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DERIVATION OF A TWO-PHASE FLOW MODEL ACCOUNTING FOR SURFACE TENSION

H. MATHIS

ABSTRACT. This paper presents the derivation of a two-phase flow model that incorporates surface tension effects using Hamilton’s principle of stationary action. The Lagrangian functional, which defines the action, consists of kinetic energy—accounting for interface characteristics—and potential energy.

A key feature of the model is the assumption that the interface separating the two phases possesses its own internal energy, which satisfies a Gibbs form that includes both surface tension and interfacial area. Consequently, surface tension is considered in both the kinetic and potential energy terms that define the Lagrangian functional.

By applying the stationary action principle, a set of partial differential equations governing the dynamics of the two-phase flow is derived. This includes evolution equations for the volume fraction and interfacial area, incorporating mechanical relaxation terms. The final model is proven to be well-posed, demonstrating hyperbolicity and satisfying Lax entropy conditions.

Key-words. Two-phase compressible flows, interfacial area, thermodynamics, Hamilton’s principle of stationary action, hyperbolicity

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1. INTRODUCTION

The modeling of compressible multiphase flows has been the subject of extensive literature over the past decades, particularly for practical applications such as the nuclear safety of pressurized water reactors. In the context of a loss-of-coolant accident, for instance, the liquid water coolant is exposed to high pressure and temperature conditions, so that a break in the coolant circuit could lead to the formation of vapor, inducing shock and phase transition waves [1].

Thus, the challenge is not only to capture the wave structure but also to obtain information about the various exchanges occurring at the liquid-vapor interface.

These transfers strongly depend on the interface area, even when focusing on large-scale descriptions. Several approaches have been proposed to establish the evolution of the interfacial area, mostly depending on the scale of description.

When focusing on polydisperse flows with many inclusions, bubbles, or droplets, modeling bubble pulsation requires maintaining a small-scale description. For instance in [38], The author proposes a transport equation based on heuristics from particulate suspensions, assuming that both phases evolve with distinct velocities. Then, focusing on the small scale, he introduces a second transport equation while studying the fluctuations of a small interface element. Following this approach, several models have been proposed in a series of works [8, 7, 6, 10, 39], considering the one-velocity framework. The set of bubbles/droplets is described by a probability density function, which satisfies the so-called Williams-Boltzmann equation. The distribution function then describes the probability of a bubble being present at a certain time and position, evolving with a given velocity. It also accounts for topological properties of the bubble or droplet, such as its volume or radius. In recent contributions [39, 40], the authors propose to make the density distribution dependent on the topological properties of the interface as well. In particular, they consider the level set of the interface and its local mean curvature. By doing so, they obtain information about the interfacial area and provide a complete partial differential equation (PDE) model for the fluid-interface dynamics, using Hamilton's principle of stationary action.

When focusing on the mesoscopic scale, one enters the framework of diffuse interface models. These equations involve van der Waals' gradient energy, corresponding to the capillary Korteweg tensor, and heat dissipative fluxes (see, for instance, [44] and references therein). In this approach, the fluid-interface system is treated as a single continuous medium, with the double-well potential acting as an equation of state for both the fluid phases and the interface. In [22] the derivation of these equations is obtained from Hamiltonian variational principles. More recently a reformulation of the barotropic Navier-Stokes-Korteweg system has been proposed in [9]. The derivation is again based on variational Hamiltonian methods, using an augmented Lagrangian that enables to recover nonlinear dispersive terms.

In the application we have in mind, a precise description of the interface topology is not mandatory. Given the high heterogeneity of the flows, the strong temperature and pressure conditions suggest to consider averaged models where the interface is represented implicitly. However, the interface between the two phases is the locus of all the thermodynamical exchanges, and the relaxation towards the thermodynamical equilibrium depends strongly on its area, especially the relaxation time scales. The objective is, then, to derive an evolution equation of the interfacial area at the macroscopic scale.

Such an averaged model is proposed in [3], where a convection equation for the interfacial area is coupled to a barotropic three-phase flow model of Baer-Nunziato type. The equation includes a source term that vanishes when the Weber number (ratio of momentum to surface tension) exceeds a given threshold or when the relative velocity between the two phases is zero. The interfacial area equation is inspired by the modeling proposed in [41] for steam explosion simulations and is based on heuristics.

In the aforementioned references, the surface tension is perceived as a geometrical feature that only plays a role in defining kinetic energy of the bulk and the interface. In all proposed derivations, the thermodynamic behaviour of the interface is not considered; in particular, surface tension does not impact the potential energy of the system. It is precisely this perspective that we develop here.

The derivation of the averaged fluid-interface model is achieved by adapting the Stationary Action Principle as detailed in [4]. The core of our model lies in the rigorous derivation of the potential energy within the Lagrangian functional, while the kinetic energy accounts for small scales, in the spirit of [8]. See also [45, 43] for the derivation of compressible two-phase models with surface tension, which enter the so-called class of Symmetric Hyperbolic Thermodynamically Compatible Systems.

The originality of our approach is to consider the surface tension not only as a dynamical feature but also as a thermodynamics one. To this end, we return to classical extensive thermodynamics in the sense of Gibbs [5, 35] to describe, as rigorously as possible, the thermodynamic behaviour of the fluid-interface system. In the context of two-phase flows, this methodology has been used, for instance, in [29, 28, 14, 15] and the references cited therein, resulting in thermodynamically consistent multiphase flow models for immiscible and miscible two-phase systems. The novelty here is that the interface is assumed to be described by an extensive internal energy function, under the assumption that the interface has no mass, occupies no volume, but is characterized by its temperature and interfacial area. Thus, the interface is fully described by its internal energy, which satisfies a Gibbs form involving not only temperature and entropy but also surface tension and interfacial area variations. This approach of the thermodynamics of fluid-interface systems originates from [36, 35, 27] and was further developed in [47] to model (multiphase) flows in porous media. Regarding Hamiltonian variational methods, there is a vast literature on the derivation of porous media and fluid-interface models. For instance, see [12] for immiscible mixtures in porous media without surface tension and the extension to two-temperature models in [26]. Additionally, [46] addresses two-phase poroelasticity models, and [16] discusses the dynamics of a porous medium filled with incompressible fluid.

The paper is organized as follows. Section 2 is dedicated to the thermodynamic modeling of the fluid-interface system, relying on the thermodynamics of irreversible processes based on local equilibrium assumptions. By focusing on the extensive and intensive descriptions of the two phases and the interface, we define the internal energy of the fluid-interface system as well as its temperature, pressure and chemical potential. This initial framework ensures that the interfacial area and surface tension naturally contribute in the pressure. When analyzing thermodynamic equilibrium, several interesting properties arise, notably that mechanical equilibrium is

described by a differential form involving the volume fraction and the interfacial area.

The thermodynamic modeling of the fluid-interface system is then used to construct an averaged two-fluid model, capturing the evolution of macroscopic quantities in space and time, with the interface implicitly represented. In Section 3 we use the fluid-interface internal energy to define the potential energy of the Lagrangian functional that defines the Action. A brief review of the so-called small-scale kinetic energy models available in the literature is presented to motivate our choice of kinetic energy.

Applying the Hamilton's principle on least action, we derive a set of PDEs describing the dynamics of the fluid-interface system. Since the stationary action principle qualifies reversible processes, it ensures conservation of momentum and total energy, as well as the model's hyperbolicity and a symmetrization property. These properties are discussed in Section 4.

2. THERMODYNAMICAL MODELING

The purpose of this section is to provide an accurate description of the thermodynamic behavior the system occupying an elementary domain, composed of two immiscible fluid phases separated by an interface. The modeling of the fluid-interface system is based on the thermodynamics of irreversible processes and relies on local equilibrium assumptions.

Assumptions 1 (Fluid-interface thermodynamics system). *The fluid-interface thermodynamics system is characterized by the following assumptions.*

- (1) *It occupies a volume $V \geq 0$, has a mass $M \geq 0$ and an internal entropy $S \geq 0$.*
- (2) *It is composed of the two immiscible fluids or phases $k = 1, 2$ with indices $k = 1, 2$, described by their volume $0 \leq V_k \leq V$, mass $0 \leq M_k \leq M$ and entropy $0 \leq S_k \leq S$.*
- (3) *The two phases are immiscible and separated by an interface, with index i , of area $A_i \geq 0$ and internal entropy S_i .*
- (4) *At each point of this system, local equilibrium is reached so that each part of the system is depicted by its own Equation of State (EoS).*

Paragraph 2.1 specifies the thermodynamic properties of the fluid phases, in both extensive and intensive variables. The thermodynamics of the interface is detailed in Paragraph 2.2. In Section 2.3, the second law of thermodynamics is used to characterize the fluid-interface internal energy and the thermodynamic equilibrium of the fluid-interface system.

2.1. Fluid phases. A fluid phase $k = 1, 2$ is characterized by its volume $V_k \geq 0$, its entropy $S_k \geq 0$ and its mass $M_k \geq 0$.

Assumptions 2 (Fluid phase thermodynamics). *The fluid phase $k = 1, 2$ is entirely described by its extensive internal energy function E_k such that:*

- $(M_k, V_k, S_k) \mapsto E_k(M_k, V_k, S_k)$ is \mathcal{C}^2 on $(\mathbb{R}_+)^3$,
- $(M_k, V_k, S_k) \mapsto E_k(M_k, V_k, S_k)$ is convex,
- The extensive internal energy E_k is positively homogeneous of degree 1 (PH1), that is

$$\forall \lambda \in \mathbb{R}_+^*, \forall (M_k, V_k, S_k) \in (\mathbb{R}_+)^3, \quad E_k(\lambda M_k, \lambda V_k, \lambda S_k) = \lambda E_k(M_k, V_k, S_k).$$

The last assumption corresponds to the extensive character of the internal energy function : when doubling the volume, mass and entropy of the system, its extensive internal energy is doubled as well. The extensive internal energy E_k is convex but cannot be strictly convex, since it is PH1.

Some intensive parameters are defined as partial derivatives of E_k :

- the pressure $p_k(M_k, V_k, S_k) = -\partial E_k/\partial V_k(M_k, V_k, S_k)$,
- the temperature $T_k(M_k, V_k, S_k) = \partial E_k/\partial S_k(M_k, V_k, S_k) > 0$,
- the chemical potential $\mu_k(M_k, V_k, S_k) = \partial E_k/\partial M_k(M_k, V_k, S_k)$,

leading to the total differential form

$$(1) \quad dE_k = T_k dS_k - p_k dV_k + \mu_k dM_k,$$

referred as extensive (phasic) Gibbs form in the sequel. Since the internal energy is extensive, it satisfies the Euler relation

$$(2) \quad E_k = T_k S_k - p_k V_k + \mu_k M_k.$$

Some intensive variables and potentials can be defined while considering the extensive ones relatively to the mass of the phase k . We introduce the specific volume $\tau_k = V_k/M_k$ and the specific entropy $s_k = S_k/M_k$ of the phase $k = 1, 2$. Then the specific internal energy $e_k(\tau_k, s_k)$ corresponds to a restriction of the extensive energy:

$$(3) \quad e_k(\tau_k, s_k) = E_k(1, \tau_k, s_k).$$

The phasic pressure and temperature can be defined as functions of the intensive variables as well (while keeping the same notations):

$$(4) \quad p_k(\tau_k, s_k) = -\partial e_k/\partial \tau_k(\tau_k, s_k), \quad T_k(\tau_k, s_k) = \partial e_k/\partial s_k(\tau_k, s_k).$$

The intensive potentials comply thus with an intensive differential (phasic) Gibbs form:

$$(5) \quad de_k = T_k ds_k - p_k d\tau_k.$$

Note that scaling the extensive Euler relation (2) with respect to the mass M_k gives another definition of the chemical potential μ_k , which turns to be the Legendre transform of the internal energy e_k :

$$(6) \quad \mu_k = e_k - T_k s_k + p_k \tau_k.$$

Example 1 (Stiffened gas law). *Any equation of state that satisfies Assumptions 2 can be used in the following sections. As an illustration, we provide an example. The stiffened gas equation of state is the simplest example of law that captures the main physical properties of pure fluids, see [37, 15] and references therein. The intensive internal energy reads*

$$e_k(\tau_k, s_k) = \tau^{\gamma_k-1} \exp\left(\frac{s_k - s_{0,k}}{c_{v,k}}\right) + q_k + \pi_k \tau_k,$$

where $\gamma_k > 1$ is the adiabatic coefficient, $s_{0,k}$ is the reference entropy, $c_{v,k}$ is the heat capacity, $-\pi_k$ is minimal pressure and q_k is a reference enthalpy. The case of a perfect gas equation of state is recovered by setting $\pi_k = q_k = 0$. This equation of

state complies with the Gibbs form (5), leading to

$$p_k(\tau_k, s_k) = (\gamma_k - 1)\tau_k^{-\gamma_k} \exp\left(\frac{s_k - s_{0,k}}{c_{v,k}}\right) - \pi_k,$$

$$T_k(\tau_k, s_k) = \frac{\tau_k^{(1-\gamma_k)}}{c_{v,k}} \exp\left(\frac{s_k - s_{0,k}}{c_{v,k}}\right).$$

2.2. The interface. The interface separating the two phases is supposed to be sharp and to have no volume and no mass. Adopting an extensive description [42], it is characterized by its entropy $S_i \geq 0$ and its area $A_i \geq 0$.

Assumptions 3 (Interface thermodynamics). *The interface i is entirely described by its extensive internal energy function E_i such that:*

- $(S_i, A_i) \mapsto E_i(S_i, A_i)$ is \mathcal{C}^2 on $(\mathbb{R}_+)^2$,
- $(S_i, A_i) \mapsto E_i(S_i, A_i)$ is convex,
- The extensive internal energy E_i is positively homogeneous of degree 1 (PH1), that is

$$\forall \lambda \in \mathbb{R}_+^*, \forall (S_i, A_i) \in (\mathbb{R}_+)^2, \quad E_i(\lambda S_i, \lambda A_i) = \lambda E_i(S_i, A_i).$$

According to the first principle of thermodynamics, it holds

$$(7) \quad dE_i = T_i dS_i + \gamma_i dA_i,$$

where $\gamma_i(S_i, A_i)$ is the surface tension and $T_i(S_i, A_i)$ the interfacial temperature. Since the internal energy $E_i \in \mathcal{C}^2((\mathbb{R}_+)^2)$ is PH1, it follows that the temperature T_i and the surface tension γ_i are \mathcal{C}^1 on $(\mathbb{R}_+)^2$. The internal energy E_i being an PH1 quantity, it complies with the Euler relation

$$(8) \quad E_i = T_i S_i + \gamma_i A_i,$$

which yields, after differentiating and subtracting (7), the so-called Gibbs–Duhem relation

$$(9) \quad 0 = S_i dT_i + A_i d\gamma_i.$$

Since the interface has no mass, a way to deduce intensive potentials is to scale with respect to the volume V of the fluid-interface system, see Assumptions 1. This way we introduce the interfacial density area

$$(10) \quad a_i = A_i/V,$$

while scaling the area A_i by the volume V of the fluid-interface system.

Now, scaling the extensive variables with respect to the interface area A_i defines the interfacial intensive entropy $s_i = S_i/A_i$ and the interfacial intensive energy $e_i = E_i/A_i$. Scaling the Euler relation (8) with respect to the volume V of the fluid-interface system, one deduces

$$(11) \quad e_i = T_i s_i + \gamma_i,$$

and doing so with the interfacial Gibbs relation (7) gives

$$(12) \quad e'_i(s_i) = T_i,$$

and

$$(13) \quad d(a_i e_i) = T_i d(a_i s_i) + \gamma_i da_i.$$

By the definition (10) of the interfacial density area a_i , observe that the relation (9) gives

$$(14) \quad \gamma'_i(T_i) = -s_i(T_i).$$

This derivative relation is not often mentioned in the literature, but it can be found in [36, 35] for instance.

Example 2 (Katayama-Guggenheim surface tension law). *Any equation of state that satisfies Assumptions 3 can be used in the following sections. As an illustration, we provide an example. The thermodynamic behavior of interfaces is typically described by a surface tension law expressed as a function of temperature. In general, observed surface tension values of liquids decrease as temperature increases [42]. One famous law is given by the Katayama-Guggenheim formula [13]*

$$\gamma_i(T_i) = \gamma_0 \left(1 - \frac{T_i}{T_c} \right)^\kappa,$$

where $\gamma_0 > 0$ is a constant, $T_c > 0$ is a reference temperature (usually the critical temperature) and $\kappa > 1$, parameters for ordinary water substance being available in [17]. Hence using (14) and (12), one deduces that the associated internal energy and temperature read,

$$e_i(s_i) = T_c s_i - \gamma_0 (\kappa - 1) \left(\frac{T_c s_i}{\gamma_0 \kappa} \right)^{\frac{\kappa}{\kappa-1}},$$

$$T_i(s_i) = T_c \left(1 - \left(\frac{T_c s_i}{\gamma_0 \kappa} \right)^{\frac{1}{\kappa-1}} \right).$$

2.3. Thermodynamic equilibrium of the fluid-interface system. The aim is now to describe the thermodynamic behaviour of the fluid-interface system as a whole. Following [36, 35], and considering the fluid-interface system as isolated, it is fully described by an internal energy E , which is a function of (M, V, S) ; see Assumptions 1. The two immiscible fluid phases $k = 1, 2$, with extensive states (M_k, V_k, S_k) , are described by equations of state consistent with Assumptions 2. The interface, with the extensive state (S_i, A_i) , is depicted by an internal energy that satisfies Assumptions 3. The state of the fluid-interface system evolves towards equilibrium due to irreversible processes. This equilibrium state is a time-invariant state in which no further physical or chemical changes occur. At equilibrium, the irreversible processes vanish.

The purpose of this section is to define the internal energy of the fluid-interface system and the extended Gibbs form it satisfies, see Proposition 1. Then, by imposing thermodynamic equilibrium, we derive the algebraic and differential relationships that characterize the system's thermodynamic equilibrium, see Proposition 2.

For a given state (M, V, S, A_i) of the system, different modeling constraints have to be set. As mentioned before, the two phases are supposed to be immiscible and that no vacuum appears, such that the total volume is the sum of the phasic volumes

$$(15) \quad V = V_1 + V_2,$$

since the interface has no volume. As the mass conservation of the system is concerned, it holds

$$(16) \quad M = M_1 + M_2,$$

since the interface has no mass and only mass transfer can occur between the two phases (and not with the interface). Finally the homogeneity property of the system entropy states that

$$(17) \quad S = S_1 + S_2 + S_i.$$

It is convenient to provide the intensive counterpart of these constraints while introducing the fractions of presence of each phase, namely the volume fraction $\alpha_k = V_k/V \in [0, 1]$, the mass fraction $y_k = M_k/M \in [0, 1]$ and the entropy fraction $z_k = S_k/S \in [0, 1]$, such that

$$(18) \quad y_k \tau_k = \alpha_k \tau, \quad y_k s_k = z_k s.$$

Then the intensive counterpart of the extensive constraints (15)-(17) reads

$$(19) \quad \begin{cases} 1 = \alpha_1 + \alpha_2, \\ 1 = y_1 + y_2, \\ 1 = z_1 + z_2 + z_i, \end{cases}$$

where $z_i = S_i/S \in [0, 1]$ stands for the entropy fraction of the interface.

We now turn to the definition of the extensive energy of the whole system. It corresponds to the sum of the energies of each part, namely

$$(20) \quad E(M, V, S, A_i) = E_1(M_1, V_1, S_1) + E_2(M_2, V_2, S_2) + E_i(S_i, A_i).$$

Using the fractions definitions, the total derivative of E reads

$$\begin{aligned} dE &= \sum_{k=1}^2 (T_k dS_k - p_k dV_k + \mu_k dM_k) + T_i dS_i + \gamma_i dA_i \\ &= \sum_{k=1}^2 (T_k z_k dS + S T_k dz_k - p_k \alpha_k dV_k - V p_k d\alpha_k \\ &\quad + y_k \mu_k dM_k + M \mu_k dy_k) \\ &\quad + T_i z_i dS + S T_i dz_i + \gamma_i a_i dV + \gamma_i V da_i. \end{aligned}$$

Reorganizing the terms and using the intensive constraints (19), one obtains

Proposition 1. *The extensive energy satisfies*

$$(21) \quad \begin{aligned} dE &= (z_1 T_1 + z_2 T_2 + z_i T_i) dS - (\alpha_1 p_1 + \alpha_2 p_2 - a_i \gamma_i) dV \\ &\quad + y_1 (\mu_1 - \mu_2) dM \\ &\quad + S ((T_1 - T_i) dz_1 + (T_2 - T_i) dz_2) \\ &\quad - V ((p_1 - p_2) d\alpha_1 - \gamma_i da_i) \\ &\quad + M (\mu_1 - \mu_2) dy_1. \end{aligned}$$

As a consequence, the temperature, pressure and chemical potential of the fluid-interface system have natural definitions in terms of the phasic and interfacial

quantities:

$$(22) \quad \begin{cases} T := z_1 T_1 + z_2 T_2 + z_i T_i, \\ p := \alpha_1 p_1 + \alpha_2 p_2 - a_i \gamma_i, \\ \mu := y_1 \mu_1 + y_2 \mu_2. \end{cases}$$

In absence of the surface tension, the fluid-interface system pressure coincides with the pressure of classic bi-fluid or two-phase models [33]. When accounting for surface tension, the fluid-interface pressure is exactly the one of the two-phase flow model, derived in [30] by homogenization techniques. This pressure also appears in jump conditions of Euler-Korteweg system, see [31]. The pressure we get is also close to the pressure term derived in [8, 7] in the context of two-phase flows with surface tension.

For a given state (M, V, E, A_i) , and according to the second principle of thermodynamics, the thermodynamic equilibrium corresponds to a minimum of the energy E defined in (20) under the extensive constraints (15)-(17). Thus, in the interior of the constraint set, the derivatives of E with respect to independant variables cancel, leading to a characterization of the thermodynamic equilibrium in terms of phasic potentials.

Proposition 2. *According to the differential form (21), the thermodynamic equilibrium is characterized by*

$$(23) \quad \begin{cases} \mu_1 = \mu_2, \\ T_1 = T_2 = T_i, \\ \gamma_i da_i - (p_1 - p_2) d\alpha_1 = 0. \end{cases}$$

The characterization (23) of the thermodynamic equilibrium holds pointwise in the elementary domain. The two first equalities of (23) are classic: they denote the thermal equilibrium in the fluid-interface system and the mass transfer between the two fluid phases. The last (differential) relation represents the mechanical equilibrium and brings out some comments:

- The case $\gamma_i = 0$ corresponds to a situation where there is no interface. Either only the two fluid phases are present, leading to the standard modeling of two-phase systems, see [5, 32, 29, 28, 15], with equilibrium achieved when the phasic pressures are equal, $p_1 = p_2$. Alternatively, the system is in a supercritical configuration: the fluid phases are indistinguishable, and the interface no longer exists; see [5] for a detailed description.
- Assume that the phase 1 occupies a spherical bubble of radius R . Then its volume is $V_1 = 4\pi R^3/3$ and the interfacial area is $A_i = 4\pi R^2$. On the other hand the differential relation in (23) gives

$$\gamma_i d\left(\frac{A_i}{V}\right) - (p_1 - p_2) d\left(\frac{V_1}{V}\right) = 0$$

Expressing this latter formula in terms of the radius R , it leads to the Young-Laplace law

$$p_1 - p_2 = \frac{2\gamma_i}{R}.$$

Typically the Young-Laplace law involves the mean curvature which corresponds here to the inverse of the radius. When the radius tends to $+\infty$,

the surface becomes planar and one recovers the equality of the phasic pressures.

Remark 1. *In the context of two-phase flows in porous media, Smaï proposed in [47] minimizing the free energy of the fluid-interface system instead of minimizing its internal energy. The advantage of this approach is that the free energy of the system (also known as canonical grand potential in the framework of porous media) depends solely on the temperature of the fluid-interface system and the phasic pressures. By using this method, one can recover all information related to the interface without explicitly computing the interfacial area.*

This approach is not developed here, as our goal is specifically to derive an evolution equation for the interfacial area.

2.3.1. *Potential energy candidate.* We now turn to the definition of the potential energy to be used in the Lagrangian formulation. A natural proposition would be to consider the intensive internal energy of the fluid-interface system by scaling the extensive energy (20) with the total mass M . Thus, for a given intensive state (τ, s, a_i) , while accounting for the intensive constraints (19), the intensive energy would be expressed as follows:

$$(24) \quad e(\tau, s, a_i, (y_k)_k, (\alpha_k)_k, (z_k)_k, z_i) = y_1 e_1(\tau_1, s_1) + y_2(\tau_2, s_2) + a_i \tau e_i \left(\frac{z_i}{a} \frac{s}{\tau} \right),$$

with notations (18) of the phasic quantities.

However, it turns out that this choice of variables is not convenient for computations. Following [25, 20], it is more appropriate to express the intensive fluid-interface internal energy as a function of the intensive entropies s and s_k , $k = 1, 2$, rather than using the entropy fractions z_k as in [21].

In the sequel, we choose to express the intensive energy as a function of

$$(25) \quad \tilde{\mathbf{B}} = \{\rho, s, s_1, s_2, a_i, y, \alpha\},$$

where $\rho = 1/\tau$ denotes the system density and $y := y_1$ and $\alpha := \alpha_1$. It reads then

$$(26) \quad e(\tilde{\mathbf{B}}) = y e_1 \left(\frac{\alpha}{y\rho}, s_1 \right) + (1-y) e_2 \left(\frac{1-\alpha}{(1-y)\rho}, s_2 \right) + \frac{a_i}{\rho} e_i \left(\frac{s - y s_1 - (1-y) s_2}{a_i} \rho \right).$$

Observe that one makes use of the extensive relation (17) on the entropies to express the interfacial entropy s_i as a function of s , s_1 and s_2 , namely

$$(27) \quad s = y s_1 + (1-y) s_2 + \frac{a_i}{\rho} s_i.$$

Remark 2. *In [6, paragraph 2.1.3.3], the author highlights the importance of the choice of variables on which the Lagrangian functional depends. This point is also emphasized in the work of Gavriljuk [19]. Indeed, while specific entropies are convenient variables for computations, their conservation along trajectories prevents any interaction between the phases. The fluid-interface entropy will also be conserved, as only reversible processes can be described by the stationary action principle. However, it is possible to introduce relaxation source terms a posteriori, in accordance with the second law of thermodynamics. See [10, Paragraph 3.5] for a presentation*

of this method when dissipation is due to pulsating behaviour of bubbles in two-phase flows.

Example 3. *If one consider that the fluid phases are depicted by stiffened gas laws, see Example 1, and the interface follows a Katayama-Guggenheim law, see Example 2, then the associated intensive energy of the fluid-interface system reads*

$$\begin{aligned}
 e(\tilde{\mathbf{B}}) = & y \left[(\gamma_1 - 1) \left(\frac{\alpha}{\rho y} \right)^{-\gamma_1} \exp \left(\frac{s_1 - s_{0,1}}{c_{v,1}} \right) - \pi_1 \right] \\
 & + (1 - y) \left[(\gamma_2 - 1) \left(\frac{1 - \alpha}{\rho(1 - y)} \right)^{-\gamma_2} \exp \left(\frac{s_2 - s_{0,2}}{c_{v,2}} \right) - \pi_2 \right] \\
 & + \frac{a_i}{\rho} \left[T_c \frac{\rho(s - y s_1 - (1 - y) s_2)}{a_i} \right. \\
 & \quad \left. - \gamma_0 (\kappa - 1) \left(\frac{T_c \rho(s - y s_1 - (1 - y) s_2)}{\gamma_0 \kappa a_i} \right)^{\frac{\kappa}{\kappa - 1}} \right].
 \end{aligned}$$

3. DERIVATION OF THE EVOLUTION EQUATIONS BY MEANS OF STATIONARY ACTION PRINCIPLE

Accounting for the previous characterization of the thermodynamic equilibrium, we now turn to modeling the fluid dynamics. The objective is to derive the Euler-type equations governing the fluid-interface system using Hamilton's principle on least action (or stationary action), following the serie of works [25, 20, 11, 8, 7, 10, 40].

We focus on homogeneous two-phase flows, meaning that the two phases evolve with the same velocity field $\mathbf{u} \in \mathbb{R}^3$. Note that considering distinct velocities for each phase is also possible, as discussed in [20].

The variational approach and the Hamilton's principle of stationary action rely on the definition of an appropriate Lagrangian L . This Lagrangian is the difference of a kinetic energy and a potential energy. The potential energy we consider has been derived in the previous section, see (26). Regarding the kinetic energy, a brief review of recent models is provided in Section 3.1, focusing on the so-called *two-scale kinetic* modeling proposed in [11, 6, 39].

In Section 3.2, we outline the main elements of Hamilton's principle on least action, along with the additional assumptions we make (such as total and partial mass conservation). The resulting equations are then presented in their preliminary form.

3.1. A non-exhaustive review of kinetic energy. Recent references address the derivation of the kinetic energy, building on the initial works of Gavrilyuk and coauthors [25, 20]. In these contributions, the kinetic energy L_{kin} is composed of a classic bulk energy associated with the translational motion of the fluid, and a small-scale contribution. This small-scale kinetic energy is expressed as a quadratic form of the material derivative of the volume fraction, $D_t \alpha$. Here and throughout the following, the material derivative is defined with the common velocity field \mathbf{u}

$$D_t \cdot = \partial_t \cdot + \mathbf{u} \cdot \nabla_{\mathbf{x}} \cdot .$$

In [11, 8, 6], others quadratic forms are considered, depending on α and a_i as well. Recently in [10, 39], the authors propose a reduced-order modeling of a small-scale kinetic equation to derive geometric variables (Gauss and mean curvatures for instance) and an associated small-scale kinetic energy.

Among all the proposed approaches, it is essential to make the kinetic energy L_{kin} depend on $D_t\alpha$; otherwise there will be no way to derive an evolution equation for α . For the same reason, and because we aim to obtain an evolution equation for the interfacial area density, we propose to consider a term involving $D_t a_i$ as well:

$$(28) \quad L_{kin} = \frac{1}{2}\rho|\mathbf{u}|^2 + \frac{m}{2}|D_t\alpha|^2 + \frac{\nu}{2}|D_t a_i|^2,$$

where m and ν are constants (with the appropriate dimensions so that L_{kin} has the right unit, namely $m[kg \cdot m^{-1}]$ and $\nu[kg \cdot m]$). By doing so, we ensure the derivation of an evolution equation for the interfacial area density, without the need to introduce any additional quantities such as local curvature or interface displacement, as in [6, 8]. It is important to note that any positive definite quadratic form of variables $D_t\alpha$ and $D_t a_i$ could serve as a kinetic energy relative to the so-called small-scale contributions. The choice made in equation (28) allows for comparison with models derived in [20, 11, 8, 6, 39].

3.2. The Lagrangian functional and additional assumptions. We introduce the vector of variables \mathbf{B}

$$(29) \quad \mathbf{B} := \{\rho, s, s_1, s_2, a_i, y, \alpha, \mathbf{u}, D_t\alpha, D_t a_i\},$$

which corresponds to the vector $\tilde{\mathbf{B}}$, defined in (25), completed by the variables involved in the kinetic energy L_{kin} , that are \mathbf{u} , $D_t\alpha$ and $D_t a_i$.

The Lagrangian L , function of \mathbf{B} , is the difference between the kinetic and the potential contribution

$$(30) \quad L(\mathbf{B}) = L_{kin} - L_{pot},$$

where $L_{kin}(\mathbf{B})$ is defined in (28) and $L_{pot}(\mathbf{B}) = \rho e(\tilde{\mathbf{B}})$, with $e(\tilde{\mathbf{B}})$ defined in (26).

Before proceeding with the variational method, we make additional assumptions that govern the fluid-interface system. First we assume mass conservation, meaning that

$$(31) \quad \begin{aligned} \partial_t \rho + \operatorname{div}_{\mathbf{x}}(\rho \mathbf{u}) &= 0, \\ D_t y &= 0. \end{aligned}$$

It is important to emphasize that, although the modeling presented in Section 2.3 allows for mass exchange between the two phases, this is not the case here. This arises because the stationary action principle applies only for reversible processes. Similarly, for the same reasons, we assume that the specific entropies are conserved along trajectories

$$(32) \quad D_t s = 0, \quad D_t s_k = 0, \quad k = 1, 2,$$

following [25, 20, 6]. Note that, since the specific phasic entropies are conserved, the interface intensive entropy (which is relative to the interfacial area A_i and not to the mass M) is not conserved along trajectories but satisfies

$$D_t(s_i a_i \tau) = 0,$$

which means that, when expressed in terms of extensive variables, $s_i a_i \tau = S_i/M$ remains constant along trajectories.

3.3. Variational principle. This paragraph recalls the classic lines of the stationary action principle, whose application to the two-phase flow modeling has been the subject of numerous works, including [12, 2, 23, 24, 4]. See also [10] for a synthetic presentation of the method and an overview of the technic in the two-fluid framework.

Consider a volume $\omega(t) \in \mathbb{R}^3$ occupied by the fluid-interface system for time $t \in [t_1, t_2]$ and denote $\Omega = \{(t, \mathbf{x}) \in \times [t_1, t_2] \times \mathbb{R}^d \mid \mathbf{x} \in \omega(t), t_1 \leq t \leq t_2\}$. Following Section 3.2, we assume the flow to be fully characterized by the quantities $(t, \mathbf{x}) \mapsto \mathbf{B}$ and by the constitutive constraints (31)-(32). We now define the Hamiltonian action as the space-time integral of the Lagrangian functional (30)

$$(33) \quad A(\mathbf{B}) = \int_{\Omega} L(\mathbf{B})(\mathbf{x}, t) d\mathbf{x}dt,$$

and apply the stationary action principle. If $(t, \mathbf{x}) \mapsto \bar{\mathbf{B}}$ is a physically relevant transformation of the system, it is the solution of a variational problem leading to a PDE system. The methodology is to consider a family of perturbation $(t, \mathbf{x}, \zeta) \mapsto \mathbf{B}_{\zeta}$ of $\bar{\mathbf{B}}$, parametrized by $\zeta \in [0, 1]$ such that

- the physical path is obtained when $\zeta = 0$:

$$\mathbf{B}_{\zeta}(t, \mathbf{x}, \zeta = 0) = \bar{\mathbf{B}}(t, \mathbf{x}),$$

- \mathbf{B}_{ζ} satisfies the conservation constraints (31) and (32) for all $\zeta \in [0, 1]$,
- $\mathbf{B}_{\zeta}(t, \mathbf{x}, \zeta) = \bar{\mathbf{B}}(t, \mathbf{x})$ for $(t, \mathbf{x}, \zeta) \in \partial\Omega \times [0, 1]$.

The stationary action principle states that $\bar{\mathbf{B}}$ is physically relevant if it is a stationary point of $\zeta \mapsto A(\mathbf{B}_{\zeta})$, that is

$$(34) \quad \frac{dA(\mathbf{B}_{\zeta})}{d\zeta}(0) = 0.$$

This stationary condition leads to the governing set of PDEs of motion for the system without dissipative processes. For $b \in \bar{\mathbf{B}}$, denoting

$$\delta_{\zeta} b(t, \mathbf{x}) = \left(\frac{\partial b_{\zeta}}{\partial \zeta} \right)_{|t, \mathbf{x}}(t, \mathbf{x}, \zeta = 0)$$

a family of infinitesimal transformations, the identity (34) reads

$$(35) \quad \frac{dA(\mathbf{B}_{\zeta})}{d\zeta}(\zeta = 0) = \int_{\Omega} \sum_{b \in \mathbf{B}} \frac{\partial L}{\partial b} \delta_{\zeta} b d\mathbf{x}dt.$$

Infinitesimal variations are related through the conservation principles (31) and (32) (see [18] and [4] for detailed computations)

- variation of density

$$(36) \quad \delta \rho = -\operatorname{div}_{\mathbf{x}}(\rho \delta \mathbf{x}),$$

- variation of velocity

$$(37) \quad \delta \mathbf{u} = D_t(\delta \mathbf{x}) - \nabla_{\mathbf{x}} \mathbf{u} \cdot \delta \mathbf{x},$$

- conservation along trajectories of the fluid specific entropies and the mass fraction

$$(38) \quad \delta b = -\nabla_{\mathbf{x}} b \cdot \delta \mathbf{x}, \quad \text{for } b \in \{s, y, z_1, z_2\}.$$

We now list all the contributions in (35).

- Density contribution: using the mass conservation (31), one has

$$(39) \quad \begin{aligned} \int_{\Omega} \frac{\partial L}{\partial \rho} \delta \rho \, d\mathbf{x} \, dt &= - \int_{\Omega} \frac{\partial L}{\partial \rho} \operatorname{div}_{\mathbf{x}}(\rho \delta \mathbf{x}) \, d\mathbf{x} \, dt \\ &= \int_{\Omega} \rho \nabla_{\mathbf{x}} \left(\frac{\partial L}{\partial \rho} \right) \cdot \delta \mathbf{x} \, d\mathbf{x} \, dt, \end{aligned}$$

by integration by parts. In order to make the partial Legendre transform of L with respect to ρ (written here as a function of \mathbf{B})

$$(40) \quad L^{*,\rho}(\mathbf{B}) = \rho \frac{\partial L}{\partial \rho} - L(\mathbf{B})$$

appear, one develops

$$(41) \quad \begin{aligned} &\int_{\Omega} \frac{\partial L}{\partial \rho} \delta \rho \, d\mathbf{x} \, dt \\ &= \int_{\Omega} \left[\nabla_{\mathbf{x}} \left(\rho \frac{\partial L}{\partial \rho} \right) - \frac{\partial L}{\partial \rho} \nabla_{\mathbf{x}} \rho \right] \cdot \delta \mathbf{x} \, d\mathbf{x} \, dt \\ &= \int_{\Omega} \left[\nabla_{\mathbf{x}} \left(\rho \frac{\partial L}{\partial \rho} - L \right) + \nabla_{\mathbf{x}} L - \frac{\partial L}{\partial \rho} \nabla_{\mathbf{x}} \rho \right] \cdot \delta \mathbf{x} \, d\mathbf{x} \, dt \\ &= \int_{\Omega} \left(\nabla_{\mathbf{x}} L^{*,\rho} + \sum_{\substack{b \in \mathbf{B} \\ b \neq \rho}} \nabla_{\mathbf{x}} b \frac{\partial L}{\partial b} \right) \cdot \delta \mathbf{x} \, d\mathbf{x} \, dt. \end{aligned}$$

- Velocity contribution: according to (37), it holds

$$(42) \quad \int_{\Omega} \frac{\partial L}{\partial \mathbf{u}} \delta \mathbf{u} \, d\mathbf{x} \, dt = \int_{\Omega} \frac{\partial L}{\partial \mathbf{u}} (\mathbf{D}_t(\delta \mathbf{x}) - \nabla_{\mathbf{x}} \mathbf{u} \cdot \delta \mathbf{x}) \, d\mathbf{x} \, dt.$$

By definition of the material derivative $\mathbf{D}_t \cdot$, and using an integration by part, it holds

$$(43) \quad \begin{aligned} &\int_{\Omega} \frac{\partial L}{\partial \mathbf{u}} \delta \mathbf{u} \, d\mathbf{x} \, dt \\ &= \int_{\Omega} \frac{\partial L}{\partial \mathbf{u}} [\partial_t(\delta \mathbf{x}) + \mathbf{u} \cdot \nabla_{\mathbf{x}}(\delta \mathbf{x}) - \nabla_{\mathbf{x}} \mathbf{u} \cdot \delta \mathbf{x}] \, d\mathbf{x} \, dt \\ &= - \int_{\Omega} \left(\partial_t \left(\frac{\partial L}{\partial \mathbf{u}} \right) + \operatorname{div}_{\mathbf{x}} \left(\mathbf{u} \frac{\partial L}{\partial \mathbf{u}} \right) + \frac{\partial L}{\partial \mathbf{u}} \nabla_{\mathbf{x}} \mathbf{u} \right) \cdot \delta \mathbf{x} \, d\mathbf{x} \, dt. \end{aligned}$$

- Contributions of conserved quantities along trajectories: using (38), it holds for $b \in \{s, s_1, s_2, y\}$

$$(44) \quad \int_{\Omega} \frac{\partial L}{\partial b} \delta b \, d\mathbf{x} \, dt = - \int_{\Omega} \frac{\partial L}{\partial b} \nabla_{\mathbf{x}} b \cdot \delta \mathbf{x} \, d\mathbf{x} \, dt.$$

- Contributions in α and a_i : the variation of the volume fraction α is not subject to any constraint. This ensures the derivation of an evolution equation for α . Therefore, the variation $\delta \alpha$, associated with the family of transformations of the medium, is arbitrary. The same applies to the interfacial area density a_i . Furthermore, the fact that these quantities evolve independently will lead to separate equations for the volume fraction and the interfacial area density.

- Contributions in $D_t\alpha$ and $D_t a_i$: the variations of $D_t\alpha$ (resp. $D_t a_i$) is related to the variation of α (resp. a_i). According to [34], it holds, for any functions f and g ,

$$(45) \quad \int_{\Omega} g \delta(D_t f) \, d\mathbf{x} dt = - \int_{\Omega} (\partial_t g + \operatorname{div}_{\mathbf{x}}(\mathbf{u}g)) \delta f \, d\mathbf{x} dt \\ - \int_{\Omega} [(\partial_t g + \operatorname{div}_{\mathbf{x}}(\mathbf{u}g)) \nabla_{\mathbf{x}} f + g \nabla_{\mathbf{x}}(D_t f)] \cdot \delta \mathbf{x} \, d\mathbf{x} dt.$$

Thus using (45) with $g = \frac{\partial L}{\partial(D_t\alpha)} =: M$ and $f = D_t\alpha$ gives

$$(46) \quad \int_{\Omega} \frac{\partial L}{\partial(D_t\alpha)} \delta(D_t\alpha) \, d\mathbf{x} dt \\ = - \int_{\Omega} (\partial_t M + \operatorname{div}_{\mathbf{x}}(M\mathbf{u})) \delta\alpha \, d\mathbf{x} dt \\ - \int_{\Omega} ((\partial_t M + \operatorname{div}_{\mathbf{x}}(M\mathbf{u})) \nabla_{\mathbf{x}}\alpha + M \nabla_{\mathbf{x}}(D_t\alpha)) \cdot \delta \mathbf{x} \, d\mathbf{x} dt.$$

Analogously it holds with $g = \frac{\partial L}{\partial(D_t a_i)} =: P$ and $f = D_t a_i$

$$(47) \quad \int_{\Omega} \frac{\partial L}{\partial(D_t a_i)} \delta(D_t a_i) \, d\mathbf{x} dt \\ = - \int_{\Omega} (\partial_t P + \operatorname{div}_{\mathbf{x}}(P\mathbf{u})) \delta a_i \, d\mathbf{x} dt \\ - \int_{\Omega} ((\partial_t P + \operatorname{div}_{\mathbf{x}}(P\mathbf{u})) \nabla_{\mathbf{x}} a_i + P \nabla_{\mathbf{x}}(D_t a_i)) \cdot \delta \mathbf{x} \, d\mathbf{x} dt.$$

Finally gathering (41), (43), (46) and (47) gives

$$\int_{\Omega} [A_{\alpha} \delta\alpha + A_{a_i} \delta a_i + A_{\mathbf{u}} \delta \mathbf{x}] \, d\mathbf{x} dt = 0.$$

where

$$(48) \quad \begin{cases} A_{\alpha} = \partial_t M + \operatorname{div}_{\mathbf{x}}(M\mathbf{u}) - \frac{\partial L}{\partial\alpha}, & \text{with } M = \frac{\partial L}{\partial(D_t\alpha)}, \\ A_{a_i} = \partial_t P + \operatorname{div}_{\mathbf{x}}(P\mathbf{u}) - \frac{\partial L}{\partial a_i}, & \text{with } P = \frac{\partial L}{\partial(D_t a_i)}, \\ A_{\mathbf{u}} = \partial_t K + \operatorname{div}_{\mathbf{x}}(K\mathbf{u}) - \nabla_{\mathbf{x}} L^{*,\rho}, & \text{with } K = \frac{\partial L}{\partial \mathbf{u}}. \end{cases}$$

Note that to express the term $A_{\mathbf{u}}$, one makes use of the terms A_{α} and A_{a_i} .

Since it is assumed that the infinitesimal displacement and the variations of volume fraction and interfacial area density, are independent, the stationary action principle applied to the Lagrangian energy L yields the equations of motion given by

$$A_{\alpha} = 0, \quad A_{a_i} = 0, \quad A_{\mathbf{u}} = 0.$$

4. FINAL SYSTEM AND PROPERTIES

As a result of the Hamilton's principle on least action, the following set of equations is obtained, describing the time evolution of the fluid-interface system governed by the Lagrangian L . It reads

$$(49) \quad \begin{cases} \partial_t M + \operatorname{div}_{\mathbf{x}}(M\mathbf{u}) - \frac{\partial L}{\partial\alpha} = 0, \\ \partial_t P + \operatorname{div}_{\mathbf{x}}(P\mathbf{u}) - \frac{\partial L}{\partial a_i} = 0, \\ \partial_t K + \operatorname{div}_{\mathbf{x}}(K\mathbf{u}) - \nabla_{\mathbf{x}} L^{*,\rho} = 0, \end{cases}$$

where $L^{*,\rho}$ is the partial Legendre transform of L defined in (40). The system is completed by the mass conservation laws (31) and the entropies evolution equations (32). Noether's theorem states that the governing equations for M , P and K correspond to the conservation of total momentum and total energy, arising from the invariance of the Lagrangian under space and time shifts, as discussed in [4]. The quantity K refers to the bulk momentum, while the equations governing M and P describe the evolution of the small scales within the fluid-interface system.

Let \mathcal{E} be the partial Legendre transform of the Lagrangian L with respect to the kinetic variables \mathbf{u} , $D_t\alpha$ and $D_t a_i$. It reads

$$(50) \quad \mathcal{E}(\rho, K, M, P, \alpha, a_i, s, s_1, s_2, y) = \mathbf{u}K + D_t\alpha M + D_t a_i P - L(\mathbf{B})$$

or analogously

$$(51) \quad \mathcal{E}(\mathbf{B}) = L_{kin}(\mathbf{B}) + L_{pot}(\mathbf{B}),$$

with notations (30). If the latter formula is more classic, the definition (50) has the advantage of simplifying the following computations.

Proposition 3 (Hyperbolicity). *The energy \mathcal{E} , defined by (50), satisfies the additional scalar conservation equation*

$$(52) \quad \partial_t \mathcal{E} + \operatorname{div}_{\mathbf{x}}((\mathcal{E} - L^{*,\rho})\mathbf{u}) = 0.$$

If the energy $\mathcal{E}(\rho, K, M, P, \alpha, a_i, s, s_1, s_2, y)$ is convex, then the system (31)-(32)-(49) is hyperbolic and it is symmetrizable.

Proof. Using that \mathcal{E} is the partial Legendre transform of the Lagrangian L with respect to the kinetic variables, it holds (dropping the dependency of \mathcal{E} and L for readability)

$$\begin{aligned} D_t \mathcal{E} &= D_t \left(\sum_{b \in \{\mathbf{u}, D_t \alpha, D_t a_i\}} b \frac{\partial L}{\partial b} - L \right) \\ &= \sum_{b \in \{\mathbf{u}, D_t \alpha, D_t a_i\}} \left(D_t b \frac{\partial L}{\partial b} + b D_t \left(\frac{\partial L}{\partial b} \right) \right) - D_t L. \end{aligned}$$

Using the notations K, M and P , given in (48), and the transport of the specific entropies (32) and of the mass fraction (31), it holds

$$D_t \mathcal{E} = \mathbf{u} D_t K + D_t \alpha D_t M + D_t a_i D_t P - \frac{\partial L}{\partial \rho} D_t \rho - \frac{\partial L}{\partial \alpha} D_t \alpha - \frac{\partial L}{\partial a_i} D_t a_i.$$

Then using the evolution equations (49), it yields

$$\begin{aligned} D_t \mathcal{E} &= \mathbf{u} \cdot (-K \operatorname{div}_{\mathbf{x}}(\mathbf{u}) + \nabla_{\mathbf{x}} L^{*,\rho}) + D_t \alpha \left(-M \operatorname{div}_{\mathbf{x}}(\mathbf{u}) + \frac{\partial L}{\partial \alpha} \right) \\ &\quad + D_t a_i \left(-P \operatorname{div}_{\mathbf{x}}(\mathbf{u}) + \frac{\partial L}{\partial a_i} \right) - \frac{\partial L}{\partial \rho} D_t \rho - \frac{\partial L}{\partial \alpha} D_t \alpha - \frac{\partial L}{\partial a_i} D_t a_i \\ &= -\operatorname{div}_{\mathbf{x}}(\mathbf{u}) \left(K \mathbf{u} + M D_t \alpha + P D_t a_i - \rho \frac{\partial L}{\partial \rho} \right) + \mathbf{u} \cdot \nabla_{\mathbf{x}} \left(\rho \frac{\partial L}{\partial \rho} - L \right). \end{aligned}$$

Using the definition (50) of \mathcal{E} , it gives

$$\begin{aligned} D_t \mathcal{E} &= -\operatorname{div}_{\mathbf{x}}(\mathbf{u}) \left(\mathcal{E} + L - \rho \frac{\partial L}{\partial \rho} \right) + \mathbf{u} \cdot \nabla_{\mathbf{x}} L^{*,\rho} \\ &= -\operatorname{div}_{\mathbf{x}}(\mathbf{u}) (\mathcal{E} - \nabla_{\mathbf{x}} L^{*,\rho}) + \mathbf{u} \cdot \nabla_{\mathbf{x}} L^{*,\rho}, \end{aligned}$$

which coincides with (52). Now if \mathcal{E} is supposed to be convex with respect to the variables $(\rho, K, M, P, \alpha, a_i, s, s_1, s_2, y)$, then it is a Lax entropy of the system which can be symmetrized in the sense of Godunov-Mock. \square

The convexity constraint on the E is quite restrictive since it is not clear the \mathcal{E} is strictly convex, since $L_{pot}(\mathbf{B}) = \rho e(\tilde{\mathbf{B}})$ is not necessarily strictly convex.

Example 4. *Following on the Example 3, if one considers that the fluid phases follow stiffened gas laws (see Example 1) and the interface a Katayama-Guggenheim-type equation (see Example 2), then the energy $\mathcal{E}(\mathbf{B})$ is given by equation (51) with the kinetic contribution L_{kin} given in (28) and $L_{pot} = \rho e(\tilde{\mathbf{B}})$ with $e(\tilde{\mathbf{B}})$ given in (26) and in Example 3. It reads*

$$\begin{aligned} \mathcal{E}(\mathbf{B}) = & \frac{1}{2}\rho|\mathbf{u}|^2 + \frac{m}{2}|D_t\alpha|^2 + \frac{\nu}{2}|D_t a_i|^2 \\ & + \rho y \left[(\gamma_1 - 1) \left(\frac{\alpha}{\rho y} \right)^{-\gamma_1} \exp\left(\frac{s_1 - s_{0,1}}{c_{v,1}}\right) - \pi_1 \right] \\ & + \rho(1 - y) \left[(\gamma_2 - 1) \left(\frac{1 - \alpha}{\rho(1 - y)} \right)^{-\gamma_2} \exp\left(\frac{s_2 - s_{0,2}}{c_{v,2}}\right) - \pi_2 \right] \\ & + a_i \left[T_c \frac{\rho(s - y s_1 - (1 - y)s_2)}{a_i} \right. \\ & \quad \left. - \gamma_0(\kappa - 1) \left(\frac{T_c}{\gamma_0 \kappa} \frac{\rho(s - y s_1 - (1 - y)s_2)}{a_i} \right)^{\frac{\kappa}{\kappa - 1}} \right]. \end{aligned}$$

4.1. Extended final set. In this paragraph, we present the final set of equations using the Lagrangian functional in equation (30).

4.1.1. Momentum equation. By the definition (40), the Legendre transform of L with respect to the density $L^{*\rho}$ is

$$(53) \quad L^{*\rho}(\mathbf{B}) = - \left(\frac{m}{2}|D_t\alpha|^2 + \frac{\nu}{2}|D_t a_i|^2 + p \right),$$

where

$$(54) \quad p = \alpha p_1 + (1 - \alpha)p_2 - a_i \gamma_i,$$

is the fluid-interface pressure derived first in (22). Here one uses $p_1 := p_1\left(\frac{\alpha}{y\rho}, s_1\right)$ and $p_2 := p_2\left(\frac{1-\alpha}{(1-y)\rho}, s_2\right)$. Then the equation for $K = \partial L / \partial \mathbf{u} = \rho \mathbf{u}$ gives the momentum equation, namely

$$\partial_t(\rho \mathbf{u}) + \operatorname{div}_{\mathbf{x}}(\rho \mathbf{u}^\top \mathbf{u}) + \nabla_{\mathbf{x}} \left(p + \frac{m}{2}|D_t\alpha|^2 + \frac{\nu}{2}|D_t a_i|^2 \right) = 0.$$

This equation is similar to the one obtained in [11] or [8], except that, in this latter reference, the pressure term includes $\nabla_{\mathbf{x}}\alpha$. When the small-scale terms $D_t\alpha$ and $D_t a_i$ are neglected, the equation flux reduces to the momentum flux derived in [30] for bubbly flows using an homogenization approach. The pressure term p arises from the potential energy L_{pot} which defines the pressure in the momentum equation.

4.1.2. *Evolution equations on α and a_i .* Since $M = mD_t\alpha$ and $P = \nu D_t a_i$, the equations on M and P involve second order derivatives in time on α and a_i respectively. Using the definition (30) of L , and relations (11)-(12), direct computations give

$$(55) \quad \frac{\partial L}{\partial \alpha} = p_1 - p_2, \quad \frac{\partial L}{\partial a_i} = \gamma_i,$$

which lead to

$$(56) \quad \begin{cases} \partial_t(D_t\alpha) + \operatorname{div}_{\mathbf{x}}(\mathbf{u}D_t\alpha) = \frac{p_1 - p_2}{m}, \\ \partial_t(D_t a_i) + \operatorname{div}_{\mathbf{x}}(\mathbf{u}D_t a_i) = \frac{\gamma_i}{\nu}. \end{cases}$$

Following [11, 8], the approach is to decompose these second-order equations into a pair of two first-order time derivative equations by introducing additional unknowns.

Concerning the equation for $M = mD_t\alpha$, we fix

$$(57) \quad D_t\alpha = \frac{\rho y w}{\sqrt{m}},$$

where w is a new unknown. Then it holds

$$(58) \quad \begin{cases} \partial_t\alpha + \mathbf{u} \cdot \nabla_{\mathbf{x}}\alpha = \frac{\rho y w}{\sqrt{m}}, \\ \partial_t w + \mathbf{u} \cdot \nabla_{\mathbf{x}} w = \frac{1}{\sqrt{m\rho y}}(p_2 - p_1). \end{cases}$$

Doing so for the equation for $P = \nu D_t a_i$, we introduce the unknown n , which satisfies

$$(59) \quad D_t a_i = \frac{\rho y n}{\sqrt{\nu}},$$

and it yields

$$(60) \quad \begin{cases} \partial_t a_i + \mathbf{u} \cdot \nabla_{\mathbf{x}} a_i = \frac{\rho y n}{\sqrt{\nu}}, \\ \partial_t n + \mathbf{u} \cdot \nabla_{\mathbf{x}} n = \frac{\gamma_i}{\sqrt{\nu\rho y}}. \end{cases}$$

According to [8, 6, 10], the equations for the quantities w and n , as defined here, correspond to small-scale momentum equations. In this context, the equations for α and a_i link the small scales and the large scales.

4.1.3. *Energy equations.* The transport equations for the specific entropies are not practical, particularly for numerical computations. Therefore, we replace them with energy equations, using the Gibbs relations given in Section 2.

The total energy equation $\mathcal{E} = L_{kin} + L_{pot}$ has already been given in (52), see Proposition 3, and its developed form reads

$$(61) \quad \partial_t \mathcal{E} + \operatorname{div}_{\mathbf{x}}((\mathcal{E} + p)\mathbf{u}) = 0,$$

where p refers to the fluid-interface pressure (54).

For sake of completeness, we provide the phasic (nonconservative) internal energy equations which read, for $k = 1, 2$,

$$(62) \quad \begin{aligned} \partial_t \left(\alpha_k \rho_k \left(e_k + \frac{|\mathbf{u}|^2}{2} \right) \right) + \operatorname{div}_{\mathbf{x}} \left(\left(\alpha_k \rho_k \left(e_k + \frac{|\mathbf{u}|^2}{2} \right) + \alpha_k p_k \right) \mathbf{u} \right) \\ = \alpha_k p_k \operatorname{div}_{\mathbf{x}} \mathbf{u} - y_k \mathbf{u} \cdot \nabla_{\mathbf{x}} \tilde{p} - p_k \frac{\rho y w}{\sqrt{m}}, \end{aligned}$$

where $\tilde{p} = p + \frac{m}{2}|D_t\alpha|^2 + \frac{\nu}{2}|D_t a_i|^2$.

Now using the transport equations of the specific entropies (32) and the mass conservation equations (31), it can be deduced that the interfacial entropy satisfies

$$(63) \quad \partial_t(a_i s_i) + \operatorname{div}_{\mathbf{x}}(a_i s_i \mathbf{u}) = 0.$$

Then combining (13) and (11) leads to the following interfacial energy evolution equation

$$(64) \quad \partial_t(a_i e_i) + \operatorname{div}_{\mathbf{x}}(a_i e_i \mathbf{u}) - a_i \gamma_i \nabla_{\mathbf{x}} \cdot \mathbf{u} = \gamma_i \frac{\rho y n}{\sqrt{\nu}}.$$

4.1.4. *Summary.* Using the definitions (57) and (59) the final set of equations reads

$$(65) \quad \begin{cases} \partial_t \rho + \operatorname{div}_{\mathbf{x}}(\rho \mathbf{u}) = 0, \\ \partial_t(\rho y) + \operatorname{div}_{\mathbf{x}}(\rho y \mathbf{u}) = 0, \\ \partial_t(\rho \mathbf{u}) + \operatorname{div}_{\mathbf{x}}(\rho \mathbf{u}^\top \mathbf{u} + (p + \frac{m}{2}(\rho y w)^2 + \frac{\nu}{2}(\rho y n)^2) \mathbf{Id}) = 0, \\ \partial_t \alpha + \mathbf{u} \cdot \nabla_{\mathbf{x}} \alpha = \frac{\rho y w}{\sqrt{m}}, \\ \partial_t a_i + \mathbf{u} \cdot \nabla_{\mathbf{x}} a_i = \frac{\rho y n}{\sqrt{\nu}}, \\ \partial_t w + \mathbf{u} \cdot \nabla_{\mathbf{x}} w = \frac{1}{\sqrt{m \rho y}}(p_1 - p_2), \\ \partial_t n + \mathbf{u} \cdot \nabla_{\mathbf{x}} n = \frac{\gamma_i}{\sqrt{\nu \rho y}}, \\ D_t s = D_t s_1 = D_t s_2 = 0. \end{cases}$$

4.2. **Hyperbolicity.** To finish we investigate the eigenstructure of the system (65), focusing on its one-dimensional version (with velocity u).

For that purpose, let consider the vector $\hat{\mathbf{B}} = (y, \alpha, a_i, w, n, s, s_1, s_2)^\top \in \mathbb{R}^8$, $(\rho, u, \hat{\mathbf{B}}) \in \mathbb{R}^+ \times \mathbb{R} \times \mathbb{R}^8$ and write the system (65) in the following quasilinear form

$$(66) \quad \partial_t \begin{pmatrix} \rho \\ u \\ \hat{\mathbf{B}} \end{pmatrix} + \mathbf{C}(\rho, u, \hat{\mathbf{B}}) \partial_x \begin{pmatrix} \rho \\ u \\ \hat{\mathbf{B}} \end{pmatrix} = \mathbf{R},$$

where $\mathbf{R} = (0, 0, 0, \frac{\rho y w}{\sqrt{m}}, \frac{\rho y n}{\sqrt{\nu}}, \frac{1}{\sqrt{m \rho y}}(p_1 - p_2), \frac{\gamma_i}{\sqrt{\nu \rho y}}, 0, 0, 0)^\top \in \mathbb{R}^{10}$, and the matrix \mathbf{C} is given by

$$(67) \quad \mathbf{C}(\rho, u, \hat{\mathbf{B}}) = \begin{pmatrix} u & \rho & \mathbf{0}_{1 \times 8} \\ \frac{\hat{p}}{\partial \rho} & u & \frac{1}{\rho} \nabla_{\hat{\mathbf{B}}} \hat{p} \\ \mathbf{0}_{8 \times 1} & \mathbf{0}_{8 \times 1} & u \mathbf{I}_{8 \times 8} \end{pmatrix},$$

with

$$\begin{aligned} \hat{p}(\hat{\mathbf{B}}) &= \alpha p_1 \left(\frac{\alpha}{y \rho}, s_1 \right) + (1 - \alpha) p_2 \left(\frac{1 - \alpha}{(1 - y) \rho}, s_2 \right) - a_i \gamma_i \\ &\quad + \frac{m}{2}(\rho y w)^2 + \frac{\nu}{2}(\rho y n)^2. \end{aligned}$$

The eigenvalues of \mathbf{C} are

$$(68) \quad \lambda_{1,2} = u \pm \rho \sqrt{y c_1^2 + (1 - y) c_2^2 + m(y w)^2 + \nu(y n)^2}, \quad \lambda_{3,\dots,10} = u,$$

where $c_k^2 = \frac{\partial p_k}{\partial \rho_k}(\rho_k, s_k)$ is the speed of sound of the phase $k = 1, 2$. The associated right eigenvectors are the following

- Eigenvectors associated to the multiple eigenvalue $\lambda_{3,\dots,10} = u$:

$$\begin{aligned} & \left(-\frac{1}{\rho} \frac{\partial \hat{p}}{\partial y} \left(\frac{\partial \hat{p}}{\partial \rho} \right)^{-1}, 0, 1, 0, 0, 0, 0, 0, 0, 0 \right)^\top, \\ & \left(-\frac{1}{\rho} \frac{\partial \hat{p}}{\partial \alpha} \left(\frac{\partial \hat{p}}{\partial \rho} \right)^{-1}, 0, 0, 1, 0, 0, 0, 0, 0, 0 \right)^\top, \\ & \left(-\frac{1}{\rho} \frac{\partial \hat{p}}{\partial a_i} \left(\frac{\partial \hat{p}}{\partial \rho} \right)^{-1}, 0, 0, 0, 1, 0, 0, 0, 0, 0 \right)^\top, \\ & \left(-\frac{1}{\rho} \frac{\partial \hat{p}}{\partial w} \left(\frac{\partial \hat{p}}{\partial \rho} \right)^{-1}, 0, 0, 0, 0, 1, 0, 0, 0, 0 \right)^\top, \\ & \left(-\frac{1}{\rho} \frac{\partial \hat{p}}{\partial n} \left(\frac{\partial \hat{p}}{\partial \rho} \right)^{-1}, 0, 0, 0, 0, 0, 1, 0, 0, 0 \right)^\top, \\ & \left(-\frac{1}{\rho} \frac{\partial \hat{p}}{\partial s} \left(\frac{\partial \hat{p}}{\partial \rho} \right)^{-1}, 0, 0, 0, 0, 0, 0, 1, 0, 0 \right)^\top, \\ & \left(-\frac{1}{\rho} \frac{\partial \hat{p}}{\partial s_1} \left(\frac{\partial \hat{p}}{\partial \rho} \right)^{-1}, 0, 0, 0, 0, 0, 0, 0, 1, 0 \right)^\top, \\ & \left(-\frac{1}{\rho} \frac{\partial \hat{p}}{\partial s_2} \left(\frac{\partial \hat{p}}{\partial \rho} \right)^{-1}, 0, 0, 0, 0, 0, 0, 0, 0, 1 \right)^\top, \end{aligned}$$

- Eigenvector associated to λ_1 :

$$\left(\sqrt{\rho} \left(\frac{\partial \hat{p}}{\partial \rho} \right)^{-1}, 1, 0, 0, 0, 0, 0, 0, 0, 0 \right)^\top,$$

- Eigenvector associated to λ_2 :

$$\left(-\sqrt{\rho} \left(\frac{\partial \hat{p}}{\partial \rho} \right)^{-1}, 1, 0, 0, 0, 0, 0, 0, 0, 0 \right)^\top.$$

All the eigenvalues are real and the right eigenvectors of \mathbf{C} constitute a basis of \mathbb{R}^{10} . This proves again the hyperbolicity of the system.

As mentioned in Remark 2, the stationary action principle describes only reversible processes. Therefore, relaxation effects must be introduced *a posteriori* in accordance with the second principle. For example, one may use source terms, as presented in [20] or [10], to model damping due to bubble pulsation. Alternatively, dissipative phase transition source terms can be designed using the framework in Section 2.3, ensuring consistency with the thermodynamic equilibrium outlined in Proposition 2.

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