 1.3 for system (9). By Lemma
206 2.1, the minimal null control time for the initial system (1) is thus also completely determined
207 by the sets \mathcal{S}_k . However, apart from \mathcal{S}_{n+1} , these sets are not explicit in terms of M , which is
208 unsatisfactory.

209 Assumption (12) is indeed satisfied in our framework because we can always find an analytic
210 solution to the kernel equations since M is constant. More precisely, we have the following result:

211 **Theorem 2.5.** *Let $m, p \geq 1$ be arbitrary. Assume that $M \in \mathbb{R}_0^{n \times n}$. For any $\delta \in \mathbb{R}$ with $\delta \neq 1$,
212 there exists a unique $K \in C^\infty(\mathbb{R}^2)^{n \times n}$ that satisfies (11) for every $(x, \xi) \in \mathbb{R}^2$ and the condition*

$$213 \quad \text{diag } K(x, \delta x) = 0, \quad \forall x \in \mathbb{R}. \quad (13)$$

214 Moreover, it satisfies the estimate

$$215 \quad \forall \text{ bounded } V \subset \mathbb{R}^2, \exists C > 0, \quad \|K\|_{C^s(\bar{V})^{n \times n}} \leq C^s, \quad \forall s \in \mathbb{N}. \quad (14)$$

216 The kernel equations (11) have been extensively studied in the literature (see e.g. [Cor+13;
 217 DVK13; HD15; Hu+16; Hu+19; CN19]) but Theorem 2.5 does not follow from the results
 218 contained therein. The most important difference is that, in Theorem 2.5, the kernel exists on
 219 a larger set than the triangle \mathcal{T} . This is crucial since we want $x \mapsto f(x) = -K(x, 0)\Lambda(1 - q)^\top$
 220 to be analytic in an interval of the form $(-\varepsilon, 1)$, $\varepsilon > 0$, which does not lie entirely in \mathcal{T} . This
 221 yields nontrivial issues in the standard fixed point approach, notably because we now have to
 222 consider points that are “on the other side” of the diagonal (x, x) , that is the condition imposed
 223 in the kernel equations at (x, x) cannot be consider as a boundary condition anymore. We have
 224 developed in Appendix B a new approach to solve the kernel equations that encompasses in
 225 particular the proof of Theorem 2.5.

226 **Remark 2.6.** Estimate (14) and Taylor’s theorem show that the solution K to (11)-(13) is in
 227 fact a power series.

228 As a consequence of Theorem 2.5, we see that, if $q_{i-1} = 0$, then $f = -K(\cdot, 0)\Lambda(1 - q)^\top$
 229 satisfies $f_i = 0$ in $(0, 1)$ if, and only if,

$$230 \quad \left\langle b, \frac{\partial^r k^c}{\partial x^r}(0, 0) \right\rangle = 0, \quad \forall r \in \mathbb{N}, \quad (15)$$

231 where $b \in \mathbb{R}^{n-1}$ is defined in (5) and $k^c = (k_{ij})_{j \neq i}$. It remains to relate the derivatives of the
 232 kernel at the origin with M . This is the purpose of the next section. This will be done only for
 233 a very well chosen particular solution to the kernel equations (i.e. for one $\delta \neq 1$) but this will be
 234 enough for our purposes as already underlined in Remark 2.2.

235 **Remark 2.7.** We emphasize that, in all the sections below and unless specifically mentioned,
 236 the number of negative speeds m is arbitrary (the orthogonality conditions (15) are studied for
 237 any nonzero $b \in \mathbb{R}^{n-1}$).

238 3 The derivatives of the kernel at the origin

239 3.1 Normalization of the equations

240 Let us first observe that a feature of the kernel equations (11)-(13) is that it does not couple
 241 different rows of K :

$$242 \quad \begin{cases} \lambda_i \frac{\partial k_{ij}}{\partial x}(x, \xi) + \frac{\partial k_{ij}}{\partial \xi}(x, \xi)\lambda_j + \sum_{r=1}^n k_{ir}(x, \xi)m_{rj} = 0, \\ \lambda_i k_{ij}(x, x) - k_{ij}(x, x)\lambda_j = m_{ij} \quad (j \neq i), \quad k_{ii}(x, \delta x) = 0. \end{cases} \quad (16)$$

243 Therefore, all along Section 3, $i \in \{1, \dots, n\}$ is fixed and we will drop the dependence on i for
 244 clarity.

245 Let us now introduce some important notations. Some of them have already been introduced
 246 in Section 1.3 but they are recalled here for the sake of the presentation.

- 247 • It is convenient to normalize the kernel equations by $\lambda_i - \lambda_j$ for $j \neq i$ and by λ_i otherwise.
 248 The kernel equations (16) become

$$249 \quad \begin{cases} \mu_j \frac{\partial k_j}{\partial x}(x, \xi) + \frac{\partial k_j}{\partial \xi}(x, \xi)\nu_j + \sum_{r=1}^n k_r(x, \xi)\alpha_{rj} = 0, \\ k_j(x, x) = \alpha_{ij} \quad (j \neq i), \quad k_i(x, \delta x) = 0, \end{cases} \quad (17)$$

where $k = (k_{i1} \ \cdots \ k_{in})^\top$ and

$$\mu_j = \frac{\lambda_i}{\lambda_i - \lambda_j}, \quad \nu_j = \frac{\lambda_j}{\lambda_i - \lambda_j}, \quad (j \neq i), \quad \mu_i = \nu_i = 1,$$

$$\alpha_{rj} = \frac{m_{rj}}{\lambda_i - \lambda_j}, \quad (j \neq i), \quad \alpha_{ri} = \frac{m_{ri}}{\lambda_i}.$$

250 Note that, with this normalization, we have in particular $\mu_j - \nu_j = 1$ for $j \neq i$.

251 • Since the component k_i plays a different role than all the other components k_j with $j \neq i$,
 252 we rewrite (17) in a matrix form separating both quantities. Let us denote by $n^c = n - 1$
 253 and introduce $k^c = (k_j)_{j \neq i}$. Then, system (17) can be written as

$$254 \quad \begin{cases} D_\mu \frac{\partial k^c}{\partial x}(x, \xi) + D_\nu \frac{\partial k^c}{\partial \xi}(x, \xi) + A^\top k^c(x, \xi) + k_i(x, \xi) \psi^0 = 0, \\ \frac{\partial k_i}{\partial x}(x, \xi) + \frac{\partial k_i}{\partial \xi}(x, \xi) + \langle w, k^c(x, \xi) \rangle = 0, \\ k^c(x, x) = \psi^0, \quad k_i(x, \delta x) = 0, \end{cases} \quad (18)$$

where $D_\mu, D_\nu, A \in \mathbb{R}^{n^c \times n^c}$ are the matrices defined by

$$D_\mu = \text{diag}(\mu_j)_{j \neq i}, \quad D_\nu = \text{diag}(\nu_j)_{j \neq i}, \quad A = (\alpha_{rj})_{r, j \neq i},$$

and $\psi^0, w \in \mathbb{R}^{n^c}$ are the vectors defined by

$$\psi^0 = (\alpha_{ij})_{j \neq i}, \quad w = (\alpha_{ji})_{j \neq i}.$$

255 Note that we used that $\alpha_{ii} = 0$ (since $M \in \mathbb{R}_0^{n \times n}$). Finally, it will also be convenient to
 256 use the matrix $D \in \mathbb{R}^{n^c \times n^c}$ and the vector $v \in \mathbb{R}^{n^c}$ defined by

$$257 \quad D = D_\mu + D_\nu, \quad v = -\frac{1}{2}w. \quad (19)$$

258 3.2 Computation of the derivatives

259 The main result of this section is the following.

260 **Theorem 3.1.** *For the solution to (18) with $\delta = -1$, we have*

$$261 \quad \frac{\partial^{\gamma+\sigma} k^c}{\partial x^\gamma \partial \xi^\sigma}(0, 0) = \sum_{r=0}^{\gamma} \sum_{s=0}^{\sigma} (-1)^r \binom{\gamma}{r} \binom{\sigma}{s} \psi_{\gamma+\sigma-(r+s), r+s}, \quad \forall \gamma, \sigma \in \mathbb{N}, \quad (20)$$

262 where $(\psi_{r,s})_{r,s \in \mathbb{N}}$ is the sequence defined by

$$263 \quad \psi_{r,0} = \psi^r, \quad \psi_{r,s} = 0 \quad \text{if } r < s, \quad \psi_{r,s} = \sum_{\ell=0}^{r-s} \langle v, \psi_{r-1-\ell, s-1} \rangle \psi_{\ell,0} \quad \text{if } r \geq s \geq 1, \quad (21)$$

264 where $(\psi^r)_{r \in \mathbb{N}}$ is the sequence defined in (4).

265 Combining this result with the estimates (14) and Taylor's theorem, we obtain an explicit
 266 formula for the solution to (18) when $\delta = -1$:

Corollary 3.2. For $\delta = -1$, the solution to (18) is given by

$$k^c(x, \xi) = \sum_{r=0}^{+\infty} \sum_{s=0}^{+\infty} \frac{(-1)^r}{r!s!} \psi_{r,s} (x - \xi)^r (x + \xi)^s, \quad k_i(x, \xi) = - \int_{\frac{x-\xi}{2}}^x \langle w, k^c(\sigma, \sigma - x + \xi) \rangle d\sigma,$$

267 for every $(x, \xi) \in \mathbb{R}^2$, where $(\psi_{r,s})_{r,s \in \mathbb{N}}$ is the sequence defined by (21) and the series is normally
268 convergent on any compact set of \mathbb{R}^2 .

269 **Remark 3.3.** Explicit solutions to the kernel equations were also obtained in [VK14, Section
270 3.4] for systems of $n = 2$ equations.

Proof of Theorem 3.1. 1. To explain the special role played by $\delta = -1$, we start the computations with an arbitrary $\delta \neq 1$. The first idea is to form a system involving only k^c by expressing k_i as a function of k^c :

$$k_i(x, \xi) = - \int_{\frac{x-\xi}{1-\delta}}^x \langle w, k^c(\sigma, \sigma - x + \xi) \rangle d\sigma.$$

As a result, k^c solves

$$\begin{cases} D_\mu \frac{\partial k^c}{\partial x}(x, \xi) + D_\nu \frac{\partial k^c}{\partial \xi}(x, \xi) + A^\top k^c(x, \xi) - \left(\int_{\frac{x-\xi}{1-\delta}}^x \langle w, k^c(\sigma, \sigma - x + \xi) \rangle d\sigma \right) \psi^0 = 0, \\ k^c(x, x) = \psi^0. \end{cases}$$

We now transform this system into a Cauchy problem by introducing the transformation

$$h(t, \theta) = k^c \left(\frac{-t + \theta}{2}, \frac{t + \theta}{2} \right).$$

271 Using that $D_\mu - D_\nu = \text{Id}_{n^c}$, we can check that h satisfies the system

$$\begin{cases} \frac{\partial h}{\partial t}(t, \theta) = D \frac{\partial h}{\partial \theta}(t, \theta) + A^\top h(t, \theta) + \left(\int_{-\frac{1+\delta}{1-\delta}t}^\theta \langle v, h(t, \eta) \rangle d\eta \right) \psi^0, \\ h(0, \theta) = \psi^0, \end{cases} \quad (22)$$

where we recall that D and v are defined in (19). Note as well that

$$\frac{\partial^{\gamma+\sigma} k^c}{\partial x^\gamma \partial \xi^\sigma}(x, \xi) = \sum_{r=0}^\gamma \sum_{s=0}^\sigma (-1)^r \binom{\gamma}{r} \binom{\sigma}{s} \frac{\partial^{\gamma+\sigma} h}{\partial t^{\gamma+\sigma-(r+s)} \partial \theta^{r+s}}(-x + \xi, x + \xi), \quad \forall \gamma, \sigma \in \mathbb{N},$$

so that the derivatives of k^c at $(0, 0)$ can be deduced from those of h . They will be computed from (22) and we see that the computations considerably simplify if the lower bound of the integral vanishes, that is if we choose $\delta = -1$. For this choice, we define

$$\psi_{r,s} = \frac{\partial^{r+s} h}{\partial t^r \partial \theta^s}(0, 0).$$

273 We are going to show that it satisfies (21).

274 2. All along the rest of the proof, we will use the notation $c_{r,s} = \langle v, \psi_{r,s} \rangle$. First observe that
 275 system (22) (with $\delta = -1$) yields the following identities:

$$276 \quad \begin{cases} \psi_{r+1,0} = D\psi_{r,1} + A^\top \psi_{r,0}, \\ \psi_{r+1,s} = D\psi_{r,s+1} + A^\top \psi_{r,s} + c_{r,s-1} \psi_{0,0}, \\ \psi_{0,s} = 0, \end{cases} \quad (23)$$

277 for every $r \geq 0$ and $s \geq 1$. The second property in (21) is easily proved by induction on
 278 $r \geq 1$. To establish the two other identities, it is sufficient to prove the following statement
 279 for any $N \geq 1$:

$$280 \quad \begin{cases} \psi_{s+q,s} = \sum_{\ell=0}^q c_{s-1+q-\ell,s-1} \psi_{\ell,0}, & \forall s \geq 1, \forall 0 \leq q \leq N, \\ \psi_{r,0} = A^\top \psi_{r-1,0} + D \sum_{\ell=0}^{r-2} c_{r-2-\ell,0} \psi_{\ell,0}, & \forall 2 \leq r \leq N+1. \end{cases} \quad (24)$$

We prove it by induction. For $N = 1$, this is clear. Indeed, for any $s \geq 1$, we have

$$\begin{aligned} \psi_{s,s} &= D\psi_{s-1,s+1} + A^\top \psi_{s-1,s} + c_{s-1,s-1} \psi_{0,0} \quad (\text{by (23)}), \\ &= c_{s-1,s-1} \psi_{0,0} \quad (\text{by the second property in (21)}), \end{aligned} \quad (25)$$

$$\begin{aligned} \psi_{s+1,s} &= D\psi_{s,s+1} + A^\top \psi_{s,s} + c_{s,s-1} \psi_{0,0} \quad (\text{by (23)}), \\ &= A^\top \psi_{s,s} + c_{s,s-1} \psi_{0,0} \quad (\text{by the second property in (21)}), \\ &= c_{s-1,s-1} \psi_{1,0} + c_{s,s-1} \psi_{0,0} \quad (\text{by (25)}), \end{aligned}$$

and

$$\begin{aligned} \psi_{2,0} &= D\psi_{1,1} + A^\top \psi_{1,0} \quad (\text{by (23)}), \\ &= c_{0,0} D\psi_{0,0} + A^\top \psi_{1,0} \quad (\text{by (25)}). \end{aligned}$$

281 Assume now that (24) holds for $N \geq 1$ and let us prove it for $N+1$. We first show that

$$282 \quad \psi_{s+N+1,s} = \sum_{\ell=0}^{N+1} c_{s+N-\ell,s-1} \psi_{\ell,0}, \quad \forall s \geq 1. \quad (26)$$

For any $s \geq 1$, we have

$$\begin{aligned} \psi_{s+N+1,s} &= D\psi_{s+N,s+1} + A^\top \psi_{s+N,s} + c_{s+N,s-1} \psi_{0,0} \quad (\text{by (23)}), \\ &= D \sum_{\ell=0}^{N-1} c_{s+N-1-\ell,s} \psi_{\ell,0} + A^\top \sum_{\ell=0}^N c_{s-1+N-\ell,s-1} \psi_{\ell,0} \\ &\quad + c_{s+N,s-1} \psi_{0,0} \quad (\text{by assumption (24)}), \\ &= D \sum_{\ell=0}^{N-1} c_{s+N-1-\ell,s} \psi_{\ell,0} + \sum_{r=2}^{N+1} c_{s+N-r,s-1} A^\top \psi_{r-1,0} \\ &\quad + c_{s-1+N,s-1} \psi_{1,0} + c_{s+N,s-1} \psi_{0,0}. \end{aligned}$$

283 Therefore, if we show the identity

$$284 \quad \sum_{\ell=0}^{N-1} c_{s+N-1-\ell,s} \psi_{\ell,0} = \sum_{r=2}^{N+1} c_{s+N-r,s-1} \left(\sum_{\ell=0}^{r-2} c_{r-2-\ell,0} \psi_{\ell,0} \right), \quad (27)$$

then we can use the second condition in (24) to obtain the desired identity (26). To establish (27), we use the first condition in (24) to deduce that, for every $0 \leq \ell \leq N-1$,

$$\begin{aligned} \langle v, \psi_{s+N-1-\ell,s} \rangle &= \sum_{j=0}^{N-1-\ell} c_{s-1+N-1-\ell-j,s-1} c_{j,0} \\ &= \sum_{r=\ell+2}^{N+1} c_{s+N-r,s-1} c_{r-2-\ell,0}. \end{aligned}$$

Finally, a simple change of order of summation leads to (27). It remains to show the second identity in (24) for $r = N+2$, namely

$$\psi_{N+2,0} = A^T \psi_{N+1,0} + D \sum_{\ell=0}^N c_{N-\ell,0} \psi_{\ell,0}.$$

We have

$$\begin{aligned} \psi_{N+2,0} &= D \psi_{N+1,1} + A^T \psi_{N+1,0} \quad (\text{by (23)}), \\ &= D \sum_{\ell=0}^N c_{N-\ell,0} \psi_{\ell,0} + A^T \psi_{N+1,0} \quad (\text{by (24)}). \end{aligned}$$

285

□

286 **Remark 3.4.** Theoretically, we can also compute all the derivatives at $(0,0)$ of the solution to
 287 (18) for arbitrary $\delta \neq 1$. This can be done by taking derivatives and inverting some matrix.
 288 However, the size of this matrix grows with the order of derivatives and computations rapidly
 289 become more and more complicated. It seems difficult with such a strategy to obtain a suitable
 290 formula for arbitrary δ . At the same time, we see from (22) that a different choice of δ means
 291 more derivatives to be computed, as for instance with $\delta = 0$ which leads to an integral of the
 292 form $\int_{-\ell}^{\theta}$. In addition to that, we recall that one choice of δ is actually sufficient for the purposes
 293 of this paper (Remark 2.2).

294 4 Study of the orthogonality conditions

295 In this section, we use the computations obtained in the previous section to study the orthogo-
 296 nality conditions (15). We start with the conclusion of the proof of our first main result.

Proof of Theorem 1.3. We recall that, from the results of the previous sections, we only have to show the equivalence between the orthogonality conditions (15) and (6). First observe that, from the definition (21) of the sequence $(\psi_{r,s})_{r,s \in \mathbb{N}}$, it is clear that (6) is equivalent to

$$\langle b, \psi_{r,s} \rangle = 0, \quad \forall r, s \in \mathbb{N}.$$

297

We can check that this condition is equivalent to (15) using (20) and (21).

□

298 **Remark 4.1.** The proof above and the analyticity of k^c in \mathbb{R}^2 shows that the following four
 299 properties are in fact equivalent:

- 300 1. $\langle b, k^c(x, \xi) \rangle = 0$ for every $(x, \xi) \in \mathbb{R}^2$.
 301 2. $\langle b, k^c(x, 0) \rangle = 0$ for every $x \in \mathbb{R}$.
 302 3. $\langle b, \psi_{r,0} \rangle = 0$ for every $r \in \mathbb{N}$.
 303 4. $\langle b, \psi_{r,s} \rangle = 0$ for every $r, s \in \mathbb{N}$.

304 We are now going to study the orthogonality conditions (6) for the sequence $(\psi^r)_{r \in \mathbb{N}}$ and
 305 prove our second main result. For the rest of Section 4, the matrices $A, D \in \mathbb{R}^{n^c \times n^c}$ and the
 306 vectors $\psi^0, v, b \in \mathbb{R}^{n^c}$ can in fact be arbitrary. We emphasize that n is also arbitrary, it is only
 307 during the proof of Theorem 1.6 that we will assume that $n = 3$ or $n = 4$.

308 4.1 Some invariant subspaces of the sequence

309 We start the general study of the orthogonality conditions (6) with the description of two simple
 310 invariant subspaces of the sequence $(\psi^r)_{r \in \mathbb{N}}$.

Proposition 4.2. *Assume that $\psi^0 \in E$ for some $E \subset \mathbb{R}^{n^c}$ satisfying one of the following two conditions:*

$$A^\top(E) \subset E, \quad E \subset \ker v^\top. \quad (28)$$

$$A^\top(E) \subset E, \quad D(E) \subset E. \quad (29)$$

311 Then, $\psi^r \in E$ for every $r \in \mathbb{N}$.

Proof. We prove the result by induction on r . For $r = 0$ this is trivial and for $r = 1$ this follows
 from the definition $\psi^1 = A^\top \psi^0$ and the property $A^\top(E) \subset E$. Assume then that $\psi^\ell \in E$ for
 every $0 \leq \ell \leq r$ for some $r \geq 1$ and let us show that $\psi^{r+1} \in E$. Since $r + 1 \geq 2$, we have

$$\psi^{r+1} = A^\top \psi^r + \sum_{\ell=0}^{r-1} \langle v, \psi^{r-1-\ell} \rangle D \psi^\ell.$$

312 Clearly, the first part $A^\top \psi^r$ belongs to E since $\psi^r \in E$ and $A^\top(E) \subset E$. The remaining part
 313 also belongs to E since either $\langle v, \psi^\ell \rangle = 0$ for every $0 \leq \ell \leq r$ (if $E \subset \ker v^\top$) or $D \psi^\ell \in E$ for
 314 every $0 \leq \ell \leq r$ (if $D(E) \subset E$). \square

315 If we can find a subspace E such as in the previous proposition and which is in addition
 316 included in $\ker b^\top$, then we see that the whole sequence will be guaranteed to stay in $\ker b^\top$.

317 4.2 Characterization of rank one sequences

318 In this section, we characterize when the rank of $(\psi^r)_{r \in \mathbb{N}}$ is equal to one, and we use it to prove
 319 Theorem 1.6 in the case $n = 3$. We recall that, by definition, $\text{rank}(\psi^r)_{r \in S} = \dim \text{span} \{ \psi^r \mid r \in S \}$
 320 for any $S \subset \mathbb{N}$.

From now on, it will be convenient to use the following notation:

$$E_s = \text{span} \{ \psi^r \mid r \leq s \}, \quad \forall s \in \mathbb{N}.$$

First of all, it is clear that

$$\text{rank}(\psi^r)_{r \in \mathbb{N}} = 1 \iff (\psi^0 \neq 0, \quad \psi^r \in E_0, \quad \forall r \geq 1).$$

321 We have the following characterization:

322 **Proposition 4.3.** *The following three conditions are equivalent:*

- 323 1. $\psi^r \in E_0$ for every $r \geq 1$.
 324 2. $\psi^r \in E_0$ for every $r \in \{1, 2\}$.
 325 3. E_0 satisfies (28) or (29).

Proof. The implication $1 \implies 2$ is trivial. The implication $3 \implies 1$ follows from Proposition 4.2. Let us show that $2 \implies 3$. We write

$$\psi^r = \alpha_r \psi^0, \quad r = 1, 2,$$

326 for some $\alpha_r \in \mathbb{R}$. The condition for $r = 1$ gives $A^\top \psi^0 = \alpha_1 \psi^0$, which is equivalent to $A^\top(E_0) \subset$
 327 E_0 . The condition for $r = 2$ gives $A^\top \psi^1 + \langle v, \psi^0 \rangle D\psi^0 = \alpha_2 \psi^0$, which implies $\langle v, \psi^0 \rangle D\psi^0 \in E_0$,
 328 that is either $E_0 \subset \ker v^\top$ or $D(E_0) \subset E_0$. This establishes the desired equivalences.
 329 □

Proof of Theorem 1.6 (case $n = 3$). Assume that

$$\langle b, \psi^r \rangle = 0, \quad \forall r \in \{0, 1, 2\}.$$

Since $n^c = 2$, we have $\dim \ker b^\top = 1$ and thus

$$\text{rank}(\psi^r)_{r \in \{0, 1, 2\}} \leq 1.$$

330 As a result, $\psi^r \in E_0$ for $r \in \{1, 2\}$ and it follows that $\psi^r \in E_0$ for every $r \geq 1$ by Proposition
 331 4.3. □

332 4.3 Characterization of rank two sequences

333 In this section, we characterize when the rank of $(\psi^r)_{r \in \mathbb{N}}$ is equal to two, and we use it to prove
 334 Theorem 1.6 in the case $n = 4$. The study of rank two sequences is considerably more difficult
 335 than for rank one.

336 We start with the following simple observation.

337 **Proposition 4.4.** *Let $\{0, 1, 2\} \subset S \subset \mathbb{N}$. We have $\text{rank}(\psi^r)_{r \in S} = 2$ if, and only if, we have one*
 338 *of the following two conditions:*

- 339 1. $\text{rank}(\psi^0|\psi^1) = 1$, $\text{rank}(\psi^0|\psi^2) = 2$ and $\psi^r \in E_2$ for every $r \in S \setminus \{0, 1, 2\}$.
 340 2. $\text{rank}(\psi^0|\psi^1) = 2$ and $\psi^r \in E_1$ for every $r \in S \setminus \{0, 1\}$.

341 *Proof.* We only need to observe that the situation $\text{rank}(\psi^0|\psi^1) = \text{rank}(\psi^0|\psi^2) = 1$ does not
 342 happen. Indeed, in that situation we have $\psi^1, \psi^2 \in E_0$ and thus $\psi^r \in E_0$ for every $r \geq 1$ by
 343 Proposition 4.3, which shows that the sequence cannot be of rank two. □

344 We now characterize both conditions of Proposition 4.4.

345 **Proposition 4.5.** *Assume that $\text{rank}(\psi^0|\psi^1) = 1$ and $\text{rank}(\psi^0|\psi^2) = 2$. Then, the following*
 346 *three conditions are equivalent:*

- 347 1. $\psi^r \in E_2$ for every $r \geq 3$.
 348 2. $\psi^r \in E_2$ for every $r \in \{3, 4\}$.

349 3. E_2 satisfies (29).

350 *Proof.* The implication $1 \implies 2$ is trivial. The implication $3 \implies 1$ follows from Proposition
 351 4.2. Let us show that $2 \implies 3$. From the definition of ψ^2 and the rank assumptions, we have
 352 $\langle v, \psi^0 \rangle \neq 0$ and $D\psi^0 \in \text{span} \{\psi^0, \psi^2\}$. Using these facts, we can first check that the condition
 353 $\psi^3 \in E_2$ gives $A^\top(E_2) \subset E_2$ and then that the condition $\psi^4 \in E_2$ yields $D(E_2) \subset E_2$. \square

354 The second condition in Proposition 4.4 is more difficult to characterize.

355 **Proposition 4.6.** *Assume that $\text{rank}(\psi^0|\psi^1) = 2$. Then, the following two conditions are equiv-*
 356 *alent:*

- 357 1. $\psi^r \in E_1$ for every $r \geq 2$.
 358 2. $\psi^r \in E_1$ for every $r \in \{2, \dots, 5\}$.

359 Before proving Proposition 4.6, we prove our second main result.

Proof of Theorem 1.6 (case $n = 4$). Assume that

$$\langle b, \psi^r \rangle = 0, \quad \forall r \in \{0, \dots, 5\}.$$

Since $n^c = 3$, we have $\dim \ker b^\top = 2$ and thus

$$\text{rank}(\psi^r)_{r \in \{0, \dots, 5\}} \leq 2.$$

360 If the rank is in fact less than or equal to 1, then we conclude as in the proof of Theorem 1.6 in
 361 the case $n = 3$. If the rank is exactly equal to 2, then the conclusion follows from Propositions
 362 4.4, 4.5 and 4.6. \square

363 We now turn to the proof of the key proposition.

364 *Proof of Proposition 4.6.* If E_1 satisfies (28) or (29), then the result follows from Proposition 4.2.
 365 Therefore, from now on, we assume that E_1 does not meet any of these conditions.

- 366 1. Let $N \geq 4$ be arbitrary and consider the property: for every $r \in \{2, \dots, N\}$, there exist
 367 $\alpha_r, \beta_r \in \mathbb{R}$ such that

$$368 \quad \psi^r = \alpha_r \psi^0 + \beta_r \psi^1. \quad (30)$$

369 Let us also set

$$370 \quad \begin{aligned} \alpha_0 &= 1, & \beta_0 &= 0, \\ \alpha_1 &= 0, & \beta_1 &= 1, \end{aligned} \quad (31)$$

so that the previous identity is always true for $r = 0, 1$. Using the definition (4) of the sequence we see that, for $r \geq 2$, identity (30) is equivalent to

$$\begin{aligned} \alpha_r \psi^0 + \beta_r \psi^1 &= \alpha_{r-1} A^\top \psi^0 + \beta_{r-1} A^\top \psi^1 \\ &+ \left(\sum_{\ell=0}^{r-2} c_{r-2-\ell} \alpha_\ell \right) D\psi^0 + \left(\sum_{\ell=0}^{r-2} c_{r-2-\ell} \beta_\ell \right) D\psi^1, \end{aligned}$$

where we introduced, for every $s \in \{0, \dots, N\}$,

$$c_s = \alpha_s \langle v, \psi^0 \rangle + \beta_s \langle v, \psi^1 \rangle.$$

371 Let us eliminate the terms $A^\top \psi^0, A^\top \psi^1$. By definition of the sequence, we have $A^\top \psi^0 = \psi^1$.
 372 On the other hand, condition (30) for $r = 2$ yields

$$373 \quad A^\top \psi^1 = \alpha_2 \psi^0 + \beta_2 \psi^1 - c_0 D \psi^0. \quad (32)$$

374 As a result, for $r \geq 3$, identity (30) is equivalent to

$$375 \quad (\psi^0 | \psi^1) u_r = (D \psi^0 | D \psi^1) f_{r-1}, \quad (33)$$

376 where $u_r, f_{r-1} \in \mathbb{R}^2$ are the vectors defined by

$$377 \quad u_r = \begin{pmatrix} \alpha_r - \beta_{r-1} \alpha_2 \\ \beta_r - (\alpha_{r-1} + \beta_{r-1} \beta_2) \end{pmatrix}, \quad f_{r-1} = \begin{pmatrix} \beta_{r-1}(-c_0) + \sum_{\ell=0}^{r-2} c_{r-2-\ell} \alpha_\ell \\ \sum_{\ell=0}^{r-2} c_{r-2-\ell} \beta_\ell \end{pmatrix}. \quad (34)$$

378 Let us make some observations.

- 379 • For any r_1, r_2 , if $\lambda f_{r_1-1} + \mu f_{r_2-1} = 0$ for some $\lambda, \mu \in \mathbb{R}$, then $\lambda u_{r_1} + \mu u_{r_2} = 0$. Indeed,
 380 denoting by $u = \lambda u_{r_1} + \mu u_{r_2}$, we have $(\psi^0 | \psi^1) u = 0$ and thus $u = 0$ since we assumed
 381 that ψ^0, ψ^1 are linearly independent.
- 382 • f_{r_1-1} and f_{r_2-1} are linearly dependent for any r_1, r_2 . Indeed, otherwise we obtain
 383 $(\psi^0 | \psi^1) U = (D \psi^0 | D \psi^1)$ for some matrix $U \in \mathbb{R}^{2 \times 2}$. This means that $D(E_1) \subset E_1$.
 384 It then follows from (32) that $A^\top(E_1) \subset E_1$ as well. Therefore, E_1 satisfies (29), but
 385 this situation has been excluded from the beginning of the proof.
- 386 • f_2 and f_3 are linearly dependent only if $c_0 \neq 0$. Indeed, if $c_0 = 0$, then $A^\top \psi^1 \in E_1$
 387 by (32), so that $A^\top(E_1) \subset E_1$. If $c_0 = 0$, we also have $\det(f_2 | f_3) = c_1^2$, which cannot
 388 be zero since otherwise $c_0 = c_1 = 0$, that is $E_1 \subset \ker v^\top$, and thus E_1 satisfies (28)
 389 (excluded).

The necessary condition $\det(f_2 | f_{r-1}) = 0$ is equivalent to the identity

$$\left(\sum_{\ell=0}^{r-2} c_{r-2-\ell} \beta_\ell \right) f_2 - c_0 f_{r-1} = 0,$$

which, by the first observation above, in turn implies that

$$\left(\sum_{\ell=0}^{r-2} c_{r-2-\ell} \beta_\ell \right) u_3 - c_0 u_r = 0.$$

390 Since $c_0 \neq 0$, this gives the following formulas:

$$391 \quad \begin{cases} f_{r-1} = \left(\sum_{\ell=0}^{r-2} \bar{c}_{r-2-\ell} \beta_\ell \right) f_2, \\ u_r = \left(\sum_{\ell=0}^{r-2} \bar{c}_{r-2-\ell} \beta_\ell \right) u_3, \end{cases} \quad (35)$$

392 where we introduced $\bar{c}_s = \frac{c_s}{c_0}$. Conversely, it is clear that if we have (35), then (33) will
 393 also hold for $r \geq 3$, provided that it holds for $r = 3$.

394 Finally, observe that the second formula in (35), combined with the definition (34) of u_r ,
 395 uniquely determines all the α_r, β_r for $r \geq 4$ as a function of α_2, β_2 and α_3, β_3 :

$$396 \quad \begin{cases} \alpha_r = \beta_{r-1}\alpha_2 + \left(\sum_{\ell=0}^{r-2} \bar{c}_{r-2-\ell}\beta_\ell \right) (\alpha_3 - \beta_2\alpha_2), \\ \beta_r = \alpha_{r-1} + \beta_{r-1}\beta_2 + \left(\sum_{\ell=0}^{r-2} \bar{c}_{r-2-\ell}\beta_\ell \right) (\beta_3 - (\alpha_2 + \beta_2^2)), \end{cases} \quad (36)$$

397 and that the first formula in (35) is equivalent to

$$398 \quad \beta_{r-1} = \sum_{\ell=0}^{r-2} \bar{c}_{r-2-\ell} (\alpha_\ell + (\beta_2 - \bar{c}_1)\beta_\ell). \quad (37)$$

399 In summary, we have shown that the property considered is equivalent to: $c_0 \neq 0$ and there
 400 exist α_2, β_2 and α_3, β_3 such that (30) holds for $r = 2, 3$ and such that the sequence defined
 401 by (36) (with (31)) satisfies (37) for every $r \in \{4, \dots, N\}$.

402 2. Let us now study the sequence (36). The proof of the result will be complete after we show
 403 that the following three conditions are equivalent:

- 404 (a) Condition (37) holds for every $r \geq 4$.
 405 (b) Condition (37) holds for $r = 4, 5$.
 406 (c) α_3 and β_3 are given by

$$407 \quad \begin{cases} \beta_3 = (\alpha_2 + \beta_2^2) - (\beta_2 - \bar{c}_1)^2 + \alpha_2 + (\beta_2 - \bar{c}_1)\beta_2, \\ \alpha_3 = \beta_2\alpha_2 - (\beta_2 - \bar{c}_1)(\beta_3 - (\alpha_2 + \beta_2^2)). \end{cases} \quad (38)$$

We start with the implication 2b \implies 2c. Condition (37) for $r = 4$ immediately gives β_3 as
 a function of α_2, β_2 :

$$\beta_3 = \sum_{\ell=0}^2 \bar{c}_{2-\ell} (\alpha_\ell + (\beta_2 - \bar{c}_1)\beta_\ell).$$

On the other hand, condition (37) for $r = 5$ gives

$$\begin{aligned} \beta_4 &= \sum_{\ell=1}^2 \bar{c}_{3-\ell} (\alpha_\ell + (\beta_2 - \bar{c}_1)\beta_\ell) + \bar{c}_0 (\alpha_3 + (\beta_2 - \bar{c}_1)\beta_3) + \bar{c}_3 (\alpha_0 + (\beta_2 - \bar{c}_1)\beta_0) \\ &= 2\alpha_3 + \beta_2\beta_3 + \sum_{\ell=1}^2 \bar{c}_{3-\ell} (\alpha_\ell + (\beta_2 - \bar{c}_1)\beta_\ell), \end{aligned}$$

whereas, by definition (36),

$$\beta_4 = \alpha_3 + \beta_3\beta_2 + \left(\sum_{\ell=0}^2 \bar{c}_{2-\ell}\beta_\ell \right) (\beta_3 - (\alpha_2 + \beta_2^2)).$$

Identifying both expressions gives α_3 as a function of α_2, β_2 :

$$\alpha_3 = - \sum_{\ell=1}^2 \bar{c}_{3-\ell} (\alpha_\ell + (\beta_2 - \bar{c}_1)\beta_\ell) + \left(\sum_{\ell=0}^2 \bar{c}_{2-\ell}\beta_\ell \right) (\beta_3 - (\alpha_2 + \beta_2^2)).$$

408 We can check that the previous formulas are equivalent to (38) (we prefer the expressions
409 in (38) because they make appear some coefficients involved in (36)).

410 Let us now prove the implication 2c \implies 2a. We prove it by induction on r . For $r = 4$,
411 this holds by very definition of β_3 as we have seen above. Assume now that (37) holds for
412 some arbitrary $r \geq 4$, and let us prove it for $r + 1$, that is to prove that we have

$$413 \quad \beta_r = \sum_{\ell=0}^{r-1} \bar{c}_{r-1-\ell} (\alpha_\ell + (\beta_2 - \bar{c}_1)\beta_\ell). \quad (39)$$

By definition (36) of β_r , we have

$$\begin{aligned} \beta_r &= \alpha_{r-1} + \beta_{r-1}\bar{c}_1 + (\beta_2 - \bar{c}_1) \left(\beta_{r-1} - (\beta_2 - \bar{c}_1) \sum_{\ell=0}^{r-2} \bar{c}_{r-2-\ell}\beta_\ell \right) \\ &\quad + (\beta_3 - (\alpha_2 + \beta_2^2) + (\beta_2 - \bar{c}_1)^2) \sum_{\ell=0}^{r-2} \bar{c}_{r-2-\ell}\beta_\ell. \end{aligned}$$

Using the induction hypothesis (37) and the definition (38) of β_3 , we obtain

$$\beta_r = \bar{c}_{r-1} + \sum_{\ell=0}^{r-2} \bar{c}_{r-2-\ell} ((\beta_2 - \bar{c}_1)\alpha_\ell + (\alpha_2 + (\beta_2 - \bar{c}_1)\beta_2)\beta_\ell).$$

Now observe that, using the definition of α_3 , we have

$$\begin{aligned} \alpha_{\ell+1} + (\beta_2 - \bar{c}_1)\beta_{\ell+1} &= \beta_\ell\alpha_2 + (\beta_2 - \bar{c}_1)(\alpha_\ell + \beta_\ell\beta_2) \\ &= (\beta_2 - \bar{c}_1)\alpha_\ell + (\alpha_2 + (\beta_2 - \bar{c}_1)\beta_2)\beta_\ell. \end{aligned}$$

414 Formula (39) easily follows this identity and the previous one.

415 □

416 5 Kernel associated with the orthogonality conditions

417 In this section, we supplement our results by giving a more explicit characterization of the
418 conditions found in the previous section and that guaranteed the orthogonality conditions. Then,
419 we discuss the structure of the associated kernel.

420 5.1 Kernel associated with the invariant subspaces

Here we discuss properties related to the invariant subspaces of Section 4.1. We recall that, for any vector $h \in \mathbb{R}^{n^c}$, we denote the Kalman matrix of (A, h) by

$$\text{Kal}(A, h) = (h|Ah|A^2h|\dots|A^{n^c-1}h) \in \mathbb{R}^{n^c \times n^c}.$$

421 **Proposition 5.1.** *1. There exists E satisfying (28), $\psi^0 \in E$ and $E \subset \ker b^\top$, if, and only if,*

$$422 \quad \text{Kal}(A, v)^\top \psi^0 = \text{Kal}(A, b)^\top \psi^0 = 0. \quad (40)$$

2. Assume that $\text{Kal}(A, v)^\top \psi^0 = 0$. Then, for any $\delta \neq 1$, the solution to the kernel equations (18) is

$$k_i(x, \xi) = 0, \quad k^c(x, \xi) = e^{-A^\top(x-\xi)}\psi^0, \quad \forall (x, \xi) \in \mathbb{R}^2.$$

423 If, moreover, $\text{Kal}(A, b)^\top \psi^0 = 0$, then the orthogonality conditions (15) are satisfied.

Proof. 1. Assume that (40) holds. Let us define

$$E = \ker \text{Kal}(A, v)^\top \cap \ker \text{Kal}(A, b)^\top.$$

424 By assumption, $\psi^0 \in E$ and it is clear that $E \subset \ker v^\top$ and $E \subset \ker b^\top$. Finally, E is stable
425 by A^\top thanks to Cayley-Hamilton theorem. Conversely, assume that (28) holds for some
426 $E \subset \ker b^\top$ with $\psi^0 \in E$. Since $\psi^0 \in E$ and E is stable by A^\top , we have $(A^\top)^k \psi^0 \in E$ for
427 every $k \in \mathbb{N}$. Since $E \subset \ker v^\top$ and $E \subset \ker b^\top$, we obtain (40).

2. We see from the kernel equations (18) that $k_i = 0$ if, and only if, we have

$$\begin{cases} D_\mu \frac{\partial k^c}{\partial x}(x, \xi) + D_\nu \frac{\partial k^c}{\partial \xi}(x, \xi) + A^\top k^c(x, \xi) = 0, \\ \langle v, k^c(x, \xi) \rangle = 0, \\ k^c(x, x) = \psi^0. \end{cases}$$

428 Using that $D_\mu - D_\nu = \text{Id}_{n^c}$, it is clear that $k^c(x, \xi) = e^{-A^\top(x-\xi)}\psi^0$ satisfies the first
429 equation. The second condition follows from the assumption $\text{Kal}(A, v)^\top \psi^0 = 0$ and Cayley-
430 Hamilton theorem. The third condition is trivial. Finally, the orthogonality conditions are
431 clearly satisfied under the additional assumption $\text{Kal}(A, b)^\top \psi^0 = 0$.
432 □

433 Let us now address the second type of invariant subspaces introduced in Section 4.1.

434 **Proposition 5.2.** Assume that $\psi^0, b \neq 0$.

1. There exists E satisfying (29), $\psi^0 \in E$ and $E \subset \ker b^\top$ if, and only if, there exists a nonempty $J \subsetneq \{1, \dots, n^c\}$ such that

$$\psi_j^0 = a_{rj} = b_r = 0, \quad \forall j \notin J, \forall r \in J.$$

435 2. Assume that there exists a nonempty $J \subsetneq \{1, \dots, n^c\}$ such that $\psi_j^0 = a_{rj} = 0$ for every
436 $j \notin J$ and $r \in J$. Then, for any $\delta \neq 1$, the solution to the kernel equations (18) satisfies

$$437 \quad k_j^c = 0, \quad \forall j \notin J. \quad (41)$$

438 If, moreover, $b_r = 0$ for every $r \in J$, then the orthogonality conditions (15) are satisfied.

Proof. 1. Since D is a diagonal matrix with distinct entries, its invariant subspaces are of the form

$$E = \text{span} \{e_r \mid r \in J\},$$

439 for some $J \subset \{1, \dots, n^c\}$, where e_1, \dots, e_{n^c} are the canonical vectors of \mathbb{R}^{n^c} . Since $\psi^0 \neq 0$
440 (resp. $b \neq 0$), we have $J \neq \emptyset$ (resp. $J \neq \{1, \dots, n^c\}$). Then, we easily check that such a
441 subspace is invariant by A^\top if, and only if, $a_{rj} = 0$ for every $r \in J$ and $j \notin J$ and that it
442 is included in $\ker b^\top$ if, and only if, $b_r = 0$ for every $r \in J$.

443 2. Property (41) is a consequence of the uniqueness of the solution to the kernel equations.
 444 The orthogonality conditions are clearly satisfied under the additional assumption that
 445 $b_r = 0$ for every $r \in J$.
 446 □

447 **Remark 5.3.** The first item in the above propositions gives explicit conditions that guarantee
 448 that the orthogonality conditions (15) hold (when combined with the results of the previous
 449 sections). We found these conditions with an algebraic approach. On the other hand, once these
 450 conditions are known, the second item of the above propositions show how to use them to obtain
 451 an analytic proof of the orthogonality conditions. Observe in addition that these different proofs
 452 are valid for arbitrary $\delta \neq 1$.

453 5.2 Kernel associated with nontrivial rank two sequences

454 In the same spirit as in the previous section, we now we discuss the following property, related
 455 to Proposition 4.6:

$$456 \quad E_1 \subset \ker b^\top, \quad \psi^r \in E_1, \quad \forall r \in \{2, \dots, 5\}. \quad (42)$$

457 Below, we denote by $c_0 = \langle v, \psi^0 \rangle$.

Proposition 5.4. 1. Assume that $\text{rank}(\psi^0 | \psi^1) = 2$, E_1 satisfies neither (28) nor (29), $c_0 \neq 0$ and $\langle b, \psi^0 \rangle = 0$. Then, condition (42) holds if, and only if, there exists $j_0 \in \{1, \dots, n^c\}$ such that

$$b_{j_0} = 0, \quad \text{rank} \Delta_{j_0} = 1,$$

where $\Delta_{j_0} \in \mathbb{R}^{2n^c \times 2}$ is given by

$$\Delta_{j_0} = \begin{pmatrix} (D - d_{j_0})\psi^0 & A^\top e_{j_0} \\ A^\top \psi^0 & v_{j_0} \psi^0 - c_0 e_{j_0} \end{pmatrix},$$

458 where d_{j_0} is the j_0 -th diagonal entry of D and e_{j_0} is the j_0 -th canonical vector of \mathbb{R}^{n^c} .

2. Assume that $n = 4$, $i = 2$,

$$b = \begin{pmatrix} 1 \\ 0 \\ -\rho \end{pmatrix}, \quad \rho \neq 0,$$

459 $c_0 \neq 0$, $\alpha_{24} \neq 0$, $\langle b, \psi^0 \rangle = 0$ and $\text{rank} \Delta_2 = 1$. Then, the solution to the kernel equations
 460 (18) with $\delta = -1$ is given, for some $\sigma \in \mathbb{R} \setminus \{0\}$, by

$$461 \quad \begin{cases} k_{21} = k_{24}\rho, \\ k_{22} = \frac{1}{-\sigma\alpha_{24}} \left(\sigma\mu_3 \frac{\partial k_{24}}{\partial x} + \sigma\nu_3 \frac{\partial k_{24}}{\partial \xi} - \alpha_{32}k_{24} \right), \\ k_{23} = \frac{1}{-\sigma\alpha_{24}} \left(\frac{\partial k_{24}}{\partial x} + \frac{\partial k_{24}}{\partial \xi} - \sigma\alpha_{23}k_{24} \right), \end{cases} \quad (43)$$

462 where $k_{24} \in C^\infty(\mathbb{R}^2)$ is the solution to

$$463 \quad \begin{cases} \mu_3 \frac{\partial^2 k_{24}}{\partial x^2}(x, \xi) + (\mu_3 + \nu_3) \frac{\partial^2 k_{24}}{\partial x \partial \xi}(x, \xi) + \nu_3 \frac{\partial^2 k_{24}}{\partial \xi^2}(x, \xi) = -2c_0 k_{24}(x, \xi), \\ \mu_3 \frac{\partial k_{24}}{\partial x}(x, -x) + \nu_3 \frac{\partial k_{24}}{\partial \xi}(x, -x) = \frac{\alpha_{32}}{\sigma} k_{24}(x, -x), \\ k_{24}(x, x) = \alpha_{24}. \end{cases} \quad (44)$$

464 To see that (44) indeed has a solution, we can introduce $h = \mu_3 \frac{\partial k_{24}}{\partial x} + \nu_3 \frac{\partial k_{24}}{\partial \xi}$ and observe that
 465 (k_{24}, h) satisfies a first-order hyperbolic system similar to the kernel equations (16) and whose
 466 well-posedness can be established as in the proof of Theorem 2.5.

Proof of Proposition 5.4. 1. Under the assumptions of the proposition and from the proof
 of Proposition 4.6, we have $\psi^r \in E_1$ for every $r \in \{2, \dots, 5\}$ if, and only if, there exist
 $\alpha_2, \beta_2 \in \mathbb{R}$ such that

$$\begin{cases} \psi^2 = \alpha_2 \psi^0 + \beta_2 \psi^1, \\ \psi^3 = \alpha_3 \psi^0 + \beta_3 \psi^1, \end{cases}$$

467 where α_3, β_3 are given by (38). We can check that this is equivalent to the existence of
 468 $\rho, \theta \in \mathbb{R}$ such that

$$\begin{cases} (A^\top - \rho)\psi^1 = -c_0(D - \theta)\psi^0 + \bar{c}_1(\psi^1 - \rho\psi^0), \\ (D - \theta)(\psi^1 - \rho\psi^0) = 0, \end{cases} \quad (45)$$

470 where we recall that $\bar{c}_1 = \frac{\langle v, \psi^1 \rangle}{c_0}$. Since D is diagonal with distinct entries and ψ^0, ψ^1 are
 471 linearly independent, the second condition in (45) is equivalent to the existence of some
 472 $j_0 \in \{1, \dots, n^c\}$ and $r \in \mathbb{R} \setminus \{0\}$ such that

$$473 \quad \theta = d_{j_0}, \quad \psi^1 = \rho\psi^0 + r e_{j_0}. \quad (46)$$

Plugging the second identity in the first condition in (45) and recalling that $\psi^1 = A^\top \psi^0$,
 we see that this condition simply becomes

$$r A^\top e_{j_0} = -c_0(D - d_{j_0})\psi^0 + r \bar{c}_1 e_{j_0}.$$

474 Comparing the j_0 -th components of both quantities, using that the diagonal of A is zero
 475 and $r \neq 0$, we see that $\bar{c}_1 = 0$. Recalling (46), the condition $\bar{c}_1 = 0$ is equivalent to
 476 $\rho = -\bar{r}v_{j_0}$ with $\bar{r} = \frac{r}{c_0}$. In summary, there exist $\rho, \theta \in \mathbb{R}$ such that (45) holds if, and only
 477 if, there exist $j_0 \in \{1, \dots, n^c\}$ such that

$$478 \quad \exists \bar{r} \in \mathbb{R} \setminus \{0\}, \quad \begin{cases} (D - d_{j_0})\psi^0 + \bar{r} A^\top e_{j_0} = 0, \\ \psi^1 + \bar{r}(v_{j_0}\psi^0 - c_0 e_{j_0}) = 0. \end{cases} \quad (47)$$

479 We can check that this condition is equivalent to $\text{rank } \Delta_{j_0} = 1$. Finally, it is clear that
 480 $E_1 \subset \ker b^\top$, i.e. $\langle b, \psi^1 \rangle = 0$, if, and only if, $b_{j_0} = 0$.

2. Since $i = 2$, we have (for the notations, see Section 3.1)

$$A^\top = \begin{pmatrix} 0 & \alpha_{31} & \alpha_{41} \\ \alpha_{13} & 0 & \alpha_{43} \\ \alpha_{14} & \alpha_{34} & 0 \end{pmatrix}, \quad D = 2\text{diag}(\mu_1, \mu_3, \mu_4) - \text{Id}_3, \quad \psi^0 = \begin{pmatrix} \alpha_{21} \\ \alpha_{23} \\ \alpha_{24} \end{pmatrix}, \quad v = -\frac{1}{2} \begin{pmatrix} \alpha_{12} \\ \alpha_{32} \\ \alpha_{42} \end{pmatrix}.$$

481 Clearly, $\langle b, \psi^0 \rangle = 0$ is equivalent to

$$482 \quad \alpha_{21} = \alpha_{24}\rho. \quad (48)$$

On the other hand, using the characterization (47), we see that $\text{rank } \Delta_2 = 1$ if, and only if,

$$\alpha_{31} = \sigma(\mu_1 - \mu_3)\rho\alpha_{24}, \quad (49)$$

$$\alpha_{34} = \sigma(\mu_4 - \mu_3)\alpha_{24}, \quad (50)$$

$$\alpha_{41} = -\frac{\rho}{\sigma}(\alpha_{32} + \sigma^2(\mu_1 - \mu_3)\alpha_{23}), \quad (51)$$

$$\alpha_{42} + \alpha_{12}\rho = \sigma(\alpha_{43} + \alpha_{13}\rho), \quad (52)$$

$$\alpha_{14} = -\frac{1}{\rho\sigma}(\alpha_{32} + \sigma^2(\mu_4 - \mu_3)\alpha_{23}), \quad (53)$$

483 where $\sigma = -\frac{2}{\tau}$. Using these conditions we easily check that $((k^c)^\top, k_2) = (k_{21}, k_{23}, k_{24}, k_{22})$
 484 defined by (43)-(44) satisfies the kernel equations (18) with $\delta = -1$.
 485 □

486 To conclude this section we will present a method which shows how conditions (48)-(53) can
 487 also be found from an analytic point of view.

Another proof of Proposition 5.4, item 2. 1. For every j , let us denote by \mathcal{P}_j the first-order linear partial differential operator

$$\mathcal{P}_j = \mu_j \frac{\partial}{\partial x} + \frac{\partial}{\partial \xi} \nu_j.$$

Since we want the orthogonality condition $\langle b, k^c(\cdot, 0) \rangle = 0$, we look for a solution satisfying

$$k_{21} = k_{24}\rho,$$

488 (recall also Remark 4.1). In particular, we assume (48). Then, the problem is to find a
 489 solution to

$$490 \left\{ \begin{array}{l} k_{22}\alpha_{21} + k_{23}\alpha_{31} = (-\rho\mathcal{P}_1 - \alpha_{41})k_{24}, \\ \mathcal{P}_2 k_{22} + k_{23}\alpha_{32} + k_{24}(\alpha_{42} + \alpha_{12}\rho) = 0, \\ \mathcal{P}_3 k_{23} + k_{22}\alpha_{23} + k_{24}(\alpha_{43} + \alpha_{13}\rho) = 0, \\ k_{22}\alpha_{24} + k_{23}\alpha_{34} = (-\mathcal{P}_4 - \alpha_{14}\rho)k_{24}. \end{array} \right. \quad (54)$$

Let us denote by

$$\omega = \det \begin{pmatrix} \alpha_{21} & \alpha_{31} \\ \alpha_{24} & \alpha_{34} \end{pmatrix}.$$

491 Assume that $\omega \neq 0$ (this will follow a posteriori from (49), (50), using also that $\sigma, \rho, \alpha_{24} \neq$
 492 0). Then, the first and fourth equations in (54) give

$$493 \left\{ \begin{array}{l} k_{22} = \frac{1}{\omega}(-\alpha_{34}\rho\mathcal{P}_1 + \alpha_{31}\mathcal{P}_4 - \alpha_{34}\alpha_{41} + \alpha_{31}\alpha_{14}\rho)k_{24}, \\ k_{23} = \frac{1}{\omega}(\alpha_{24}\rho\mathcal{P}_1 - \alpha_{21}\mathcal{P}_4 + \alpha_{24}\alpha_{41} - \alpha_{21}\alpha_{14}\rho)k_{24}. \end{array} \right. \quad (55)$$

Plugging these relations in the second and third equations in (54) give the following two second-order partial differential equations for k_{24} :

$$\mathcal{Q}k_{24} = 0, \quad \tilde{\mathcal{Q}}k_{24} = 0,$$

where $\mathcal{Q} = \mathcal{Q}^{(2)} + \mathcal{Q}^{(1)} + \mathcal{Q}^{(0)}$, with

$$\begin{aligned}\mathcal{Q}^{(2)} &= \mathcal{P}_2(-\alpha_{34}\rho\mathcal{P}_1 + \alpha_{31}\mathcal{P}_4), \\ \mathcal{Q}^{(1)} &= (-\alpha_{34}\alpha_{41} + \alpha_{31}\alpha_{14}\rho)\mathcal{P}_2 + (\alpha_{24}\rho\mathcal{P}_1 - \alpha_{21}\mathcal{P}_4)\alpha_{32}, \\ \mathcal{Q}^{(0)} &= (\alpha_{24}\alpha_{41} - \alpha_{21}\alpha_{14}\rho)\alpha_{32} + \omega(\alpha_{42} + \alpha_{12}\rho),\end{aligned}$$

and $\tilde{\mathcal{Q}} = \tilde{\mathcal{Q}}^{(2)} + \tilde{\mathcal{Q}}^{(1)} + \tilde{\mathcal{Q}}^{(0)}$, with

$$\begin{aligned}\tilde{\mathcal{Q}}^{(2)} &= \mathcal{P}_3(\alpha_{24}\rho\mathcal{P}_1 - \alpha_{21}\mathcal{P}_4), \\ \tilde{\mathcal{Q}}^{(1)} &= (\alpha_{24}\alpha_{41} - \alpha_{21}\alpha_{14}\rho)\mathcal{P}_3 + (-\alpha_{34}\rho\mathcal{P}_1 + \alpha_{31}\mathcal{P}_4)\alpha_{23}, \\ \tilde{\mathcal{Q}}^{(0)} &= (-\alpha_{34}\alpha_{41} + \alpha_{31}\alpha_{14}\rho)\alpha_{23} + \omega(\alpha_{43} + \alpha_{13}\rho).\end{aligned}$$

494 2. We are going to find conditions to guarantee that these two equations are compatible. To
495 this end, it is for instance sufficient to have

$$496 \mathcal{Q}^{(r)} = \sigma\tilde{\mathcal{Q}}^{(r)}, \quad r = 0, 1, 2, \quad (56)$$

497 for some $\sigma \in \mathbb{R}$. We first look at the operators of highest order. Using the identities

$$498 \alpha_{21} = \alpha_{24}\rho, \quad \mu_j - \nu_j = 1 \quad (j \neq 2), \quad \mu_2 = \nu_2 = 1, \quad (57)$$

we have

$$\begin{aligned}\alpha_{24}\rho\mathcal{P}_1 - \alpha_{21}\mathcal{P}_4 &= \alpha_{24}\rho(\mathcal{P}_1 - \mathcal{P}_4) \\ &= \alpha_{24}\rho(\mu_1 - \mu_4)\mathcal{P}_2.\end{aligned} \quad (58)$$

499 It follows that

$$500 \tilde{\mathcal{Q}}^{(2)} = \alpha_{24}\rho(\mu_1 - \mu_4)\mathcal{P}_3\mathcal{P}_2. \quad (59)$$

501 Consequently, we see that (56) holds for $r = 2$ if we have

$$502 -\alpha_{34}\rho\mathcal{P}_1 + \alpha_{31}\mathcal{P}_4 = \sigma\alpha_{24}\rho(\mu_1 - \mu_4)\mathcal{P}_3. \quad (60)$$

This identity holds if $(\alpha_{34}, \alpha_{31})$ satisfies

$$\begin{pmatrix} -\rho\mu_1 & \mu_4 \\ -\rho\nu_1 & \nu_4 \end{pmatrix} \begin{pmatrix} \alpha_{34} \\ \alpha_{31} \end{pmatrix} = \sigma\alpha_{24}\rho(\mu_1 - \mu_4) \begin{pmatrix} \mu_3 \\ \nu_3 \end{pmatrix},$$

503 which is equivalent to (50)-(49) (using $\rho \neq 0$ and (57)).

3. Let us now compute the first-order differential operators. We have

$$\begin{aligned}\mathcal{Q}^{(1)} &= (-\alpha_{34}\alpha_{41} + \alpha_{31}\alpha_{14}\rho + \alpha_{24}\rho(\mu_1 - \mu_4)\alpha_{32})\mathcal{P}_2 \quad (\text{by (58)}), \\ \tilde{\mathcal{Q}}^{(1)} &= (\alpha_{24}\alpha_{41} - \alpha_{21}\alpha_{14}\rho + \sigma\alpha_{24}\rho(\mu_1 - \mu_4)\alpha_{23})\mathcal{P}_3 \quad (\text{by (60)}).\end{aligned}$$

504 As a result, we have (56) for $r = 1$, if $\mathcal{Q}^{(1)} = \tilde{\mathcal{Q}}^{(1)} = 0$, that is, if

$$505 \begin{cases} -\alpha_{34}\alpha_{41} + \alpha_{31}\alpha_{14}\rho = -\alpha_{24}\rho(\mu_1 - \mu_4)\alpha_{32}, \\ \alpha_{24}\alpha_{41} - \alpha_{21}\alpha_{14}\rho = -\sigma\alpha_{24}\rho(\mu_1 - \mu_4)\alpha_{23}. \end{cases} \quad (61)$$

This holds if $(\alpha_{41}, \alpha_{14})$ satisfies

$$\begin{pmatrix} -\alpha_{34} & \alpha_{31}\rho \\ \alpha_{24} & -\alpha_{21}\rho \end{pmatrix} \begin{pmatrix} \alpha_{41} \\ \alpha_{14} \end{pmatrix} = -\alpha_{24}\rho(\mu_1 - \mu_4) \begin{pmatrix} \alpha_{32} \\ \sigma\alpha_{23} \end{pmatrix},$$

506 which is equivalent to (51) and (53) (using (48), (49), (50) and $\alpha_{24}, \rho \neq 0$).

4. Let us now compute the zero order terms. Using (61), we immediately see that

$$\begin{aligned}\tilde{Q}^{(0)} &= -\alpha_{24}\rho(\mu_1 - \mu_4)\alpha_{32}\alpha_{23} + \omega(\alpha_{43} + \alpha_{13}\rho), \\ Q^{(0)} &= -\sigma\alpha_{24}\rho(\mu_1 - \mu_4)\alpha_{23}\alpha_{32} + \omega(\alpha_{42} + \alpha_{12}\rho).\end{aligned}$$

507 As a result, we see that (56) holds for $r = 0$ if we have condition (52). Moreover, using
508 (48), (49) and (50), we have

$$\omega = -\sigma\alpha_{24}^2\rho(\mu_1 - \mu_4), \quad (62)$$

509

so that, using again (48) and the definition of c_0 , we obtain

$$Q^{(0)} = \sigma\alpha_{24}\rho(\mu_1 - \mu_4)(2c_0).$$

510 It follows that k_{24} indeed satisfies the first equation in (44) (recall that $Q^{(2)} = \sigma\tilde{Q}^{(2)}$ with
511 (59) and $Q^{(1)} = 0$).

5. Using (60), (61), (58) and (62), we can simplify the expressions in (55) to obtain

$$\begin{cases} k_{22} = \frac{1}{-\sigma\alpha_{24}} (\sigma\mathcal{P}_3 - \alpha_{32}) k_{24}, \\ k_{23} = \frac{1}{-\sigma\alpha_{24}} (\mathcal{P}_2 - \sigma\alpha_{23}) k_{24}. \end{cases}$$

512 In addition, it follows from these formula that the remaining conditions are satisfied. In-
513 deed, the condition $k_{22}(x, -x) = 0$ is exactly the condition that we require for k_{24} at
514 $(x, -x)$ in (44) and the condition $k_{23}(x, x) = \alpha_{23}$ follows from the above expression since
515 $k_{24}(x, x) = \alpha_{24}$ and $(\mathcal{P}_2 k_{24})(x, x) = \frac{d}{dx} k_{24}(x, x) = 0$.

516 □

Remark 5.5. In [VK14, Section 3.3], the authors showed that we can solve a kernel system of two equations of the form

$$\begin{cases} \frac{\partial k_{21}}{\partial x} - \frac{\partial k_{21}}{\partial \xi} + k_{22}\alpha_{21} = 0, \\ \frac{\partial k_{22}}{\partial x} + \frac{\partial k_{22}}{\partial \xi} + k_{21}\alpha_{12} = 0, \\ k_{21}(x, x) = \alpha_{21}, \quad k_{22}(x, 0) = 0, \end{cases}$$

517 with $\alpha_{21} \neq 0$ by first expressing k_{22} from the first equation and then showing that the resulting
518 second order equation for k_{21} indeed has a solution. The method we introduced in the second
519 proof of Proposition 5.4, item 2, can be seen as an extension of the method of [VK14] where,
520 instead of dividing by a scalar (namely, α_{21}), we invert a matrix.

521 Acknowledgements

522 This project was supported by National Natural Science Foundation of China (Nos. 12122110 and
523 12071258) and National Science Centre, Poland UMO-2020/39/D/ST1/01136. For the purpose
524 of Open Access, the authors have applied a CC-BY public copyright licence to any Author
525 Accepted Manuscript (AAM) version arising from this submission.

A Controllability of the equivalent system

In this appendix, we give a simple and direct proof of Corollary 2.4. We recall that it can be deduced from Theorem 2.3 but this result is based on the Titchmarsh convolution theorem (see [HO21b]) and we show here how to directly prove the corollary without resorting to this difficult result.

Proof of Corollary 2.4. It is enough to show that, if $(q, f) \in \mathcal{S}_k \setminus \mathcal{S}_{k+1}$ for some $k \in \{2, \dots, n+1\}$, then system (9) (with $m = 1$) is null controllable in time T if, and only if, $T \geq \tau_k$.

1. We first observe that system (9) is equivalent to the same system with $f_1 = 0$. This follows from the invertible spatial transformation

$$\hat{y}_1(t, x) = \tilde{y}_1(t, x) - \int_0^x h(x - \xi) \tilde{y}_1(t, \xi) d\xi,$$

where the kernel h is the solution to

$$h(x)\lambda_1 + \int_0^x h(x - \xi) f_1(\xi) d\xi = f_1(x), \quad 0 < x < 1.$$

Therefore, for the rest of the proof, we assume that $f_1 = 0$.

2. Assume now that $(q, f) \in \mathcal{S}_k \setminus \mathcal{S}_{k+1}$ for some $k \in \{2, \dots, n\}$ (the result for $k = n+1$ is trivial). It will be convenient to use the notation $\hat{q}_i = q_{i-1}$ for $2 \leq i \leq n$. Let us write system (9) (with $f_1 = 0$) component-wise:

$$\begin{cases} \frac{\partial \tilde{y}_1}{\partial t}(t, x) + \lambda_1 \frac{\partial \tilde{y}_1}{\partial x}(t, x) = 0, & \begin{cases} \frac{\partial \tilde{y}_i}{\partial t}(t, x) + \lambda_i \frac{\partial \tilde{y}_i}{\partial x}(t, x) = f_i(x)v(t), \\ \tilde{y}_i(t, 0) = \hat{q}_i v(t), \\ \tilde{y}_i(0, x) = \tilde{y}_i^0(x), \end{cases} \\ \tilde{y}_1(t, 1) = \tilde{u}(t), \\ \tilde{y}_1(0, x) = \tilde{y}_1^0(x), \end{cases}$$

for $i \in \{2, \dots, n\}$, and where we introduced $v(t) = \tilde{y}_1(t, 0)$. It is clear that this system is null controllable in any time $T \geq \tau_k = \max\{T_1 + T_k, T_2\}$ since in this case taking $\tilde{u} = 0$ in $(T - (T_1 + T_k), T)$ does the job. It is the necessary part that requires more work.

3. First of all, we recall that the condition $T \geq \max\{T_1, T_2\}$ is always necessary (see e.g. the proof of [HO21b, Lemma 3.3]). Under this condition and by mimicking the second step in the proof of [HO21b, Theorem 3.1], we see that the null controllability condition $\tilde{y}_k(T, x) = 0$ is equivalent to

$$\hat{q}_k \alpha(\tau) + \int_0^\tau \beta(\tau - \sigma) \alpha(\sigma) d\sigma = 0, \quad 0 < \tau < T_k, \quad (63)$$

where $\alpha(\theta) = v(-\theta + T)$ and $\beta(\theta) = f_k(\lambda_k \theta)$ for $0 < \theta < T_k$.

4. We now have two possibilities for (63).

- (a) Case $\hat{q}_k \neq 0$. Then, by uniqueness of the solution to the Volterra equation of the second kind (63), we obtain $\alpha = 0$ in $(0, T_k)$. This means that $v = 0$ in $(T - T_k, T)$. Since this is true for any \tilde{y}_0^1 , it is possible only if $T_1 \leq T - T_k$, which is the desired condition.

(b) Case $\hat{q}_k = 0$. Since $(q, f) \in \mathcal{S}_k \setminus \mathcal{S}_{k+1}$, we necessarily have $f_k \neq 0$. Since f_k is analytic in a neighborhood of $[0, 1)$, this implies in particular that there exists $N \geq 1$ such that

$$f_k^{(N-1)}(0) \neq 0, \quad f_k^{(\ell)}(0) = 0, \quad \forall \ell < N - 1.$$

Then, taking N times the derivative with respect to τ in (63) (with $\hat{q}_k = 0$), we obtain the new Volterra equation

$$c\alpha(\tau) + \int_0^\tau \beta^{(N)}(\tau - \sigma)\alpha(\sigma) d\sigma = 0, \quad 0 < \tau < T_k,$$

548 where $c = \beta^{(N-1)}(0) = f_k^{(N-1)}(0)\lambda_k^{N-1}$. Therefore, $c \neq 0$ and the situation is now
 549 identical to the previous case.

550

□

551 B Solution to the kernel equations

552 In this appendix, we present a new approach to solve the kernel equations that encompasses
 553 in particular the proof of Theorem 2.5. We recall that, when considering the kernel equations
 554 in the triangle $\mathcal{T} = \{(x, \xi) \in \mathbb{R}^2 \mid 0 < \xi < x < 1\}$, the approach used in all current results
 555 in the literature ([Cor+13; DVK13; HD15; Hu+16; Hu+19; CN19], etc.) consists in adding
 556 “artificial boundary conditions” to close the system of kernel equations. In our approach, we will
 557 not consider the condition at (x, x) as a boundary condition but rather as an initial condition.
 558 We will simply let propagate this condition along the characteristics and find the corresponding
 559 so-called domain of determinacy, much in the spirit of the reference books [LY85; Bre00]. Then,
 560 another idea of our method is also to solve the equation for $j = i$ and plug it into the other
 561 equations of the system to obtain a new system with initial conditions at (x, x) only (as in the
 562 proof of Theorem 3.1). Moreover, this gives a natural bound in $|x - \xi|$ for the estimates needed
 563 to prove the contraction of the mapping defining the integral equations corresponding to the new
 564 system (rather than $|x - (1 - \varepsilon)\xi|$ as in [Hu+16; Hu+19]).

565 All along this appendix, $i \in \{1, \dots, n\}$ is fixed and we continue using the notation $k =$
 566 $(k_{ij})_{1 \leq j \leq n}$ to denote the transpose of the i -th row of K . We also emphasize that $m \geq 1$ is
 567 arbitrary.

568 First of all, it will be more convenient to work with the kernel equations normalized by λ_i :

$$569 \begin{cases} \frac{\partial k_j}{\partial x}(x, \xi) + \bar{\lambda}_j \frac{\partial k_j}{\partial \xi}(x, \xi) + \sum_{r=1}^n k_r(x, \xi) \bar{m}_{rj} = 0, \\ k_j(x, x) = f_j \quad (j \neq i), \quad k_i(x, \delta x) = 0, \end{cases} \quad (64)$$

where

$$\bar{\lambda}_j = \frac{\lambda_j}{\lambda_i}, \quad \bar{m}_{rj} = \frac{m_{rj}}{\lambda_i}, \quad f_j = \frac{m_{ij}}{\lambda_i - \lambda_j}.$$

570 From now on, we will assume for instance that $i \geq m + 1$, so that $\lambda_i > 0$ and thus, from (1b),

571

572

$$\bar{\lambda}_1 < \dots < \bar{\lambda}_{i-1} < 1 < \bar{\lambda}_{i+1} < \dots < \bar{\lambda}_n. \quad (65)$$

For every $(x, \xi) \in \mathbb{R} \times \mathbb{R}$, we denote by $s \mapsto \zeta_j(s; x, \xi)$ the solution to

$$\begin{cases} \frac{d}{ds} \zeta_j(s; x, \xi) = \bar{\lambda}_j, \quad \forall s \in \mathbb{R}, \\ \zeta_j(x; x, \xi) = \xi. \end{cases}$$

Let us now consider the more general condition

$$k_j(x, x) = f_j(x) \quad (j \neq i),$$

where f_j is a function defined on an interval of the form $[a, b]$ with $a < 0 < b$. Even if f_j is constant in (64), we will need to consider space-dependent data to deduce the existence of smooth solutions by an inductive argument. We will describe the largest domain $D \subset \mathbb{R}^2$ where the system can then be solved along the characteristics. We first take care of the characteristics for $j \neq i$. Recalling the ordering (65), we introduce

$$D^c = \left\{ (x, \xi) \in \mathbb{R}^2 \mid \begin{array}{l} \zeta_{i-1}(x; a, a) < \xi < \zeta_{i-1}(x; b, b) \\ \zeta_{i+1}(x; b, b) < \xi < \zeta_{i+1}(x; a, a) \end{array} \right\},$$

(see Figure 1). Above, we use the usual conventions for $i = 1$ and $i = n$. We now take care of the characteristic for $j = i$. We can check that the line $\{(x, \delta x) \mid a < x < b\}$ intersects the boundary of D^c at exactly two points $(c, \delta c)$ and $(d, \delta d)$, with $c, d \in (a, b)$ and $c < 0 < d$ if $\delta < 1$ or $d < 0 < c$ if $\delta > 1$. Let then

$$D = \{(x, \xi) \in D^c \mid \zeta_i(x; d, \delta d) < \xi < \zeta_i(x; c, \delta c)\},$$

573 (see Figure 2 with $\delta = -1$) and define $I = (c, d)$. Here and in what follows, it will be convenient
 574 to use the notation (α, β) to denote the interval $(\min\{\alpha, \beta\}, \max\{\alpha, \beta\})$, whatever $\alpha, \beta \in \mathbb{R}$ are
 575 (we use a similar notation for $[\alpha, \beta]$).

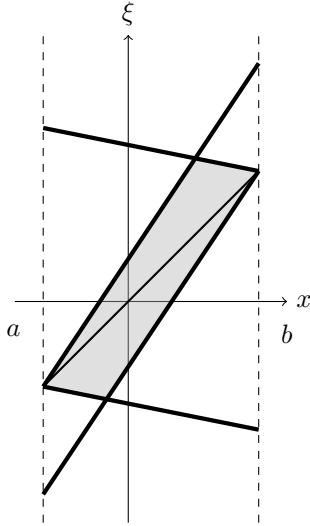


Figure 1: Domain D^c

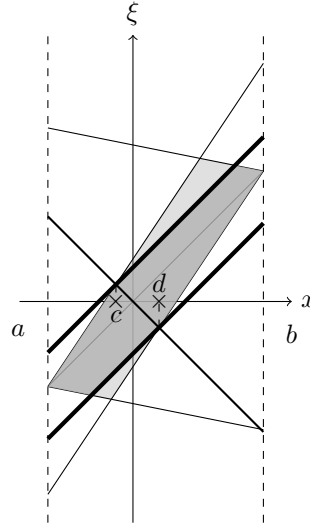


Figure 2: Domain D (in dark gray)

576 We will prove the following result.

577 **Theorem B.1.** *Let $a < 0 < b$ and $s \in \mathbb{N}$ be fixed. For any $(f_j)_{j \neq i} \in C^s([a, b])^{n-1}$ and $f_i \in C^s(\bar{I})$,*
 578 *there exists a unique solution $k = (k_j)_{1 \leq j \leq n} \in C^s(\bar{D})^n$ to*

$$579 \quad \begin{cases} \frac{\partial k_j}{\partial x}(x, \xi) + \bar{\lambda}_j \frac{\partial k_j}{\partial \xi}(x, \xi) + \sum_{r=1}^n k_r(x, \xi) \bar{m}_{rj} = 0, & (x, \xi) \in D, \\ k_j(x, x) = f_j(x), & x \in (a, b) \quad (j \neq i), \\ k_i(x, \delta x) = f_i(x), & x \in I, \end{cases} \quad (66)$$

580 Moreover, we have the estimate

$$581 \quad \|k\|_{C^s(\bar{D})^n} \leq C \max \left\{ \max_{j \neq i} \|f_j\|_{C^s([a,b])}, \quad \|f_i\|_{C^s(\bar{I})} \right\}, \quad (67)$$

582 for some $C > 0$ that does not depend on any f_j .

583 For $s = 0$, by solution we mean “solution along the characteristics”, see below.

584 The first part of Theorem 2.5 follows from the previous result and the following simple
585 observation:

$$586 \quad \forall V \subset \mathbb{R}^2, \exists a < 0 < b, \quad V \subset D. \quad (68)$$

587 On the other hand, using that the coefficients of the system are constant and arguing as in
588 the proof of [CN19, Lemma 6.2], we can show that (67) holds with $C = R^s$ for some $R > 0$ that
589 does not depend on s . This establishes the estimate in Theorem 2.5.

590 Let us now prove Theorem B.1. We start with a description of the key properties satisfied
591 by the point where the j -th characteristic intersects the corresponding data line.

592 **Lemma B.2.** For every $j \in \{1, \dots, n\}$, there exists $\sigma_j \in C^\infty(\bar{D})$ such that, for every $(x, \xi) \in \bar{D}$,
593 we have:

594 • $\zeta_j(\sigma_j(x, \xi); x, \xi) = \sigma_j(x, \xi)$ with $\sigma_j(x, \xi) \in [a, b]$ for $j \neq i$ and $\zeta_i(\sigma_i(x, \xi); x, \xi) = \delta\sigma_i(x, \xi)$
595 with $\sigma_i(x, \xi) \in \bar{I}$.

596 • $(s, \zeta_j(s; x, \xi)) \in \bar{D}$ for every $s \in [\sigma_j(x, \xi), x]$.

597 • For every $j \neq i$, we have

$$598 \quad |\sigma_j(x, \xi) - x| \leq C|x - \xi|, \quad (69)$$

599 for some $C > 0$ that does not depend on j, x, ξ .

600 We point out that ζ_j and σ_j are explicit. In particular, this is how we prove estimate (69).

601 Now, instead of writing (66) along all the characteristics (as it is usually done), we first
602 replace k_i by formally solving the corresponding equation (recall that $\bar{m}_{ii} = 0$):

$$603 \quad k_i(x, \xi) = f_i(\sigma_i(x, \xi)) - \int_{\sigma_i(x, \xi)}^x \sum_{r \neq i} k_r(\eta, \zeta_i(\eta; x, \xi)) \bar{m}_{ri} d\eta. \quad (70)$$

Let us introduce the following notations to exclude the i -th components: $k^\mathfrak{c} = (k_j)_{j \neq i}$, $f^\mathfrak{c} = (f_j)_{j \neq i}$, $\sigma^\mathfrak{c} = (\sigma_j)_{j \neq i}$, $\zeta^\mathfrak{c} = (\zeta_j)_{j \neq i}$, $M^\mathfrak{c} = (\bar{m}_{rj})_{r, j \neq i}$, $\psi = (\bar{m}_{ij})_{j \neq i}$ and $w = (\bar{m}_{ji})_{j \neq i}$. Then, plugging the previous expression of k_i in (66) and integrating along the characteristics, we can transform this system into the following system of integral equations for $k^\mathfrak{c}$:

$$\begin{aligned} k_\ell^\mathfrak{c}(x, \xi) &= f_\ell^\mathfrak{c}(\sigma_\ell^\mathfrak{c}(x, \xi)) - \int_{\sigma_\ell^\mathfrak{c}(x, \xi)}^x \sum_{r=1}^{n-1} k_r^\mathfrak{c}(s, \zeta_\ell^\mathfrak{c}(s; x, \xi)) m_{r\ell}^\mathfrak{c} ds \\ &\quad - \int_{\sigma_\ell^\mathfrak{c}(x, \xi)}^x f_i(\sigma_i^\mathfrak{c}(s, \zeta_\ell^\mathfrak{c}(s; x, \xi))) \psi_\ell ds \\ &\quad + \int_{\sigma_\ell^\mathfrak{c}(x, \xi)}^x \left(\int_{\sigma_i^\mathfrak{c}(s, \zeta_\ell^\mathfrak{c}(s; x, \xi))}^s \sum_{r=1}^{n-1} k_r^\mathfrak{c}(\eta, \zeta_i(\eta; s, \zeta_\ell^\mathfrak{c}(s; x, \xi))) w_r d\eta \right) \psi_\ell ds, \end{aligned} \quad (71)$$

604 for every $\ell \in \{1, \dots, n-1\}$ and $(x, \xi) \in \bar{D}$.

All the quantities in (70) and (71) are well defined thanks to Lemma B.2. It remains to prove the existence and uniqueness of a C^s solution k^c to this system of integral equations. We start with $s = 0$. As usual, we use the Banach fixed point theorem and suitable estimates. A solution to this system is a fixed point of the map $F(k^c) = u^0 + \Phi k^c$, where

$$u_\ell^0(x, \xi) = f_\ell^c(\sigma_\ell^c(x, \xi)) - \int_{\sigma_\ell^c(x, \xi)}^x f_i(\sigma_i^c(s, \zeta_\ell^c(s; x, \xi))) \psi_\ell ds,$$

and Φ is the linear map $\Phi = \Phi_1 + \Phi_2$ with

$$(\Phi_1 k^c)_\ell(x, \xi) = - \int_{\sigma_\ell^c(x, \xi)}^x \sum_{r=1}^{n-1} k_r^c(s, \zeta_\ell^c(s; x, \xi)) m_{r\ell}^c ds,$$

and

$$(\Phi_2 k^c)_\ell(x, \xi) = \int_{\sigma_\ell^c(x, \xi)}^x \left(\int_{\sigma_i^c(s, \zeta_\ell^c(s; x, \xi))}^s \sum_{r=1}^{n-1} k_r^c(\eta, \zeta_i(\eta; s, \zeta_\ell^c(s; x, \xi))) w_r d\eta \right) \psi_\ell ds,$$

605 for every $\ell \in \{1, \dots, n-1\}$ and $(x, \xi) \in \bar{D}$.

606 Let us now precisely set the functional framework. Let $B = C^0(\bar{D})^{n-1}$ and consider the
607 standard norm $\|k^c\|_B = \max_{1 \leq \ell \leq n-1} \max_{(x, \xi) \in \bar{D}} |k_\ell^c(x, \xi)|$. Clearly, B is a Banach space and
608 $F(B) \subset B$. Let us now prove that F^N is a contraction for $N \in \mathbb{N}^*$ large enough. This is
609 equivalent to show that Φ^N is a contraction. To this end, it is sufficient to prove the following
610 key estimate:

Lemma B.3. *There exists $C > 0$ such that, for every $N \in \mathbb{N}^*$, we have*

$$|(\Phi^N k^c)_\ell(x, \xi)| \leq \frac{C^N |x - \xi|^N}{N!} \|k^c\|_B,$$

611 for every $k^c \in B$, $\ell \in \{1, \dots, n-1\}$ and $(x, \xi) \in \bar{D}$.

Proof. We prove the property by induction on N . Let us first consider $N = 1$. We have

$$|(\Phi_1 k^c)_\ell(x, \xi)| \leq C_1 |x - \sigma_\ell^c(x, \xi)| \|k^c\|_B,$$

with $C_1 = \max_\ell \sum_r |m_{r\ell}^c| \geq 0$. Similarly,

$$|(\Phi_2 k^c)_j(x, \xi)| \leq C_2 |x - \sigma_\ell^c(x, \xi)| \|k^c\|_B,$$

612 with $C_2 = \max_{\ell, (x, \xi), s} |s - \sigma_i^c(s, \zeta_\ell^c(s; x, \xi))| \sum_r |w_r| |\psi_\ell| \geq 0$. Finally, we have

$$613 |x - \sigma_\ell^c(x, \xi)| \leq C_3 |x - \xi|, \tag{72}$$

614 for some $C_3 > 0$ that does not depend on ℓ, x, ξ (see (69)). This proves the property for $N = 1$.

Let us now assume that the property holds for N and let us prove it for $N + 1$. We have

$$|(\Phi_1 \Phi^N k^c)_\ell(x, \xi)| \leq \int_{[\sigma_\ell^c(x, \xi), x]} \sum_{r=1}^{n-1} |(\Phi^N k^c)_r(s, \zeta_\ell^c(s; x, \xi))| |m_{r\ell}^c| ds,$$

Using the induction assumption, we get

$$|(\Phi_1 \Phi^N k^c)_\ell(x, \xi)| \leq C_1 \frac{C^N}{N!} \|k^c\|_B \int_{[\sigma_\ell^c(x, \xi), x]} |s - \zeta_\ell^c(s; x, \xi)|^N ds.$$

Similarly, noting that $\eta - \zeta_i(\eta; s, \zeta_\ell^c(s; x, \xi)) = s - \zeta_\ell^c(s; x, \xi)$, we get

$$|(\Phi_2 \Phi^N k^c)_\ell(x, \xi)| \leq C_2 \frac{C^N}{N!} \|k^c\|_B \int_{[\sigma_\ell^c(x, \xi), x]} |s - \zeta_\ell^c(s; x, \xi)|^N ds.$$

Now observe that $|s - \zeta_\ell^c(s; x, \xi)| \leq C_4 |s - \sigma_\ell^c(x, \xi)|$ for some $C_4 > 0$ that does not depend on ℓ, s, x, ξ . It follows that

$$\int_{[\sigma_\ell^c(x, \xi), x]} |s - \zeta_\ell^c(s; x, \xi)|^N ds \leq C_4 \int_{[\sigma_\ell^c(x, \xi), x]} |s - \sigma_\ell^c(x, \xi)|^N ds = C_4 \frac{|x - \sigma_\ell^c(x, \xi)|^{N+1}}{N+1}.$$

615 We conclude thanks to the estimate (72). □

616

617 Finally, the estimate (67) can be deduced from the identities $k^c = F^N(k^c) - F^N(0) + F^N(0) =$
 618 $\Phi^N(k^c) - \Phi^N(0) + \sum_{r=0}^N \Phi^r u^0$, combined with the fact that Φ^N is a contraction and that u^0 can
 619 be estimated by the right-hand side of (67) (with $s = 0$). This concludes the proof of Theorem
 620 B.1 for $s = 0$.

621 To prove the result for $s \geq 1$ we can argue as in the proof of [Bre00, Theorem 3.6] and then
 622 use an induction argument.

623 **Remark B.4.** The proof above can be adapted to deal with space-dependent systems, i.e. when
 624 λ_j and m_{rj} depend on x . The additional condition for k_i has to be modified though, but we
 625 can for instance consider $k_i(x, 0) = f_i(x)$. Note that we still have explicit formulas for the
 626 corresponding ζ_j and σ_j .

627 **Remark B.5.** Our approach can be used to recover existence results in the triangle \mathcal{T} . To this
 628 end, we simply extend the parameters λ_j and m_{rj} to $[a, b] \supset [0, 1]$ in a smooth way. Then, for
 629 a, b large enough, the domain D will contain the triangle \mathcal{T} (recall (68)) and we apply Theorem
 630 B.1 in this D . This approach is different from all the previous ones in the literature, which
 631 consisted in adding “artificial boundary conditions” at some parts of the boundary of \mathcal{T} . Note in
 632 addition that extending λ_j and m_{rj} outside $[0, 1]$ in a smooth way is easier than building artificial
 633 boundary conditions that satisfy compatibility conditions associated with the kernel equations.

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